

Quaternionic versus Maxwell based differential calculus

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Abstract

Two quite different forms of differential calculus exist that both have physical significance. The most simple version is quaternionic differential calculus. Maxwell based differential calculus is based on the equations that Maxwell and others have developed in order to describe electromagnetic phenomena. Both approaches can be represented by four-component “fields” and four-component differential operators. Both approaches result in a dedicated non-homogeneous wave equation. These wave equations differ and offer solutions that differ in details.

Maxwell based differential calculus uses coordinate time t , where quaternionic differential calculus uses proper time τ . The consequence is that also the interpretation of speed differs between the two approaches. A more intriguing fact is that these differences involve a different space-progression model and different charges and currents. The impact of these differences are not treated in this paper.

By adding an extra Maxwell based differential equation the conformance between the two approaches increases significantly.

The formulation of physics in Maxwell based differential calculus differs significantly from the formulation of physics in quaternionic differential calculus. It results in a different space-progression model. The choice between the two approaches influences the description of physical reality. It cannot not influence physical reality.

Quaternionic differential calculus fits better with the application of Hilbert spaces in quantum physics than Maxwell based differential calculus. However, Maxwell based differential calculus is the general trend in current physical theories.



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1 Introduction

In this paper the quaternionic differential equations are compared to Maxwell based differential equations [1][2].

In order to ease the comparison of the two approaches, we apply four-component “fields” and four-component operators. The parameter space is represented in a similar way by a similar but flat four-component “field”.

We start with a four-component “field” φ that is differentiable and we also define the corresponding four-component differential operator ∇ . The four-component approach is sometimes implemented with the help of spinors and corresponding matrices. Here we could, but will not apply that methodology. The method confuses more than that it elucidates the situation. Instead, we consider the scalar part as a separate part and we apply base vectors $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ rather than the corresponding Pauli matrices [3].

Both approaches start with a basic “field” φ . Gravitation concerns applications where this “field” φ is always and everywhere present. This kind of field is suited as continuum for embedding discrete objects. It is also suited as long range transport medium for carriers of information and energy. Electromagnetic theory concerns applications where the existence of the “field” φ is determined by a set of charges in the form of nearby point-like sources or nearby point-like drains. These two kinds of basic fields are related, but that is subject of another paper [4].

Double differentiation results in a non-homogeneous wave equation that reveals how the basic “field” φ can be deformed or vibrated and how sources and drains control its existence.

2 Notation

Italic font face without subscript indicates four-component “fields” or four-component operators. Bold italic font face indicates 3D vectors or 3D operators.

In order to support the comparison, we will apply four-component “fields”. We will use corresponding four-component differential operators and indicate these with symbol ∇ .

The four-component “fields” consist of a combination of a scalar field and a field of 3D vectors.

$$\varphi = \{\varphi_0, \varphi_1, \varphi_2, \varphi_3\} = \{\varphi_0, \boldsymbol{\varphi}\} = \{\varphi_0, \mathbf{i} \varphi_1 + \mathbf{j} \varphi_2 + \mathbf{k} \varphi_3\} \quad (1)$$

Both approaches start with a basic “field” φ that is always and everywhere present or this “field” is generated by a set of local sources or drains. All related “fields” are supposed to be derived from the basic “field”.

Both approaches use the 3D nabla operator ∇ .

$$\nabla = \left\{ \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\} = \mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z} \quad (2)$$

This vector operator can be used when the deformations of the subjected fields are not too violent.

$$\nabla = \{\nabla_0, \nabla_1, \nabla_2, \nabla_3\} = \{\nabla_0, \nabla\} \quad (3)$$

For example the four-component “field” ϕ is defined as:

$$\phi = \{\varphi_0, \boldsymbol{\varphi}\} = \nabla\varphi = \{\nabla\varphi_0, \nabla\boldsymbol{\varphi}\} \quad (4)$$

$$\nabla = \{\nabla_0, \nabla_1, \nabla_2, \nabla_3\} = \{\nabla_0, \nabla\} \quad (5)$$

The four-component differential operator differs between the two approaches. Quaternionic differential calculus uses proper time τ and partial derivative $\nabla_0 = \nabla_\tau = \frac{\partial}{\partial\tau}$ and Maxwell based differential calculus uses coordinate time t and partial derivative $\nabla_t = \frac{\partial}{\partial t}$.

We suppose that ∇_0 commutes with ∇ .

3 Parameter spaces

The parameter space is represented by a four-component flat “field”:

$$\{x_0, \mathbf{i}x_1 + \mathbf{j}x_2 + \mathbf{k}x_3\} = \{x_0, \mathbf{x} + \mathbf{y} + \mathbf{z}\}; x_0 = \tau \text{ or } x_0 = t \quad (1)$$

Infinitesimal coordinate time steps and infinitesimal proper time steps are related by:

$$\text{Coordinate time step vector} = \text{proper time step vector} + \text{spatial step vector} \quad (2)$$

Or in Pythagoras format:

$$(\Delta t)^2 = (\Delta\tau)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 \quad (3)$$

In the quaternionic model the formula indicates that the coordinate time step corresponds to the step of a full quaternion, which is a superposition of a proper time step and a pure spatial step.

An infinitesimal spacetime step Δs is usually presented as an infinitesimal proper time step $\Delta\tau$.

$$(\Delta s)^2 = (\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2 \quad (4)$$

The signs on the right side form the Minkowski signature $(+, -, -, -)$.

The quaternionic model offers a Euclidean signature $(+, +, +, +)$ as is shown in formula (3).

4 Definition of the differential

Locally, the deformation of the “field” φ is supposed to be sufficiently moderate, such that the nabla operator ∇ can be applied.

In the two approaches, the differentiations $\phi = \nabla\varphi$ have different definitions.

We do not go further than double differentiation. This double differentiation results in a non-homogeneous wave equation. In the two approaches, the non-homogeneous wave equations have a different format.

5 Mathematical facts

Here α is a real or complex valued scalar function. \mathbf{a} is a vector function.

The following formulas are just mathematical facts that generally hold for vector differential calculus:

$$\langle \nabla, \nabla \mathbf{a} \rangle \equiv \langle \nabla, \nabla \rangle \mathbf{a} \quad (1)$$

$$\langle \nabla, \nabla \alpha \rangle \equiv \langle \nabla, \nabla \rangle \alpha \quad (2)$$

$$\nabla \times \nabla \alpha = \mathbf{0} \quad (3)$$

$$\langle \nabla, \nabla \times \mathbf{a} \rangle = 0 \quad (4)$$

$$\langle \nabla \times \nabla, \mathbf{a} \rangle = \mathbf{0} \quad (5)$$

$$\nabla \times (\nabla \times \mathbf{a}) = \nabla \langle \nabla, \mathbf{a} \rangle - \langle \nabla, \nabla \rangle \mathbf{a} \quad (6)$$

$$\nabla \times (\nabla \times \nabla \alpha) = \nabla \langle \nabla, \nabla \alpha \rangle - \langle \nabla, \nabla \rangle \nabla \alpha \quad (7)$$

6 Quaternionic differential calculus

Quaternions are a combination of a real scalar a_0 and a 3D vector \mathbf{a} , which forms the imaginary part. Quaternionic number systems are division rings. This means that every non-zero element has an inverse. Hilbert spaces can only cope with number systems that are division rings. Quaternionic number systems form the most elaborate division rings.

Continuous quaternionic functions represent skew fields. Quaternionic differential calculus uses proper time τ as progression parameter. For that reason all quaternionic differential equations are inherently Lorentz invariant.

Due to their four dimensions quaternionic number systems exist in 16 symmetry flavors that only differ in their discrete symmetry sets [1][4].

$$\mathbf{a} \equiv a_0 + \mathbf{a} \quad (1)$$

The quaternionic conjugate is defined as:

$$\mathbf{a}^* \equiv a_0 - \mathbf{a} \quad (2)$$

The norm is defined as:

$$|\mathbf{a}| \equiv \sqrt{\mathbf{a}^* \mathbf{a}} \quad (3)$$

The norm of a quaternionic function φ is defined as

$$\|\varphi\| \equiv \sqrt{\int_V \varphi^* \varphi dV} \quad (4)$$

The quaternionic product is defined as:

$$c = ab = a_0 b_0 + a_0 \mathbf{b} + b_0 \mathbf{a} - \langle \mathbf{a}, \mathbf{b} \rangle \pm \mathbf{a} \times \mathbf{b} \quad (5)$$

The \pm sign indicates the freedom of choice between a left handed and a right handed external vector product. This indicates that quaternionic number systems exist in several versions.

The quaternionic nabla is defined by:

$$\nabla \equiv \left\{ \frac{\partial}{\partial \tau}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\} \equiv \frac{\partial}{\partial \tau} + \mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z} = \nabla_0 + \nabla \quad (6)$$

$$\nabla^* = \nabla_0 - \nabla \quad (7)$$

In quaternionic differential calculus ***the differential can be defined as a product.***

$$\phi = \nabla \varphi \equiv (\nabla_0 + \nabla)(\varphi_0 + \varphi) = \nabla_0 \varphi_0 - \langle \nabla, \varphi \rangle + \nabla \varphi_0 + \nabla_0 \varphi \pm \nabla \times \varphi \quad (8)$$

$$\phi_0 = \nabla_0 \varphi_0 - \langle \nabla, \varphi \rangle \quad (9)$$

$$\phi = \nabla \varphi_0 + \nabla_0 \varphi \pm \nabla \times \varphi \quad (10)$$

The second derivative delivers an non-homogeneous wave equation:

$$\zeta = \nabla^* \phi = \nabla^* \nabla \phi = (\nabla_0 - \nabla)(\nabla_0 + \nabla)(\phi_0 + \phi) \quad (11)$$

$$= \{\nabla_0 \nabla_0 + \langle \nabla, \nabla \rangle\} \phi = \frac{\partial^2 \phi}{\partial \tau^2} + \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$$

$$\zeta_0 = \nabla_0 \phi_0 - \langle \nabla, \phi \rangle \quad (12)$$

$$\zeta = \nabla \phi_0 + \nabla_0 \phi \pm \nabla \times \phi \quad (13)$$

Notice that:

$$\nabla_0 \phi = \nabla_0 \nabla_0 \phi_0 - \nabla_0 \langle \nabla, \phi \rangle + \nabla_0 \nabla \phi_0 + \nabla_0 \nabla_0 \phi \pm \nabla_0 \nabla \times \phi \quad (14)$$

$$\begin{aligned} -\nabla \phi &= -\nabla \nabla_0 \phi_0 + \nabla \langle \nabla, \phi \rangle + \langle \nabla, \nabla \rangle \phi_0 + \nabla_0 \langle \nabla, \phi \rangle \mp \nabla_0 \nabla \times \phi - \nabla \times (\nabla \times \phi) \\ &= -\nabla \nabla_0 \phi_0 + \langle \nabla, \nabla \rangle \phi_0 + \nabla_0 \langle \nabla, \phi \rangle \mp \nabla_0 \nabla \times \phi + \langle \nabla, \nabla \rangle \phi \end{aligned} \quad (15)$$

The gauge transformation $\phi \rightarrow \phi + \chi$, where $\nabla^* \chi = 0$, does not change ϕ in $\phi = \nabla \phi$.

6.1 Solutions of the quaternionic wave equation

Solutions of the wave equation depend on start and boundary conditions. The wave equation only holds for moderate discontinuity conditions.

$$\nabla^* \nabla \phi \equiv \{\nabla_0 \nabla_0 + \langle \nabla, \nabla \rangle\} \phi = \frac{\partial^2 \phi}{\partial \tau^2} + \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = \zeta \quad (1)$$

Apart from more detailed conditions the equation can be reduced to special forms. Examples are the Poisson equation, the screened Poisson equation, the Helmholtz equation and the homogeneous wave equation.

6.1.1 Poisson Equations

In the screened Poisson equation the first term is reduced to multiplication with a real constant λ :

$$\nabla_0 \nabla_0 \phi = -\lambda^2 \phi \quad (1)$$

$$\{-\lambda^2 + \langle \nabla, \nabla \rangle\} \phi = \zeta \quad (2)$$

The corresponding solution is superposition of screened Green's functions.

Green functions represent solutions for point sources.

$$\{-\lambda^2 + \langle \nabla, \nabla \rangle\}G(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}') \quad (3)$$

$$\varphi = \iiint G(\mathbf{r} - \mathbf{r}') \zeta(\mathbf{r}') d^3\mathbf{r}' \quad (4)$$

In this case the Green's function for spherical symmetric conditions ($\mathbf{r} = \mathbf{0}$) is:

$$G(r) = \frac{\exp(-\lambda r)}{r} \quad (5)$$

A zero value of λ offers the normal Poisson equation.

6.1.2 Coherent swarm of charges

A coherent swarm of charges that can be described by a continuous quaternionic location density function represents a blurred Green's function. For example, in case of an isotropic Gaussian distribution ζ_0 the N contributions of the swarm elements to the integral φ will on average equal $\mathfrak{G}(r) = \text{ERF}(r)/r$. $N \mathfrak{G}(r)$ represents the local potential.

6.1.3 The homogeneous quaternionic wave equation

Locally, the wave function is considered to act in a rather flat continuum φ .

$$\nabla^* \nabla \varphi = \nabla_0 \nabla_0 \varphi + \langle \nabla, \nabla \rangle \varphi = 0 \quad (1)$$

First we look at:

$$\nabla^* \nabla \varphi_0 = 0 \quad (2)$$

φ_0 is a scalar function. For isotropic conditions in three participating dimensions equation (2) has three dimensional spherical wave fronts as one group of its solutions.

By changing to polar coordinates it can be deduced that a general solution is given by:

$$\varphi_0(r, \tau) = \frac{f_0(\mathbf{i}r - c\tau)}{r} = \frac{f_0(\mathbf{r} - c\tau)}{|\mathbf{r}|} \quad (3)$$

where $c = \pm 1$ and \mathbf{i} represents a base vector in radial direction.

In fact the parameter $\mathbf{i}r - c\tau$ of f_0 can be considered as a complex number valued function.

We use

$$\langle \nabla, \nabla \rangle \varphi_0 \equiv \frac{1}{r^2} \left(\frac{\partial}{\partial r} \left(r^2 \frac{\partial \varphi_0}{\partial r} \right) \right) = -\frac{f''(\mathbf{i}r - c\tau)}{r} = \frac{1}{c^2} \frac{\partial^2 \varphi_0}{\partial \tau^2} \quad (4)$$

Next we consider the vector function $\boldsymbol{\varphi}$

$$\nabla^* \nabla \boldsymbol{\varphi} = 0 \quad (5)$$

Equation (5) has one dimensional wave fronts as one group of its solutions:

$$\boldsymbol{\varphi}(z, \tau) = \mathbf{f}(\mathbf{i}z - c\tau) \quad (6)$$

Again the parameter $\mathbf{i}z - c\tau$ of \mathbf{f} can be interpreted as a complex number based function.

The imaginary \mathbf{i} represents a normalized base vector in the x, y plane. Its orientation θ may be a function of z .

That orientation determines the polarization of the one dimensional wave front.

6.1 Coupling equation

The coupling equation represents a peculiar property of the quaternionic differential equation.

We start with two normalized functions ψ and φ and a normalizable function $\Phi = m \varphi$.

Here m is a fixed quaternion. Function φ can be adapted such that m becomes a real number.

$$\|\psi\| = \|\varphi\| = 1 \quad (1)$$

These normalized functions are supposed to be related by:

$$\Phi = \nabla \psi = m \varphi \quad (2)$$

$$\Phi = \nabla \psi \text{ defines the } \mathbf{differential equation}. \quad (3)$$

$$\nabla \psi = \Phi \text{ formulates a differential } \mathbf{continuity equation}. \quad (4)$$

$$\nabla \psi = m \varphi \text{ formulates the } \mathbf{coupling equation}. \quad (5)$$

The coupling equation indicates that the wave fronts carry a bit of something that can be characterized with a real number.

7 Maxwell based differential calculus

7.1 Maxwell-like equations

We start from the quaternionic differential.

$$\phi = \nabla\varphi = (\nabla_t + \nabla)(\varphi_0 + \boldsymbol{\varphi}) = \nabla_t\varphi_0 - \langle \nabla, \boldsymbol{\varphi} \rangle + \nabla\varphi_0 + \nabla_t\boldsymbol{\varphi} \pm \nabla \times \boldsymbol{\varphi} \quad (1)$$

$$\phi_0 = \nabla_t\varphi_0 - \langle \nabla, \boldsymbol{\varphi} \rangle \quad (2)$$

$$\boldsymbol{\phi} = \nabla\varphi_0 + \nabla_t\boldsymbol{\varphi} - \nabla \times \boldsymbol{\varphi} \quad (3)$$

We define new symbols:

$$\boldsymbol{\mathcal{E}} \equiv -\nabla\varphi_0 - \nabla_t\boldsymbol{\varphi} \quad (4)$$

$$\boldsymbol{\mathcal{B}} \equiv \nabla \times \boldsymbol{\varphi} \quad (5)$$

$$\kappa \equiv \alpha \nabla_t\varphi_0 + \langle \nabla, \boldsymbol{\varphi} \rangle \quad (6)$$

$$\nabla_t\boldsymbol{\mathcal{B}} = \nabla \times \nabla_t\boldsymbol{\varphi} = -\nabla \times \boldsymbol{\mathcal{E}} \quad (7)$$

$$\nabla \times \boldsymbol{\mathcal{B}} = \nabla \times (\nabla \times \boldsymbol{\varphi}) = \nabla\langle \nabla, \boldsymbol{\varphi} \rangle - \langle \nabla, \nabla \rangle\boldsymbol{\varphi} \quad (8)$$

$$\nabla_t\boldsymbol{\mathcal{E}} \equiv -\nabla_t\nabla\varphi_0 - \nabla_t\nabla_t\boldsymbol{\varphi} \quad (9)$$

$$\langle \nabla, \boldsymbol{\mathcal{E}} \rangle = -\langle \nabla, \nabla \rangle\varphi_0 - \nabla_t\langle \nabla, \boldsymbol{\varphi} \rangle \quad (10)$$

$$\nabla_t\phi_0 = \nabla_t\nabla_t\varphi_0 - \nabla_t\langle \nabla, \boldsymbol{\varphi} \rangle \quad (11)$$

$$\nabla\phi_0 \equiv \nabla_t\nabla\varphi_0 - \nabla\langle \nabla, \boldsymbol{\varphi} \rangle \quad (12)$$

$$\nabla_t\phi_0 - \langle \nabla, \boldsymbol{\mathcal{E}} \rangle = \nabla_t\nabla_t\varphi_0 - \nabla_t\langle \nabla, \boldsymbol{\varphi} \rangle + \langle \nabla, \nabla \rangle\varphi_0 + \nabla_t\langle \nabla, \boldsymbol{\varphi} \rangle \quad (13)$$

$$= \nabla_t\nabla_t\varphi_0 + \langle \nabla, \nabla \rangle\varphi_0 = (\nabla_t\nabla_t + \langle \nabla, \nabla \rangle)\varphi_0$$

$$-\nabla\phi_0 - \nabla_{\tau}\mathfrak{E} - \nabla \times \mathfrak{B} \quad (14)$$

$$= -\nabla_{\tau}\nabla\phi_0 + \nabla\langle\nabla, \phi\rangle + \nabla_{\tau}\nabla\phi_0 + \nabla_{\tau}\nabla_{\tau}\phi - \nabla\langle\nabla, \phi\rangle + \langle\nabla, \nabla\rangle\phi$$

$$= (\nabla_{\tau}\nabla_{\tau} + \langle\nabla, \nabla\rangle)\phi$$

In quaternionic differential calculus is:

$$\phi = -\mathfrak{E} \mp \mathfrak{B} \quad (15)$$

$$(\nabla_{\tau}\nabla_{\tau} + \langle\nabla, \nabla\rangle)\phi_0 = \zeta_0 = \nabla_{\tau}\phi_0 - \langle\nabla, \mathfrak{E}\rangle \quad (16)$$

$$\frac{\partial^2\phi_0}{\partial\tau^2} + \frac{\partial^2\phi_0}{\partial x^2} + \frac{\partial^2\phi_0}{\partial y^2} + \frac{\partial^2\phi_0}{\partial z^2} = \zeta_0 \quad (17)$$

$$(\nabla_{\tau}\nabla_{\tau} + \langle\nabla, \nabla\rangle)\phi = \zeta = -\nabla\phi_0 - \nabla_{\tau}\mathfrak{E} - \nabla \times \mathfrak{B} \quad (18)$$

$$\frac{\partial^2\phi}{\partial\tau^2} + \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = \zeta \quad (19)$$

$$\frac{\partial^2\phi}{\partial\tau^2} + \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = \zeta = \nabla^*\nabla\phi \quad (20)$$

This corresponds to a Euclidean signature.

$$\zeta = \nabla^*\phi_0 - \langle\nabla, \mathfrak{E}\rangle - \nabla_{\tau}\mathfrak{E} - \nabla \times \mathfrak{B} = \nabla^*(\phi_0 + \phi) \quad (21)$$

$$\nabla^*\phi = -\langle\nabla, \mathfrak{E}\rangle - \nabla_{\tau}\mathfrak{E} - \nabla \times \mathfrak{B} \quad (22)$$

7.2 Maxwell equations

The Maxwell based formulas that are used here are taken from Bo Thidé; “Electromagnetic field theory”; second edition.

We use these formulas without units. Thus $c = 1$; $\varepsilon_0 = 1$; $\mu = 1$.

The Maxwell equations use coordinate time t .

∇_t stands for $\frac{\partial}{\partial t}$.

We use symbol \mathcal{E} for the electrical field and symbol \mathcal{B} for the magnetic field.

We introduce \varkappa as a new scalar field.

$$\mathcal{E} \equiv -\nabla\varphi_0 - \nabla_t\boldsymbol{\varphi} \quad (1)$$

$$\mathcal{B} \equiv \nabla \times \boldsymbol{\varphi} \quad (2)$$

$$\varkappa \equiv \alpha \nabla_t\varphi_0 + \langle \nabla, \boldsymbol{\varphi} \rangle \quad (3)$$

In EMFT the scalar field \varkappa is taken as a gauge with

$\alpha = 1$; Lorentz gauge

$\alpha = 0$; Coulomb gauge

$\alpha = -1$; Kirchhoff gauge.

In Maxwell based differential calculus the scalar field is ignored or it is taken equal to zero. As will be shown, zeroing \varkappa is not necessary for the derivation of the wave equation.

$$\boldsymbol{\phi} = -\mathcal{E} - \mathcal{B} \quad (4)$$

$$\nabla_t\mathcal{B} = \nabla \times \nabla_t\boldsymbol{\varphi} = -\nabla \times \mathcal{E} \quad (5)$$

$$\nabla \times \mathcal{B} = \nabla \times (\nabla \times \boldsymbol{\varphi}) = \nabla\langle \nabla, \boldsymbol{\varphi} \rangle - \langle \nabla, \nabla \rangle\boldsymbol{\varphi} \quad (6)$$

$$\nabla_t\mathcal{E} \equiv -\nabla_t\nabla\varphi_0 - \nabla_t\nabla_t\boldsymbol{\varphi} \quad (7)$$

$$\langle \nabla, \mathcal{E} \rangle = -\langle \nabla, \nabla \rangle\varphi_0 - \nabla_t\langle \nabla, \boldsymbol{\varphi} \rangle \quad (8)$$

$$\nabla_t\varkappa = \alpha\nabla_t\nabla_t\varphi_0 + \nabla_t\langle \nabla, \boldsymbol{\varphi} \rangle \quad (9)$$

$$\nabla \kappa \equiv \alpha \nabla_t \nabla \varphi_0 + \nabla \langle \nabla, \varphi \rangle \quad (10)$$

$$\nabla_t \kappa + \langle \nabla, \mathcal{E} \rangle = \alpha \nabla_t \nabla_t \varphi_0 + \nabla_t \langle \nabla, \varphi \rangle - \langle \nabla, \nabla \rangle \varphi_0 - \nabla_t \langle \nabla, \varphi \rangle \quad (11)$$

$$= \alpha \nabla_t \nabla_t \varphi_0 - \langle \nabla, \nabla \rangle \varphi_0 = (\alpha \nabla_t \nabla_t - \langle \nabla, \nabla \rangle) \varphi_0$$

$$-\nabla \kappa - \alpha \nabla_t \mathcal{E} + \nabla \times \mathcal{B} \quad (12)$$

$$= -\alpha \nabla_t \nabla \varphi_0 - \nabla \langle \nabla, \varphi \rangle + \alpha \nabla_t \nabla \varphi_0 + \alpha \nabla_t \nabla_t \varphi + \nabla \langle \nabla, \varphi \rangle - \langle \nabla, \nabla \rangle \varphi$$

$$= -\alpha \nabla_t \nabla \varphi_0 + \alpha \nabla_t \nabla \varphi_0 + \nabla_t \nabla_t \varphi - \langle \nabla, \nabla \rangle \varphi$$

In quaternionic differential calculus $\alpha = -1$ and $\phi_0 = -\kappa$. If the Lorentz gauge $\alpha = 1$ is applied, then:

$$(\nabla_t \nabla_t - \langle \nabla, \nabla \rangle) \varphi_0 = \rho_0 = \nabla_t \kappa + \langle \nabla, \mathcal{E} \rangle \quad (13)$$

$$\frac{\partial^2 \varphi_0}{\partial t^2} - \frac{\partial^2 \varphi_0}{\partial x^2} - \frac{\partial^2 \varphi_0}{\partial y^2} - \frac{\partial^2 \varphi_0}{\partial z^2} = \rho_0 \quad (14)$$

$$(\nabla_t \nabla_t - \langle \nabla, \nabla \rangle) \varphi = \mathbf{J} = \nabla \times \mathcal{B} - \alpha \nabla_t \mathcal{E} - \nabla \kappa \quad (15)$$

$$\frac{\partial^2 \varphi}{\partial t^2} - \frac{\partial^2 \varphi}{\partial x^2} - \frac{\partial^2 \varphi}{\partial y^2} - \frac{\partial^2 \varphi}{\partial z^2} = \mathbf{J} \quad (16)$$

This corresponds to the Minkowski signature.

$$\{\rho_0, \mathbf{J}\} \Leftrightarrow \{\nabla_t \kappa - \langle \nabla, \mathcal{E} \rangle, -\nabla \kappa + \nabla \times \mathcal{B} - \alpha \nabla_t \mathcal{E}\}$$

$$= \{\nabla_t \kappa, -\nabla \kappa\} + \{\langle \nabla, \mathcal{E} \rangle, \nabla \times \mathcal{B} - \alpha \nabla_t \mathcal{E}\}$$

Adding equation (3) as an extra Maxwell equation would bring Maxwell equations more in conformance with the equations of quaternionic differential calculus.

Notice the difference of the Minkowski signature of these equations with the Euclidean signature of the wave function of quaternionic differential calculus. This difference is enforced by the selection of the value of α .

7.2.1 Poisson equations

The Poisson equations for the Maxwell based differential calculus are similar to the Poisson equations for the quaternionic differential calculus.

$$\frac{\partial^2 \varphi_0}{\partial x^2} + \frac{\partial^2 \varphi_0}{\partial y^2} + \frac{\partial^2 \varphi_0}{\partial z^2} = -\rho_0 = -\nabla_t \kappa - \langle \nabla, \mathcal{E} \rangle \quad (1)$$

$$\frac{\partial^2 \boldsymbol{\varphi}}{\partial x^2} + \frac{\partial^2 \boldsymbol{\varphi}}{\partial y^2} + \frac{\partial^2 \boldsymbol{\varphi}}{\partial z^2} = -\mathbf{J} = \nabla \kappa + \nabla_t \mathcal{E} - \nabla \times \mathcal{B} \quad (2)$$

7.2.2 The Maxwell-Huygens wave equation

In Maxwell format the homogeneous wave equation uses coordinate time t . It runs as:

$$\frac{\partial^2 \varphi_0}{\partial t^2} - \frac{\partial^2 \varphi_0}{\partial x^2} - \frac{\partial^2 \varphi_0}{\partial y^2} - \frac{\partial^2 \varphi_0}{\partial z^2} = 0 \quad (1)$$

Papers on Huygens principle work with the homogeneous version of this formula or it uses the version with polar coordinates.

For isotropic conditions in three participating dimensions the general solution runs:

$$\varphi_0 = f(r - ct)/r, \text{ where } c = \pm 1; f \text{ is real} \quad (2)$$

This follows from

$$\langle \nabla, \nabla \rangle \varphi_0 \equiv \frac{1}{r^2} \left(\frac{\partial}{\partial r} \left(r^2 \frac{\partial \varphi_0}{\partial r} \right) \right) = \frac{f''(r - ct)}{r} = \frac{1}{c^2} \partial^2 \varphi_0 / \partial t^2 \quad (3)$$

In a single participating dimension the general solution runs:

$$\varphi_0 = f(x - ct), \text{ where } c = \pm 1; f \text{ is real} \quad (4)$$

8 Conclusion

Great similarities, but also essential differences exist between quaternionic differential equations and Maxwell based differential equations. In the quaternionic differential calculus the differential can be seen as a product between the four-component differential operator ∇ and the four-component field φ . That simple interpretation is not possible in Maxwell based differential calculus. It is not possible to interpret the Maxwell field as a function of a number system that offers target values from that number system. In the Maxwell approach, the parameter space has a Minkowski signature and does not form a division ring. In quaternionic function theory the parameter space has a Euclidean signature. This shows in the structure of the wave equations of the two approaches. These wave equations have solutions that differ between the two approaches. However, the Poisson part of the two wave equations is similar. As a consequence the corresponding Green's functions are also similar. Both homogeneous wave equations have solutions in the form of one dimensional and three

dimensional fronts that keep their shape. The one dimensional fronts also keep their amplitude. Between the two approaches, these fronts have different mathematical representations. In applications, the fronts can act as carriers of information or energy.

In Maxwell based differential calculus the usage of gauges can be replaced by the introduction of an extra scalar field χ . This results in the same form of the Maxwell based wave equation as the Lorentz gauge delivers, but the non-homogeneous equation applies different charge and current distributions. The impact of the difference in charges and currents is not treated here.

The speed of information transfer c differs between the two approaches.

8.1 Extra

Maxwell based differential calculus can be implemented with complex numbers. In that way it does not have to cope so intensively with non-commutative operators. As a consequence, gauges can be implemented easily.

A disadvantage of Maxwell based differential calculus is that spacetime based dynamic geometric data must first be dismantled into real numbers or complex numbers before Hilbert spaces can handle them.

Quaternionic differential calculus must circumvent most gauges. On the other hand, the quaternionic approach offers compensating advantages.

Hilbert spaces can cope with quaternions as eigenvalues of operators. This holds for separate quaternions and for quaternionic continuums [1][4].

Since proper time is Lorentz invariant, all quaternionic differential equations are inherently Lorentz invariant.

Due to the fact that quaternions form a number system with a non-commutative product, they can implement rotations:

$$c = a b/a$$

In this way they can implement the functionality of gluons [4]. This is not possible with members of the Maxwell based "field".

9 References

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- [2] The vector differential calculus formulas and the Maxwell based equations that are used here are taken from the online textbook Bo Thidé; “Electromagnetic field theory”; second edition; <http://www.plasma.uu.se/CED/Book/>, or https://www.calvin.edu/~priebeiro/courses/engr315/EMFT_Book.pdf .
- [3] The relation between spinor notation and quaternionic formulation is shown in “The Dirac Equation in Quaternionic Format”; <http://vixra.org/abs/1505.0149>
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