Expressing the Dirac Equation as a Generalization of Maxwell's Equations

A factorization of the Klein-Gordon equation into a pair of linear differential operators was discovered by Paul Adrien Maurice Dirac in 1928 as a (quantum) field equation accurately describing all elementary matter particles of spin ℏ/2 – fermions (quarks and leptons) .
The Klein-Gordon equation expressed as:

$\left(\nabla^2 - \frac{\partial^2}{\partial t^2} - m^2\right)\boldsymbol{\psi}_D = \mathbf{0}$; with $\boldsymbol{\psi}_D \equiv \begin{pmatrix} \boldsymbol{\phi}_D^A \\ \boldsymbol{\phi}_D^B \end{pmatrix}$ a 2^M-dimensional vector, so factored may be written (in the Dirac representation):

$$\begin{pmatrix} \mathbf{I}_2\left(i\frac{\partial}{\partial t} + m\right) & i\boldsymbol{\sigma}\bullet\vec{\nabla} \\ -i\boldsymbol{\sigma}\bullet\vec{\nabla} & \mathbf{I}_2\left(-i\frac{\partial}{\partial t} + m\right) \end{pmatrix}\begin{pmatrix} \mathbf{I}_2\left(-i\frac{\partial}{\partial t} + m\right) & -i\boldsymbol{\sigma}\bullet\vec{\nabla} \\ i\boldsymbol{\sigma}\bullet\vec{\nabla} & \mathbf{I}_2\left(i\frac{\partial}{\partial t} + m\right) \end{pmatrix}\begin{pmatrix} \boldsymbol{\phi}_D^A \\ \boldsymbol{\phi}_D^B \end{pmatrix} = \mathbf{0}$$

and, since these matrix operators are commutative:

$$\begin{pmatrix} \mathbf{I}_2\left(-i\frac{\partial}{\partial t} + m\right) & -i\boldsymbol{\sigma}\bullet\vec{\nabla} \\ i\boldsymbol{\sigma}\bullet\vec{\nabla} & \mathbf{I}_2\left(i\frac{\partial}{\partial t} + m\right) \end{pmatrix}\begin{pmatrix} \mathbf{I}_2\left(i\frac{\partial}{\partial t} + m\right) & i\boldsymbol{\sigma}\bullet\vec{\nabla} \\ -i\boldsymbol{\sigma}\bullet\vec{\nabla} & \mathbf{I}_2\left(-i\frac{\partial}{\partial t} + m\right) \end{pmatrix}\begin{pmatrix} \boldsymbol{\phi}_D^A \\ \boldsymbol{\phi}_D^B \end{pmatrix} = \mathbf{0}$$

Where:

$$\boldsymbol{\sigma}\bullet\vec{\nabla} = \sum_{\nu=1}^{3} \boldsymbol{\sigma}^\nu \frac{\partial}{\partial x^\nu}$$

$$\boldsymbol{\sigma}^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I}_2$$

$$\boldsymbol{\sigma}^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad \boldsymbol{\sigma}^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad , \quad \boldsymbol{\sigma}^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\Rightarrow \boldsymbol{\sigma}\bullet\vec{\nabla} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\partial_1 + \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\partial_2 + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\partial_3$$

$$= \begin{pmatrix} \partial_3 & \partial_1 - i\partial_2 \\ \partial_1 + i\partial_2 & -\partial_3 \end{pmatrix}$$

$$\Rightarrow \left(\boldsymbol{\sigma}\bullet\vec{\nabla}\right)^2 = \begin{pmatrix} \partial_3 & \partial_1 - i\partial_2 \\ \partial_1 + i\partial_2 & -\partial_3 \end{pmatrix}\begin{pmatrix} \partial_3 & \partial_1 - i\partial_2 \\ \partial_1 + i\partial_2 & -\partial_3 \end{pmatrix} = (\partial_3^2 + \partial_1^2 + \partial_2^2)\mathbf{I}_2 = \boldsymbol{\nabla}^2\mathbf{I}_2$$

Now, continuing from [1], the transformations between the special case of the Maxwell-Cassano equations and the Dirac equation are:

$(\partial_0 \pm m) \Leftrightarrow (\overline{\partial_0} \pm m) \qquad \Rightarrow x^0 = \bar{x}^0$
$\begin{pmatrix} \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{0}_2 & \boldsymbol{\sigma}_2^0 \\ \mathbf{0}_2 & \mathbf{0}_2 & \boldsymbol{\sigma}_2^0 & 0 \\ \mathbf{0}_2 & -\boldsymbol{\sigma}_2^0 & \mathbf{0}_2 & \mathbf{0}_2 \\ -\boldsymbol{\sigma}_2^0 & \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{0}_2 \end{pmatrix}\partial_1 \Leftrightarrow \begin{pmatrix} \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{0}_2 & \boldsymbol{\sigma}_2^0 \\ \mathbf{0}_2 & \mathbf{0}_2 & \boldsymbol{\sigma}_2^0 & 0 \\ \mathbf{0}_2 & \boldsymbol{\sigma}_2^0 & \mathbf{0}_2 & \mathbf{0}_2 \\ \boldsymbol{\sigma}_2^0 & \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{0}_2 \end{pmatrix} i\overline{\partial_1} \Rightarrow x^1 = -i\begin{pmatrix} \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{0}_2 & \boldsymbol{\sigma}_2^0 \\ \mathbf{0}_2 & \mathbf{0}_2 & \boldsymbol{\sigma}_2^0 & 0 \\ \mathbf{0}_2 & \boldsymbol{\sigma}_2^0 & \mathbf{0}_2 & \mathbf{0}_2 \\ \boldsymbol{\sigma}_2^0 & \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{0}_2 \end{pmatrix}^{-1}\begin{pmatrix} \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{0}_2 & \boldsymbol{\sigma}_2^0 \\ \mathbf{0}_2 & \mathbf{0}_2 & \boldsymbol{\sigma}_2^0 & 0 \\ \mathbf{0}_2 & -\boldsymbol{\sigma}_2^0 & \mathbf{0}_2 & \mathbf{0}_2 \\ -\boldsymbol{\sigma}_2^0 & \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{0}_2 \end{pmatrix}\bar{x}^1$
$\begin{pmatrix} \mathbf{0}_2 & \mathbf{0}_2 & -\boldsymbol{\sigma}_2^0 & \mathbf{0}_2 \\ \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{0}_2 & \boldsymbol{\sigma}_2^0 \\ \boldsymbol{\sigma}_2^0 & \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{0}_2 \\ \mathbf{0}_2 & -\boldsymbol{\sigma}_2^0 & \mathbf{0}_2 & \mathbf{0}_2 \end{pmatrix}\partial_2 \Leftrightarrow \begin{pmatrix} \mathbf{0}_2 & \mathbf{0}_2 & \boldsymbol{\sigma}_2^0 & \mathbf{0}_2 \\ \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{0}_2 & -\boldsymbol{\sigma}_2^0 \\ \boldsymbol{\sigma}_2^0 & \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{0}_2 \\ \mathbf{0}_2 & -\boldsymbol{\sigma}_2^0 & \mathbf{0}_2 & \mathbf{0}_2 \end{pmatrix} i\overline{\partial_3} \Rightarrow x^2 = -i\begin{pmatrix} \mathbf{0}_2 & \mathbf{0}_2 & \boldsymbol{\sigma}_2^0 & \mathbf{0}_2 \\ \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{0}_2 & -\boldsymbol{\sigma}_2^0 \\ \boldsymbol{\sigma}_2^0 & \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{0}_2 \\ \mathbf{0}_2 & -\boldsymbol{\sigma}_2^0 & \mathbf{0}_2 & \mathbf{0}_2 \end{pmatrix}^{-1}\begin{pmatrix} \mathbf{0}_2 & \mathbf{0}_2 & -\boldsymbol{\sigma}_2^0 & \mathbf{0}_2 \\ \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{0}_2 & \boldsymbol{\sigma}_2^0 \\ \boldsymbol{\sigma}_2^0 & \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{0}_2 \\ \mathbf{0}_2 & -\boldsymbol{\sigma}_2^0 & \mathbf{0}_2 & \mathbf{0}_2 \end{pmatrix} x^3$
$\begin{pmatrix} \mathbf{0}_2 & \boldsymbol{\sigma}_2^1 & \mathbf{0}_2 & \mathbf{0}_2 \\ -\boldsymbol{\sigma}_2^1 & \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{0}_2 \\ \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{0}_2 & \boldsymbol{\sigma}_2^1 \\ \mathbf{0}_2 & \mathbf{0}_2 & -\boldsymbol{\sigma}_2^1 & \mathbf{0}_2 \end{pmatrix}\partial_3 \Leftrightarrow \begin{pmatrix} \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{0}_2 & \boldsymbol{\sigma}_2^0 \\ \mathbf{0}_2 & \mathbf{0}_2 & -\boldsymbol{\sigma}_2^0 & \mathbf{0}_2 \\ \mathbf{0}_2 & \boldsymbol{\sigma}_2^0 & \mathbf{0}_2 & \mathbf{0}_2 \\ -\boldsymbol{\sigma}_2^0 & \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{0}_2 \end{pmatrix}\overline{\partial_2} \Rightarrow x^2 = \begin{pmatrix} \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{0}_2 & \boldsymbol{\sigma}_2^0 \\ \mathbf{0}_2 & \mathbf{0}_2 & -\boldsymbol{\sigma}_2^0 & \mathbf{0}_2 \\ \mathbf{0}_2 & \boldsymbol{\sigma}_2^0 & \mathbf{0}_2 & \mathbf{0}_2 \\ -\boldsymbol{\sigma}_2^0 & \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{0}_2 \end{pmatrix}^{-1}\begin{pmatrix} \mathbf{0}_2 & \boldsymbol{\sigma}_2^1 & \mathbf{0}_2 & \mathbf{0}_2 \\ -\boldsymbol{\sigma}_2^1 & \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{0}_2 \\ \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{0}_2 & \boldsymbol{\sigma}_2^1 \\ \mathbf{0}_2 & \mathbf{0}_2 & -\boldsymbol{\sigma}_2^1 & \mathbf{0}_2 \end{pmatrix}\bar{x}^3$
[Dirac (barred)]

where:

$$\boldsymbol{\sigma}_2^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \boldsymbol{\sigma}^0 \quad , \quad \boldsymbol{\sigma}_2^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \boldsymbol{\sigma}^1 \quad , \quad \mathbf{0}_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

The correspondence between matrices

$$\begin{pmatrix} (\partial_0 - m) & 0 & 0 & \partial_3 & -\partial_2 & 0 & \partial_1 & 0 \\ 0 & (\partial_0 + m) & \partial_3 & 0 & 0 & -\partial_2 & 0 & \partial_1 \\ 0 & -\partial_3 & (\partial_0 - m) & 0 & \partial_1 & 0 & \partial_2 & 0 \\ -\partial_3 & 0 & 0 & (\partial_0 + m) & 0 & \partial_1 & 0 & \partial_2 \\ \partial_2 & 0 & -\partial_1 & 0 & (\partial_0 + m) & 0 & 0 & \partial_3 \\ 0 & \partial_2 & 0 & -\partial_1 & 0 & (\partial_0 - m) & \partial_3 & 0 \\ -\partial_1 & 0 & -\partial_2 & 0 & 0 & -\partial_3 & (\partial_0 + m) & 0 \\ 0 & -\partial_1 & 0 & -\partial_2 & -\partial_3 & 0 & 0 & (\partial_0 - m) \end{pmatrix}$$

and

$$\begin{pmatrix} (\partial_0 - m) & 0 & 0 & 0 & i\partial_3 & 0 & i(\partial_1 - i\partial_2) & 0 \\ 0 & (\partial_0 + m) & 0 & 0 & 0 & i\partial_3 & 0 & i(\partial_1 - i\partial_2) \\ 0 & 0 & (\partial_0 - m) & 0 & i(\partial_1 + i\partial_2) & 0 & -i\partial_3 & 0 \\ 0 & 0 & 0 & (\partial_0 + m) & 0 & i(\partial_1 + i\partial_2) & 0 & -i\partial_3 \\ i\partial_3 & 0 & i(\partial_1 - i\partial_2) & 0 & (\partial_0 + m) & 0 & 0 & 0 \\ 0 & i\partial_3 & 0 & i(\partial_1 - i\partial_2) & 0 & (\partial_0 - m) & 0 & 0 \\ i(\partial_1 + i\partial_2) & 0 & -i\partial_3 & 0 & 0 & 0 & (\partial_0 + m) & 0 \\ 0 & i(\partial_1 + i\partial_2) & 0 & -i\partial_3 & 0 & 0 & 0 & (\partial_0 - m) \end{pmatrix}$$

were used to determine the above coordinate transformations, the first of which was modified simply by the elementary row operations of rearranging rows, starting with:

$$\begin{aligned} (\partial_0 + m_0)Z^0 + (\partial_1 + m_1)Z^3 + (\partial_3 - m_3)Z^5 - (\partial_2 - m_2)Z^6 &= J^0 \\ (\partial_3 + m_3)Z^1 - (\partial_2 + m_2)Z^2 + (\partial_0 - m_0)Z^4 + (\partial_1 - m_1)Z^7 &= J^4 \\ (\partial_0 + m_0)Z^1 + (\partial_2 + m_2)Z^3 - (\partial_3 - m_3)Z^4 + (\partial_1 - m_1)Z^6 &= J^1 \\ -(\partial_3 + m_3)Z^0 + (\partial_1 + m_1)Z^2 + (\partial_0 - m_0)Z^5 + (\partial_2 - m_2)Z^7 &= J^5 \\ (\partial_0 + m_0)Z^2 + (\partial_3 - m_3)Z^3 + (\partial_2 - m_2)Z^4 - (\partial_1 - m_1)Z^5 &= J^2 \\ (\partial_2 + m_2)Z^0 - (\partial_1 + m_1)Z^1 + (\partial_0 - m_0)Z^6 + (\partial_3 - m_3)Z^7 &= J^6 \\ -(\partial_1 - m_1)Z^0 - (\partial_2 - m_2)Z^1 - (\partial_3 - m_3)Z^2 + (\partial_0 - m_0)Z^3 &= J^3 \\ -(\partial_1 + m_1)Z^4 - (\partial_2 + m_2)Z^5 - (\partial_3 + m_3)Z^6 + (\partial_0 + m_0)Z^7 &= J^7 \end{aligned}$$

$$\begin{pmatrix} (\partial_0 + m_0) & 0 & 0 & (\partial_3 - m_3) & 0 & -(\partial_2 - m_2) & (\partial_1 + m_1) & 0 \\ 0 & (\partial_0 - m_0) & (\partial_3 + m_3) & 0 & -(\partial_2 + m_2) & 0 & 0 & (\partial_1 - m_1) \\ 0 & -(\partial_3 - m_3) & (\partial_0 + m_0) & 0 & 0 & (\partial_1 - m_1) & (\partial_2 + m_2) & 0 \\ -(\partial_3 + m_3) & 0 & 0 & (\partial_0 - m_0) & (\partial_1 + m_1) & 0 & 0 & (\partial_2 - m_2) \\ 0 & (\partial_2 - m_2) & 0 & -(\partial_1 - m_1) & (\partial_0 + m_0) & 0 & (\partial_3 + m_3) & 0 \\ (\partial_2 + m_2) & 0 & -(\partial_1 + m_1) & 0 & 0 & (\partial_0 - m_0) & 0 & (\partial_3 - m_3) \\ (\partial_1 - m_1) & 0 & (\partial_2 - m_2) & 0 & (\partial_3 - m_3) & 0 & -(\partial_0 - m_0) & 0 \\ 0 & (\partial_1 + m_1) & 0 & (\partial_2 + m_2) & 0 & (\partial_3 + m_3) & 0 & -(\partial_0 + m_0) \end{pmatrix}$$

equivalently (multiplying the bottom two rows by -1): :

$$\begin{pmatrix} (\partial_0 + m_0) & 0 & 0 & (\partial_3 - m_3) & 0 & -(\partial_2 - m_2) & (\partial_1 + m_1) & 0 \\ 0 & (\partial_0 - m_0) & (\partial_3 + m_3) & 0 & -(\partial_2 + m_2) & 0 & 0 & (\partial_1 - m_1) \\ 0 & -(\partial_3 - m_3) & (\partial_0 + m_0) & 0 & 0 & (\partial_1 - m_1) & (\partial_2 + m_2) & 0 \\ -(\partial_3 + m_3) & 0 & 0 & (\partial_0 - m_0) & (\partial_1 + m_1) & 0 & 0 & (\partial_2 - m_2) \\ 0 & (\partial_2 - m_2) & 0 & -(\partial_1 - m_1) & (\partial_0 + m_0) & 0 & (\partial_3 + m_3) & 0 \\ (\partial_2 + m_2) & 0 & -(\partial_1 + m_1) & 0 & 0 & (\partial_0 - m_0) & 0 & (\partial_3 - m_3) \\ -(\partial_1 - m_1) & 0 & -(\partial_2 - m_2) & 0 & -(\partial_3 - m_3) & 0 & (\partial_0 - m_0) & 0 \\ 0 & -(\partial_1 + m_1) & 0 & -(\partial_2 + m_2) & 0 & -(\partial_3 + m_3) & 0 & (\partial_0 + m_0) \end{pmatrix}$$

more compactly written:

$$\begin{pmatrix} D_0 & D_3^{\Leftrightarrow} & -D_2^{\Leftrightarrow} & D_1 \\ -D_3^{\Leftrightarrow} & D_0 & D_1^{\Leftrightarrow} & D_2 \\ D_2^{\Leftrightarrow} & -D_1^{\Leftrightarrow} & D_0 & D_3 \\ -D_1^{\Updownarrow} & -D_2^{\Updownarrow} & -D_3^{\Updownarrow} & D_0^{\Updownarrow} \end{pmatrix}$$

where:

$$D_i^+ \equiv (\partial_i + m_i) \quad , \quad D_i^- \equiv (\partial_i - m_i)$$

$$D_i \equiv \begin{pmatrix} D_i^+ & 0 \\ 0 & D_i^- \end{pmatrix} \quad , \quad D_i^{\Updownarrow} \equiv \begin{pmatrix} D_i^- & 0 \\ 0 & D_i^+ \end{pmatrix} \ ,$$

$$D_i^{\Leftrightarrow} \equiv \begin{pmatrix} 0 & D_i^- \\ D_i^+ & 0 \end{pmatrix} \quad , \quad D_i^{\Leftrightarrow\Updownarrow} \equiv \begin{pmatrix} 0 & D_i^+ \\ D_i^- & 0 \end{pmatrix}$$

$$\mathbf{A}_{\Updownarrow} \equiv \mathbf{w}^{4;1}\begin{pmatrix} A_-^1 \\ A_+^1 \end{pmatrix} + \mathbf{w}^{4;2}\begin{pmatrix} A_-^2 \\ A_+^2 \end{pmatrix} + \mathbf{w}^{4;3}\begin{pmatrix} A_-^3 \\ A_+^3 \end{pmatrix} + \mathbf{w}^{4;0}\begin{pmatrix} A_-^0 \\ A_+^0 \end{pmatrix}$$

note the identities:

$$D_iDj \equiv \begin{pmatrix} D_i^+ & 0 \\ 0 & D_i^- \end{pmatrix}\begin{pmatrix} D_j^+ & 0 \\ 0 & D_j^- \end{pmatrix} = \begin{pmatrix} D_i^+D_j^+ & 0 \\ 0 & D_i^-D_j^- \end{pmatrix} = DjD_i$$

$$D_iD_j^{\Updownarrow} \equiv \begin{pmatrix} D_i^+ & 0 \\ 0 & D_i^- \end{pmatrix}\begin{pmatrix} D_j^- & 0 \\ 0 & D_j^+ \end{pmatrix} = \begin{pmatrix} D_i^+D_j^- & 0 \\ 0 & D_i^-D_j^+ \end{pmatrix} = D_j^{\Updownarrow}D_i = D_j^{\Leftrightarrow}D_i^{\Leftrightarrow}$$

$$D_iD_j^{\Leftrightarrow} \equiv \begin{pmatrix} D_i^+ & 0 \\ 0 & D_i^- \end{pmatrix}\begin{pmatrix} 0 & D_j^- \\ D_j^+ & 0 \end{pmatrix} = \begin{pmatrix} 0 & D_i^+D_j^- \\ D_i^-D_j^+ & 0 \end{pmatrix} = D_j^{\Leftrightarrow}D_i^{\Updownarrow}$$

$$D_i^{\Updownarrow}D_j \equiv \begin{pmatrix} D_i^- & 0 \\ 0 & D_i^+ \end{pmatrix}\begin{pmatrix} D_j^+ & 0 \\ 0 & D_j^- \end{pmatrix} = \begin{pmatrix} D_i^-D_j^+ & 0 \\ 0 & D_i^+D_j^- \end{pmatrix} = D_jD_i^{\Updownarrow} = D_i^{\Leftrightarrow}D_j^{\Leftrightarrow}$$

$$D_i^{\Updownarrow}D_j^{\Updownarrow} \equiv \begin{pmatrix} D_i^- & 0 \\ 0 & D_i^+ \end{pmatrix}\begin{pmatrix} D_j^- & 0 \\ 0 & D_j^+ \end{pmatrix} = \begin{pmatrix} D_i^-D_j^- & 0 \\ 0 & D_i^+D_j^+ \end{pmatrix} = D_j^{\Updownarrow}D_i^{\Updownarrow}$$

$$D_i^{\Updownarrow}D_j^{\Leftrightarrow} \equiv \begin{pmatrix} D_i^- & 0 \\ 0 & D_i^+ \end{pmatrix}\begin{pmatrix} 0 & D_j^- \\ D_j^+ & 0 \end{pmatrix} = \begin{pmatrix} 0 & D_i^-D_j^- \\ D_i^+D_j^+ & 0 \end{pmatrix} = D_i^{\Leftrightarrow}D_j = D_j^{\Leftrightarrow}D_i$$
$$= D_j^{\Updownarrow}D_i^{\Leftrightarrow}$$

$$D_i^{\Leftrightarrow}D_j \equiv \begin{pmatrix} 0 & D_i^- \\ D_i^+ & 0 \end{pmatrix}\begin{pmatrix} D_j^+ & 0 \\ 0 & D_j^- \end{pmatrix} = \begin{pmatrix} 0 & D_i^-D_j^- \\ D_i^+D_j^+ & 0 \end{pmatrix} = D_j^{\Leftrightarrow}D_i = D_i^{\Updownarrow}D_j^{\Leftrightarrow}$$
$$= D_j^{\Updownarrow}D_i^{\Leftrightarrow}$$

$$D_i^{\Leftrightarrow}D_j^{\Updownarrow} \equiv \begin{pmatrix} 0 & D_i^- \\ D_i^+ & 0 \end{pmatrix}\begin{pmatrix} D_j^- & 0 \\ 0 & D_j^+ \end{pmatrix} = \begin{pmatrix} 0 & D_i^-D_j^+ \\ D_i^+D_j^- & 0 \end{pmatrix} = D_jD_i^{\Leftrightarrow}$$

$$D_i^{\Leftrightarrow}D_j^{\Leftrightarrow} \equiv \begin{pmatrix} 0 & D_i^- \\ D_i^+ & 0 \end{pmatrix}\begin{pmatrix} 0 & D_j^- \\ D_j^+ & 0 \end{pmatrix} = \begin{pmatrix} D_i^-D_j^+ & 0 \\ 0 & D_i^+D_j^- \end{pmatrix} = D_i^{\Updownarrow}D_j = D_jD_i^{\Updownarrow}$$
$$= D_j^{\Leftrightarrow}D_i^{\Leftrightarrow}$$

The matrix shown above:

$$\begin{pmatrix} (\partial_0 - m) & 0 & 0 & 0 & i\partial_3 & 0 & i(\partial_1 - i\partial_2) & 0 \\ 0 & (\partial_0 + m) & 0 & 0 & 0 & i\partial_3 & 0 & i(\partial_1 - i\partial_2) \\ 0 & 0 & (\partial_0 - m) & 0 & i(\partial_1 + i\partial_2) & 0 & -i\partial_3 & 0 \\ 0 & 0 & 0 & (\partial_0 + m) & 0 & i(\partial_1 + i\partial_2) & 0 & -i\partial_3 \\ i\partial_3 & 0 & i(\partial_1 - i\partial_2) & 0 & (\partial_0 + m) & 0 & 0 & 0 \\ 0 & i\partial_3 & 0 & i(\partial_1 - i\partial_2) & 0 & (\partial_0 - m) & 0 & 0 \\ i(\partial_1 + i\partial_2) & 0 & -i\partial_3 & 0 & 0 & 0 & (\partial_0 + m) & 0 \\ 0 & i(\partial_1 + i\partial_2) & 0 & -i\partial_3 & 0 & 0 & 0 & (\partial_0 - m) \end{pmatrix}$$

may be more compactly written:

$$\begin{pmatrix} D_0 & 0 & iD_3 & iD_1 + D_2 \\ 0 & D_0 & iD_1 - D_2 & -iD_3 \\ iD_3 & iD_1 + D_2 & D_0^{\Updownarrow} & 0 \\ iD_1 - D_2 & -iD_3 & 0 & D_0^{\Updownarrow} \end{pmatrix}$$

The Dirac equation may be written:

$$\begin{pmatrix} \mathbf{I}_2\left(i\frac{\partial}{\partial t} - m\right) & i\boldsymbol{\sigma}\bullet\vec{\nabla} \\ i\boldsymbol{\sigma}\bullet\vec{\nabla} & \mathbf{I}_2\left(i\frac{\partial}{\partial t} + m\right) \end{pmatrix}\begin{pmatrix} \mathbf{I}_2\left(i\frac{\partial}{\partial t} + m\right) & -i\boldsymbol{\sigma}\bullet\vec{\nabla} \\ -i\boldsymbol{\sigma}\bullet\vec{\nabla} & \mathbf{I}_2\left(i\frac{\partial}{\partial t} - m\right) \end{pmatrix}\begin{pmatrix} \boldsymbol{\phi}_D^A \\ \boldsymbol{\phi}_D^B \end{pmatrix} = \mathbf{0}$$

$x^0 = it$:

$$\begin{pmatrix} \mathbf{I}_2(\partial_0 - m) & i\boldsymbol{\sigma}\bullet\vec{\nabla} \\ i\boldsymbol{\sigma}\bullet\vec{\nabla} & \mathbf{I}_2(\partial_0 + m) \end{pmatrix}\begin{pmatrix} \mathbf{I}_2(\partial_0 + m) & -i\boldsymbol{\sigma}\bullet\vec{\nabla} \\ -i\boldsymbol{\sigma}\bullet\vec{\nabla} & \mathbf{I}_2(\partial_0 - m) \end{pmatrix}\begin{pmatrix} \boldsymbol{\phi}_D^A \\ \boldsymbol{\phi}_D^B \end{pmatrix} = \mathbf{0}$$

In 4×4 matrix form: $\left[\boldsymbol{\sigma}\bullet\vec{\nabla} = \begin{pmatrix} \partial_3 & \partial_1 - i\partial_2 \\ \partial_1 + i\partial_2 & -\partial_3 \end{pmatrix} \right]$

$$\begin{pmatrix} (\partial_0 - m) & 0 & i\partial_3 & i\partial_1 + \partial_2 \\ 0 & (\partial_0 - m) & i\partial_1 - \partial_2 & -i\partial_3 \\ i\partial_3 & i\partial_1 + \partial_2 & (\partial_0 + m) & 0 \\ i\partial_1 - \partial_2 & -i\partial_3 & 0 & (\partial_0 + m) \end{pmatrix} \text{ and } \begin{pmatrix} (\partial_0 + m) & 0 & -i\partial_3 & -i\partial_1 - \partial_2 \\ 0 & (\partial_0 + m) & -i\partial_1 + \partial_2 & i\partial_3 \\ -i\partial_3 & -i\partial_1 - \partial_2 & (\partial_0 - m) & 0 \\ -i\partial_1 + \partial_2 & i\partial_3 & 0 & (\partial_0 - m) \end{pmatrix}$$

Expanding each into 8×8 matrices:

$$\begin{pmatrix} (\partial_0 - m) & 0 & 0 & 0 & i\partial_3 & 0 & i\partial_1 + \partial_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & (\partial_0 - m) & 0 & i\partial_1 - \partial_2 & 0 & -i\partial_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ i\partial_3 & 0 & i\partial_1 + \partial_2 & 0 & (\partial_0 + m) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ i\partial_1 - \partial_2 & 0 & -i\partial_3 & 0 & 0 & 0 & (\partial_0 + m) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \text{ and }$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & (\partial_0 + m) & 0 & 0 & 0 & -i\partial_3 & 0 & -i\partial_1 - \partial_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & (\partial_0 + m) & 0 & -i\partial_1 + \partial_2 & 0 & i\partial_3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -i\partial_3 & 0 & -i\partial_1 - \partial_2 & 0 & (\partial_0 - m) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -i\partial_1 + \partial_2 & 0 & i\partial_3 & 0 & 0 & 0 & (\partial_0 - m) \end{pmatrix}$$

and adding:

$$\begin{pmatrix} (\partial_0 - m) & 0 & 0 & 0 & i\partial_3 & 0 & i\partial_1 + \partial_2 & 0 \\ 0 & (\partial_0 + m) & 0 & 0 & 0 & -i\partial_3 & 0 & -i\partial_1 - \partial_2 \\ 0 & 0 & (\partial_0 - m) & 0 & i\partial_1 - \partial_2 & 0 & -i\partial_3 & 0 \\ 0 & 0 & 0 & (\partial_0 + m) & 0 & -i\partial_1 + \partial_2 & 0 & i\partial_3 \\ i\partial_3 & 0 & i\partial_1 + \partial_2 & 0 & (\partial_0 + m) & 0 & 0 & 0 \\ 0 & -i\partial_3 & 0 & -i\partial_1 - \partial_2 & 0 & (\partial_0 - m) & 0 & 0 \\ i\partial_1 - \partial_2 & 0 & -i\partial_3 & 0 & 0 & 0 & (\partial_0 + m) & 0 \\ 0 & -i\partial_1 + \partial_2 & 0 & i\partial_3 & 0 & 0 & 0 & (\partial_0 - m) \end{pmatrix}$$

Multiplying row 1 by -1 & column 1 by -1 :

$$\begin{pmatrix} (\partial_0 - m) & 0 & 0 & 0 & i\partial_3 & 0 & i\partial_1 + \partial_2 & 0 \\ 0 & (\partial_0 + m) & 0 & 0 & 0 & i\partial_3 & 0 & i\partial_1 + \partial_2 \\ 0 & 0 & (\partial_0 - m) & 0 & i\partial_1 - \partial_2 & 0 & -i\partial_3 & 0 \\ 0 & 0 & 0 & (\partial_0 + m) & 0 & -i\partial_1 + \partial_2 & 0 & i\partial_3 \\ i\partial_3 & 0 & i\partial_1 + \partial_2 & 0 & (\partial_0 + m) & 0 & 0 & 0 \\ 0 & i\partial_3 & 0 & -i\partial_1 - \partial_2 & 0 & (\partial_0 - m) & 0 & 0 \\ i\partial_1 - \partial_2 & 0 & -i\partial_3 & 0 & 0 & 0 & (\partial_0 + m) & 0 \\ 0 & i\partial_1 - \partial_2 & 0 & i\partial_3 & 0 & 0 & 0 & (\partial_0 - m) \end{pmatrix}$$

Multiplying row 3 by -1 & column 3 by -1 :

$$\begin{pmatrix} (\partial_0 - m) & 0 & 0 & 0 & i\partial_3 & 0 & i\partial_1 + \partial_2 & 0 \\ 0 & (\partial_0 + m) & 0 & 0 & 0 & i\partial_3 & 0 & i\partial_1 + \partial_2 \\ 0 & 0 & (\partial_0 - m) & 0 & i\partial_1 - \partial_2 & 0 & -i\partial_3 & 0 \\ 0 & 0 & 0 & (\partial_0 + m) & 0 & i\partial_1 - \partial_2 & 0 & -i\partial_3 \\ i\partial_3 & 0 & i\partial_1 + \partial_2 & 0 & (\partial_0 + m) & 0 & 0 & 0 \\ 0 & i\partial_3 & 0 & i\partial_1 + \partial_2 & 0 & (\partial_0 - m) & 0 & 0 \\ i\partial_1 - \partial_2 & 0 & -i\partial_3 & 0 & 0 & 0 & (\partial_0 + m) & 0 \\ 0 & i\partial_1 - \partial_2 & 0 & -i\partial_3 & 0 & 0 & 0 & (\partial_0 - m) \end{pmatrix}$$

This is the same result as in [1]. Performing the appropriate operations on the corresponding product matrix will yield the coresponding product 8×8 product matrix of the Dirac equation product.

Expanding each into 8×8 matrices, alternatively (as they are commutative):

$$\begin{pmatrix} (\partial_0 - m) & 0 & i\partial_3 & i\partial_1 + \partial_2 \\ 0 & (\partial_0 - m) & i\partial_1 - \partial_2 & -i\partial_3 \\ i\partial_3 & i\partial_1 + \partial_2 & (\partial_0 + m) & 0 \\ i\partial_1 - \partial_2 & -i\partial_3 & 0 & (\partial_0 + m) \end{pmatrix} \text{ and } \begin{pmatrix} (\partial_0 + m) & 0 & -i\partial_3 & -i\partial_1 - \partial_2 \\ 0 & (\partial_0 + m) & -i\partial_1 + \partial_2 & i\partial_3 \\ -i\partial_3 & -i\partial_1 - \partial_2 & (\partial_0 - m) & 0 \\ -i\partial_1 + \partial_2 & i\partial_3 & 0 & (\partial_0 - m) \end{pmatrix}$$

$$\begin{pmatrix} (\partial_0+m) & 0 & 0 & 0 & -i\partial_3 & 0 & -i\partial_1-\partial_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & (\partial_0+m) & 0 & -i\partial_1+\partial_2 & 0 & i\partial_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -i\partial_3 & 0 & -i\partial_1-\partial_2 & 0 & (\partial_0-m) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -i\partial_1+\partial_2 & 0 & i\partial_3 & 0 & 0 & 0 & (\partial_0-m) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \text{ and}$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & (\partial_0-m) & 0 & 0 & 0 & i\partial_3 & 0 & i\partial_1+\partial_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & (\partial_0-m) & 0 & i\partial_1-\partial_2 & 0 & -i\partial_3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & i\partial_3 & 0 & i\partial_1+\partial_2 & 0 & (\partial_0+m) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & i\partial_1-\partial_2 & 0 & -i\partial_3 & 0 & 0 & 0 & (\partial_0+m) \end{pmatrix}$$

and adding:

$$\begin{pmatrix} (\partial_0+m) & 0 & 0 & 0 & -i\partial_3 & 0 & -i\partial_1-\partial_2 & 0 \\ 0 & (\partial_0-m) & 0 & 0 & 0 & i\partial_3 & 0 & i\partial_1+\partial_2 \\ 0 & 0 & (\partial_0+m) & 0 & -i\partial_1+\partial_2 & 0 & i\partial_3 & 0 \\ 0 & 0 & 0 & (\partial_0-m) & 0 & i\partial_1-\partial_2 & 0 & -i\partial_3 \\ -i\partial_3 & 0 & -i\partial_1-\partial_2 & 0 & (\partial_0-m) & 0 & 0 & 0 \\ 0 & i\partial_3 & 0 & i\partial_1+\partial_2 & 0 & (\partial_0+m) & 0 & 0 \\ -i\partial_1+\partial_2 & 0 & i\partial_3 & 0 & 0 & 0 & (\partial_0-m) & 0 \\ 0 & i\partial_1-\partial_2 & 0 & -i\partial_3 & 0 & 0 & 0 & (\partial_0+m) \end{pmatrix}$$

Multiplying row 1 by -1 & column 1 by -1 :

$$\begin{pmatrix} (\partial_0+m) & 0 & 0 & 0 & -i\partial_3 & 0 & -i\partial_1-\partial_2 & 0 \\ 0 & (\partial_0-m) & 0 & 0 & 0 & -i\partial_3 & 0 & -i\partial_1-\partial_2 \\ 0 & 0 & (\partial_0+m) & 0 & -i\partial_1+\partial_2 & 0 & i\partial_3 & 0 \\ 0 & 0 & 0 & (\partial_0-m) & 0 & i\partial_1-\partial_2 & 0 & -i\partial_3 \\ -i\partial_3 & 0 & -i\partial_1-\partial_2 & 0 & (\partial_0-m) & 0 & 0 & 0 \\ 0 & -i\partial_3 & 0 & i\partial_1+\partial_2 & 0 & (\partial_0+m) & 0 & 0 \\ -i\partial_1+\partial_2 & 0 & i\partial_3 & 0 & 0 & 0 & (\partial_0-m) & 0 \\ 0 & -i\partial_1+\partial_2 & 0 & -i\partial_3 & 0 & 0 & 0 & (\partial_0+m) \end{pmatrix}$$

Multiplying row 3 by -1 & column 3 by -1 :

$$\begin{pmatrix} (\partial_0+m) & 0 & 0 & 0 & -i\partial_3 & 0 & -i\partial_1-\partial_2 & 0 \\ 0 & (\partial_0-m) & 0 & 0 & 0 & -i\partial_3 & 0 & -i\partial_1-\partial_2 \\ 0 & 0 & (\partial_0+m) & 0 & -i\partial_1+\partial_2 & 0 & i\partial_3 & 0 \\ 0 & 0 & 0 & (\partial_0-m) & 0 & -i\partial_1+\partial_2 & 0 & i\partial_3 \\ -i\partial_3 & 0 & -i\partial_1-\partial_2 & 0 & (\partial_0-m) & 0 & 0 & 0 \\ 0 & -i\partial_3 & 0 & -i\partial_1-\partial_2 & 0 & (\partial_0+m) & 0 & 0 \\ -i\partial_1+\partial_2 & 0 & i\partial_3 & 0 & 0 & 0 & (\partial_0-m) & 0 \\ 0 & -i\partial_1+\partial_2 & 0 & i\partial_3 & 0 & 0 & 0 & (\partial_0+m) \end{pmatrix}$$

These two matrices may be written more compactly as:

$$\begin{pmatrix} D_0 & 0 & iD_3 & iD_1+D_2 \\ 0 & D_0 & iD_1-D_2 & -iD_3 \\ iD_3 & iD_1+D_2 & D_0^{\Updownarrow} & 0 \\ iD_1-D_2 & -iD_3 & 0 & D_0^{\Updownarrow} \end{pmatrix} \text{ and } \begin{pmatrix} D_0^{\Updownarrow} & 0 & -iD_3^{\Updownarrow} & -iD_1^{\Updownarrow}-D_2^{\Updownarrow} \\ 0 & D_0^{\Updownarrow} & -iD_1^{\Updownarrow}+D_2^{\Updownarrow} & iD_3^{\Updownarrow} \\ -iD_3^{\Updownarrow} & -iD_1^{\Updownarrow}-D_2^{\Updownarrow} & D_0 & 0 \\ -iD_1^{\Updownarrow}+D_2^{\Updownarrow} & iD_3^{\Updownarrow} & 0 & D_0 \end{pmatrix}$$

These matrices are the commutative left and right Dirac product matrices, respectively.
The Helmholtzian product matrices of the Maxwell-Cassano equations are:

$$\begin{pmatrix} D_0 & D_3^{\Leftrightarrow} & -D_2^{\Leftrightarrow} & D_1 \\ -D_3^{\Leftrightarrow} & D_0 & D_1^{\Leftrightarrow} & D_2 \\ D_2^{\Leftrightarrow} & -D_1^{\Leftrightarrow} & D_0 & D_3 \\ D_1^{\Updownarrow} & D_2^{\Updownarrow} & D_3^{\Updownarrow} & -D_0^{\Updownarrow} \end{pmatrix} \text{ and } \begin{pmatrix} D_0^{\Updownarrow} & -D_3^{\Leftrightarrow} & D_2^{\Leftrightarrow} & D_1 \\ D_3^{\Leftrightarrow} & D_0^{\Updownarrow} & -D_1^{\Leftrightarrow} & D_2 \\ -D_2^{\Leftrightarrow} & D_1^{\Leftrightarrow} & D_0^{\Updownarrow} & D_3 \\ D_1^{\Updownarrow} & D_2^{\Updownarrow} & D_3^{\Updownarrow} & -D_0 \end{pmatrix}$$

Multiplying row 3 of the leftmost by -1 & column 3 of the rightmost by -1 yields:

$$\begin{pmatrix} D_0 & D_3^{\Leftrightarrow} & -D_2^{\Leftrightarrow} & D_1 \\ -D_3^{\Leftrightarrow} & D_0 & D_1^{\Leftrightarrow} & D_2 \\ D_2^{\Leftrightarrow} & -D_1^{\Leftrightarrow} & D_0 & D_3 \\ -D_1^{\Updownarrow} & -D_2^{\Updownarrow} & -D_3^{\Updownarrow} & D_0^{\Updownarrow} \end{pmatrix} \text{ and } \begin{pmatrix} D_0^{\Updownarrow} & -D_3^{\Leftrightarrow} & D_2^{\Leftrightarrow} & -D_1 \\ D_3^{\Leftrightarrow} & D_0^{\Updownarrow} & -D_1^{\Leftrightarrow} & -D_2 \\ -D_2^{\Leftrightarrow} & D_1^{\Leftrightarrow} & D_0^{\Updownarrow} & -D_3 \\ D_1^{\Updownarrow} & D_2^{\Updownarrow} & D_3^{\Updownarrow} & D_0 \end{pmatrix}$$

[note that these are commutative, although the initial Maxwell-Cassano product matrices are not]

The leftmost is the same as the initial 8×8 matrix above, expanding 8×8 to:

$$\begin{pmatrix} (\partial_0 - m) & 0 & 0 & \partial_3 & 0 & -\partial_2 & \partial_1 & 0 \\ 0 & (\partial_0 + m) & \partial_3 & 0 & -\partial_2 & 0 & 0 & \partial_1 \\ 0 & -\partial_3 & (\partial_0 - m) & 0 & 0 & \partial_1 & \partial_2 & 0 \\ -\partial_3 & 0 & 0 & (\partial_0 + m) & \partial_1 & 0 & 0 & \partial_2 \\ 0 & \partial_2 & 0 & -\partial_1 & (\partial_0 - m) & 0 & \partial_3 & 0 \\ \partial_2 & 0 & -\partial_1 & 0 & 0 & (\partial_0 + m) & 0 & \partial_3 \\ -\partial_1 & 0 & -\partial_2 & 0 & -\partial_3 & 0 & (\partial_0 + m) & 0 \\ 0 & -\partial_1 & 0 & -\partial_2 & 0 & -\partial_3 & 0 & (\partial_0 - m) \end{pmatrix}$$

Instead, performing the elementary operations of interchanging rows 4 & 5 ; and of interchanging columns 4 & 5, yields:

$$\begin{pmatrix} (\partial_0 - m) & 0 & 0 & \partial_3 & -\partial_2 & 0 & \partial_1 & 0 \\ 0 & (\partial_0 + m) & \partial_3 & 0 & 0 & -\partial_2 & 0 & \partial_1 \\ 0 & -\partial_3 & (\partial_0 - m) & 0 & \partial_1 & 0 & \partial_2 & 0 \\ -\partial_3 & 0 & 0 & (\partial_0 + m) & 0 & \partial_1 & 0 & \partial_2 \\ \partial_2 & 0 & -\partial_1 & 0 & (\partial_0 + m) & 0 & 0 & \partial_3 \\ 0 & \partial_2 & 0 & -\partial_1 & 0 & (\partial_0 - m) & \partial_3 & 0 \\ \partial_1 & 0 & \partial_2 & 0 & 0 & \partial_3 & -(\partial_0 + m) & 0 \\ 0 & \partial_1 & 0 & \partial_2 & \partial_3 & 0 & 0 & -(\partial_0 - m) \end{pmatrix}$$

Is the matrix in [1] corresponding to the leftmost Dirac product matrix yielding the transformations between the special case of the Maxwell-Cassano equations and the Dirac equation.

More compactly written:

$$\begin{pmatrix} D_0 & D_3^{\Leftrightarrow} & -D_2 & D_1 \\ -D_3^{\Leftrightarrow} & D_0 & D_1 & D_2 \\ D_2 & -D_1 & D_0^{\Updownarrow} & D_3^{\Leftrightarrow} \\ D_1^{\Updownarrow} & D_2^{\Updownarrow} & D_3^{\Leftrightarrow} & -D_0^{\Updownarrow} \end{pmatrix} \overset{T}{\Leftrightarrow} \begin{pmatrix} D_0 & 0 & iD_3 & iD_1 + D_2 \\ 0 & D_0 & iD_1 - D_2 & -iD_3 \\ iD_3 & iD_1 + D_2 & D_0^{\Updownarrow} & 0 \\ iD_1 - D_2 & -iD_3 & 0 & D_0^{\Updownarrow} \end{pmatrix}$$

The matrix:

$$\begin{pmatrix} (\partial_0 - m) & 0 & 0 & \partial_3 & -\partial_2 & 0 & \partial_1 & 0 \\ 0 & (\partial_0 + m) & \partial_3 & 0 & 0 & -\partial_2 & 0 & \partial_1 \\ 0 & -\partial_3 & (\partial_0 - m) & 0 & \partial_1 & 0 & \partial_2 & 0 \\ -\partial_3 & 0 & 0 & (\partial_0 + m) & 0 & \partial_1 & 0 & \partial_2 \\ \partial_2 & 0 & -\partial_1 & 0 & (\partial_0 + m) & 0 & 0 & \partial_3 \\ 0 & \partial_2 & 0 & -\partial_1 & 0 & (\partial_0 - m) & \partial_3 & 0 \\ -\partial_1 & 0 & -\partial_2 & 0 & 0 & -\partial_3 & (\partial_0 + m) & 0 \\ 0 & -\partial_1 & 0 & -\partial_2 & -\partial_3 & 0 & 0 & (\partial_0 - m) \end{pmatrix}$$

is more compactly written:

$$\begin{pmatrix} D_0 & D_3^{\Leftrightarrow} & -D_2 & D_1 \\ -D_3^{\Leftrightarrow} & D_0 & D_1 & D_2 \\ D_2 & -D_1 & D_0^{\Updownarrow} & D_3^{\Leftrightarrow} \\ -D_1^{\Updownarrow} & -D_2^{\Updownarrow} & -D_3^{\Leftrightarrow} & D_0^{\Updownarrow} \end{pmatrix}$$

Given this transformation from [1], since:

$$\begin{pmatrix} D_0 & D_3^{\Leftrightarrow} & -D_2^{\Leftrightarrow} & D_1 \\ -D_3^{\Leftrightarrow} & D_0 & D_1^{\Leftrightarrow} & D_2 \\ D_2^{\Leftrightarrow} & -D_1^{\Leftrightarrow} & D_0 & D_3 \\ -D_1^{\Updownarrow} & -D_2^{\Updownarrow} & -D_3^{\Updownarrow} & D_0^{\Updownarrow} \end{pmatrix} \text{ and } \begin{pmatrix} D_0^{\Updownarrow} & -D_3^{\Leftrightarrow} & D_2^{\Leftrightarrow} & -D_1 \\ D_3^{\Leftrightarrow} & D_0^{\Updownarrow} & -D_1^{\Leftrightarrow} & -D_2 \\ -D_2^{\Leftrightarrow} & D_1^{\Leftrightarrow} & D_0^{\Updownarrow} & -D_3 \\ D_1^{\Updownarrow} & D_2^{\Updownarrow} & D_3^{\Updownarrow} & D_0 \end{pmatrix}$$

are commutative, and:

$$\begin{pmatrix} D_0 & 0 & iD_3 & iD_1+D_2 \\ 0 & D_0 & iD_1-D_2 & -iD_3 \\ iD_3 & iD_1+D_2 & D_0^{\Updownarrow} & 0 \\ iD_1-D_2 & -iD_3 & 0 & D_0^{\Updownarrow} \end{pmatrix} \text{ and } \begin{pmatrix} D_0^{\Updownarrow} & 0 & -iD_3^{\Updownarrow} & -iD_1^{\Updownarrow}-D_2^{\Updownarrow} \\ 0 & D_0^{\Updownarrow} & -iD_1^{\Updownarrow}+D_2^{\Updownarrow} & iD_3^{\Updownarrow} \\ -iD_3^{\Updownarrow} & -iD_1^{\Updownarrow}-D_2^{\Updownarrow} & D_0 & 0 \\ -iD_1^{\Updownarrow}+D_2^{\Updownarrow} & iD_3^{\Updownarrow} & 0 & D_0 \end{pmatrix}$$

are commutative, then:

$$\begin{pmatrix} D_0^{\Updownarrow} & -D_3^{\Leftrightarrow} & D_2^{\Leftrightarrow} & D_1 \\ D_3^{\Leftrightarrow} & D_0^{\Updownarrow} & -D_1^{\Leftrightarrow} & D_2 \\ -D_2^{\Leftrightarrow} & D_1^{\Leftrightarrow} & D_0^{\Updownarrow} & D_3 \\ D_1^{\Updownarrow} & D_2^{\Updownarrow} & D_3^{\Updownarrow} & -D_0 \end{pmatrix}$$

would be expected to correspond the rightmost Dirac product matrix under the same transformations; and performing the operations upon the rightmost Dirac product matrix .

That is, the following have been shown, above:

$$\begin{pmatrix} D_0 & 0 & iD_3 & iD_1+D_2 \\ 0 & D_0 & iD_1-D_2 & -iD_3 \\ iD_3 & iD_1+D_2 & D_0^{\Updownarrow} & 0 \\ iD_1-D_2 & -iD_3 & 0 & D_0^{\Updownarrow} \end{pmatrix} \text{ and } \begin{pmatrix} D_0^{\Updownarrow} & 0 & -iD_3^{\Updownarrow} & -iD_1^{\Updownarrow}-D_2^{\Updownarrow} \\ 0 & D_0^{\Updownarrow} & -iD_1^{\Updownarrow}+D_2^{\Updownarrow} & iD_3^{\Updownarrow} \\ -iD_3^{\Updownarrow} & -iD_1^{\Updownarrow}-D_2^{\Updownarrow} & D_0 & 0 \\ -iD_1^{\Updownarrow}+D_2^{\Updownarrow} & iD_3^{\Updownarrow} & 0 & D_0 \end{pmatrix}$$

$$\Updownarrow T$$

$$\begin{pmatrix} D_0 & D_3^{\Leftrightarrow} & -D_2 & D_1 \\ -D_3^{\Leftrightarrow} & D_0 & D_1 & D_2 \\ D_2 & -D_1 & D_0^{\Updownarrow} & D_3^{\Leftrightarrow} \\ -D_1^{\Updownarrow} & -D_2^{\Updownarrow} & -D_3^{\Leftrightarrow} & D_0^{\Updownarrow} \end{pmatrix}$$

$$\Updownarrow$$

$$\begin{pmatrix} D_0 & D_3^{\Leftrightarrow} & -D_2^{\Leftrightarrow} & D_1 \\ -D_3^{\Leftrightarrow} & D_0 & D_1^{\Leftrightarrow} & D_2 \\ D_2^{\Leftrightarrow} & -D_1^{\Leftrightarrow} & D_0 & D_3 \\ D_1^{\Updownarrow} & D_2^{\Updownarrow} & D_3^{\Updownarrow} & -D_0^{\Updownarrow} \end{pmatrix} \text{ and } \begin{pmatrix} D_0^{\Updownarrow} & -D_3^{\Leftrightarrow} & D_2^{\Leftrightarrow} & D_1 \\ D_3^{\Leftrightarrow} & D_0^{\Updownarrow} & -D_1^{\Leftrightarrow} & D_2 \\ -D_2^{\Leftrightarrow} & D_1^{\Leftrightarrow} & D_0^{\Updownarrow} & D_3 \\ D_1^{\Updownarrow} & D_2^{\Updownarrow} & D_3^{\Updownarrow} & -D_0 \end{pmatrix} \text{ Maxwell-Cassano}$$

$$\Updownarrow \qquad\qquad \Updownarrow$$

$$\begin{pmatrix} D_0 & D_3^{\Leftrightarrow} & -D_2^{\Leftrightarrow} & D_1 \\ -D_3^{\Leftrightarrow} & D_0 & D_1^{\Leftrightarrow} & D_2 \\ D_2^{\Leftrightarrow} & -D_1^{\Leftrightarrow} & D_0 & D_3 \\ -D_1^{\Updownarrow} & -D_2^{\Updownarrow} & -D_3^{\Updownarrow} & D_0^{\Updownarrow} \end{pmatrix} \text{ and } \begin{pmatrix} D_0^{\Updownarrow} & -D_3^{\Leftrightarrow} & D_2^{\Leftrightarrow} & -D_1 \\ D_3^{\Leftrightarrow} & D_0^{\Updownarrow} & -D_1^{\Leftrightarrow} & -D_2 \\ -D_2^{\Leftrightarrow} & D_1^{\Leftrightarrow} & D_0^{\Updownarrow} & -D_3 \\ D_1^{\Updownarrow} & D_2^{\Updownarrow} & D_3^{\Updownarrow} & D_0 \end{pmatrix} \text{ commutative}$$

What is left to do is verify by performing the appropriate elementary operations on the Maxwell-Cassano rightmost product matrix that were made on the letmost product matrix.

$$\begin{pmatrix} D_0^{\Updownarrow} & -D_3^{\Leftrightarrow} & D_2^{\Leftrightarrow} & D_1 \\ D_3^{\Leftrightarrow} & D_0^{\Updownarrow} & -D_1^{\Leftrightarrow} & D_2 \\ -D_2^{\Leftrightarrow} & D_1^{\Leftrightarrow} & D_0^{\Updownarrow} & D_3 \\ D_1^{\Updownarrow} & D_2^{\Updownarrow} & D_3^{\Updownarrow} & -D_0 \end{pmatrix} \Rightarrow \begin{pmatrix} D_0 & D_3^{\Leftrightarrow} & -D_2 & D_1 \\ -D_3^{\Leftrightarrow} & D_0 & D_1 & D_2 \\ D_2 & -D_1 & D_0^{\Updownarrow} & D_3^{\Leftrightarrow} \\ -D_1^{\Updownarrow} & -D_2^{\Updownarrow} & -D_3^{\Leftrightarrow} & D_0^{\Updownarrow} \end{pmatrix}$$

$$\begin{pmatrix} D_0^{\Updownarrow} & -D_3^{\Leftrightarrow} & D_2^{\Leftrightarrow} & D_1 \\ D_3^{\Leftrightarrow} & D_0^{\Updownarrow} & -D_1^{\Leftrightarrow} & D_2 \\ -D_2^{\Leftrightarrow} & D_1^{\Leftrightarrow} & D_0^{\Updownarrow} & D_3 \\ D_1^{\Updownarrow} & D_2^{\Updownarrow} & D_3^{\Updownarrow} & -D_0 \end{pmatrix} \Rightarrow \ ?$$

If the same operations are made on each, the transformations don't produce the proper matrix - not even a Klein-Gordon product; so clearly this is not the appropriate set of operations.

By appropriate, what is meant is the operations producing a Klein-Gordon product from the left product matrix.

Then, applying the transformations T of [1].

Going from the leftmost Maxwell-Cassano product matrix to the transformation matrix of [1]:

Exchange rows 4 & 5 & columns 4 & 5 & multiply columns 7 & 8 by -1 yielding:

$$\begin{pmatrix} D_0^{\Updownarrow} & -D_3^{\Leftrightarrow} & D_2^{\Leftrightarrow} & D_1 \\ D_3^{\Leftrightarrow} & D_0^{\Updownarrow} & -D_1^{\Leftrightarrow} & D_2 \\ -D_2^{\Leftrightarrow} & D_1^{\Leftrightarrow} & D_0^{\Updownarrow} & D_3 \\ D_1^{\Updownarrow} & D_2^{\Updownarrow} & D_3^{\Updownarrow} & -D_0 \end{pmatrix}$$

$$\Downarrow \textit{into}$$

$$\begin{pmatrix} D_0^{\Updownarrow} & -D_3^{\Leftrightarrow} & D_2 & -D_1 \\ D_3^{\Leftrightarrow} & D_0^{\Updownarrow} & -D_1 & -D_2 \\ -D_2 & D_1 & D_0 & -D_3 \\ D_1^{\Updownarrow} & D_2^{\Updownarrow} & D_3^{\Leftrightarrow} & D_0 \end{pmatrix} = \begin{pmatrix} D_0^{\Updownarrow} & 0 & 0 & 0 \\ 0 & D_0^{\Updownarrow} & 0 & 0 \\ 0 & 0 & D_0 & 0 \\ 0 & 0 & 0 & D_0 \end{pmatrix} +$$

$$+ \begin{pmatrix} 0 & 0 & 0 & -D_1 \\ 0 & 0 & -D_1 & 0 \\ 0 & D_1 & 0 & 0 \\ D_1^{\Updownarrow} & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & D_2 & 0 \\ 0 & 0 & 0 & -D_2 \\ -D_2 & 0 & 0 & 0 \\ 0 & D_2^{\Updownarrow} & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -D_3^{\Leftrightarrow} & 0 & 0 \\ D_3^{\Leftrightarrow} & 0 & 0 & 0 \\ 0 & 0 & 0 & -D_3 \\ 0 & 0 & D_3^{\Leftrightarrow} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \boldsymbol{\sigma}_2^0 & 0 & 0 & 0 \\ 0 & \boldsymbol{\sigma}_2^0 & 0 & 0 \\ 0 & 0 & \boldsymbol{\sigma}_2^0 & 0 \\ 0 & 0 & 0 & \boldsymbol{\sigma}_2^0 \end{pmatrix} D_0 +$$

$$+ \begin{pmatrix} 0 & 0 & 0 & -\boldsymbol{\sigma}_2^0 \\ 0 & 0 & -\boldsymbol{\sigma}_2^0 & 0 \\ 0 & \boldsymbol{\sigma}_2^0 & 0 & 0 \\ \boldsymbol{\sigma}_2^0 & 0 & 0 & 0 \end{pmatrix} D_1 + \begin{pmatrix} 0 & 0 & \boldsymbol{\sigma}_2^0 & 0 \\ 0 & 0 & 0 & -\boldsymbol{\sigma}_2^0 \\ -\boldsymbol{\sigma}_2^0 & 0 & 0 & 0 \\ 0 & \boldsymbol{\sigma}_2^0 & 0 & 0 \end{pmatrix} D_2 + \begin{pmatrix} 0 & -\boldsymbol{\sigma}_2^1 & 0 & 0 \\ \boldsymbol{\sigma}_2^1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\boldsymbol{\sigma}_2^1 \\ 0 & 0 & \boldsymbol{\sigma}_2^1 & 0 \end{pmatrix} D_3$$

And:

$$\begin{pmatrix} D_0^{\Updownarrow} & 0 & -iD_3^{\Updownarrow} & -iD_1^{\Updownarrow} - D_2^{\Updownarrow} \\ 0 & D_0^{\Updownarrow} & -iD_1^{\Updownarrow} + D_2^{\Updownarrow} & iD_3^{\Updownarrow} \\ -iD_3^{\Updownarrow} & -iD_1^{\Updownarrow} - D_2^{\Updownarrow} & D_0 & 0 \\ -iD_1^{\Updownarrow} + D_2^{\Updownarrow} & iD_3^{\Updownarrow} & 0 & D_0 \end{pmatrix} = \begin{pmatrix} D_0^{\Updownarrow} & 0 & 0 & 0 \\ 0 & D_0^{\Updownarrow} & 0 & 0 \\ 0 & 0 & D_0 & 0 \\ 0 & 0 & 0 & D_0 \end{pmatrix} +$$

$$+ \begin{pmatrix} 0 & 0 & 0 & -iD_1^{\Updownarrow} \\ 0 & 0 & -iD_1^{\Updownarrow} & 0 \\ 0 & -iD_1^{\Updownarrow} & 0 & 0 \\ -iD_1^{\Updownarrow} & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & -iD_3^{\Updownarrow} & 0 \\ 0 & 0 & 0 & iD_3^{\Updownarrow} \\ -iD_3^{\Updownarrow} & 0 & 0 & 0 \\ 0 & iD_3^{\Updownarrow} & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & -D_2^{\Updownarrow} \\ 0 & 0 & D_2^{\Updownarrow} & 0 \\ 0 & -D_2^{\Updownarrow} & 0 & 0 \\ D_2^{\Updownarrow} & 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \boldsymbol{\sigma}_2^0 & 0 & 0 & 0 \\ 0 & \boldsymbol{\sigma}_2^0 & 0 & 0 \\ 0 & 0 & \boldsymbol{\sigma}_2^0 & 0 \\ 0 & 0 & 0 & \boldsymbol{\sigma}_2^0 \end{pmatrix} D_0 +$$

$$+ \begin{pmatrix} 0 & 0 & 0 & \boldsymbol{\sigma}_2^0 \\ 0 & 0 & \boldsymbol{\sigma}_2^0 & 0 \\ 0 & \boldsymbol{\sigma}_2^0 & 0 & 0 \\ \boldsymbol{\sigma}_2^0 & 0 & 0 & 0 \end{pmatrix} \left(-iD_1^{\Updownarrow}\right) + \begin{pmatrix} 0 & 0 & -\boldsymbol{\sigma}_2^0 & 0 \\ 0 & 0 & 0 & \boldsymbol{\sigma}_2^0 \\ -\boldsymbol{\sigma}_2^0 & 0 & 0 & 0 \\ 0 & \boldsymbol{\sigma}_2^0 & 0 & 0 \end{pmatrix} iD_3^{\Updownarrow} + \begin{pmatrix} 0 & 0 & 0 & -\boldsymbol{\sigma}_2^0 \\ 0 & 0 & \boldsymbol{\sigma}_2^0 & 0 \\ 0 & -\boldsymbol{\sigma}_2^0 & 0 & 0 \\ \boldsymbol{\sigma}_2^0 & 0 & 0 & 0 \end{pmatrix} D_2^{\Updownarrow}$$

$\Rightarrow$

$\begin{pmatrix} 0 & 0 & 0 & -\boldsymbol{\sigma}_2^0 \\ 0 & 0 & -\boldsymbol{\sigma}_2^0 & 0 \\ 0 & \boldsymbol{\sigma}_2^0 & 0 & 0 \\ \boldsymbol{\sigma}_2^0 & 0 & 0 & 0 \end{pmatrix} D_1 \Rightarrow \begin{pmatrix} 0 & 0 & 0 & \boldsymbol{\sigma}_2^0 \\ 0 & 0 & \boldsymbol{\sigma}_2^0 & 0 \\ 0 & \boldsymbol{\sigma}_2^0 & 0 & 0 \\ \boldsymbol{\sigma}_2^0 & 0 & 0 & 0 \end{pmatrix} \left(-i\overline{D_1}^{\Updownarrow}\right)$	$\begin{pmatrix} \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{0}_2 & \boldsymbol{\sigma}_2^0 \\ \mathbf{0}_2 & \mathbf{0}_2 & \boldsymbol{\sigma}_2^0 & 0 \\ \mathbf{0}_2 & -\boldsymbol{\sigma}_2^0 & \mathbf{0}_2 & \mathbf{0}_2 \\ -\boldsymbol{\sigma}_2^0 & \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{0}_2 \end{pmatrix} \partial_1 \Leftrightarrow \begin{pmatrix} \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{0}_2 & \boldsymbol{\sigma}_2^0 \\ \mathbf{0}_2 & \mathbf{0}_2 & \boldsymbol{\sigma}_2^0 & 0 \\ \mathbf{0}_2 & \boldsymbol{\sigma}_2^0 & \mathbf{0}_2 & \mathbf{0}_2 \\ \boldsymbol{\sigma}_2^0 & \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{0}_2 \end{pmatrix} i\overline{\partial_1}$
$\begin{pmatrix} 0 & 0 & \boldsymbol{\sigma}_2^0 & 0 \\ 0 & 0 & 0 & -\boldsymbol{\sigma}_2^0 \\ -\boldsymbol{\sigma}_2^0 & 0 & 0 & 0 \\ 0 & \boldsymbol{\sigma}_2^0 & 0 & 0 \end{pmatrix} D_2 \Rightarrow \begin{pmatrix} 0 & 0 & -\boldsymbol{\sigma}_2^0 & 0 \\ 0 & 0 & 0 & \boldsymbol{\sigma}_2^0 \\ -\boldsymbol{\sigma}_2^0 & 0 & 0 & 0 \\ 0 & \boldsymbol{\sigma}_2^0 & 0 & 0 \end{pmatrix} i\overline{D_3}^{\Updownarrow}$	$\begin{pmatrix} \mathbf{0}_2 & \mathbf{0}_2 & -\boldsymbol{\sigma}_2^0 & \mathbf{0}_2 \\ \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{0}_2 & \boldsymbol{\sigma}_2^0 \\ \boldsymbol{\sigma}_2^0 & \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{0}_2 \\ \mathbf{0}_2 & -\boldsymbol{\sigma}_2^0 & \mathbf{0}_2 & \mathbf{0}_2 \end{pmatrix} \partial_2 \Leftrightarrow \begin{pmatrix} \mathbf{0}_2 & \mathbf{0}_2 & \boldsymbol{\sigma}_2^0 & \mathbf{0}_2 \\ \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{0}_2 & -\boldsymbol{\sigma}_2^0 \\ \boldsymbol{\sigma}_2^0 & \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{0}_2 \\ \mathbf{0}_2 & -\boldsymbol{\sigma}_2^0 & \mathbf{0}_2 & \mathbf{0}_2 \end{pmatrix} i\overline{\partial_3}$
$\begin{pmatrix} 0 & -\boldsymbol{\sigma}_2^1 & 0 & 0 \\ \boldsymbol{\sigma}_2^1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\boldsymbol{\sigma}_2^1 \\ 0 & 0 & \boldsymbol{\sigma}_2^1 & 0 \end{pmatrix} D_3 \Rightarrow \begin{pmatrix} 0 & 0 & 0 & -\boldsymbol{\sigma}_2^0 \\ 0 & 0 & \boldsymbol{\sigma}_2^0 & 0 \\ 0 & -\boldsymbol{\sigma}_2^0 & 0 & 0 \\ \boldsymbol{\sigma}_2^0 & 0 & 0 & 0 \end{pmatrix} \overline{D_2}^{\Updownarrow}$	$\begin{pmatrix} \mathbf{0}_2 & \boldsymbol{\sigma}_2^1 & \mathbf{0}_2 & \mathbf{0}_2 \\ -\boldsymbol{\sigma}_2^1 & \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{0}_2 \\ \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{0}_2 & \boldsymbol{\sigma}_2^1 \\ \mathbf{0}_2 & \mathbf{0}_2 & -\boldsymbol{\sigma}_2^1 & \mathbf{0}_2 \end{pmatrix} \partial_3 \Leftrightarrow \begin{pmatrix} \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{0}_2 & \boldsymbol{\sigma}_2^0 \\ \mathbf{0}_2 & \mathbf{0}_2 & -\boldsymbol{\sigma}_2^0 & \mathbf{0}_2 \\ \mathbf{0}_2 & \boldsymbol{\sigma}_2^0 & \mathbf{0}_2 & \mathbf{0}_2 \\ -\boldsymbol{\sigma}_2^0 & \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{0}_2 \end{pmatrix} \overline{\partial_2}$

Thus, the transformations [1] yield the appropriate results:

$$\begin{pmatrix} D_0 & 0 & iD_3 & iD_1 + D_2 \\ 0 & D_0 & iD_1 - D_2 & -iD_3 \\ iD_3 & iD_1 + D_2 & D_0^{\Updownarrow} & 0 \\ iD_1 - D_2 & -iD_3 & 0 & D_0^{\Updownarrow} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} D_0^{\Updownarrow} & 0 & -iD_3^{\Updownarrow} & -iD_1^{\Updownarrow} - D_2^{\Updownarrow} \\ 0 & D_0^{\Updownarrow} & -iD_1^{\Updownarrow} + D_2^{\Updownarrow} & iD_3^{\Updownarrow} \\ -iD_3^{\Updownarrow} & -iD_1^{\Updownarrow} - D_2^{\Updownarrow} & D_0 & 0 \\ -iD_1^{\Updownarrow} + D_2^{\Updownarrow} & iD_3^{\Updownarrow} & 0 & D_0 \end{pmatrix}$$

$$\Updownarrow T \qquad\qquad \Updownarrow T$$

$$\begin{pmatrix} D_0 & D_3^{\Leftrightarrow} & -D_2 & D_1 \\ -D_3^{\Leftrightarrow} & D_0 & D_1 & D_2 \\ D_2 & -D_1 & D_0^{\Updownarrow} & D_3^{\Leftrightarrow} \\ -D_1^{\Updownarrow} & -D_2^{\Updownarrow} & -D_3^{\Leftrightarrow} & D_0^{\Updownarrow} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} D_0^{\Updownarrow} & -D_3^{\Leftrightarrow} & D_2^{\Updownarrow} & -D_1 \\ D_3^{\Leftrightarrow} & D_0^{\Updownarrow} & -D_1^{\Updownarrow} & -D_2 \\ -D_2^{\Updownarrow} & D_1^{\Updownarrow} & D_0 & -D_3 \\ D_1^{\Updownarrow} & D_2^{\Updownarrow} & D_3^{\Leftrightarrow} & D_0 \end{pmatrix} \text{ commutative}$$

$$\Updownarrow \qquad\qquad \Updownarrow$$

$$\begin{pmatrix} D_0 & D_3^{\Leftrightarrow} & -D_2^{\Leftrightarrow} & D_1 \\ -D_3^{\Leftrightarrow} & D_0 & D_1^{\Leftrightarrow} & D_2 \\ D_2^{\Leftrightarrow} & -D_1^{\Leftrightarrow} & D_0 & D_3 \\ D_1^{\Updownarrow} & D_2^{\Updownarrow} & D_3^{\Updownarrow} & -D_0^{\Updownarrow} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} D_0^{\Updownarrow} & -D_3^{\Leftrightarrow} & D_2^{\Leftrightarrow} & D_1 \\ D_3^{\Leftrightarrow} & D_0^{\Updownarrow} & -D_1^{\Leftrightarrow} & D_2 \\ -D_2^{\Leftrightarrow} & D_1^{\Leftrightarrow} & D_0^{\Updownarrow} & D_3 \\ D_1^{\Updownarrow} & D_2^{\Updownarrow} & D_3^{\Updownarrow} & -D_0 \end{pmatrix} \text{ Maxwell-Cassano}$$

$$\Updownarrow \qquad\qquad \Updownarrow$$

$$\begin{pmatrix} D_0 & D_3^{\Leftrightarrow} & -D_2^{\Leftrightarrow} & D_1 \\ -D_3^{\Leftrightarrow} & D_0 & D_1^{\Leftrightarrow} & D_2 \\ D_2^{\Leftrightarrow} & -D_1^{\Leftrightarrow} & D_0 & D_3 \\ -D_1^{\Updownarrow} & -D_2^{\Updownarrow} & -D_3^{\Updownarrow} & D_0^{\Updownarrow} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} D_0^{\Updownarrow} & -D_3^{\Leftrightarrow} & D_2^{\Leftrightarrow} & -D_1 \\ D_3^{\Leftrightarrow} & D_0^{\Updownarrow} & -D_1^{\Leftrightarrow} & -D_2 \\ -D_2^{\Leftrightarrow} & D_1^{\Leftrightarrow} & D_0^{\Updownarrow} & -D_3 \\ D_1^{\Updownarrow} & D_2^{\Updownarrow} & D_3^{\Updownarrow} & D_0 \end{pmatrix} \text{ commutative}$$

The advantage of the initial Maxwell-Cassano product matrices is that the pair of right products with the 4-vectors may be written as a sum and difference of $\mathbf{E}$ & $\mathbf{B}_{\Updownarrow}$ field strengths with gauge term (based on potentials); which, in turn, yield the Maxwell-Cassano equations mass-generalization of Maxwell's equations for a Klein-Gordon nuclear field with source. The same does not so readily appear with this single commutative matrix pair.

Now, since the Maxwell-Cassano rightmost product matrix applied to the 8-vector/4-vector-doublet may be written in terms of potentials, the Dirac rightmost product matrix applied to it's corresponding 8-vector may also (although not necessarily as a sum and difference of $\mathbf{E}$ & $\mathbf{B}_{\Updownarrow}$ potentials with gauge term, but something more complicated.

Via an understanding of [3], the 8-vector components themselves are the potential components. definitions of $\mathbf{E}$ & $\mathbf{B}$ may be made by composition of appropriate elementary operations and transformations [1], in which case sums and differences of the factors:

$$\begin{pmatrix} D_0^{\Updownarrow} & 0 & -iD_3^{\Updownarrow} & -iD_1^{\Updownarrow} - D_2^{\Updownarrow} \\ 0 & D_0^{\Updownarrow} & -iD_1^{\Updownarrow} + D_2^{\Updownarrow} & iD_3^{\Updownarrow} \\ -iD_3^{\Updownarrow} & -iD_1^{\Updownarrow} - D_2^{\Updownarrow} & D_0 & 0 \\ -iD_1^{\Updownarrow} + D_2^{\Updownarrow} & iD_3^{\Updownarrow} & 0 & D_0 \end{pmatrix} \begin{pmatrix} \boldsymbol{\theta}^1 \\ \boldsymbol{\theta}^2 \\ \boldsymbol{\theta}^3 \\ \boldsymbol{\theta}^0 \end{pmatrix} =$$

$$= \begin{pmatrix} D_0^{\Updownarrow}\boldsymbol{\theta}^1 - iD_3^{\Updownarrow}\boldsymbol{\theta}^3 + \left(-iD_1^{\Updownarrow} - D_2^{\Updownarrow}\right)\boldsymbol{\theta}^0 \\ D_0^{\Updownarrow}\boldsymbol{\theta}^2 + \left(-iD_1^{\Updownarrow} + D_2^{\Updownarrow}\right)\boldsymbol{\theta}^3 + iD_3^{\Updownarrow}\boldsymbol{\theta}^0 \\ -iD_3^{\Updownarrow}\boldsymbol{\theta}^1 + \left(-iD_1^{\Updownarrow} - D_2^{\Updownarrow}\right)\boldsymbol{\theta}^2 + D_0\boldsymbol{\theta}^3 \\ \left(-iD_1^{\Updownarrow} + D_2^{\Updownarrow}\right)\boldsymbol{\theta}^1 + iD_3^{\Updownarrow}\boldsymbol{\theta}^2 + D_0\boldsymbol{\theta}^0 \end{pmatrix} = \begin{pmatrix} D_0^{\Updownarrow}\boldsymbol{\theta}^1 - iD_1^{\Updownarrow}\boldsymbol{\theta}^0 - iD_3^{\Updownarrow}\boldsymbol{\theta}^3 - D_2^{\Updownarrow}\boldsymbol{\theta}^0 \\ D_0^{\Updownarrow}\boldsymbol{\theta}^2 + iD_3^{\Updownarrow}\boldsymbol{\theta}^0 - iD_1^{\Updownarrow}\boldsymbol{\theta}^3 + D_2^{\Updownarrow}\boldsymbol{\theta}^3 \\ D_0\boldsymbol{\theta}^3 - D_2^{\Updownarrow}\boldsymbol{\theta}^2 - iD_1^{\Updownarrow}\boldsymbol{\theta}^2 - iD_3^{\Updownarrow}\boldsymbol{\theta}^1 \\ D_0\boldsymbol{\theta}^0 - iD_1^{\Updownarrow}\boldsymbol{\theta}^1 + iD_3^{\Updownarrow}\boldsymbol{\theta}^2 + D_2^{\Updownarrow}\boldsymbol{\theta}^1 \end{pmatrix}$$

and:

$$\begin{pmatrix} D_0 & 0 & iD_3 & iD_1 + D_2 \\ 0 & D_0 & iD_1 - D_2 & -iD_3 \\ iD_3 & iD_1 + D_2 & D_0^{\Updownarrow} & 0 \\ iD_1 - D_2 & -iD_3 & 0 & D_0^{\Updownarrow} \end{pmatrix} \begin{pmatrix} \boldsymbol{\theta}^1 \\ \boldsymbol{\theta}^2 \\ \boldsymbol{\theta}^3 \\ \boldsymbol{\theta}^0 \end{pmatrix} =$$

$$= \begin{pmatrix} D_0\boldsymbol{\theta}^1 + iD_3\boldsymbol{\theta}^3 + (iD_1 + D_2)\boldsymbol{\theta}^0 \\ D_0\boldsymbol{\theta}^2 + (iD_1 - D_2)\boldsymbol{\theta}^3 - iD_3\boldsymbol{\theta}^0 \\ iD_3\boldsymbol{\theta}^1 + (iD_1 + D_2)\boldsymbol{\theta}^2 + D_0^{\Updownarrow}\boldsymbol{\theta}^3 \\ (iD_1 - D_2)\boldsymbol{\theta}^1 - iD_3\boldsymbol{\theta}^2 + D_0^{\Updownarrow}\boldsymbol{\theta}^0 \end{pmatrix} = \begin{pmatrix} D_0\boldsymbol{\theta}^1 + iD_1\boldsymbol{\theta}^0 + iD_3\boldsymbol{\theta}^3 + D_2\boldsymbol{\theta}^0 \\ D_0\boldsymbol{\theta}^2 - iD_3\boldsymbol{\theta}^0 - iD_1\boldsymbol{\theta}^3 - D_2\boldsymbol{\theta}^3 \\ D_0^{\Updownarrow}\boldsymbol{\theta}^3 + D_2\boldsymbol{\theta}^2 + iD_1\boldsymbol{\theta}^2 + iD_3\boldsymbol{\theta}^1 \\ D_0^{\Updownarrow}\boldsymbol{\theta}^0 + iD_1\boldsymbol{\theta}^1 - iD_3\boldsymbol{\theta}^2 - D_2\boldsymbol{\theta}^1 \end{pmatrix}$$

don't offer much.

Note that:

$$\mathbf{G} = iD_1^{\Updownarrow}\boldsymbol{\theta}^1 - iD_3^{\Updownarrow}\boldsymbol{\theta}^2 - D_2^{\Updownarrow}\boldsymbol{\theta}^1 = -\left(-iD_1^{\Updownarrow}\boldsymbol{\theta}^1 + iD_3^{\Updownarrow}\boldsymbol{\theta}^2 + D_2^{\Updownarrow}\boldsymbol{\theta}^1\right)$$

so either $\mathbf{G}$ or either of: $D_0^{\Updownarrow}\boldsymbol{\theta}^0 + \mathbf{G}$ or $D_0\boldsymbol{\theta}^0 - \mathbf{G}$

may be appropriate choices of guage

(analogous to the Lorentz guage and Coulomb or Radiation guage).

Defining $\mathbf{E}$ & $\mathbf{B}$ by these products as in [2]:

$$\begin{pmatrix} D_0^{\Updownarrow}\boldsymbol{\theta}^1 - iD_1^{\Updownarrow}\boldsymbol{\theta}^0 - iD_3^{\Updownarrow}\boldsymbol{\theta}^3 - D_2^{\Updownarrow}\boldsymbol{\theta}^0 \\ D_0^{\Updownarrow}\boldsymbol{\theta}^2 + iD_3^{\Updownarrow}\boldsymbol{\theta}^0 - iD_1^{\Updownarrow}\boldsymbol{\theta}^3 + D_2^{\Updownarrow}\boldsymbol{\theta}^3 \\ D_0\boldsymbol{\theta}^3 - D_2^{\Updownarrow}\boldsymbol{\theta}^2 - iD_1^{\Updownarrow}\boldsymbol{\theta}^2 - iD_3^{\Updownarrow}\boldsymbol{\theta}^1 \\ D_0\boldsymbol{\theta}^0 - iD_1^{\Updownarrow}\boldsymbol{\theta}^1 + iD_3^{\Updownarrow}\boldsymbol{\theta}^2 + D_2^{\Updownarrow}\boldsymbol{\theta}^1 \end{pmatrix} = \begin{pmatrix} \mathbf{E}^1_{\Updownarrow *} - \mathbf{B}^1_{\Updownarrow *} \\ \mathbf{E}^2_{\Updownarrow *} - \mathbf{B}^2_{\Updownarrow *} \\ \mathbf{E}^3_{*} - \mathbf{B}^3_{\Updownarrow *} \\ D_0\boldsymbol{\theta}^0 - \mathbf{G} \end{pmatrix}$$

$$\begin{pmatrix} D_0\boldsymbol{\theta}^1 + iD_1\boldsymbol{\theta}^0 + iD_3\boldsymbol{\theta}^3 + D_2\boldsymbol{\theta}^0 \\ D_0\boldsymbol{\theta}^2 - iD_3\boldsymbol{\theta}^0 - iD_1\boldsymbol{\theta}^3 - D_2\boldsymbol{\theta}^3 \\ D_0^{\Updownarrow}\boldsymbol{\theta}^3 + D_2\boldsymbol{\theta}^2 + iD_1\boldsymbol{\theta}^2 + iD_3\boldsymbol{\theta}^1 \\ D_0^{\Updownarrow}\boldsymbol{\theta}^0 + iD_1\boldsymbol{\theta}^1 - iD_3\boldsymbol{\theta}^2 - D_2\boldsymbol{\theta}^1 \end{pmatrix} = \begin{pmatrix} \mathbf{E}^1_{*} + \mathbf{B}^1_{*} \\ \mathbf{E}^2_{*} + \mathbf{B}^2_{*} \\ \mathbf{E}^3_{\Updownarrow *} + \mathbf{B}^3_{*} \\ D_0^{\Updownarrow}\boldsymbol{\theta}^0 + \mathbf{G} \end{pmatrix}$$

The signs of the component parts of the vector elements indicate that appropriate definitions of $\mathbf{E}$ & $\mathbf{B}$ would be:

$$\mathbf{E}_* = \mathbf{u}_1(D_0\boldsymbol{\theta}^1) + \mathbf{u}_2(D_0\boldsymbol{\theta}^2 - iD_1\boldsymbol{\theta}^3) + \mathbf{u}_3(D_0\boldsymbol{\theta}^3)$$
$$\mathbf{B}_* = \mathbf{u}_1(iD_1\boldsymbol{\theta}^0 + iD_3\boldsymbol{\theta}^3 + D_2\boldsymbol{\theta}^0) + \mathbf{u}_2(-iD_3\boldsymbol{\theta}^0 - D_2\boldsymbol{\theta}^3) + \mathbf{u}_3(D_2\boldsymbol{\theta}^2 + iD_1\boldsymbol{\theta}^2 + iD_3\boldsymbol{\theta}^1)$$
$$\mathbf{E}_{\Updownarrow *} = \mathbf{u}_1\left(D_0^{\Updownarrow}\boldsymbol{\theta}^1\right) + \mathbf{u}_2\left(D_0^{\Updownarrow}\boldsymbol{\theta}^2 - iD_1^{\Updownarrow}\boldsymbol{\theta}^3\right) + \mathbf{u}_3\left(D_0^{\Updownarrow}\boldsymbol{\theta}^3\right)$$
$$\mathbf{B}_{\Updownarrow *} = \mathbf{u}_1\left(iD_1^{\Updownarrow}\boldsymbol{\theta}^0 + iD_3^{\Updownarrow}\boldsymbol{\theta}^3 + D_2^{\Updownarrow}\boldsymbol{\theta}^0\right) + \mathbf{u}_2\left(-iD_3^{\Updownarrow}\boldsymbol{\theta}^0 - D_2^{\Updownarrow}\boldsymbol{\theta}^3\right) + \mathbf{u}_3\left(D_2^{\Updownarrow}\boldsymbol{\theta}^2 + iD_1^{\Updownarrow}\boldsymbol{\theta}^2 + iD_3^{\Updownarrow}\boldsymbol{\theta}^1\right)$$

Thus, consistent, effective definitions in the Dirac iso-spin space for field strengths $\mathbf{E}_*$ & $\mathbf{B}_*$ based on potentials have been established.

But, subtracting & adding, as in [4]:

$$(\Box - m^2)\begin{pmatrix} \boldsymbol{\theta}^1 \\ \boldsymbol{\theta}^2 \\ \boldsymbol{\theta}^3 \\ \boldsymbol{\theta}^0 \end{pmatrix} = \begin{pmatrix} D_0^{\Updownarrow} & 0 & -iD_3^{\Updownarrow} & -iD_1^{\Updownarrow} - D_2^{\Updownarrow} \\ 0 & D_0^{\Updownarrow} & -iD_1^{\Updownarrow} + D_2^{\Updownarrow} & iD_3^{\Updownarrow} \\ -iD_3^{\Updownarrow} & -iD_1^{\Updownarrow} - D_2^{\Updownarrow} & D_0 & 0 \\ -iD_1^{\Updownarrow} + D_2^{\Updownarrow} & iD_3^{\Updownarrow} & 0 & D_0 \end{pmatrix}\begin{pmatrix} D_0\boldsymbol{\theta}^1 + iD_1\boldsymbol{\theta}^0 + iD_3\boldsymbol{\theta}^3 + D_2\boldsymbol{\theta}^0 \\ D_0\boldsymbol{\theta}^2 - iD_3\boldsymbol{\theta}^0 - iD_1\boldsymbol{\theta}^3 - D_2\boldsymbol{\theta}^3 \\ D_0^{\Updownarrow}\boldsymbol{\theta}^3 + D_2\boldsymbol{\theta}^2 + iD_1\boldsymbol{\theta}^2 + iD_3\boldsymbol{\theta}^1 \\ D_0^{\Updownarrow}\boldsymbol{\theta}^0 + iD_1\boldsymbol{\theta}^1 - iD_3\boldsymbol{\theta}^2 - D_2\boldsymbol{\theta}^1 \end{pmatrix}$$

$$= \begin{pmatrix} D_0^{\Updownarrow} & 0 & -iD_3^{\Updownarrow} & -iD_1^{\Updownarrow} - D_2^{\Updownarrow} \\ 0 & D_0^{\Updownarrow} & -iD_1^{\Updownarrow} + D_2^{\Updownarrow} & iD_3^{\Updownarrow} \\ -iD_3^{\Updownarrow} & -iD_1^{\Updownarrow} - D_2^{\Updownarrow} & D_0 & 0 \\ -iD_1^{\Updownarrow} + D_2^{\Updownarrow} & iD_3^{\Updownarrow} & 0 & D_0 \end{pmatrix}\begin{pmatrix} \mathbf{E}^1_{*} + \mathbf{B}^1_{*} \\ \mathbf{E}^2_{*} + \mathbf{B}^2_{*} \\ \mathbf{E}^3_{\Updownarrow *} + \mathbf{B}^3_{*} \\ D_0\boldsymbol{\theta}^0 + \mathbf{G} \end{pmatrix}$$

$$= \begin{pmatrix} D_0 & 0 & iD_3 & iD_1 + D_2 \\ 0 & D_0 & iD_1 - D_2 & -iD_3 \\ iD_3 & iD_1 + D_2 & D_0^{\Updownarrow} & 0 \\ iD_1 - D_2 & -iD_3 & 0 & D_0^{\Updownarrow} \end{pmatrix}\begin{pmatrix} D_0^{\Updownarrow}\boldsymbol{\theta}^1 - iD_1^{\Updownarrow}\boldsymbol{\theta}^0 - iD_3^{\Updownarrow}\boldsymbol{\theta}^3 - D_2^{\Updownarrow}\boldsymbol{\theta}^0 \\ D_0^{\Updownarrow}\boldsymbol{\theta}^2 + iD_3^{\Updownarrow}\boldsymbol{\theta}^0 - iD_1^{\Updownarrow}\boldsymbol{\theta}^3 + D_2^{\Updownarrow}\boldsymbol{\theta}^3 \\ D_0\boldsymbol{\theta}^3 - D_2^{\Updownarrow}\boldsymbol{\theta}^2 - iD_1^{\Updownarrow}\boldsymbol{\theta}^2 - iD_3^{\Updownarrow}\boldsymbol{\theta}^1 \\ D_0\boldsymbol{\theta}^0 - iD_1^{\Updownarrow}\boldsymbol{\theta}^1 + iD_3^{\Updownarrow}\boldsymbol{\theta}^2 + D_2^{\Updownarrow}\boldsymbol{\theta}^1 \end{pmatrix}$$

$$= \begin{pmatrix} D_0 & 0 & iD_3 & iD_1 + D_2 \\ 0 & D_0 & iD_1 - D_2 & -iD_3 \\ iD_3 & iD_1 + D_2 & D_0^{\Updownarrow} & 0 \\ iD_1 - D_2 & -iD_3 & 0 & D_0^{\Updownarrow} \end{pmatrix}\begin{pmatrix} \mathbf{E}^1_{\Updownarrow *} - \mathbf{B}^1_{\Updownarrow *} \\ \mathbf{E}^2_{\Updownarrow *} - \mathbf{B}^2_{\Updownarrow *} \\ \mathbf{E}^3_{*} - \mathbf{B}^3_{\Updownarrow *} \\ D_0^{\Updownarrow}\boldsymbol{\theta}^0 - \mathbf{G} \end{pmatrix}$$

Recall that:

$D_i^{\Updownarrow} = D_i \quad , \quad (i \neq 0) \quad , \quad D_0^{\Updownarrow} - D_0 = 2m\boldsymbol{\sigma}^3$ & $D_0^{\Updownarrow} + D_0 = 2\boldsymbol{\sigma}^0\partial_0$

because $m_1 = m_1 = m_1 = 0 : \mathbf{B}^i_{\Updownarrow *} = \mathbf{B}^i_*$

and: $D_i^{\Updownarrow}D_j = D_jD_i^{\Updownarrow} \Rightarrow D_0\mathbf{E}^1_{\Updownarrow *} = D_0^{\Updownarrow}\mathbf{E}^1_* \quad , \quad D_0\mathbf{E}^3_{\Updownarrow *} = D_0^{\Updownarrow}\mathbf{E}^3_*$

$$\mathbf{E}_* = \mathbf{u}_1(D_0\boldsymbol{\theta}^1) + \mathbf{u}_2(D_0\boldsymbol{\theta}^2 - iD_1\boldsymbol{\theta}^3) + \mathbf{u}_3(D_0\boldsymbol{\theta}^3)$$
$$\mathbf{E}_{\Updownarrow *} = \mathbf{u}_1\left(D_0^{\Updownarrow}\boldsymbol{\theta}^1\right) + \mathbf{u}_2\left(D_0^{\Updownarrow}\boldsymbol{\theta}^2 - iD_1^{\Updownarrow}\boldsymbol{\theta}^3\right) + \mathbf{u}_3\left(D_0^{\Updownarrow}\boldsymbol{\theta}^3\right)$$
$$\mathbf{B}_* = \mathbf{u}_1(iD_1\boldsymbol{\theta}^0 + iD_3\boldsymbol{\theta}^3 + D_2\boldsymbol{\theta}^0) + \mathbf{u}_2(-iD_3\boldsymbol{\theta}^0 - D_2\boldsymbol{\theta}^3) + \mathbf{u}_3(D_2\boldsymbol{\theta}^2 + iD_1\boldsymbol{\theta}^2 + iD_3\boldsymbol{\theta}^1)$$
$$\mathbf{B}_{\Updownarrow *} = \mathbf{u}_1\left(iD_1^{\Updownarrow}\boldsymbol{\theta}^0 + iD_3^{\Updownarrow}\boldsymbol{\theta}^3 + D_2^{\Updownarrow}\boldsymbol{\theta}^0\right) + \mathbf{u}_2\left(-iD_3^{\Updownarrow}\boldsymbol{\theta}^0 - D_2^{\Updownarrow}\boldsymbol{\theta}^3\right) + \mathbf{u}_3\left(D_2^{\Updownarrow}\boldsymbol{\theta}^2 + iD_1^{\Updownarrow}\boldsymbol{\theta}^2 + iD_3^{\Updownarrow}\boldsymbol{\theta}^1\right)$$

Subtracting:

$$\Rightarrow 0 = \begin{pmatrix} D_0 & 0 & iD_3 & iD_1 + D_2 \\ 0 & D_0 & iD_1 - D_2 & -iD_3 \\ iD_3 & iD_1 + D_2 & D_0^{\Updownarrow} & 0 \\ iD_1 - D_2 & -iD_3 & 0 & D_0^{\Updownarrow} \end{pmatrix}\begin{pmatrix} \mathbf{E}^1_{\Updownarrow *} - \mathbf{B}^1_{\Updownarrow *} \\ \mathbf{E}^2_{\Updownarrow *} - \mathbf{B}^2_{\Updownarrow *} \\ \mathbf{E}^3_{*} - \mathbf{B}^3_{\Updownarrow *} \\ D_0^{\Updownarrow}\boldsymbol{\theta}^0 - \mathbf{G} \end{pmatrix} +$$
$$- \begin{pmatrix} D_0^{\Updownarrow} & 0 & -iD_3^{\Updownarrow} & -iD_1^{\Updownarrow} - D_2^{\Updownarrow} \\ 0 & D_0^{\Updownarrow} & -iD_1^{\Updownarrow} + D_2^{\Updownarrow} & iD_3^{\Updownarrow} \\ -iD_3^{\Updownarrow} & -iD_1^{\Updownarrow} - D_2^{\Updownarrow} & D_0 & 0 \\ -iD_1^{\Updownarrow} + D_2^{\Updownarrow} & iD_3^{\Updownarrow} & 0 & D_0 \end{pmatrix}\begin{pmatrix} \mathbf{E}^1_{*} + \mathbf{B}^1_{*} \\ \mathbf{E}^2_{*} + \mathbf{B}^2_{*} \\ \mathbf{E}^3_{\Updownarrow *} + \mathbf{B}^3_{*} \\ D_0\boldsymbol{\theta}^0 + \mathbf{G} \end{pmatrix}$$

$$
= \begin{pmatrix}
D_0\left(\mathbf{E}^1_{\Updownarrow *} - \mathbf{B}^1_{\Updownarrow *}\right) - D_0^{\Updownarrow}(\mathbf{E}^1_{*} + \mathbf{B}^1_{*}) + iD_3\left(\mathbf{E}^3_{*} - \mathbf{B}^3_{\Updownarrow *}\right) + iD_3^{\Updownarrow}\left(\mathbf{E}^3_{\Updownarrow *} + \mathbf{B}^3_{*}\right) + (iD_1 + D_2)\left(D_0^{\Updownarrow}\boldsymbol{\theta}^0 - \mathbf{G}\right) + \left(iD_1^{\Updownarrow} + D_2^{\Updownarrow}\right)(D_0\boldsymbol{\theta}^0 + \mathbf{G}) \\
D_0\left(\mathbf{E}^2_{\Updownarrow *} - \mathbf{B}^2_{\Updownarrow *}\right) - D_0^{\Updownarrow}(\mathbf{E}^2_{*} + \mathbf{B}^2_{*}) + (iD_1 - D_2)\left(\mathbf{E}^3_{*} - \mathbf{B}^3_{\Updownarrow *}\right) + \left(iD_1^{\Updownarrow} - D_2^{\Updownarrow}\right)\left(\mathbf{E}^3_{\Updownarrow *} + \mathbf{B}^3_{*}\right) - iD_3\left(D_0^{\Updownarrow}\boldsymbol{\theta}^0 - \mathbf{G}\right) - iD_3^{\Updownarrow}(D_0\boldsymbol{\theta}^0 + \mathbf{G}) \\
iD_3\left(\mathbf{E}^1_{\Updownarrow *} - \mathbf{B}^1_{\Updownarrow *}\right) + iD_3^{\Updownarrow}(\mathbf{E}^1_{*} + \mathbf{B}^1_{*}) + (iD_1 + D_2)\left(\mathbf{E}^2_{\Updownarrow *} - \mathbf{B}^2_{\Updownarrow *}\right) + (iD_1 + D_2)(\mathbf{E}^2_{*} + \mathbf{B}^2_{*}) + D_0^{\Updownarrow}\left(\mathbf{E}^3_{*} - \mathbf{B}^3_{\Updownarrow *}\right) - D_0\left(\mathbf{E}^3_{\Updownarrow *} + \mathbf{B}^3_{*}\right) \\
(iD_1 - D_2)\left(\mathbf{E}^1_{\Updownarrow *} - \mathbf{B}^1_{\Updownarrow *}\right) + \left(iD_1^{\Updownarrow} - D_2^{\Updownarrow}\right)(\mathbf{E}^1_{*} + \mathbf{B}^1_{*}) - iD_3\left(\mathbf{E}^2_{\Updownarrow *} - \mathbf{B}^2_{\Updownarrow *}\right) - iD_3^{\Updownarrow}(\mathbf{E}^2_{*} + \mathbf{B}^2_{*}) + D_0^{\Updownarrow}\left(D_0^{\Updownarrow}\boldsymbol{\theta}^0 - \mathbf{G}\right) - D_0(D_0\boldsymbol{\theta}^0 + \mathbf{G})
\end{pmatrix}
$$

doesn’t seem to yield a set of simple-powerful equations linear in the field strengths alone,
as before.

With this comprehensive look at the linearization of the four-vector-Klein-Gordon equation, distinctions have arisen between the various representations obtained.

The Dirac matrix operators have the symmetry of commutivity, but there is an inherent asymmetry in that the 1 & 2 -coordinates are paired - while coordinate-3 stands apart from them.

(of course, each may in-turn be paired, bu then the system grows by a factor of 3)

However, as seen above, two other factorizations are commutative, as well.

Although the other commutative factorizations have not been delved deeply into, the Helholtzian factorization has already yielded substantially.

Although the Helmholtzian factorization is nor commutative it yields guage-invariant symmetric Maxwell-Cassano equations linear in the field-strenghts alone via the adding & subtracting process.

Further, the transformation shown demonstrates that the Helmholtzian factorization is a generalization of the Dirac matrix operator factorization, which implies the fermion architecture and appropriate observed group structure characteristics as demonstrated by the hadrons.

The Dirac equation has yielded knowledge of the fundamental particles, importantly that the field is a four-vector-doublet (four-vector-complex equivalent) and that they have particle-anti-particle nature.
However, just as polar coordinates may simplify a description; using the Helmholtzian factorization yields not only the generalization of Maxwell’s equations to massive particles, but also generalizes the field to the fermion architecture, expanding knowledge beyond what the Dirac equation provided.

It is sad to be lost within a Glashow-Salam-Weinberg + Higgs fairytale and acquiesce to confinement within a cell of which an open door to which has been revealed.
The truth will leave these in the dust as it moves on and beyond.

With this & [1] & [2], I have killed not only the Glashow-Salam-Weinberg + Higgs model, but the quantum mechanics probabilistic model; because probability amplitudes are no longer the fundamentals, but field potentials akin to the field potentials of Maxwell’s electromagnetic field theory.
(Note that any function with a finite-valued definite-integral (over it’s full limits) can be normalized and cast as a "probability density/distribution function". It’s applicability to a given situation is a question. There are a number of probability density/distribution functions; such as the Binomial distribution, Normal/Gaussian distribution, Pareto distribution, Student’s t-distribution, etc. Thus, merely asserting an applicability of a probability density/distribution function to a given situation does not make it so.)

No longer are physicists and engineers fumbling in the dark, but actual engineering can be done again - free from "interpretations" - following in the footsteps of the likes of Hertz, Ampère, Oersted, and Marconi.

I have shot them between the eyes, decaptated them, dismembered them, run them through a wood-chipper, baked them, and fed them to swine.
Only Jason Voorhees could return, again.
Why? Because it’s out there, now. And sooner or later - and likely sooner than late - it’ll be used to engineer projects, which will vanquish those who ignore this knowledge, suffering domination.

The Glashow-Salam-Weinberg + Higgs model, and the quantum mechanics probabilistic model are zombies and will eat your brain!
Already they are eating the brains of physicists, and have eaten the brains of countless physicists over recent decades!!

It is sad to be lost within a Glashow-Salam-Weinberg + Higgs fairytale and acquiesce to confinement within a cell of which an open door to which has been revealed.
The truth will leave these in the dust as it moves on and beyond.

References
and further readings

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