The Dirac equation in quaternionic format

By J.A.J. van Leunen

Last modified: 22 mei 2015

ABSTRACT
In its original form the Dirac equation for the free electron and the free positron is formulated by using complex number based spinors and matrices. That equation can be split into two equations, one for the electron and one for the positron. These equations can easily be converted to their quaternionic format. The equation for the electron and the equation for the positron differ in the sign of a curl term. This results in a special way for constructing the corresponding wave equation.

Contents

The Dirac equation in original format

In its original form the Dirac equation is a complex equation that uses spinors, matrices and partial derivatives.

Instead of the usual $\left\{i\frac{\partial f}{\partial \tau}, i\frac{\partial f}{\partial x}, i\frac{\partial f}{\partial y}, i\frac{\partial f}{\partial z}\right\}$ we use operators $\nabla = \{\nabla_0, \boldsymbol{\nabla}\}$

In that case the Dirac equation runs

$$\nabla_0\{\psi\} + \boldsymbol{\nabla\alpha}\{\psi\} = m\beta\{\psi\} \tag{1}$$

$\boldsymbol{\alpha}$ and β represent the matrices that implement the quaternion behavior including the sign flavors of quaternionic number systems and continuums.

$$\alpha_1 = \begin{bmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \boldsymbol{i} \\ -\boldsymbol{i} & 0 \end{bmatrix} \tag{2}$$

$$\alpha_2 = \begin{bmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \boldsymbol{j} \\ -\boldsymbol{j} & 0 \end{bmatrix} \tag{3}$$

$$\alpha_3 = \begin{bmatrix} 0 & \sigma_3 \\ -\sigma_3 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \boldsymbol{k} \\ -\boldsymbol{k} & 0 \end{bmatrix} \tag{4}$$

$$\beta = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{5}$$

The Pauli[1] matrices $\sigma_1, \sigma_2, \sigma_3$ are given by:

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \qquad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{6}$$

They relate to the quaternionic base vectors $1, \boldsymbol{i}, \boldsymbol{j}$ and $\boldsymbol{k}$

$$1 \mapsto I, \qquad \boldsymbol{i} \mapsto \sigma_1, \qquad \boldsymbol{j} \mapsto \sigma_2, \qquad \boldsymbol{k} \mapsto \sigma_3 \tag{7}$$

Splitting into two equations

The four component spinors $\{\psi\}$ can be converted in two component spinors $\{\psi_R\}$ and $\{\psi_L\}$. Transferring the matrix form of the Dirac equation into quaternionic format delivers two quaternionic fields ψ_R and ψ_L that replace the spinors and that couple two equations of motion. One of these fields is right handed and the other is left handed.

$$\nabla_0 \psi_R + \boldsymbol{\nabla} \psi_R = m \psi_L \tag{1}$$

$$\nabla_0 \psi_L - \boldsymbol{\nabla} \psi_L = m \psi_R \tag{2}$$

The factor m couples ψ_L and ψ_R.
These fields are each other's quaternionic conjugate.

$$\psi_R = \psi_L^* = \psi_0 + \boldsymbol{\psi} \tag{3}$$

Due to the four dimensions of quaternions, these fields represent two different sign flavors of one and the same quaternionic field that exist in 16 versions that only differ in their discrete symmetry set.

The quaternionic format

Reformulating the quaternionic equation for the free electron gives

$$\nabla \psi = m\, \psi^* \tag{1}$$

$$\nabla_0 \psi_0 - \langle \boldsymbol{\nabla}, \boldsymbol{\psi} \rangle = m\, \psi_0 \tag{2}$$

$$\nabla_0 \boldsymbol{\psi} + \boldsymbol{\nabla} \psi_0 + \boldsymbol{\nabla} \times \boldsymbol{\psi} = -m\, \boldsymbol{\psi} \tag{3}$$

For the antiparticle holds

$$\nabla^* \psi^* = m\, \psi \tag{4}$$

$$\nabla_0 \psi_0 - \langle \boldsymbol{\nabla}, \boldsymbol{\psi} \rangle = m\, \psi_0 \tag{5}$$

$$-\nabla_0 \boldsymbol{\psi} - \boldsymbol{\nabla} \psi_0 + \boldsymbol{\nabla} \times \boldsymbol{\psi} = +m\, \boldsymbol{\psi} \tag{6}$$

[1] http://en.wikipedia.org/wiki/Pauli_matrices

These equations differ in the sign of the $\boldsymbol{\nabla} \times \boldsymbol{\psi}$ term.

The wave equation

In general the non-homogeneous wave equation is given by

$$\nabla^* \nabla \chi \equiv \nabla_0^2 \chi + \langle \boldsymbol{\nabla}, \boldsymbol{\nabla} \rangle \chi = \rho \tag{1}$$

With $\nabla \chi = \varphi$ follows $\nabla^* \varphi = \rho$ (2)

Here χ represents the embedding continuum and ρ represents a location density distribution of triggers. Equation 2 represents two continuity equations.

By using the Dirac equation for the electron as the first continuity equation and the Dirac equation of the positron as the second continuity equation a sensible wave equation is obtained.
Thus, the non-homogeneous wave equation for the combined electron and positron is given by:

$$\nabla^* \nabla \psi \equiv \nabla_0^2 \psi + \langle \boldsymbol{\nabla}, \boldsymbol{\nabla} \rangle \psi = m \nabla^* \psi^* = m^2 \psi \tag{3}$$

The coupling equation

The Dirac equation is a more specific form of the coupling equation.
The coupling equation holds more generally:

$$\phi = \nabla \chi = m\, \varphi; \; \|\chi\| = \|\varphi\| = 1 \tag{1}$$

By adapting φ, the coupling factor m can become a real positive number.