Supersymmetric equations of massive and massless fields

Victor L. Mironov and Sergey V. Mironov

Institute for physics of microstructures RAS, Nizhny Novgorod, Russia

E-mail: mironov@ipmras.ru

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Abstract

We present the sypersymmetric scalar-vector equations for massive and massless fields. The gauge invariance for the potentials described by second-order and first-order wave equations and for the field strengths described by the systems of Maxwell-like equations is demonstrated.

1. Introduction

In classical electrodynamics the electromagnetic field is described by scalar φ and vector $\vec{A}$ potentials [1]. The strengths of electric and magnetic fields are defined as:

$$\begin{aligned} &\vec{E} = -\partial\vec{A} - \vec{\nabla}\varphi, \\ &\vec{H} = \left[\vec{\nabla}\times\vec{A}\right]. \end{aligned} \tag{1.1}$$

Here $\vec{\nabla}$ is the Hamilton operator (nabla-operator) and we use the following notation for the time differential operator:

$$\partial = \frac{1}{c}\frac{\partial}{\partial t}, \tag{1.2}$$

where c is the speed of light. The electromagnetic field potentials satisfy the Lorentz gauge condition

$$\partial\varphi + \left(\vec{\nabla}\cdot\vec{A}\right) = 0\,. \tag{1.3}$$

The equations for electromagnetic field are gauge-invariant. The substitutions

$$\begin{aligned} &\varphi \to \varphi + \partial\alpha, \\ &\vec{A} \to \vec{A} - \vec{\nabla}\alpha, \end{aligned} \tag{1.4}$$

do not change the electric and magnetic fields. Here $\alpha(\vec{r},t)$ is arbitrary scalar function satisfying homogeneous wave equation (because of the Lorentz gauge (1.3)). The gauge invariance is a cornerstone of modern field theory [2]. However, if the mass of a field quantum is nonzero (massive field), there is a problem with the violation of gauge invariance [2, 3].

On the other hand, the Gibbs-Heaviside vector algebra, which usually used for the description of fields, does not adequately specify the space-time properties of physical fields with respect to the spatial and time inversions. From this point of view the algebras taking into account the space-time symmetries are more appropriate. Particularly, in recent years many attempts have been made to generalize the second-order wave equation for massive field using different algebras of hypercomplex numbers, such as four-component quaternions (including scalar and vector) and eight-component octonions (including scalar, vector (polar vector), pseudoscalar and pseudovector (axial vector)). The authors discussed the possibility of constructing the field equations similar to the equations of electrodynamics but with a massive ”photon”. In particular they tried to represent the wave equation as the system of first-order Maxwell-like equations. However, the resulting Proca-Maxwell equations enclosing field’s strengths and potentials are not gauge invariant [4-6]. Besides, a consistent relativistic approach implies equally the space and time symmetries that require the consideration of the extended sixteen-component space-time algebras. There are a few approaches in the development of field theory on the basis of sixteen-component structures. One of them is the application of hypernumbers sedenions, which are obtained from octonions by Cayley-Dickson extension procedure [7, 8]. But the essential imperfection of sedenions is their nonssociativity. Another approach is based on the application of hypercomplex multivectors generating associative space-time Clifford algebras [4, 9]. However, the application of such sixteen-component structures is considered in general as the abstract algebraic schemes enabling the factorization of Klein-Gordon operator.

Recently we proposed the space-time algebra of sixteen-component sedeons generating noncommutative associative scalar-vector Clifford algebra [10, 11]. The sedeons take into account the properties of physical values with respect to the space-time inversion and realize the scalar-vector representation of Poincare group. In present paper, we use the sedeonic approach for the consideration of massive fields described by sedeonic second-order and first-order wave equations within a unified field conception. The gauge invariance of supersymmetric sedeonic field equations is demonstrated.

2. Space-time sedeons

The sedeonic algebra [10] encloses four groups of values, which are differed with respect to spatial and time inversion.

- Absolute scalars (V) and absolute vectors ($\vec{V}$) are not transformed under spatial and time inversion.
- Time scalars ($V_{\mathbf{t}}$) and time vectors ($\vec{V}_{\mathbf{t}}$) are changed (in sign) under time inversion and are not transformed under spatial inversion.
- Space scalars ($V_{\mathbf{r}}$) and space vectors ($\vec{V}_{\mathbf{r}}$) are changed under spatial inversion and are not transformed under time inversion.
- Space-time scalars ($V_{\mathbf{tr}}$) and space-time vectors ($\vec{V}_{\mathbf{tr}}$) are changed under spatial and time inversion.

Here indexes $\mathbf{t}$ and $\mathbf{r}$ indicate the transformations ($\mathbf{t}$ for time inversion and $\mathbf{r}$ for spatial inversion), which change the corresponding values. All introduced values can be integrated into one space-time sedeon $\tilde{\mathbf{V}}$, which is defined by the following expression:

$$\tilde{\mathbf{V}} = V + \vec{V} + V_{\mathbf{t}} + \vec{V}_{\mathbf{t}} + V_{\mathbf{r}} + \vec{V}_{\mathbf{r}} + V_{\mathbf{tr}} + \vec{V}_{\mathbf{tr}} . \tag{2.1}$$

Let us introduce a scalar-vector basis $\mathbf{a}_0$, $\vec{\mathbf{a}}_1$, $\vec{\mathbf{a}}_2$, $\vec{\mathbf{a}}_3$, where the element $\mathbf{a}_0$ is an absolute scalar unit ($\mathbf{a}_0 \equiv 1$), and the values $\vec{\mathbf{a}}_1$, $\vec{\mathbf{a}}_2$, $\vec{\mathbf{a}}_3$ are absolute unit vectors generating the right Cartesian basis. Further we will indicate the absolute unit vectors by symbols without arrows as $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$. We also introduce the four space-time units $\mathbf{e}_0$, $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, where $\mathbf{e}_0$ is an absolute scalar unit ($\mathbf{e}_0 \equiv 1$); $\mathbf{e}_1$ is a time scalar unit ($\mathbf{e}_1 \equiv \mathbf{e}_{\mathbf{t}}$); $\mathbf{e}_2$ is a space scalar unit ($\mathbf{e}_2 \equiv \mathbf{e}_{\mathbf{r}}$); $\mathbf{e}_3$ is a space-time scalar unit ($\mathbf{e}_3 \equiv \mathbf{e}_{\mathbf{tr}}$). Using space-time basis $\mathbf{e}_\alpha$ and scalar-vector basis $\mathbf{a}_\beta$ (Greek indexes $\alpha, \beta = 0, 1, 2, 3$), we can introduce unified sedeonic components $V_{\alpha\beta}$ in accordance with following relations:

$$\begin{aligned}
&V = \mathbf{e}_0 V_{00} \mathbf{a}_0 , \\
&\vec{V} = \mathbf{e}_0 \left(V_{01}\mathbf{a}_1 + V_{02}\mathbf{a}_2 + V_{03}\mathbf{a}_3 \right), \\
&V_{\mathbf{t}} = \mathbf{e}_1 V_{10} \mathbf{a}_0 , \\
&\vec{V}_{\mathbf{t}} = \mathbf{e}_1 \left(V_{11}\mathbf{a}_1 + V_{12}\mathbf{a}_2 + V_{13}\mathbf{a}_3 \right), \\
&V_{\mathbf{r}} = \mathbf{e}_2 V_{20} \mathbf{a}_0 , \\
&\vec{V}_{\mathbf{r}} = \mathbf{e}_2 \left(V_{21}\mathbf{a}_1 + V_{22}\mathbf{a}_2 + V_{23}\mathbf{a}_3 \right), \\
&V_{\mathbf{tr}} = \mathbf{e}_3 V_{30} \mathbf{a}_0 , \\
&\vec{V}_{\mathbf{tr}} = \mathbf{e}_3 \left(V_{31}\mathbf{a}_1 + V_{32}\mathbf{a}_2 + V_{33}\mathbf{a}_3 \right).
\end{aligned} \tag{2.2}$$

Then sedeon (2.1) can be written in the following expanded form:

$$\begin{aligned}
\tilde{\mathbf{V}} = {} & \mathbf{e}_0 \left(V_{00}\mathbf{a}_0 + V_{01}\mathbf{a}_1 + V_{02}\mathbf{a}_2 + V_{03}\mathbf{a}_3 \right) \\
& + \mathbf{e}_1 \left(V_{10}\mathbf{a}_0 + V_{11}\mathbf{a}_1 + V_{12}\mathbf{a}_2 + V_{13}\mathbf{a}_3 \right) \\
& + \mathbf{e}_2 \left(V_{20}\mathbf{a}_0 + V_{21}\mathbf{a}_1 + V_{22}\mathbf{a}_2 + V_{23}\mathbf{a}_3 \right) \\
& + \mathbf{e}_3 \left(V_{30}\mathbf{a}_0 + V_{31}\mathbf{a}_1 + V_{32}\mathbf{a}_2 + V_{33}\mathbf{a}_3 \right).
\end{aligned} \tag{2.3}$$

The sedeonic components $V_{\alpha\beta}$ are numbers (complex in general). Further we will omit units $\mathbf{a}_0$ and $\mathbf{e}_0$ for the simplicity. The important property of sedeons is that the equality of two sedeons means the equality of all sixteen components $V_{\alpha\beta}$.

Let us consider the multiplication rules for the basis elements $\mathbf{a}_n$ and $\mathbf{e}_k$ (Latin indexes **n**, **k** = 1, 2, 3). The vectors $\mathbf{a}_n$ and space-time units $\mathbf{e}_k$ satisfy the following rules:

$$\mathbf{a}_n\mathbf{a}_n = \mathbf{a}_n^2 = 1, \tag{2.4}$$

$$\mathbf{a}_n\mathbf{a}_k = -\mathbf{a}_k\mathbf{a}_n \text{ (for } \mathbf{n} \neq \mathbf{k}\text{)}. \tag{2.5}$$

$$\mathbf{a}_1\mathbf{a}_2 = i\mathbf{a}_3, \quad \mathbf{a}_2\mathbf{a}_3 = i\mathbf{a}_1, \quad \mathbf{a}_3\mathbf{a}_1 = i\mathbf{a}_2. \tag{2.6}$$

$$\mathbf{e}_k\mathbf{e}_k = \mathbf{e}_k^2 = 1, \tag{2.7}$$

$$\mathbf{e}_n\mathbf{e}_k = -\mathbf{e}_k\mathbf{e}_n \text{ (for } \mathbf{n} \neq \mathbf{k}\text{)}, \tag{2.8}$$

$$\mathbf{e}_1\mathbf{e}_2 = i\mathbf{e}_3, \quad \mathbf{e}_2\mathbf{e}_3 = i\mathbf{e}_1, \quad \mathbf{e}_3\mathbf{e}_1 = i\mathbf{e}_2. \tag{2.9}$$

Here and further the value i is imaginary unit ($i^2 = -1$). The multiplication and commutation rules for sedeonic absolute unit vectors $\mathbf{a}_n$ and space-time units $\mathbf{e}_k$ can be presented for obviousness as the tables 1 and 2.

Table 1. Multiplication rules for absolute unit vectors $\mathbf{a}_n$.

	$\mathbf{a}_1$	$\mathbf{a}_2$	$\mathbf{a}_3$
$\mathbf{a}_1$	1	$i\mathbf{a}_3$	$-i\mathbf{a}_2$
$\mathbf{a}_2$	$-i\mathbf{a}_3$	1	$i\mathbf{a}_1$
$\mathbf{a}_3$	$i\mathbf{a}_2$	$-i\mathbf{a}_1$	1

Table 2. Multiplication rules for space-time units $\mathbf{e}_k$.

	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$
$\mathbf{e}_1$	1	$i\mathbf{e}_3$	$-i\mathbf{e}_2$
$\mathbf{e}_2$	$-i\mathbf{e}_3$	1	$i\mathbf{e}_1$
$\mathbf{e}_3$	$i\mathbf{e}_2$	$-i\mathbf{e}_1$	1

Note that units $\mathbf{e}_k$ commute with vectors $\mathbf{a}_n$:

$$\mathbf{a}_n\mathbf{e}_k = \mathbf{e}_k\mathbf{a}_n \tag{2.10}$$

for any $\mathbf{n}$ and $\mathbf{k}$.

In sedeonic algebra we assume the Clifford multiplication of vectors. The sedeonic product of two vectors $\vec{A}$ and $\vec{B}$ can be presented in the following form:

$$\vec{A}\vec{B} = \left(\vec{A} \cdot \vec{B}\right) + \left[\vec{A} \times \vec{B}\right]. \tag{2.11}$$

Here we denote the sedeonic scalar multiplication of two vectors (internal product) by symbol "$\cdot$" and round brackets

$$\left(\vec{A} \cdot \vec{B}\right) = A_1B_1 + A_2B_2 + A_3B_3, \tag{2.12}$$

and sedeonic vector multiplication (external product) by symbol "$\times$" and square brackets

$$\left[\vec{A} \times \vec{B}\right] = i\left(A_2B_3 - A_3B_2\right) + i\left(A_3B_1 - A_1B_3\right) + i\left(A_1B_2 - A_2B_1\right). \tag{2.13}$$

Note that in sedeonic algebra the expression for the vector product differs from analogous expression in Gibbs vector algebra.

3. Lorentz transformations

In the frames of sedeonic algebra the transformation of values from one inertial coordinate system to another are carried out with the following sedeons:

$$\tilde{\mathbf{L}} = \cosh\vartheta - \mathbf{e}_{\mathbf{tr}}\vec{m}\sinh\vartheta,$$
$$\tilde{\mathbf{L}}^* = \cosh\vartheta + \mathbf{e}_{\mathbf{tr}}\vec{m}\sinh\vartheta, \tag{3.1}$$

where $\tanh(2\vartheta) = v/c$; c is the speed of light; v is the speed of uniform motion of the system along the absolute vector $\vec{m}$. Note, that

$$\tilde{\mathbf{L}}^*\tilde{\mathbf{L}} = \tilde{\mathbf{L}}\tilde{\mathbf{L}}^* = 1. \tag{3.2}$$

Let us consider the Lorentz transformation of the sedeon $\tilde{\mathbf{V}}$. The transformed sedeon $\tilde{\mathbf{V}}'$ can be written as sedeonic product

$$\tilde{\mathbf{V}}' = \tilde{\mathbf{L}}^*\tilde{\mathbf{V}}\tilde{\mathbf{L}}. \tag{3.3}$$

The transformed sedeon $\tilde{\mathbf{V}}'$ have the following components:

$$\begin{aligned}
V'_{\mathbf{t}} &= V_{\mathbf{t}}\cosh(2\vartheta) + \mathbf{e}_{\mathbf{tr}}\left(\vec{m}\cdot\vec{V}_{\mathbf{r}}\right)\sinh(2\vartheta),\\
V'_{\mathbf{r}} &= V_{\mathbf{r}}\cosh(2\vartheta) + \mathbf{e}_{\mathbf{tr}}\left(\vec{m}\cdot\vec{V}_{\mathbf{t}}\right)\sinh(2\vartheta),\\
\vec{V}'_{\mathbf{t}} &= \vec{V}_{\mathbf{t}} + \left(\vec{m}\cdot\vec{V}_{\mathbf{t}}\right)\vec{m}\left(\cosh 2\vartheta - 1\right) + \mathbf{e}_{\mathbf{tr}}V_{\mathbf{r}}\vec{m}\sinh(2\vartheta),\\
\vec{V}'_{\mathbf{r}} &= \vec{V}_{\mathbf{r}} + \left(\vec{m}\cdot\vec{V}_{\mathbf{r}}\right)\vec{m}\left(\cosh 2\vartheta - 1\right) + \mathbf{e}_{\mathbf{tr}}V_{\mathbf{t}}\vec{m}\sinh(2\vartheta).
\end{aligned} \tag{3.4}$$

$$\begin{aligned}
V' &= V,\\
V'_{\mathbf{tr}} &= V_{\mathbf{tr}},\\
\vec{V}' &= \vec{V}\cosh(2\vartheta) - \left(\vec{m}\cdot\vec{V}\right)\vec{m}\left(\cosh 2\vartheta - 1\right) + \mathbf{e}_{\mathbf{tr}}\left[\vec{m}\times\vec{V}_{\mathbf{tr}}\right]\sinh(2\vartheta),\\
\vec{V}'_{\mathbf{tr}} &= \vec{V}_{\mathbf{tr}}\cosh(2\vartheta) - \left(\vec{m}\cdot\vec{V}_{\mathbf{tr}}\right)\vec{m}\left(\cosh 2\vartheta - 1\right) + \mathbf{e}_{\mathbf{tr}}\left[\vec{m}\times\vec{V}\right]\sinh(2\vartheta).
\end{aligned} \tag{3.5}$$

The Lorentz transforms (3.4) coincide with the common used transformations of field potentials in classical electrodynamics, while the transformations (3.5) are valid for the electromagnetic field strengths.

4. Second-order equation for massive field

Let us consider the sedeonic second-order wave equation for massive field [12]:

$$\left(i\mathbf{e}_{\mathbf{t}}\partial - \mathbf{e}_{\mathbf{r}}\vec{\nabla} - i\mathbf{e}_{\mathbf{tr}}m\right)\left(i\mathbf{e}_{\mathbf{t}}\partial - \mathbf{e}_{\mathbf{r}}\vec{\nabla} - i\mathbf{e}_{\mathbf{tr}}m\right)\tilde{\mathbf{W}}_{\mathbf{m}} = \tilde{\mathbf{J}}_{\mathbf{m}}. \tag{4.1}$$

where $\tilde{\mathbf{W}}_{\mathbf{m}}$ is a sedeonic potential, $\tilde{\mathbf{J}}_{\mathbf{m}}$ is a phenomenological sedeonic source of massive field (index **m**). We use the following operators:

$$\begin{aligned}
\partial &= \frac{1}{c}\frac{\partial}{\partial t},\\
\vec{\nabla} &= \frac{\partial}{\partial x}\mathbf{a}_1 + \frac{\partial}{\partial y}\mathbf{a}_2 + \frac{\partial}{\partial z}\mathbf{a}_3,\\
m &= \frac{m_0 c}{\hbar}.
\end{aligned} \tag{4.2}$$

Let us choose the potential as

$$\tilde{\mathbf{W}}_{\mathbf{m}} = ia_1\mathbf{e}_{\mathbf{t}} - ia_2\mathbf{e}_{\mathbf{r}} + a_3 - ia_4\mathbf{e}_{\mathbf{tr}} + \vec{A}_1\mathbf{e}_{\mathbf{r}} + \vec{A}_2\mathbf{e}_{\mathbf{t}} - \vec{A}_3\mathbf{e}_{\mathbf{tr}} + i\vec{A}_4, \tag{4.3}$$

where components a_{S} and $\vec{A}_{\mathrm{S}}$ are real functions of coordinates and time. Here and further the index S = 1, 2, 3, 4. Also we take the source in the following form:

$$\tilde{\mathbf{J}}_{\mathbf{m}} = -i\rho_1\mathbf{e}_{\mathbf{t}} + i\rho_2\mathbf{e}_{\mathbf{r}} - \rho_3 + i\rho_4\mathbf{e}_{\mathbf{tr}} - \vec{j}_1\mathbf{e}_{\mathbf{r}} - \vec{j}_2\mathbf{e}_{\mathbf{t}} + \vec{j}_3\mathbf{e}_{\mathbf{tr}} - \vec{j}_4 i, \tag{4.4}$$

where $\rho_{\mathrm{S}} = 4\pi\rho'_{\mathrm{S}}$ (ρ'_{k} is the volume density of charge) and $\vec{j}_{\mathrm{S}} = \frac{4\pi}{c}\vec{j}'_{\mathrm{S}}$ ($\vec{j}'_{\mathrm{S}}$ is volume density of current). Multiplying the operators in the left part of equation (4.1) we obtain the following wave equations for the components of potentials:

$$\begin{aligned}&\left(\partial^2-\Delta+m^2\right)a_S=\rho_S,\\&\left(\partial^2-\Delta+m^2\right)\vec{A}_S=\vec{j}_S.\end{aligned}\tag{4.7}$$

Let us introduce the scalar g_S and vector $\vec{G}_S$ field strengths according the following definitions:

$$\begin{aligned}
g_1&=\partial a_1+\left(\vec{\nabla}\cdot\vec{A}_1\right)+ma_4,\\
g_2&=\partial a_2+\left(\vec{\nabla}\cdot\vec{A}_2\right)-ma_3,\\
g_3&=\partial a_3+\left(\vec{\nabla}\cdot\vec{A}_3\right)+ma_2,\\
g_4&=\partial a_4+\left(\vec{\nabla}\cdot\vec{A}_4\right)-ma_1,\\
\vec{G}_1&=-\partial\vec{A}_1-\vec{\nabla}a_1+i\left[\vec{\nabla}\times\vec{A}_2\right]+m\vec{A}_4,\\
\vec{G}_2&=-\partial\vec{A}_2-\vec{\nabla}a_2-i\left[\vec{\nabla}\times\vec{A}_1\right]-m\vec{A}_3,\\
\vec{G}_3&=-\partial\vec{A}_3-\vec{\nabla}a_3-i\left[\vec{\nabla}\times\vec{A}_4\right]+m\vec{A}_2,\\
\vec{G}_4&=-\partial\vec{A}_4-\vec{\nabla}a_4+i\left[\vec{\nabla}\times\vec{A}_3\right]-m\vec{A}_1.
\end{aligned}\tag{4.8}$$

The definitions of field strengths (4.8) have the specific gauge invariance. It is easy to verify that g_S and $\vec{G}_S$ are not changed under the following substitutions for the potentials:

$$\begin{aligned}
a_1&\Rightarrow a_1+\partial\varepsilon_1-m\varepsilon_4,\\
a_2&\Rightarrow a_2+\partial\varepsilon_2+m\varepsilon_3,\\
a_3&\Rightarrow a_3+\partial\varepsilon_3-m\varepsilon_2,\\
a_4&\Rightarrow a_4+\partial\varepsilon_4+m\varepsilon_1,\\
\vec{A}_1&\Rightarrow\vec{A}_1-\vec{\nabla}\varepsilon_1,\\
\vec{A}_2&\Rightarrow\vec{A}_2-\vec{\nabla}\varepsilon_2,\\
\vec{A}_3&\Rightarrow\vec{A}_3-\vec{\nabla}\varepsilon_3,\\
\vec{A}_4&\Rightarrow\vec{A}_4-\vec{\nabla}\varepsilon_4.
\end{aligned}\tag{4.9}$$

Here ε_1, ε_2, ε_3, ε_4 are arbitrary scalar functions satisfying the homogeneous Klein-Gordon wave equation. Taking into account (4.8) we get that

$$\begin{aligned}&\left(i\mathbf{e_t}\partial-\mathbf{e_r}\vec{\nabla}-i\mathbf{e_{tr}}m\right)\left(ia_1\mathbf{e_t}-ia_2\mathbf{e_r}+a_3-ia_4\mathbf{e_{tr}}+\vec{A}_1\mathbf{e_r}+\vec{A}_2\mathbf{e_t}-\vec{A}_3\mathbf{e_{tr}}+i\vec{A}_4\right)\\&=-g_1+ig_2\mathbf{e_{tr}}+ig_3\mathbf{e_t}-ig_4\mathbf{e_r}+\vec{G}_1\mathbf{e_{tr}}-i\vec{G}_2+\vec{G}_3\mathbf{e_r}+\vec{G}_4\mathbf{e_t},\end{aligned}\tag{4.10}$$

and the initial wave equation (4.1) is reduced to the following equation:

$$\begin{aligned}&\left(i\mathbf{e_t}\partial-\mathbf{e_r}\vec{\nabla}-i\mathbf{e_{tr}}m\right)\left(-g_1+ig_2\mathbf{e_{tr}}+ig_3\mathbf{e_t}-ig_4\mathbf{e_r}+\vec{G}_1\mathbf{e_{tr}}-i\vec{G}_2+\vec{G}_3\mathbf{e_r}+\vec{G}_4\mathbf{e_t}\right)\\&=-i\rho_1\mathbf{e_t}+i\rho_2\mathbf{e_r}-\rho_3+i\rho_4\mathbf{e_{tr}}-\vec{j}_1\mathbf{e_r}-\vec{j}_2\mathbf{e_t}+\vec{j}_3\mathbf{e_{tr}}-\vec{j}_4 i.\end{aligned}\tag{4.11}$$

Producing the action of the operator on the left side of equation (4.11) and separating the values with different space-time properties, we obtain a system of equations for the field strengths, similar to the system of Maxwell equations in electrodynamics:

$$
\begin{aligned}
&\partial g_1+\left(\vec{\nabla}\cdot\vec{G}_1\right)-mg_4=\rho_1\ ,\\
&\partial g_2+\left(\vec{\nabla}\cdot\vec{G}_2\right)+mg_3=\rho_2,\\
&\partial g_3+\left(\vec{\nabla}\cdot\vec{G}_3\right)-mg_2=\rho_3,\\
&\partial g_4+\left(\vec{\nabla}\cdot\vec{G}_4\right)+mg_1=\rho_4,\\
&\partial\vec{G}_1+\vec{\nabla}g_1+i\left[\vec{\nabla}\times\vec{G}_2\right]+m\vec{G}_4=-\vec{j}_1,\\
&\partial\vec{G}_2+\vec{\nabla}g_2-i\left[\vec{\nabla}\times\vec{G}_1\right]-m\vec{G}_3=-\vec{j}_2,\\
&\partial\vec{G}_3+\vec{\nabla}g_3-i\left[\vec{\nabla}\times\vec{G}_4\right]+m\vec{G}_2=-\vec{j}_3,\\
&\partial\vec{G}_4+\vec{\nabla}g_4+i\left[\vec{\nabla}\times\vec{G}_3\right]-m\vec{G}_1=-\vec{j}_4.
\end{aligned}
\tag{4.12}
$$

The system (4.12) is also invariant with respect to the following substitutions:

$$
\begin{aligned}
&g_1\Rightarrow g_1+\partial\varepsilon_1-m\varepsilon_4,\\
&g_2\Rightarrow g_2-\partial\varepsilon_2-m\varepsilon_3,\\
&g_3\Rightarrow g_3+\partial\varepsilon_3-m\varepsilon_2,\\
&g_4\Rightarrow g_4-\partial\varepsilon_4-m\varepsilon_1,\\
&\vec{G}_1\Rightarrow\vec{G}_1-\vec{\nabla}\varepsilon_1,\\
&\vec{G}_2\Rightarrow\vec{G}_2+\vec{\nabla}\varepsilon_2,\\
&\vec{G}_3\Rightarrow\vec{G}_3-\vec{\nabla}\varepsilon_3,\\
&\vec{G}_4\Rightarrow\vec{G}_4+\vec{\nabla}\varepsilon_4,
\end{aligned}
\tag{4.13}
$$

Multiplying each of the equations (4.12) to the corresponding field strength and adding these equations to each other, we obtain:

$$
\begin{aligned}
&\frac{1}{2}\partial\left(g_1^2+g_2^2+g_3^2+g_4^2+\vec{G}_1^2+\vec{G}_2^2+\vec{G}_3^2+\vec{G}_4^2\right)\\
&+g_1\left(\vec{\nabla}\cdot\vec{G}_1\right)+g_2\left(\vec{\nabla}\cdot\vec{G}_2\right)+g_3\left(\vec{\nabla}\cdot\vec{G}_3\right)+g_4\left(\vec{\nabla}\cdot\vec{G}_4\right)\\
&+\left(\vec{G}_1\cdot\vec{\nabla}g_1\right)+\left(\vec{G}_2\cdot\vec{\nabla}g_2\right)+\left(\vec{G}_3\cdot\vec{\nabla}g_3\right)+\left(\vec{G}_4\cdot\vec{\nabla}g_4\right)\\
&+i\left(\vec{G}_1\cdot\left[\vec{\nabla}\times\vec{G}_2\right]\right)-i\left(\vec{G}_2\cdot\left[\vec{\nabla}\times\vec{G}_1\right]\right)-i\left(\vec{G}_3\cdot\left[\vec{\nabla}\times\vec{G}_4\right]\right)+i\left(\vec{G}_4\cdot\left[\vec{\nabla}\times\vec{G}_3\right]\right)\\
&=g_1\rho_1+g_2\rho_2+g_3\rho_3+g_4\rho_4-\left(\vec{G}_1\cdot\vec{j}_1\right)-\left(\vec{G}_2\cdot\vec{j}_2\right)-\left(\vec{G}_3\cdot\vec{j}_3\right)-\left(\vec{G}_4\cdot\vec{j}_4\right).
\end{aligned}
\tag{4.14}
$$

This expression is the analog of Poynting's theorem for massive field. The term

$$
w=\frac{1}{8\pi}\left(g_1^2+g_2^2+g_3^2+g_4^2+\vec{G}_1^2+\vec{G}_2^2+\vec{G}_3^2+\vec{G}_4^2\right)
\tag{4.15}
$$

plays the role of field energy density, while the term

$$
\vec{p}=\frac{c}{4\pi}\left(g_1\vec{G}_1+g_2\vec{G}_2+g_3\vec{G}_3+g_4\vec{G}_4-i\left[\vec{G}_1\times\vec{G}_2\right]+i\left[\vec{G}_3\times\vec{G}_4\right]\right)
\tag{4.16}
$$

plays the role of energy flux density.

On the other hand, applying the operator $\left(i\mathbf{e_t}\partial-\mathbf{e_r}\vec{\nabla}-i\mathbf{e_{tr}}m\right)$ to the equation (4.11) we obtain the following wave equation for the field strengths:

$$
\begin{aligned}
&\left(i\mathbf{e_t}\partial-\mathbf{e_r}\vec{\nabla}-i\mathbf{e_{tr}}m\right)\left(i\mathbf{e_t}\partial-\mathbf{e_r}\vec{\nabla}-i\mathbf{e_{tr}}m\right)\left(-g_1+ig_2\mathbf{e_{tr}}+ig_3\mathbf{e_t}-ig_4\mathbf{e_r}+\vec{G}_1\mathbf{e_{tr}}-i\vec{G}_2+\vec{G}_3\mathbf{e_r}+\vec{G}_4\mathbf{e_t}\right)\\
&=\left(i\mathbf{e_t}\partial-\mathbf{e_r}\vec{\nabla}-i\mathbf{e_{tr}}m\right)\left(-i\rho_1\mathbf{e_t}+i\rho_2\mathbf{e_r}-\rho_3+i\rho_4\mathbf{e_{tr}}-\vec{j}_1\mathbf{e_r}-\vec{j}_2\mathbf{e_t}+\vec{j}_3\mathbf{e_{tr}}-\vec{j}_4 i\right).
\end{aligned}
\tag{4.17}
$$

Separating the terms with different space-time properties we get the following wave equation for the field strength components g_S and $\vec{G}_S$:

$$
\begin{aligned}
&\left(\partial^2-\Delta+m^2\right)g_1=-\partial\rho_1-\left(\vec{\nabla}\cdot\vec{j}_1\right)-m\rho_4,\\
&\left(\partial^2-\Delta+m^2\right)g_2=-\partial\rho_2-\left(\vec{\nabla}\cdot\vec{j}_2\right)+m\rho_3,\\
&\left(\partial^2-\Delta+m^2\right)g_3=-\partial\rho_3-\left(\vec{\nabla}\cdot\vec{j}_3\right)-m\rho_2,\\
&\left(\partial^2-\Delta+m^2\right)g_4=-\partial\rho_4-\left(\vec{\nabla}\cdot\vec{j}_4\right)+m\rho_1,\\
&\left(\partial^2-\Delta+m^2\right)\vec{G}_1=\vec{\nabla}\rho_1+\partial\vec{j}_1-i\left[\vec{\nabla}\times\vec{j}_2\right]-m\vec{j}_4,\\
&\left(\partial^2-\Delta+m^2\right)\vec{G}_2=\vec{\nabla}\rho_2+\partial\vec{j}_2+i\left[\vec{\nabla}\times\vec{j}_1\right]+m\vec{j}_3,\\
&\left(\partial^2-\Delta+m^2\right)\vec{G}_3=\vec{\nabla}\rho_3+\partial\vec{j}_3+i\left[\vec{\nabla}\times\vec{j}_4\right]-m\vec{j}_2,\\
&\left(\partial^2-\Delta+m^2\right)\vec{G}_4=\vec{\nabla}\rho_4+\partial\vec{j}_4-i\left[\vec{\nabla}\times\vec{j}_3\right]+m\vec{j}_1.
\end{aligned}
\tag{4.18}
$$

It can be seen that equations (4.18) are invariant with respect to the following substitutions:

$$
\begin{aligned}
&\rho_1\Rightarrow\rho_1+\partial\varepsilon_1-m\varepsilon_4,\\
&\rho_2\Rightarrow\rho_2+\partial\varepsilon_2+m\varepsilon_3,\\
&\rho_3\Rightarrow\rho_3+\partial\varepsilon_3-m\varepsilon_2,\\
&\rho_4\Rightarrow\rho_4+\partial\varepsilon_4+m\varepsilon_1,\\
&\vec{j}_1\Rightarrow\vec{j}_1-\vec{\nabla}\varepsilon_1,\\
&\vec{j}_2\Rightarrow\vec{j}_2-\vec{\nabla}\varepsilon_2,\\
&\vec{j}_3\Rightarrow\vec{j}_3-\vec{\nabla}\varepsilon_3,\\
&\vec{j}_4\Rightarrow\vec{j}_4-\vec{\nabla}\varepsilon_4.
\end{aligned}
\tag{4.19}
$$

As an example, let us consider the fields produced by a one type of sources ρ_1 and $\vec{j}_1$. In this case the massive field is described by a_1 and $\vec{A}_1$ potentials:

$$
\tilde{\mathbf{W}}_{\mathbf{m}}=ia_1\mathbf{e}_{\mathbf{t}}+\vec{A}_1\mathbf{e}_{\mathbf{r}}\,. \tag{4.20}
$$

Then we have only the following nonzero field's strengths:

$$
\begin{aligned}
&g_1=\partial a_1+\left(\vec{\nabla}\cdot\vec{A}_1\right),\\
&g_4=-ma_1,\\
&\vec{G}_1=-\partial\vec{A}_1-\vec{\nabla}a_1,\\
&\vec{G}_2=-i\left[\vec{\nabla}\times\vec{A}_1\right],\\
&\vec{G}_4=-m\vec{A}_1,
\end{aligned}
\tag{4.21}
$$

and the wave equation (4.4) takes the following form:

$$
\begin{aligned}
&\left(i\mathbf{e}_{\mathbf{t}}\partial-\mathbf{e}_{\mathbf{r}}\vec{\nabla}-i\mathbf{e}_{\mathbf{tr}}m\right)\left(-g_1-ig_4\mathbf{e}_{\mathbf{r}}+\vec{G}_1\mathbf{e}_{\mathbf{tr}}-i\vec{G}_2+\vec{G}_4\mathbf{e}_{\mathbf{t}}\right)\\
&=-i\rho_1\mathbf{e}_{\mathbf{t}}-\vec{j}_1\mathbf{e}_{\mathbf{r}}.
\end{aligned}
\tag{4.22}
$$

Then the system (4.12) can be rewritten as

$$
\begin{aligned}
&\partial g_1+\left(\vec{\nabla}\cdot\vec{G}_1\right)-mg_4=\rho_1\,,\\
&\left(\vec{\nabla}\cdot\vec{G}_2\right)=0,\\
&\partial g_4+\left(\vec{\nabla}\cdot\vec{G}_4\right)+mg_1=0,\\
&\partial\vec{G}_1+\vec{\nabla}g_1+i\left[\vec{\nabla}\times\vec{G}_2\right]+m\vec{G}_4=-\vec{j}_1,\\
&\partial\vec{G}_2-i\left[\vec{\nabla}\times\vec{G}_1\right]=0,\\
&-i\left[\vec{\nabla}\times\vec{G}_4\right]+m\vec{G}_2=0,\\
&\partial\vec{G}_4+\vec{\nabla}g_4-m\vec{G}_1=0.
\end{aligned}
\tag{4.23}
$$

The system (4.23) is the analog of Proca-Maxwell equations. In addition, we have the following wave equations for the field strengths:

$$\begin{aligned}
&\left(\partial^2-\Delta+m^2\right)g_1 = \partial\rho_1+\left(\vec{\nabla}\cdot\vec{j}_1\right),\\
&\left(\partial^2-\Delta+m^2\right)g_4 = -m\rho_1,\\
&\left(\partial^2-\Delta+m^2\right)\vec{G}_1 = -\vec{\nabla}\rho_1-\partial\vec{j}_1,\\
&\left(\partial^2-\Delta+m^2\right)\vec{G}_2 = -i\left[\vec{\nabla}\times\vec{j}_1\right],\\
&\left(\partial^2-\Delta+m^2\right)\vec{G}_4 = -m\vec{j}_1.
\end{aligned} \tag{4.24}$$

Assuming the charge conservation

$$\partial\rho_1+\left(\vec{\nabla}\cdot\vec{j}_1\right)=0, \tag{4.25}$$

we can choose the scalar field strength g_1 equal to zero. This is equivalent to the following gauge condition:

$$\partial a_1+\left(\vec{\nabla}\cdot\vec{A}_1\right)=0, \tag{4.26}$$

similar to the Lorentz gauge in electrodynamics.

Let us consider the stationary field of point scalar source. In the static case $\vec{j}_1=0$, and potential of the field can be chosen as

$$\tilde{\mathbf{W}}_{\mathbf{m}} = i\mathbf{e}_{\mathbf{t}}a_1\left(\vec{r}\right). \tag{4.27}$$

Then we have only two nonzero field components**:**

$$\begin{aligned}
g_4 &= -ma_1,\\
\vec{G}_1 &= -\vec{\nabla}a_1,
\end{aligned} \tag{4.28}$$

and the following filed equations:

$$\begin{aligned}
&\left(\vec{\nabla}\cdot\vec{G}_1\right)-mg_4=\rho_1,\\
&-i\left[\vec{\nabla}\times\vec{G}_1\right]=0,\\
&\vec{\nabla}g_4-m\vec{G}_1=0.
\end{aligned} \tag{4.29}$$

As an example, let us consider the field produced by scalar point source. In this case the charge density can be presented as

$$\rho_1 = q_1\delta\left(\vec{r}\right), \tag{4.30}$$

where q_1 is the point charge and $\delta\left(\vec{r}\right)$ is delta function. Then stationary wave equation can be written in the spherical coordinates as

$$\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)-m^2\right)a_1\left(\vec{r}\right)=-\,q_1\delta\left(\vec{r}\right). \tag{4.31}$$

The partial solution of the equation (4.31), which decays at $r\to\infty$, is

$$a_1=\frac{q_1}{r}\exp\left(-mr\right). \tag{4.32}$$

Thus, the stationary field has scalar and vector components

$$g_4=-m\frac{q_1}{r}\exp\left(-mr\right), \tag{4.33}$$

$$\vec{G}_1=\left(\frac{1}{r}+m\right)\frac{q_1}{r}\exp\left(-mr\right)\vec{r}_0, \tag{4.34}$$

where $\vec{r}_0$ is a unit radial vector.

Let us consider the interaction of two point charges q_{11} and q_{12} due to the overlap of their fields. Taking into account that the field in this case is the sum of the two fields $g_4 = g_{41} + g_{42}$ and $\vec{G}_1 = \vec{G}_{11} + \vec{G}_{12}$, the energy of interaction is equal (see expression (4.15))

$$W_{12} = \frac{1}{4\pi}\int\left\{g_{41}g_{42} + \left(\vec{G}_{11}\cdot\vec{G}_{12}\right)\right\} dV\,, \tag{4.35}$$

where the integral is over all space. Substituting (4.33) and (4.34), we obtain

$$W_{12} = \frac{q_{11}q_{12}}{R}\exp\left(-mR\right), \tag{4.36}$$

where R is the distance between the point charges.

5. Second-order equation for massless field

In the case of massless field the equation (4.1) takers the following form [13]:

$$\left(i\mathbf{e_t}\partial - \mathbf{e_r}\vec{\nabla}\right)\left(i\mathbf{e_t}\partial - \mathbf{e_r}\vec{\nabla}\right)\tilde{\mathbf{W}}_\mathbf{0} = \tilde{\mathbf{J}}_\mathbf{0}\,, \tag{5.1}$$

where we choose the potential $\tilde{\mathbf{W}}_\mathbf{0}$ and source $\tilde{\mathbf{J}}_\mathbf{0}$ of massless field (index **0**) in the form of (4.3) and (4.4) as before

$$\tilde{\mathbf{W}}_\mathbf{0} = ib_1\mathbf{e_t} - ib_2\mathbf{e_r} + b_3 - ib_4\mathbf{e_{tr}} + \vec{B}_1\mathbf{e_r} + \vec{B}_2\mathbf{e_t} - \vec{B}_3\mathbf{e_{tr}} + i\vec{B}_4\,, \tag{5.2}$$

$$\tilde{\mathbf{J}}_\mathbf{0} = -i\beta_1\mathbf{e_t} + i\beta_2\mathbf{e_r} - \beta_3 + i\beta_4\mathbf{e_{tr}} - \vec{l}_1\mathbf{e_r} - \vec{l}_2\mathbf{e_t} + \vec{l}_3\mathbf{e_{tr}} - \vec{l}_4 i\,, \tag{5.3}$$

where $\beta_s = 4\pi\beta_s'$ (β_s' is the volume density of charge) and $\vec{l}_s = \frac{4\pi}{c}\vec{l}_s'$ ($\vec{l}_s'$ is volume density of current). We introduce the scalar and vector field strengths according following definitions:

$$\begin{aligned}
h_1 &= \partial b_1 + \left(\vec{\nabla}\cdot\vec{B}_1\right),\\
h_2 &= \partial b_2 + \left(\vec{\nabla}\cdot\vec{B}_2\right),\\
h_3 &= \partial b_3 + \left(\vec{\nabla}\cdot\vec{B}_3\right),\\
h_4 &= \partial b_4 + \left(\vec{\nabla}\cdot\vec{B}_4\right)_1,\\
\vec{H}_1 &= -\partial\vec{B}_1 - \vec{\nabla}b_1 + i\left[\vec{\nabla}\times\vec{B}_2\right],\\
\vec{H}_2 &= -\partial\vec{B}_2 - \vec{\nabla}b_2 - i\left[\vec{\nabla}\times\vec{B}_1\right],\\
\vec{H}_3 &= -\partial\vec{B}_3 - \vec{\nabla}b_3 - i\left[\vec{\nabla}\times\vec{B}_4\right],\\
\vec{H}_4 &= -\partial\vec{B}_4 - \vec{\nabla}b_4 + i\left[\vec{\nabla}\times\vec{B}_3\right].
\end{aligned} \tag{5.4}$$

Note that the definitions (5.4) are invariant with respect to the following substitutions:

$$\begin{aligned}
b_1 &\Rightarrow b_1 + \partial\varepsilon_1,\\
b_2 &\Rightarrow b_2 + \partial\varepsilon_2,\\
b_3 &\Rightarrow b_3 + \partial\varepsilon_3,\\
b_4 &\Rightarrow b_4 + \partial\varepsilon_4,\\
\vec{B}_1 &\Rightarrow \vec{B}_1 - \vec{\nabla}\varepsilon_1,\\
\vec{B}_2 &\Rightarrow \vec{B}_2 - \vec{\nabla}\varepsilon_2,\\
\vec{B}_3 &\Rightarrow \vec{B}_3 - \vec{\nabla}\varepsilon_3,\\
\vec{B}_4 &\Rightarrow \vec{B}_4 - \vec{\nabla}\varepsilon_4.
\end{aligned} \tag{5.5}$$

Taking into account (5.4) we get

$$\left(i\mathbf{e_t}\partial-\mathbf{e_r}\vec{\nabla}\right)\left(ib_1\mathbf{e_t}-ib_2\mathbf{e_r}+b_3-ib_4\mathbf{e_{tr}}+\vec{B}_1\mathbf{e_r}+\vec{B}_2\mathbf{e_t}-\vec{B}_3\mathbf{e_{tr}}+i\vec{B}_4\right)$$
$$=-h_1+ih_2\mathbf{e_{tr}}+ih_3\mathbf{e_t}-ih_4\mathbf{e_r}+\vec{H}_1\mathbf{e_{tr}}-i\vec{H}_2+\vec{H}_3\mathbf{e_r}+\vec{H}_4\mathbf{e_t}, \tag{5.6}$$

and wave equation (5.1) can be rewritten as

$$\left(i\mathbf{e_t}\partial-\mathbf{e_r}\vec{\nabla}\right)\left(-h_1+ih_2\mathbf{e_{tr}}+ih_3\mathbf{e_t}-ih_4\mathbf{e_r}+\vec{H}_1\mathbf{e_{tr}}-i\vec{H}_2+\vec{H}_3\mathbf{e_r}+\vec{H}_4\mathbf{e_t}\right)$$
$$=-i\beta_1\mathbf{e_t}+i\beta_2\mathbf{e_r}-\beta_3+i\beta_4\mathbf{e_{tr}}-\vec{l}_1\mathbf{e_r}-\vec{l}_2\mathbf{e_t}+\vec{l}_3\mathbf{e_{tr}}-\vec{l}_4 i. \tag{5.7}$$

Producing the action of the operator on the left side of equation (5.7) and separating the terms with different space-time properties, we obtain two independent systems of the equations for the field strengths, similar to the system of Maxwell equations in electrodynamics. The first system is

$$\begin{aligned}
&\partial h_1+\left(\vec{\nabla}\cdot\vec{H}_1\right)=\beta_1,\\
&\partial h_2+\left(\vec{\nabla}\cdot\vec{H}_2\right)=\beta_2,\\
&\partial\vec{H}_1+\vec{\nabla}h_1+i\left[\vec{\nabla}\times\vec{H}_2\right]=-\vec{l}_1,\\
&\partial\vec{H}_2+\vec{\nabla}h_2-i\left[\vec{\nabla}\times\vec{H}_1\right]=-\vec{l}_2.
\end{aligned} \tag{5.8}$$

This system is invariant with respect to the following substitutions:

$$\begin{aligned}
&h_1\Rightarrow h_1+\partial\varepsilon_1,\\
&h_2\Rightarrow h_2-\partial\varepsilon_2,\\
&\vec{H}_1\Rightarrow\vec{H}_1-\vec{\nabla}\varepsilon_1,\\
&\vec{H}_2\Rightarrow\vec{H}_2+\vec{\nabla}\varepsilon_2.
\end{aligned} \tag{5.9}$$

Multiplying each of the equations (5.8) to the corresponding field strength and adding these equations to each other, we obtain:

$$\begin{aligned}
&\frac{1}{2}\partial\left(h_1^2+h_2^2+\vec{H}_1^2+\vec{H}_2^2\right)\\
&+h_1\left(\vec{\nabla}\cdot\vec{H}_1\right)+h_2\left(\vec{\nabla}\cdot\vec{H}_2\right)\\
&+\left(\vec{H}_1\cdot\vec{\nabla}h_1\right)+\left(\vec{H}_2\cdot\vec{\nabla}h_2\right)\\
&+i\left(\vec{H}_1\cdot\left[\vec{\nabla}\times\vec{H}_2\right]\right)-i\left(\vec{H}_2\cdot\left[\vec{\nabla}\times\vec{H}_1\right]\right)\\
&=h_1\beta_1+h_2\beta_2-\left(\vec{H}_1\cdot\vec{l}_1\right)-\left(\vec{H}_2\cdot\vec{l}_2\right).
\end{aligned} \tag{5.10}$$

This expression is the analog of Poynting's theorem for first type of massless field. The term

$$w=\frac{1}{8\pi}\left(h_1^2+h_2^2+\vec{H}_1^2+\vec{H}_2^2\right) \tag{5.11}$$

plays the role of field energy density, while the term

$$\vec{p}=\frac{c}{4\pi}\left(h_1\vec{H}_1+h_2\vec{H}_2-i\left[\vec{H}_1\times\vec{H}_2\right]\right) \tag{5.12}$$

plays the role of energy flux density.

The second system is

$$\begin{aligned}
&\partial h_3+\left(\vec{\nabla}\cdot\vec{H}_3\right)=\beta_3,\\
&\partial h_4+\left(\vec{\nabla}\cdot\vec{H}_4\right)=\beta_4,\\
&\partial\vec{H}_3+\vec{\nabla}h_3-i\left[\vec{\nabla}\times\vec{H}_4\right]=-\vec{l}_3,\\
&\partial\vec{H}_4+\vec{\nabla}h_4+i\left[\vec{\nabla}\times\vec{H}_3\right]=-\vec{l}_4.
\end{aligned} \tag{5.13}$$

This system is invariant with respect to the following substitutions:

$$\begin{aligned} &h_3 \Rightarrow h_3 + \partial\varepsilon_3, \\ &h_4 \Rightarrow h_4 - \partial\varepsilon_4, \\ &\vec{H}_3 \Rightarrow \vec{H}_3 - \vec{\nabla}\varepsilon_3, \\ &\vec{H}_4 \Rightarrow \vec{H}_4 + \vec{\nabla}\varepsilon_4. \end{aligned} \tag{5.14}$$

Multiplying each of the equations (5.13) to the corresponding field strength and adding these equations to each other, we obtain:

$$\begin{aligned} &\frac{1}{2}\partial\left(h_3^2 + h_4^2 + \vec{H}_3^2 + \vec{H}_4^2\right) \\ &+h_3\left(\vec{\nabla}\cdot\vec{H}_3\right) + h_4\left(\vec{\nabla}\cdot\vec{H}_4\right) \\ &+\left(\vec{H}_3\cdot\vec{\nabla}h_3\right) + \left(\vec{H}_4\cdot\vec{\nabla}h_4\right) \\ &+i\left(\vec{H}_3\cdot\left[\vec{\nabla}\times\vec{H}_4\right]\right) - i\left(\vec{H}_4\cdot\left[\vec{\nabla}\times\vec{H}_3\right]\right) \\ &= h_3\beta_3 + h_4\beta_4 - \left(\vec{H}_3\cdot\vec{l}_3\right) - \left(\vec{H}_4\cdot\vec{l}_4\right). \end{aligned} \tag{5.15}$$

This expression is the analog of Poynting's theorem for second type of massless field. The term

$$w = \frac{1}{8\pi}\left(h_3^2 + h_4^2 + \vec{H}_3^2 + \vec{H}_4^2\right) \tag{5.16}$$

plays the role of field energy density, while the term

$$\vec{p} = \frac{c}{4\pi}\left(h_3\vec{H}_3 + h_4\vec{H}_4 - i\left[\vec{H}_3\times\vec{H}_4\right]\right) \tag{5.17}$$

plays the role of energy flux density.

Accordingly, the wave equations for the massless field strengths are also divided into two independent systems. The first system combines the potentials and sources, which are transformed in accordance with Lorentz transformations of type I (see (3.4))

$$\begin{aligned} &\left(\partial^2 - \Delta\right)h_1 = -\partial\beta_1 - \left(\vec{\nabla}\cdot\vec{l}_1\right), \\ &\left(\partial^2 - \Delta\right)h_2 = -\partial\beta_2 - \left(\vec{\nabla}\cdot\vec{l}_2\right), \\ &\left(\partial^2 - \Delta\right)\vec{H}_1 = \vec{\nabla}\beta_1 + \partial\vec{l}_1 - i\left[\vec{\nabla}\times\vec{l}_2\right], \\ &\left(\partial^2 - \Delta\right)\vec{H}_2 = \vec{\nabla}\beta_2 + \partial\vec{l}_2 + i\left[\vec{\nabla}\times\vec{l}_1\right]. \end{aligned} \tag{5.18}$$

The second system combines the fields and sources, which are transformed in accordance with Lorentz transformations of type II (see (3.5))

$$\begin{aligned} &\left(\partial^2 - \Delta\right)h_3 = -\partial\beta_3 - \left(\vec{\nabla}\cdot\vec{l}_3\right), \\ &\left(\partial^2 - \Delta\right)h_4 = -\partial\beta_4 - \left(\vec{\nabla}\cdot\vec{l}_4\right), \\ &\left(\partial^2 - \Delta\right)\vec{H}_3 = \vec{\nabla}\beta_3 + \partial\vec{l}_3 + i\left[\vec{\nabla}\times\vec{l}_4\right], \\ &\left(\partial^2 - \Delta\right)\vec{H}_4 = \vec{\nabla}\beta_4 + \partial\vec{l}_4 - i\left[\vec{\nabla}\times\vec{l}_3\right]. \end{aligned} \tag{5.19}$$

The equations (5.18) and (5.19) are invariant with respect to the substitutions

$$
\begin{aligned}
&\beta_1 \Rightarrow \beta_1 + \partial\varepsilon_1,\\
&\beta_2 \Rightarrow \beta_2 + \partial\varepsilon_2,\\
&\beta_3 \Rightarrow \beta_3 + \partial\varepsilon_3,\\
&\beta_4 \Rightarrow \beta_4 + \partial\varepsilon_4,\\
&\vec{l}_1 \Rightarrow \vec{l}_1 - \vec{\nabla}\varepsilon_1,\\
&\vec{l}_2 \Rightarrow \vec{l}_2 - \vec{\nabla}\varepsilon_2,\\
&\vec{l}_3 \Rightarrow \vec{l}_3 - \vec{\nabla}\varepsilon_3,\\
&\vec{l}_4 \Rightarrow \vec{l}_4 - \vec{\nabla}\varepsilon_4.
\end{aligned}
\tag{5.20}
$$

The system of equations (5.8) corresponds to the usual system of Maxwell equations. Let us show it. If we assume the charge conservation

$$
\begin{aligned}
&\partial\beta_1 + \left(\vec{\nabla}\cdot\vec{l}_1\right) = 0,\\
&\partial\beta_2 + \left(\vec{\nabla}\cdot\vec{l}_2\right) = 0,
\end{aligned}
\tag{5.21}
$$

then as it follows from (5.18) we can choose the scalar fields h_1 and h_2 equal to zero and obtain the following system:

$$
\begin{aligned}
&\left(\vec{\nabla}\cdot\vec{H}_1\right) = \beta_1,\\
&\left(\vec{\nabla}\cdot\vec{H}_2\right) = \beta_2,\\
&\partial\vec{H}_1 + i\left[\vec{\nabla}\times\vec{H}_2\right] = -\vec{l}_1,\\
&\partial\vec{H}_2 - i\left[\vec{\nabla}\times\vec{H}_1\right] = -\vec{l}_2.
\end{aligned}
\tag{5.22}
$$

Here $\vec{H}_1$ is the electric field strength; $\vec{H}_2$ is the magnetic field strength; β_1 is the volume density of electrical charge; β_2 is the volume density of magnetic charge; $\vec{l}_1$ is the volume density of electrical current; $\vec{l}_2$ is the volume density of magnetic current. Taking into account the experimental fact that in our part of the universe there are no magnetic charges and currents, we obtain the system of equations

$$
\begin{aligned}
&\left(\vec{\nabla}\cdot\vec{H}_1\right) = \beta_1,\\
&\left(\vec{\nabla}\cdot\vec{H}_2\right) = 0,\\
&\partial\vec{H}_1 + i\left[\vec{\nabla}\times\vec{H}_2\right] = -\vec{l}_1,\\
&\partial\vec{H}_2 - i\left[\vec{\nabla}\times\vec{H}_1\right] = 0,
\end{aligned}
\tag{5.23}
$$

which coincides with the conventional system of Maxwell's equations.

6. First-order equation for massive field

Let us consider a massive field, which is described by the sedeonic first-order equation [12]:

$$
\left(i\mathbf{e_t}\partial - \mathbf{e_r}\vec{\nabla} - i\mathbf{e_{tr}}m\right)\tilde{\mathbf{W}}_\mathbf{m} = \tilde{\mathbf{I}}_\mathbf{m} \quad . \tag{6.1}
$$

Here $\tilde{\mathbf{I}}_\mathbf{m}$ is the phenomenological field source, which can be chosen in the following sedeonic form:

$$
\tilde{\mathbf{I}}_\mathbf{m} = -d_1 + id_2\mathbf{e_{tr}} + id_3\mathbf{e_t} - id_4\mathbf{e_r} + \vec{f}_1\mathbf{e_{tr}} - i\vec{f}_2 + \vec{f}_3\mathbf{e_r} + \vec{f}_4\mathbf{e_t} \tag{6.2}
$$

where $d_k = 4\pi d'_k$ (d'_k are the volume density of charges) and $\vec{f}_k = \frac{4\pi}{c}\vec{f}'_k$ ($\vec{f}'_k$ are the corresponding volume density of currents). Choosing the potential $\tilde{\mathbf{W}}_\mathbf{m}$ in the form of (4.3) we can rewrite the equation (6.1) in the following expanded form

$$
\begin{aligned}
&\left(i\mathbf{e_t}\partial - \mathbf{e_r}\vec{\nabla} - i\mathbf{e_{tr}}m\right)\left(ia_1\mathbf{e_t} - ia_2\mathbf{e_r} + a_3 - ia_4\mathbf{e_{tr}} + \vec{A}_1\mathbf{e_r} + \vec{A}_2\mathbf{e_t} - \vec{A}_3\mathbf{e_{tr}} + i\vec{A}_4\right)\\
&= -d_1 + id_2\mathbf{e_{tr}} + id_3\mathbf{e_t} - id_4\mathbf{e_r} + \vec{f}_1\mathbf{e_{tr}} - i\vec{f}_2 + \vec{f}_3\mathbf{e_r} + \vec{f}_4\mathbf{e_t}.
\end{aligned}
\tag{6.3}
$$

This sedeonic equation is equivalent to the following system:

$$
\begin{aligned}
&\partial a_1 + \left(\vec{\nabla}\cdot\vec{A}_1\right) + m a_4 = d_1, \\
&\partial a_2 + \left(\vec{\nabla}\cdot\vec{A}_2\right) - m a_3 = d_2, \\
&\partial a_3 + \left(\vec{\nabla}\cdot\vec{A}_3\right) + m a_2 = d_3, \\
&\partial a_4 + \left(\vec{\nabla}\cdot\vec{A}_4\right) - m a_1 = d_4, \\
&-\partial\vec{A}_1 - \vec{\nabla} a_1 + i\left[\vec{\nabla}\times\vec{A}_2\right] + m\vec{A}_4 = \vec{f}_1, \\
&-\partial\vec{A}_2 - \vec{\nabla} a_2 - i\left[\vec{\nabla}\times\vec{A}_1\right] - m\vec{A}_3 = \vec{f}_2, \\
&-\partial\vec{A}_3 - \vec{\nabla} a_3 - i\left[\vec{\nabla}\times\vec{A}_4\right] + m\vec{A}_2 = \vec{f}_3, \\
&-\partial\vec{A}_4 - \vec{\nabla} a_4 + i\left[\vec{\nabla}\times\vec{A}_3\right] - m\vec{A}_1 = \vec{f}_4.
\end{aligned} \tag{6.4}
$$

On the other hand, introducing the massless field strengths according the definitions (5.2) we get

$$
\begin{aligned}
&-g_1 + i g_2 \mathbf{e}_{\mathbf{tr}} + i g_3 \mathbf{e}_{\mathbf{t}} - i g_4 \mathbf{e}_{\mathbf{r}} + \vec{G}_1 \mathbf{e}_{\mathbf{tr}} - i\vec{G}_2 + \vec{G}_3 \mathbf{e}_{\mathbf{r}} + \vec{G}_4 \mathbf{e}_{\mathbf{t}} \\
&= -d_1 + i d_2 \mathbf{e}_{\mathbf{tr}} + i d_3 \mathbf{e}_{\mathbf{t}} - i d_4 \mathbf{e}_{\mathbf{r}} + \vec{f}_1 \mathbf{e}_{\mathbf{tr}} - i\vec{f}_2 + \vec{f}_3 \mathbf{e}_{\mathbf{r}} + \vec{f}_4 \mathbf{e}_{\mathbf{t}}.
\end{aligned} \tag{6.5}
$$

It means that in fact the field strengths are non-zero only in the regions of the field sources.

Applying the operator $\left(i\mathbf{e}_{\mathbf{t}}\partial - \mathbf{e}_{\mathbf{r}}\vec{\nabla} - i\mathbf{e}_{\mathbf{tr}} m\right)$ to the equation (6.3) we obtain the following second-order wave equation:

$$
\begin{aligned}
&\left(i\mathbf{e}_{\mathbf{t}}\partial - \mathbf{e}_{\mathbf{r}}\vec{\nabla} - i\mathbf{e}_{\mathbf{tr}} m\right)\left(i\mathbf{e}_{\mathbf{t}}\partial - \mathbf{e}_{\mathbf{r}}\vec{\nabla} - i\mathbf{e}_{\mathbf{tr}} m\right)\left(i a_1 \mathbf{e}_{\mathbf{t}} - i a_2 \mathbf{e}_{\mathbf{r}} + a_3 - i a_4 \mathbf{e}_{\mathbf{tr}} + \vec{A}_1 \mathbf{e}_{\mathbf{r}} + \vec{A}_2 \mathbf{e}_{\mathbf{t}} - \vec{A}_3 \mathbf{e}_{\mathbf{tr}} + i\vec{A}_4\right) \\
&= \left(i\mathbf{e}_{\mathbf{t}}\partial - \mathbf{e}_{\mathbf{r}}\vec{\nabla} - i\mathbf{e}_{\mathbf{tr}} m\right)\left(-d_1 + i d_2 \mathbf{e}_{\mathbf{tr}} + i d_3 \mathbf{e}_{\mathbf{t}} - i d_4 \mathbf{e}_{\mathbf{r}} + \vec{f}_1 \mathbf{e}_{\mathbf{tr}} - i\vec{f}_2 + \vec{f}_3 \mathbf{e}_{\mathbf{r}} + \vec{f}_4 \mathbf{e}_{\mathbf{t}}\right),
\end{aligned} \tag{6.6}
$$

which is equivalent to the following system:

$$
\begin{aligned}
&\left(\partial^2 - \Delta + m^2\right) a_1 = \partial d_1 + \left(\vec{\nabla}\cdot\vec{f}_1\right) - m d_4, \\
&\left(\partial^2 - \Delta + m^2\right) a_2 = \partial d_2 + \left(\vec{\nabla}\cdot\vec{f}_2\right) + m d_3, \\
&\left(\partial^2 - \Delta + m^2\right) a_3 = \partial d_3 + \left(\vec{\nabla}\cdot\vec{f}_3\right) - m d_2, \\
&\left(\partial^2 - \Delta + m^2\right) a_4 = \partial d_4 + \left(\vec{\nabla}\cdot\vec{f}_4\right) + m d_1, \\
&\left(\partial^2 - \Delta + m^2\right) \vec{A}_1 = -\partial\vec{f}_1 - \vec{\nabla} d_1 - i\left[\vec{\nabla}\times\vec{f}_2\right] - m\vec{f}_4, \\
&\left(\partial^2 - \Delta + m^2\right) \vec{A}_2 = -\partial\vec{f}_2 - \vec{\nabla} d_2 + i\left[\vec{\nabla}\times\vec{f}_1\right] + m\vec{f}_3, \\
&\left(\partial^2 - \Delta + m^2\right) \vec{A}_3 = -\partial\vec{f}_3 - \vec{\nabla} d_3 + i\left[\vec{\nabla}\times\vec{f}_4\right] - m\vec{f}_2, \\
&\left(\partial^2 - \Delta + m^2\right) \vec{A}_4 = -\partial\vec{f}_4 - \vec{\nabla} d_4 - i\left[\vec{\nabla}\times\vec{f}_3\right] + m\vec{f}_1.
\end{aligned} \tag{6.7}
$$

It can be seen that equations (6.7) are invariant with respect to the following substitutions for the sources:

$$
\begin{aligned}
&d_1 \Rightarrow d_1 + \partial\varepsilon_1 + m\varepsilon_4, \\
&d_2 \Rightarrow d_2 + \partial\varepsilon_2 - m\varepsilon_3, \\
&d_3 \Rightarrow d_3 + \partial\varepsilon_3 + m\varepsilon_2, \\
&d_4 \Rightarrow d_4 + \partial\varepsilon_4 - m\varepsilon_1, \\
&\vec{f}_1 \Rightarrow \vec{f}_1 - \vec{\nabla}\varepsilon_1, \\
&\vec{f}_2 \Rightarrow \vec{j}_2 - \vec{\nabla}\varepsilon_2, \\
&\vec{f}_3 \Rightarrow \vec{f}_3 - \vec{\nabla}\varepsilon_3, \\
&\vec{f}_4 \Rightarrow \vec{f}_4 - \vec{\nabla}\varepsilon_4.
\end{aligned} \tag{6.8}
$$

As an example, let us consider the fields produced by a one type of sources d_4 and $\vec{f}_4$:

$$\tilde{\mathbf{I}}_{\mathbf{m}} = -id_4\mathbf{e}_{\mathbf{r}} + \vec{f}_4\mathbf{e}_{\mathbf{t}}, \tag{6.9}$$

In this case the equation (6.5) is rewritten as

$$-ig_4\mathbf{e}_{\mathbf{r}} + \vec{G}_4\mathbf{e}_{\mathbf{t}} = -id_4\mathbf{e}_{\mathbf{r}} + \vec{f}_4\mathbf{e}_{\mathbf{t}}. \tag{6.10}$$

Applying the operator $\left(i\mathbf{e}_1\partial - \mathbf{e}_2\vec{\nabla} - i\mathbf{e}_3 m\right)$ to the equation (6.10) and separating the values with different space-time properties we obtain the following equations for the field strengths:

$$\begin{gathered} g_4 = d_4, \\ \vec{G}_4 = \vec{f}_4, \\ \partial g_4 + \left(\vec{\nabla}\cdot\vec{G}_4\right) = \partial d_4 + \frac{1}{c}\left(\vec{\nabla}\cdot\vec{f}_4\right), \\ \left[\vec{\nabla}\times\vec{G}_4\right] = \left[\vec{\nabla}\times\vec{f}_4\right], \\ \partial\,\vec{G}_4 + \vec{\nabla}g_4 = \partial\vec{f}_4 + \vec{\nabla}d_4. \end{gathered} \tag{6.11}$$

Assuming the charge conservation

$$\partial d_4 + \left(\vec{\nabla}\cdot\vec{f}_4\right) = 0, \tag{6.12}$$

we have the following gauge condition:

$$\partial g_4 + \left(\vec{\nabla}\cdot\vec{G}_4\right) = 0, \tag{6.13}$$

which is similar to conventional Lorentz gauge, but for field strengths here.

Let us consider a stationary field generated by a scalar point source. In this case we can choose the source as

$$\tilde{\mathbf{I}}_{\mathbf{m}} = -i\mathbf{e}_{\mathbf{r}}4\pi d_4', \tag{6.14}$$

and density of charge as

$$d_4' = \theta_4\delta(\vec{r}). \tag{6.15}$$

where θ_4 is the point charge. Then the strength of the scalar field is

$$g_4\left(\vec{r}\right) = 4\pi\theta_4\delta\left(\vec{r}\right). \tag{6.16}$$

This field is non-zero only in the region of source. In particular, it indicates that two point charges interact only if they are at the same point of space. The interaction energy for two point charges θ_{41} and θ_{42} is equal

$$W_{12} = \frac{1}{4\pi}\int_V g_{41}g_{42}\,dV = 4\pi\theta_{41}\theta_{42}\delta\left(\vec{R}\right), \tag{6.17}$$

where $\vec{R}$ is the vector of distance between point charges.

7. First-order equation for massless field

In massless case the first-order wave equation can be presented as

$$\left(i\mathbf{e}_{\mathbf{t}}\partial - \mathbf{e}_{\mathbf{r}}\vec{\nabla}\right)\tilde{\mathbf{W}}_{\mathbf{0}} = \tilde{\mathbf{I}}_{\mathbf{0}}, \tag{7.1}$$

where the potential $\tilde{\mathbf{W}}_{\mathbf{0}}$ and phenomenological source $\tilde{\mathbf{I}}_{\mathbf{0}}$ have the following form:

$$\tilde{\mathbf{W}}_{\mathbf{0}} = ib_1\mathbf{e}_{\mathbf{t}} - ib_2\mathbf{e}_{\mathbf{r}} + b_3 - ib_4\mathbf{e}_{\mathbf{tr}} + \vec{B}_1\mathbf{e}_{\mathbf{r}} + \vec{B}_2\mathbf{e}_{\mathbf{t}} - \vec{B}_3\mathbf{e}_{\mathbf{tr}} + i\vec{B}_4, \tag{7.2}$$

$$\tilde{\mathbf{I}}_{\mathbf{0}} = -\upsilon_1 + i\upsilon_2\mathbf{e}_{\mathbf{tr}} + i\upsilon_3\mathbf{e}_{\mathbf{t}} - i\upsilon_4\mathbf{e}_{\mathbf{r}} + \vec{\gamma}_1\mathbf{e}_{\mathbf{tr}} - i\vec{\gamma}_2 + \vec{\gamma}_3\mathbf{e}_{\mathbf{r}} + \vec{\gamma}_4\mathbf{e}_{\mathbf{t}}. \tag{7.3}$$

Here $\upsilon_s = 4\pi\upsilon'_s$ (υ'_s is the volume density of charge) and $\vec{\gamma}_s = \frac{4\pi}{c}\vec{\gamma}'_s$ ($\vec{\gamma}'_s$ is volume density of current). The equation (7.1) is equivalent to the following system:

$$\begin{aligned}
&\partial b_1 + \left(\vec{\nabla}\cdot\vec{B}_1\right) = \upsilon_1, \\
&\partial b_2 + \left(\vec{\nabla}\cdot\vec{B}_2\right) = \upsilon_2, \\
&\partial b_3 + \left(\vec{\nabla}\cdot\vec{B}_3\right) = \upsilon_3, \\
&\partial b_4 + \left(\vec{\nabla}\cdot\vec{B}_4\right) = \upsilon_4, \\
&-\partial\vec{B}_1 - \vec{\nabla}b_1 + i\left[\vec{\nabla}\times\vec{B}_2\right] = \vec{\gamma}_1, \\
&-\partial\vec{B}_2 - \vec{\nabla}b_2 - i\left[\vec{\nabla}\times\vec{B}_1\right] = \vec{\gamma}_2, \\
&-\partial\vec{B}_3 - \vec{\nabla}b_3 - i\left[\vec{\nabla}\times\vec{B}_4\right] = \vec{\gamma}_3, \\
&-\partial\vec{B}_4 - \vec{\nabla}b_4 + i\left[\vec{\nabla}\times\vec{B}_3\right] = \vec{\gamma}_4.
\end{aligned} \tag{7.4}$$

The equations (7.4) are invariant with respect to the substitutions (5.5).

As an example, let us consider the massless field generated by scalar point source. In this case we can choose the scalar source in the form

$$\tilde{\mathbf{I}}_{\mathbf{0}} = -4\pi\upsilon'_1, \tag{7.2}$$

It follows that only scalar field strength h_1 (see definition (5.2) for massless field) is nonzero:

$$h_1 = 4\pi\upsilon'_1. \tag{6.10}$$

This field is non-zero only in the region of source. The density of charge for point source is equal

$$\upsilon'_1 = \sigma_1\delta(\vec{r}), \tag{6.11}$$

where σ_1 is the point charge. Then the interaction energy of two point charges can be presented as follows (see (5.11)):

$$W_{12} = \frac{1}{4\pi}\int h_{11}h_{12}dV. \tag{6.12}$$

Substituting (6.10) and (6.11), we obtain

$$W_{12} = 4\pi\sigma_{11}\sigma_{12}\delta(\vec{R}), \tag{6.13}$$

where $\vec{R}$ is the vector of distance between first and second charges. It indicates that two point charges interact only if they are at the same point of space.

7. Discussion

The gradient gauge invariance of the sedeonic equations describing the massive fields is a property of the operator $\left(i\mathbf{e}_t\partial - \mathbf{e}_r\vec{\nabla} - i\mathbf{e}_{tr}m\right)$ and can be generalized to a wider class of scalar-vector substitutions. Indeed, let us denote

$$\left(i\mathbf{e}_t\partial - \mathbf{e}_r\vec{\nabla} - i\mathbf{e}_{tr}m\right) \equiv \hat{\hat{\nabla}}, \tag{7.1}$$

then the wave equation (4.1) takes the following form:

$$\hat{\hat{\nabla}}\hat{\hat{\nabla}}\tilde{\mathbf{W}}_{\mathbf{m}} = \tilde{\mathbf{J}}_{\mathbf{m}}. \tag{7.2}$$

This equation is not changed under the following replacement of potential:

$$\tilde{\mathbf{W}}_{\mathbf{m}} \Rightarrow \tilde{\mathbf{W}}_{\mathbf{m}} + \tilde{\mathbf{F}} + \hat{\hat{\nabla}}\tilde{\mathbf{E}}, \tag{7.3}$$

where $\tilde{\mathbf{F}}$ and $\tilde{\mathbf{E}}$ are arbitrary sedeons satisfy the following conditions:

$$\hat{\vec{\nabla}}\tilde{\mathbf{F}} = 0 , \qquad (7.4)$$

$$\hat{\vec{\nabla}}\hat{\vec{\nabla}}\tilde{\mathbf{E}} = 0 . \qquad (7.5)$$

The condition (7.4) indicates that the potential $\tilde{\mathbf{W}}_{\mathbf{m}}$ is defined up to an additive function $\tilde{\mathbf{F}}$ satisfying the homogeneous first-order wave equation, while expression (7.5) means that $\tilde{\mathbf{E}}$ satisfy the homogeneous second-order wave equation. Let us consider the generalized gradient gauge condition. For the potential determined by the expression (4.3) the function $\tilde{\mathbf{E}}$ can be chosen as follows:

$$\tilde{\mathbf{E}} = \varepsilon_1 - i\varepsilon_2\mathbf{e}_{\mathbf{tr}} - i\varepsilon_3\mathbf{e}_{\mathbf{t}} + i\varepsilon_4\mathbf{e}_{\mathbf{r}} + \vec{E}_1\mathbf{e}_{\mathbf{tr}} - i\vec{E}_2 + \vec{E}_3\mathbf{e}_{\mathbf{r}} + \vec{E}_4\mathbf{e}_{\mathbf{t}}, \qquad (7.6)$$

where components ε_{S} and $\vec{E}_{\mathrm{S}}$ are arbitrary real functions of coordinates and time. Then the replacement (7.3) leads us to the following system of substitutions:

$$\begin{aligned}
a_1 &\Rightarrow a_1 + \partial\varepsilon_1 - \left(\vec{\nabla}\cdot\vec{E}_1\right) - m\varepsilon_4,\\
a_2 &\Rightarrow a_2 + \partial\varepsilon_2 - \left(\vec{\nabla}\cdot\vec{E}_2\right) + m\varepsilon_3,\\
a_3 &\Rightarrow a_3 + \partial\varepsilon_3 - \left(\vec{\nabla}\cdot\vec{E}_3\right) - m\varepsilon_2,\\
a_4 &\Rightarrow a_4 + \partial\varepsilon_4 - \left(\vec{\nabla}\cdot\vec{E}_4\right) + m\varepsilon_1,\\
\vec{A}_1 &\Rightarrow \vec{A}_1 + \partial\vec{E}_1 - \vec{\nabla}\varepsilon_1 + i\left[\vec{\nabla}\times\vec{E}_2\right] + m\vec{E}_4,\\
\vec{A}_2 &\Rightarrow \vec{A}_2 + \partial\vec{E}_2 - \vec{\nabla}\varepsilon_2 - i\left[\vec{\nabla}\times\vec{E}_1\right] - m\vec{E}_3,\\
\vec{A}_3 &\Rightarrow \vec{A}_3 + \partial\vec{E}_3 - \vec{\nabla}\varepsilon_3 - i\left[\vec{\nabla}\times\vec{E}_4\right] + m\vec{E}_2,\\
\vec{A}_4 &\Rightarrow \vec{A}_4 + \partial\vec{E}_4 - \vec{\nabla}\varepsilon_4 + i\left[\vec{\nabla}\times\vec{E}_3\right] - m\vec{E}_1.
\end{aligned} \qquad (7.7)$$

If we chose the vector part equal to zero ($\vec{E}_{\mathrm{S}} = 0$), then the substitutions (7.7) are reduced to (4.9) and to (5.5) for the zero mass quantum. Analogous substitutions for the field strengths have the following form:

$$\begin{aligned}
g_1 &\Rightarrow g_1 + \partial\varepsilon_1 + \left(\vec{\nabla}\cdot\vec{E}_1\right) - m\varepsilon_4,\\
g_2 &\Rightarrow g_2 - \partial\varepsilon_2 - \left(\vec{\nabla}\cdot\vec{E}_2\right) - m\varepsilon_3,\\
g_3 &\Rightarrow g_3 + \partial\varepsilon_3 - \left(\vec{\nabla}\cdot\vec{E}_3\right) - m\varepsilon_2,\\
g_3 &\Rightarrow g_3 - \partial\varepsilon_4 + \left(\vec{\nabla}\cdot\vec{E}_4\right) - m\varepsilon_1,\\
\vec{G}_1 &\Rightarrow \vec{G}_1 - \partial\vec{E}_1 - \vec{\nabla}\varepsilon_1 + i\left[\vec{\nabla}\times\vec{E}_2\right] + m\vec{E}_4,\\
\vec{G}_2 &\Rightarrow \vec{G}_2 + \partial\vec{E}_2 + \vec{\nabla}\varepsilon_2 + i\left[\vec{\nabla}\times\vec{E}_1\right] + m\vec{E}_3,\\
\vec{G}_3 &\Rightarrow \vec{G}_3 + \partial\vec{E}_3 - \vec{\nabla}\varepsilon_3 + i\left[\vec{\nabla}\times\vec{E}_4\right] - m\vec{E}_2,\\
\vec{G}_4 &\Rightarrow \vec{G}_4 - \partial\vec{E}_4 + \vec{\nabla}\varepsilon_4 + i\left[\vec{\nabla}\times\vec{E}_3\right] - m\vec{E}_1.
\end{aligned} \qquad (7.8)$$

If we chose the vector part equal to zero, then the substitutions (7.8) are reduced to (4.13) and to (5.5) for the zero mass quantum.

8. Conclusion

Thus we have presented the sypersymmetric scalar-vector equations for massive and massless fields. The gauge invariance for the potentials described by second-order and first-order wave equations and for the field strengths described by the systems of Maxwell-like equations has been demonstrated. As the example we considered the interaction of different point charges caused by the fields overlapping.

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