The Mathematical Structure of Quantum Nambu Mechanics and Neutrino Oscillations

Tadafumi Ohsaku

today

Abstract

Some Lie-algebraic structures of three-dimensional quantum Nambu mechanics are studied. From our result, we argue that the three-dimensional quantum Nambu mechanics is a natural extension of the ordinary Heisenberg quantum theory, and we give our insight that we can construct several candidates "beyond the Heisenberg quantum theory".

1 Introduction

Needless to say, the framework of Hamiltonian mechanics [2] is the basis for theories of modern physics, including quantum mechanics, quantum field theory, statistical physics, ..., so forth. The Nambu mechanics [3,9,14,19,24] is a famous example for a generalization of the framework of Hamiltonian mechanics, in which the preservation of a volume element expressed by the canonical variables of a dynamical system is the main motive. One of the important fact in the Hamiltonian mechanics from the context of this paper is its symplectic structure expressed by a pair of canonical conjugate variables (x, p_x), namely $dx \wedge dp_x$ or the Poisson bracket $\{x, p_x\}$, and its algebraic extension gives a Heisenberg algebra $[x, p_x] = -[p_x, x] = \pm i\hbar$. The Nambu mechanics introduces a canonical conjugate variables more than two, namely a triplet (X, Y, Z), a quartet (X, Y, Z, W), ..., so on. Thus, the algebraic nature of the canonical variables in the framework of Nambu mechanics is changed from that of the Hamiltonian mechanics, and then a Nambu bracket, a generalization of a Poisson bracket, was introduced. Until now, a large part of several works on the Nambu mechanics in literature mainly concentrated on algebras of the Nambu-Poisson brackets of classical Nambu mechanics. In this paper, we investigate some mathematical aspects of algebraic nature of the Nambu mechanics, especially its quantization.

This paper is organized as follows. In Sec. 2, we summarize some mathematical properties of the classical Nambu mechanics from the perspective of dynamical systems. Sec. 3 contains the main results of this paper: The quantization condition of quantum Nambu mechanics introduced by Nambu himself is examined in detail. Sec. 4 will be devoted for some possible applications of the quantum Nambu mechanics, especially for a treatment of neutrino oscillation. The summary of this paper and some interesting problems for our future works is given in Sec. 5.

2 Classical Theory

In this section, we give a short discussion on some mathematical structures of classical Nambu mechanics. Main purpose here is to prepare a comparison between classical and quantum theories of the Nambu mechanics. It is an obvious fact that several aspects of theory of classical Hamilton (conservative) and dissipative mechanics are also valid in the framework of Nambu mechanics [2,3,10,14,24]. Especially the general theory of dynamical systems is useful for us to understand the structure of Nambu mechanics: The framework of classical Nambu mechanics was firstly given by a generalization of the classical Hamilton mechanics, while several notions of dissipative system can also be introduced in the framework. A volume element $dx_1 \wedge \cdots \wedge dx_N$ of a phase space is conserved in a Hamiltonian dynamics, while it shrinks in a dissipative system under a time evolution. In the ordinary Hamiltonian mechanical system, N must be an even number, while the Nambu mechanics can have both the cases of even and odd numbers. These conditions are written by the following Jacobian:

$$\frac{\partial(\tilde{x}_1, \cdots, \tilde{x}_N)}{\partial(x_1, \cdots, x_N)} \leq 1. \tag{1}$$

(Here, we give a Jacobian of a case of continuous dynamics, though we can also consider a discrete dynamical system = cellular automata, under the same manner.) Here, the left-hand side is smaller than 1 in a dissipative case, while it is exactly kept as 1 in a conservative case. Equivalently, a continuity equation $\partial_\nu \dot{x}^\nu = 0$ holds in a conservative system. Usually, a transformation (time evolution) of conservative case is caused by a group action of symplectic map $Sp(V)$ (V: a vector space) which preserves area and orientation. While, a Nambu mechanics can takes a set of canonical variables of odd number, thus a symplectic group cannot be utilized for a transformation of volume element naively. The Jacobian given above is known as a Nambu bracket in literature [9,14,24]. Usually, a chaotic behavior will be found in a dissipative case under a bifurcation with respect to the variation of a parameter of the dynamical system: Such a bifurcation can takes place also in a dissipative Nambu mechanics [3]. The general form of classical Nambu mechanics, containing both conservative and dissipative cases, is given by

$$\frac{d}{dt}F = \frac{\partial}{\partial t}F + \frac{\partial(F, H_1 \cdots, H_{N-1})}{\partial(x_1, x_2, \cdots, x_N)}. \tag{2}$$

(F: a function) and especially [14],

$$\frac{d}{dt}x_l = \frac{\partial(x_l, H_1 \cdots, H_{N-1})}{\partial(x_1, x_2, \cdots, x_N)}. \tag{3}$$

Then, an equilibrium (fixed point) is defined by

$$0 = \frac{\partial(x_l, H_1 \cdots, H_{N-1})}{\partial(x_1, x_2, \cdots, x_N)}. \tag{4}$$

Here, $(H_1, \cdots, H_{N-1})$ are usually called as Hamiltonians.

For example, the types of orbits are classified as (i) (multiply) periodic and (ii) aperiodic, and the phase space of a dynamical system is divided by those orbits [2,10]. In both of the cases of Hamilton and dissipative dynamical systems, a region of chaos consists with unstable aperiodic orbits (non-quasiperiodic system = chaos): In a dissipative case, the phase space has a basin B and an orbit inside it approaches to an attractor A (a kind of invariant/symmetric set at $t \to \infty$ such that $f^t \cdot A = A$, $f^t \in G$, such as a fixed point, a limit cycle, a torus, and G can be a Lie group) at the limit $t \to \infty$ of a time-evolution of a sequence of a mapping: $A = \bigcap_{n=0}^{\infty} f^{(n)} B$ (f is a mapping of the dynamical system). In a conservative case, invariant tori are embedded into chaotic regions. An N-dimensional Nambu mechanics may have an invariant torus $\mathbf{T}^N$ in its phase space. For example, a three-dimensional Nambu mechanics has three canonical variables (X, Y, Z), and a (quasi)periodic orbit of them becomes a "hyper-torus" $\mathbf{T}^3$, while the chaotic region will be found in the region between tori in the phase space. One can consider the Thom's toral automorphism for such a torus, $M : \mathbf{T}^N \to \mathbf{T}^N$ (M: an $N \times N$ matrix). If M has an eigenvalue μ with $|\mu| > 1$, then a chaotic behavior takes place in the automorphism. A Lyapunov exponent λ is defined by [10]

$$\lambda = \lim_{T\to\infty} \frac{1}{T} \ln\Big|(D_X f^{(T)})\delta X\Big| \tag{5}$$

(δX: a small deviation between initial states), and it corresponds to an eigenvalue of the Jacobian matrix defined above: The case where a Lyapunov exponent is nonzero (an eigenvalue of Jacobian does not have its value in a unit circle) is called as hyperbolic, and a chaos may take place if the Lyapunov exponent is positive. For example, in a three-dimensional case (X, Y, Z), the Jacobian is a 3×3 matrix, and its eigenvalues are given by one real number with two real or a pair of complex conjugates. Thus, if we express eigenvalues of the Jacobian in the three-dimensional case such that $(a, re^{i\theta}, re^{-i\theta})$ ($a, r, \theta \in \mathbf{R}^1$), then a torus will be obtained by $a = r = 1$, while the eigenvalues must deviate from them to give a chaotic orbit. In general, $\mathbf{G}$ of a transformation of canonical variables $\mathbf{GX} = \mathbf{X}'$ belongs to $SL(N, \mathbf{R})$ in a conservative case ($\mathbf{X}$ is an N-dimensional linear space), and the mapping of $\mathbf{G}$ can cause a chaos when it is hyperbolic. From the Pesin formula, we know that the production rate of Kolmogolov-Sinai entropy is given by a sum of positive Lyapunov exponents: This must valid also in the classical Nambu mechanics. It is a known fact that a topological entropy $h_{top}(f)$ of the map $f : M \to M$ (M: a manifold) gives the volume growth rate. The definition of ergodicity of classical Nambu mechanics is also the same with the usual dynamical system:

$$\rho(f) = \int f(\mathbf{X})\rho(d\mathbf{X}) = \lim_{T\to\infty} \frac{1}{T} \int_0^T f(\mathbf{X}(t))dt. \tag{6}$$

Namely, if the measure ρ is indecomposable, the system is ergodic. Let $\mathbf{X}^{(l)} = f^{(l)}\mathbf{X}^{(0)}$ be a chaotic orbit of a Nambu mechanics, and let $\mathcal{F}(\mathbf{X}^{(l)})$ be a function of the chaotic orbit. An observable of a chaotic system of the classical Nambu mechanics is defined by the same manner of theory of dynamical systems:

$$\langle \mathcal{F}(\mathbf{X}) \rangle = \lim_{N \to \infty} \frac{1}{N} \sum_{l=0}^{N-1} \mathcal{F}(\mathbf{X}^{(l)}). \tag{7}$$

3 Quantum Theory

In this section, we will examine quantum Nambu mechanics, especially its three-dimensional case. It is given by canonical variables (X_1, X_2, X_3) with the following quantization condition called as a quantum Nambu bracket [9,14,24]:

$$[X_{\sigma(1)}, X_{\sigma(3)}, X_{\sigma(3)}] = \mathrm{sgn}(\sigma)[X_1, X_2, X_3], \tag{8}$$

$$[X_1, X_2, X_3] = [X_1, X_2]X_3 + [X_2, X_3]X_1 + [X_3, X_1]X_2 = \pm i\hbar. \tag{9}$$

Namely, it is defined as a total sum of cyclic permutations (a Galois group) of $[X_{\sigma(1)}, X_{\sigma(2)}]X_{\sigma(3)}$ in the three-dimension. To investigate this algebraic relation, we assume the triplet (X, Y, Z) gives a three-dimensional Lie algebra Lie(G) (G: a Lie group). Then the algebra which the Nambu bracket belongs is determined by the Lie brackets $[X, Y]$, $[Y, Z]$, and $[Z, X]$. Then the Nambu bracket acquires the following expression of structure constants of the Lie algebra, especially in Lie$(SU(2))$ or Lie$(SO(3))$:

$$[X_1, X_2, X_3] = if_{123} + if_{231} + if_{312}, \tag{10}$$

for $[X_i, X_j] = if_{ijk}X_k$ (here, a summation convention has been used). Namely, this is an adjoint representation. Let us write the Nambu bracket as $[X_1, X_2, X_3] = D$, where the case $D \in$Lie(G). From the following relation,

$$\mathrm{Ad}(G)D = \mathrm{Ad}(G)([X_1, X_2, X_3]) = [\mathrm{Ad}(G)X_1, \mathrm{Ad}(G)X_2, \mathrm{Ad}(G)X_3], \tag{11}$$

we find $\mathrm{Ad}(G)D = D$ must holds for a consistent quantization. This fact indicates that the quantization of Nambu bracket is a choice of an adjoint orbit in G, and it defines a homogeneous space $G/(\mathrm{Ad}(G)D = D)$: In that case, the dynamical system is defined over the homogeneous space, and one can say it is a kind of symmetry breaking. (Namely, a symmetry breaking by a quantization condition.) Note that D given here does not have to be a c-number, can take a matrix of Lie(G). It should also be mentioned that a quantum theory gives an observable after taking an expectation value of a (Nambu-)Heisenberg equation of motion, not an algebra itself. Another choice of D is a Casimir element of the universal enveloping algebra of Lie(G) [14]. A characteristic feature of the quantum Nambu bracket is its symplectic structure. Since

$$\eta_{XY} = [X, Y] = -[Y, X], \quad \eta_{YZ} = [Y, Z] = -[Z, Y], \quad \eta_{ZX} = [Z, X] = -[X, Z], \tag{12}$$

the quantum Nambu bracket contains three symplectic structures. (Note that this fact is also valid in the classical Nambu mechanics, since it consists with the algebra of $dX \wedge dY \wedge dZ$ (for example, the Nambu-Poisson brackets, the volume form, ...). On the contrary, the canonical Hamiltonian mechanics only has $dX \wedge dY$ pairwisely in the space of total canonical variables.) Let us introduce the notion of symplectic homogeneous space (G, H, Ω), in which G is a semisimple connected Lie group, and H is a closed connected subgroup, while Ω is a G-invariant symplectic form [1,7,8]. It is known fact that a symplectic homogeneous space can be constructed by an adjoint orbit [1,7]:

$$O_G(Z) \;=\; \{g \in G | \mathrm{Ad}(g)Z = Z\}, \tag{13}$$

with a symplectic form given by the Killing form defined as follows:

$$\Omega_Z \;=\; -K(Z, [X, Y]). \tag{14}$$

Here, Ω_Z is defined for a fixed Z, and G-invariant from its definition. Then the coset $G/O_G(Z)$ becomes a symplectic homogeneous space, with the following group extension:

$$1 \to O_G(Z) \to G \to G/O_G(Z) \to 1. \tag{15}$$

Usually, Z is chosen from the Cartan subalgebra of Lie(G), though we can consider other generators due to isotropy of the space of Lie algebra. For example [7], in the case of $G = SL(2, \mathbf{R})$, a triple $(G, H, \Omega) = (SL(2, \mathbf{R}), SO(2), \Omega_Z)$ is obtained for $Z = i\sigma_2$, while $(G, H, \Omega) = (SL(2, \mathbf{R}), SO(1, 1), \Omega_Z)$ is obtained for $Z = \sigma_3$. Therefore, the three-dimensional quantum Nambu bracket implicitly determines three symplectic forms,

$$\Omega_X = -K(X, [Y, Z]), \quad \Omega_Y = -K(Y, [Z, X]), \quad \Omega_Z = -K(Z, [X, Y]), \tag{16}$$

and three homogeneous spaces by the projections,

$$\pi_X : G \to G/O_G(X), \quad \pi_Y : G \to G/O_G(Y), \quad \pi_Z : G \to G/O_G(Z). \tag{17}$$

Thus, a three-dimensional vectorial nature of the quantum Nambu bracket is understood. In fact, by the definition, the Nambu bracket

$$[X, Y, Z] = [X, Y]Z + [Y, Z]X + [Z, X]Y \tag{18}$$

contains three Lie brackets along with three orthogonal coordinates X, Y, Z. The Nambu bracket is given by an inner product of those Lie brackets and coordinates, namely $\mathbf{X} \cdot \mathbf{X} \wedge \mathbf{X}$ ($\mathbf{X} = (X, Y, Z)$): The geometric meaning of this quantity is well-known.

From the observation of classical and quantum cases of three-dimensional Nambu brackets, we have understood that they have three symplectic structures we have to consider simultaneously for a canonical triplet. On the contrary, the ordinary Hamiltonian mechanics has only one symplectice structure for a canonical pair. There are several famous works for deformation quantizations of Poisson structures (namely, Hamiltonian mechanical systems) in literature [6,11]. Thus, one can consider the following *-product for a deformation quantization:

$$\begin{aligned} & f(X, Y, Z) * g(X, Y, Z) \\ & \quad = f(X, Y, Z) \exp \Big[\nu \Big(\overleftarrow{\partial_X} \overrightarrow{\partial_Y} - \overleftarrow{\partial_Y} \overrightarrow{\partial_X} \Big) \\ & \qquad + \nu \Big(\overleftarrow{\partial_Y} \overrightarrow{\partial_Z} - \overleftarrow{\partial_Z} \overrightarrow{\partial_Y} \Big) + \nu \Big(\overleftarrow{\partial_Z} \overrightarrow{\partial_X} - \overleftarrow{\partial_X} \overrightarrow{\partial_Z} \Big) \Big] g(X, Y, Z). \end{aligned} \tag{19}$$

Namely, the quantum structures will be introduced into the three directions (they are orthogonal with each other) indicated by the projections of symplectic structures discussed above. It is interesting for us to investigate an equivalence class of our *-product: In the usual case of deformation quantization of a Poisson manifold, a Hochschild cohomology provides a main tool to express an equivalence class. Such a systematic investigation is our plan of future work, and it might give us several variants of quantum theory, as we will mention soon in this paper. Since the three-dimensional quantum Nambu mechanical system (X, Y, Z) has three symplectic structures in the Nambu bracket, one could say it is isotropic in S^2. While, due to the fact that the Heisenberg-type quantum mechanics shares some mathematical nature with the canonical Hamiltonian mechanics, the set of dynamical variables is pairwisely decomposed: Thus it could be said that the Heisenberg quantum theory is anisotropically (S^1), embedded in the quantum Nambu mechanics.

According to the work of Nambu himself, we choose the following dynamical system as the three-dimensional quantum Nambu mechanics [14]:

$$i\frac{d}{dt}F \;=\; [F, H_1, H_2]. \tag{20}$$

An equilibrium (fixed point) of quantum Nambu mechanics in the phase space is determined by

$$0 \;=\; [\mathbf{X}, H_1, H_2]. \tag{21}$$

In classical Nambu mechanics, a second-order Casimir element is frequently used for H_1 or H_2, though one of (H_1, H_2) must not be a Casimir element to make the quantum theory of Nambu mechanics non-trivial. Since we do not assume it is the case that the quantum theory of Nambu mechanics always has its classical counterpart, we have a large freedom for our choice for Hamiltonians H_1 and H_2. (Of course, it is natural to assume $[H_1, H_2] = 0$ for a physical theory.) Our logic for constructing a system of quantum Nambu mechanics is summarized as follows: (1) Find a three-dimensional Lie algebra Lie(G) which can satisfy the quantization of the Nambu bracket $[X, Y, Z]$, $\mathbf{X} = (X, Y, Z) \in$ Lie(G). (2) Then we construct a Hamiltonian H and a Casimir element $\mathcal{C}$ defined by $\mathbf{X} = (X, Y, Z)$. (3) The dynamical system is, for example, defined by

$$i\hbar\frac{d}{dt}\mathbf{X} \;=\; [\mathbf{X}, H(\mathbf{X}), \mathcal{C}(\mathbf{X})]. \tag{22}$$

Namely, we have arrived at the notion that a quantum theory is given by a Lie algebra (such as Lie($SU(2)$), Lie($SL(2,\mathbf{R})$), Lie(H_3), ...) which has the same dimension with the number of canonical variables. The usual quantum mechanics of Heisenberg can be interpreted as a special case of those possibilities of quantum theories, i.e., it chooses Lie(H_3) for constructing a theory. An interesting example is the Holstein-Primakoff transformation, which gives an approximation of spin variables of Lie($SU(2)$) by creation/annihilation operators $a, a^\dagger$ of the Heisenberg algebra Lie(H_3). For example, the conformal algebra of Lie($SL(2,\mathbf{R})$) is given by,

$$[L, T] = T, \quad [H, L] = H, \quad [H, T] = \frac{1}{2}L. \tag{23}$$

Here, L, T and H are algebra generators of dilatation, special conformal transformation, and translation, respectively. It can satisfy the quantization condition of the three-dimensional Nambu bracket. Moreover, if the VEV of them becomes

$$\langle [L,T] \rangle = 0, \quad \langle [H,L] \rangle = 0, \quad \langle [H,T] \rangle = \frac{1}{2}\langle L \rangle \neq 0 \tag{24}$$

in a quantum field theory, a Heisenberg algebra is obtained and it is a situation of the anomalous Nambu-Goldstone theorem [18,19].

An interesting viewpoint will be obtained from the perspective of quaternions: What we will argue here is that the map $\tilde{q} : (X,Y,Z) \in \mathbf{R}^3 \to (X,Y,Z) \in \mathbf{H}^1$ is just a quantization. A quaternion q is defined by

$$q \;=\; a + ib + jc + kd, \quad \mathbf{H} \equiv \{a + ib + jc + kd | a,b,c,d \in \mathbf{R}\}. \tag{25}$$

Here, the following relations are satisfied:

$$i^2 = j^2 = k^2 = -1, \quad ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j. \tag{26}$$

Thus, the Nambu bracket of quaternions becomes

$$[i,j,k] \;=\; 2(i^2 + j^2 + k^2) = -6. \tag{27}$$

Thus, the algebra of quantization condition of the Nambu bracket $[X,Y,Z] = \pm i\hbar$ is isomorphic with the quaternion algebra of (i,j,k). (An interesting fact is that both of the Hamiltonian mechanics and quaternions ware discovered by the Irish genius mathematician William Rowan Hamilton.) If we express the algebra of Nambu bracket by Lie$SU(2)$ (similar results will be obtained by Lie$(SO(3))$, or a conformal algebra of Lie$(SL(2,\mathbf{R}))$),

$$[X,Y,Z] = i(X^2 + Y^2 + Z^2) = \pm i\hbar. \tag{28}$$

Thus, the Nambu bracket defines a unit sphere $X^2 + Y^2 + Z^2 = 1$, in the case of positive sign with $\hbar > 0$. Hence, a three-dimensional quantum Nambu mechanics naturally has a structure of conformal geometry. Thus, if a three-dimensional quantum Nambu mechanics gives a "quantum chaos", it might be found as an unstable aperiodic orbit over a conformal geometry S^2. An interesting subject is how we can understand the Ehrenfest theorem in the three-dimensional quantum Nambu mechanics, which might be useful to understand "quantum chaos" and dissipative dynamics of quantum Nambu mechanics. The equation of unit sphere can be regarded as a projective space, and we obtain a homogeneous equation $U^2 - X^2 - Y^2 - Z^2 = 0$. Namely, it is a "light-cone" in theory of relativity. Thus, we can say $SO(3,1)$ or $SO(4)$ acts on the quantization condition of Nambu bracket implicitly. If we consider a multiple Nambu triplets $((X_1, Y_1, Z_1), \cdots, (X_N, Y_N, Z_N))$, then the coset $SO(4N)/(\overbrace{SO(4) \otimes \cdots \otimes SO(4)}^{N})$ is found via a set of homogeneous equations. An interesting fact is that each factor of $SO(4)$ of this coset contains a set of triple-symplectic structures. It is known from the Ostrowski theorem that only $\mathbf{R}$, $\mathbf{C}$, $\mathbf{F}_q$ and $\mathbf{Q}_p$ are locally compact topological field, and $\mathbf{H}$ is not contained in it. Thus, a translation from classical to quantum Nambu mechanics by quaternions seems to demand us to change the nature of topological characters of number fields. (A quantum theory may change an algebra and/or the nature of topological space of the classical counterpart.) Since the quantization condition of Nambu mechanics has a nature of quaternions, it is interesting for us to find a Fourier transform defined over $\mathbf{H}$, to give an appropriate definition of quantum numbers of quantum variables via a Pontrjagin-type duality, and also to find a Weyl representation for the quantum Nambu mechanics. (Such a Fourier transform (integral transform) over $\mathbf{H}$ relates with Weil representations [25], harmonic analysis, automorphic representations, and arithmetic of quadratic forms [26].) A Weyl representation might be found from the "group element",

$$g(\alpha, \beta, \gamma) \quad = \quad e^{-i\alpha X} e^{-j\beta Y} e^{-k\gamma Z}. \tag{29}$$

A systematic investigation of quaternionic aspects of the algebra and group of quantum Nambu brackets will be given by our future work, possibly by employing methods of quaternionic geometry, hyper-Kähler geometry, and quaternionic geometry. Since quaternions are utilized for expressions of rotations of $\mathbf{R}^3$ and $\mathbf{R}^4$, it seems natural that an Euler top [14] appears as a typical example of the Nambu mechanics. Moreover, it should be mentioned that the Clifford algebra $\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu}$ ($g^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$) is isomorphic with the algebra defined for a quaternion, by the following correspondences:

$$\gamma^0 \leftrightarrow 1, \quad \gamma^1 \leftrightarrow i, \quad \gamma^2 \leftrightarrow j, \quad \gamma^3 \leftrightarrow k, \tag{30}$$

$$q = v_0 1 + v_1 i + v_2 j + v_3 k = \gamma^\nu v_\nu. \tag{31}$$

Thus, a spin group also be introduced in our prescription of quantization of Nambu mechanics. It should be mentioned that the quaternion-like algebra of the quantum Nambu bracket can also be expressed by Lie($Sp(1)$). We know the following well-known relation, $Sp(1) \simeq Spin(3) \subset Sp(1) \otimes Sp(1) \simeq Sp(4)$, and $Spin(3)$ is a covering group of $SO(3)$. Thus, $Sp(1) \simeq SO(3)$ acts on the quaternion representation of the Nambu triplet (X, Y, Z). Needless to say, $SO(3)$ is a Wigner little group of the Lorentz group $SO(3,1)$, and isomorphic with a subalgebra of the Clifford algebra. Another mathematically interesting fact is found from the fact that $\mathbf{H}^1$ (or more generally, $\mathbf{H}^1 \backslash \{0\}/\Gamma$ where Γ is a finite subgroup of $SU(2)$) is a hyper-Kähler manifold. Because we mainly need to satisfy the quantization condition $[X, Y, Z] = \pm i\hbar$ for constructing a quantum Nambu mechanical system, and it can be achieved by the quaternionic algebra, the domain of (X, Y, Z) has various choices: An example for it is the K3 surface. When a $4n$-dimensional manifold M is almost quaternionic, a subbundle of End(TM) is expressed by a basis defined by the triplet of almost complex structures (I, J, K) at any point $x \in M$. The triplet forms the quaternionic algebra. Hence one can take a Nambu triplet (X, Y, Z) over a Kähler manifold of almost quaternionic structure. From our observation, the isomorphism $(X_1, X_2, Y_3) \sim (i, j, k)$ can be utilized for constructing quantum theory of Nambu mechanics. Thus,

$$
\begin{aligned}
X_1 &= x_1 i + y_1 j + z_1 k, && (32)\\
X_2 &= x_2 i + y_2 j + z_2 k, && (33)\\
X_3 &= x_3 i + y_3 j + z_3 k, && (34)\\
H_1(\mathbf{H}) &= h_0^1 1 + h_1^1 i + h_2^1 j + h_3^1 k, && (35)\\
H_2(\mathbf{H}) &= h_0^2 1 + h_1^2 i + h_2^2 j + h_3^2 k, && (36)\\
i\hbar \frac{d}{dt} X_l &= [X_l, H_1(\mathbf{H}), H_2(\mathbf{H})]. && (37)
\end{aligned}
$$

The right-hand side of the Nambu-Heisenberg dynamical equation can be expanded further by utilizing the algebra of quaternions. A Hilbert space for this dynamical system may be subjected to an action of quaternion algebra. In the axiomatic quantum field theory, a local field operator is defined by

$$
\begin{aligned}
\psi(f) &= \int d^3\mathbf{x} \psi(\mathbf{x}) f(\mathbf{x}), \quad \text{(nonrelativistic)}, && (38)\\
\psi(f) &= \int d^4 x \psi(x) f(x), \quad \text{(relativistic)}. && (39)
\end{aligned}
$$

Here, f is a cutoff function defined over three- (nonrelativistic) and four- (relativistic) dimensional supports. Because a statistical algebra of a bosonic case is commutative,

$$\psi(\mathbf{x}) \propto \phi(\mathbf{x}) \otimes \mathbf{H}, \tag{40}$$

$$\psi(x) \propto \phi(x) \otimes \mathbf{H}, \tag{41}$$

can satisfy the quantization condition of Nambu bracket. In this case, a quantum field is given in terms of a triplet:

$$\Phi(x) \;=\; \phi_1(x)i + \phi_2(x)j + \phi_3 k. \tag{42}$$

A partition function of classical Nambu mechanics takes the following form [14,19]:

$$\mathcal{Z} = \int \frac{dXdYdZ}{(2\pi)^{3/2}} e^{-\beta_1 H_1 - \beta_2 H_2}, \quad dXdYdZ \in \mathbf{R}^3. \tag{43}$$

If we use the quaternion representation of the quantum Nambu triplet $(X, Y, Z) \in \mathbf{H}$, then the quantum theory might be given by the following path integral:

$$\mathcal{Z} \sim \int \mathcal{D}X\mathcal{D}Y\mathcal{D}Z \exp(-S(H_1(\mathbf{H}) + H_2(\mathbf{H}))), \quad dXdYdZ \in \mathbf{H}. \tag{44}$$

Here, S indicates an action derived from the Hamiltonians (H_1, H_2) defined over the quaternion $\mathbf{H}$. In our perspective, a transform from $dXdYdZ \in \mathbf{R}^3 \to dXdYdZ \in \mathbf{H}$ is a quantization. (A possibility of deformation quantization of a function space over $\mathbf{H}$ was considered by the author in Ref. [16]. It was indicated that an associativity of the algebra of deformation quantization seems to be broken.)

Let us examine the uncertainty relation in the three-dimensional quantum Nambu mechanics. By the following definitions,

$$(\Delta A)^2 \;=\; \langle (A - \langle A\rangle)\Psi | (A - \langle A\rangle)\Psi\rangle, \tag{45}$$

$$(\Delta B)^2 \;=\; \langle (B - \langle B\rangle)\Psi | (B - \langle B\rangle)\Psi\rangle, \tag{46}$$

$$(\Delta C)^2 \;=\; \langle (C - \langle C\rangle)\Psi | (C - \langle C\rangle)\Psi\rangle, \tag{47}$$

($A, B, C \in \mathrm{Lie}(G)$; canonical variables), we get

$$\Delta A \Delta B \Delta C \geq \sqrt{\frac{1}{(2i)^3}\langle [A, B]\rangle \langle [B, C]\rangle \langle [C, A]\rangle}. \tag{48}$$

Here, we have used the ordinary Heisenberg uncertainty relations (we can call them as the two-dimensional uncertainty relation from the context of this paper) for any pair of (A, B, C) obtained by using the Cauchy-Schwarz inequality:

$$\Delta A \Delta B \geq \frac{1}{2i}\langle [A, B] \rangle, \tag{49}$$

$$\Delta B \Delta C \geq \frac{1}{2i}\langle [B, C] \rangle, \tag{50}$$

$$\Delta C \Delta A \geq \frac{1}{2i}\langle [C, A] \rangle. \tag{51}$$

For example, if we apply the ferromagnetic case $\langle S_1 \rangle = \langle S_2 \rangle = 0$, $\langle S_3 \rangle \neq 0$, then we yield $\Delta S_1 \Delta S_2 \Delta S_3 \geq 0$. This indicates that $\Delta X_1 \Delta X_2 \Delta X_3 \geq 0$ generically holds in a three-dimensional Heisenberg algebra: This fact is a direct result of symplectic structure of Hamiltonian mechanics where a set of dynamical variables are decomposed into conjugate pairs. Since the Nambu mechanics is given by a triplet of canonical variables, there may be another quantum uncertainty relation. There is a systematic classification of three-dimensional Poisson-Lie algebra in literature [4], and the result is

$$A_{3,1}: \quad [e_1, e_2] = 0, \quad [e_2, e_3] = e_1, \quad [e_3, e_1] = 0, \tag{52}$$

$$A_{3,2}: \quad [e_1, e_2] = 0, \quad [e_2, e_3] = e_1 + e_2, \quad [e_3, e_1] = -e_1, \tag{53}$$

$$A_{3,3}: \quad [e_1, e_2] = 0, \quad [e_2, e_3] = e_2, \quad [e_3, e_1] = -e_1, \tag{54}$$

$$A_{3,4}: \quad [e_1, e_2] = 0, \quad [e_2, e_3] = -e_2, \quad [e_3, e_1] = -e_1, \tag{55}$$

$$A_{3,5}: \quad [e_1, e_2] = 0, \quad [e_2, e_3] = \rho e_2, \quad [e_3, e_1] = -e_1, \quad (0 < |\rho| < 1), \tag{56}$$

$$A_{3,6}: \quad [e_1, e_2] = 0, \quad [e_2, e_3] = e_1, \quad [e_3, e_1] = e_2, \tag{57}$$

$$A_{3,7}: \quad [e_1, e_2] = 0, \quad [e_2, e_3] = e_1 + \mu e_2, \quad [e_3, e_1] = -\mu e_1 + e_2, \quad (\mu > 0) \tag{58}$$

$$A_{3,8}: \quad [e_1, e_2] = e_1, \quad [e_2, e_3] = e_3, \quad [e_3, e_1] = 2e_2, \tag{59}$$

$$A_{3,9}: \quad [e_1, e_2] = e_3, \quad [e_2, e_3] = e_1, \quad [e_3, e_1] = e_2. \tag{60}$$

Here, $A_{3,1}$ corresponds to the three-dimensional Heisenberg algebra Lie(H_3), while $A_{3,8}$ and $A_{3,9}$ are included in Lie$(SL(2, \mathbf{R}))$ and Lie$(SO(3))$, respectively. Thus, at least from the result of the classification, we recognize that only Lie$(SL(2, \mathbf{R}))$ and Lie$(SO(3))$ (Lie$SU(2)$) can have three non-vanishing Lie brackets $[X, Y] \neq 0$, $[Y, Z] \neq 0$, and $[Z, X] \neq 0$, can have finite (non-zero) uncertainty relations, $\Delta X_1 \Delta X_2 \Delta X_3 \geq$ const. > 0. Of course, this result does *not* imply that a state of minimal uncertainty (such as a coherent state of harmonic oscillator in the ordinary Heisenberg quantum mechanics) in a quantum theory of *three* canonical variables expressed by a Lie algebra $A_{3,l}$ ($l = 1, \cdots, 7$) has $\Delta X = 0$, $\Delta Y = 0$, $\Delta Z = 0$. For example, in the case of $A_{3,2}$,

$$\Delta e_1 \Delta e_3 \geq \frac{1}{2}|\langle e_1 \rangle|, \quad \Delta e_2 \Delta e_3 \geq \frac{1}{2}|\langle e_1 + e_2 \rangle|, \quad \Delta e_3 \Delta e_1 \geq 0, \tag{61}$$

are obtained. The relations of $[X, Y] \neq 0$, $[Y, Z] \neq 0$, and $[Z, X] \neq 0$ give a condition for a manifold in which the set of canonical variables (X, Y, Z) is defined. From these observations, we argue the quantization via quaternions is a quite natural method for a three-dimensional (the canonical dimension is three) quantum Nambu mechanical system, and it puts the quantum Nambu mechanics on a place where it is not a simple generalization of the ordinary Heisenberg-type quantum theory, with obtaining the perspective to construct several quantum theories different from the ordinary Heisenberg quantum theory. The dynamical equation consistent with the Heisenberg algebra Lie(H_3) (namely, $A_{3,1}$ given above) is

$$i\langle e_1 \rangle \frac{d}{dt} e_2 = [e_2, H(e_2, e_3)], \tag{62}$$

$$i\langle e_1 \rangle \frac{d}{dt} e_3 = [e_3, H(e_2, e_3)]. \tag{63}$$

By using the result of Poisson-Lie algebra, a systematic reduction of three-dimensional quantum Nambu mechanics can be performed, and we may yield several theories of quantum mechanics. Then, we may find new perspectives on quantum information theory, quantum communications, quantum entanglement, quantum algorithm, and quantum teleportation.

The quantum uncertainty is important for us to understand the ground state of a spin system. In a spin system of magnetism, we frequently meet with a situation where a quantum uncertainty of dynamical degrees of freedom gives a ground state which is more "randomized" (quantum fluctuation), and it has the energy lower than that of a classically ordered state. If there is a physical system which is described by the framework of the quantum Nambu mechanics, and which shows a phase transition, then the Nambu-mechanical quantum uncertainty might affect the nature of the ground state of the system.

4 On Dynamical Models of Neutrino Oscillations

First, we discuss general mathematical aspects of neutrino oscillations. Quite often, the Schrödinger equation for neutrino oscillation is defined as follows:

$$
\begin{aligned}
i\frac{d}{dt}|\psi\rangle &= \mathcal{H}|\psi\rangle, && (64)\\
|\psi\rangle &= (\nu_e, \nu_\mu, \nu_\tau), && (65)\\
\mathcal{H} &= U^{-1}\mathcal{M}U + \mathcal{V}, && (66)\\
\mathcal{M} &= \mathrm{diag}(E_1, E_2, E_3), && (67)\\
\mathcal{V} &= \mathrm{diag}(V(x), 0, 0). && (68)
\end{aligned}
$$

Here, U is the so-called PMNS (Pontecorvo-Maki-Nakagawa-Sakata) matrix [22,12] of a flavor mixing of neutrino sector which belongs to $SO(3)$ as a group element, and $V(x)$ is a potential depends on the number of electrons at a point x of the environment. In a more mathematical form, we can rewrite the essential part of them, with taking into account a possible collective neutrino oscillation (neutrino many-body system) as follows:

$$
\begin{aligned}
i\frac{d}{dt}|\Psi\rangle &= H|\Psi\rangle, && (69)\\
H &= A + g^{-1}Bg + \lambda(g^{-1}Cg)(g^{-1}Cg), \quad g \in G, && (70)\\
A &= \mathrm{diag}(a_1, \cdots, a_N), \quad A^\dagger = A, && (71)\\
B &= \mathrm{diag}(b_1, \cdots, b_N), \quad B^\dagger = B. && (72)
\end{aligned}
$$

where, λ is a coupling constant for neutrino-neutrino interaction in a dense neutrino gas, A indicates an external potential of environment, B may gives the energy eigenvalues of the neutrino sector, and C is a Hermitian matrix. All of (A, B, C) belong to Lie(G) and they are expanded such as $\sum c_j T_j$ (T_j: generators of Lie(G)), and thus the Hamiltonian H takes its value in Lie(G). This Hamiltonian has some similarity with a nonlinear sigma model of a local spin system such as a Heisenberg model. The form of the Hamiltonian of neutrino oscillation is interpreted as an expansion by elements of adjoint orbits, and the group element g is regarded as a transformation between two coordinate systems. This form of neutrino oscillation model also has a similarity with a mass matrix the generalized Nambu-Goldstone (GNG) theorem [17], and the matrices (A, B, C) look like order parameters. After taking expectation values of H, one yields

$$H = A + \mathbf{B} + \lambda \mathbf{C} \cdot \mathbf{C}, \tag{73}$$

because both $g^{-1}Bg$ and $g^{-1}Cg$ give three-dimensional vectors and group orbits defined on $SO(3)$. A dynamical equation for CP-violating kaon oscillation system also takes a similar form, by using the notation of Lie($SU(2)$):

$$i\frac{d}{dt}|\Phi\rangle = \begin{pmatrix} \bar{m} - \frac{i}{2}\bar{\gamma} & \delta m - \frac{i}{2}\delta\gamma \\ \delta m^* - \frac{i}{2}\delta\gamma^* & \bar{m} - \frac{i}{2}\bar{\gamma} \end{pmatrix} |\Phi\rangle \tag{74}$$

$$= \sigma_3(\alpha_1\sigma_1 + \alpha_2\sigma_2 + \alpha_3\sigma_3)|\Phi\rangle. \tag{75}$$

Here, all of $(m, \bar{m}, \bar{\gamma}, \delta\gamma)$ are complex numbers, and α_j ($j = 1, 2, 3$) are matrices. Needless to say, Lie($SU(2)$) is locally isomorphic with Lie($SO(3)$), and thus this form of the Hamiltonian of kaon oscillation can be converted into an $SO(3)$ vector model like the neutrino oscillation model without an interaction term between kaons. While, the essential difference between the neutrino and kaon systems is the numbers of dimensions of the eigenvector spaces (three for the former, two for the latter). We meet a lot of examples of Hamiltonians they take their values on Lie algebras: The BCS (Bardeen-Cooper-Schrieffer) Hamiltonian [5] of Nambu notation [13] is also an example, and a Bloch Hamiltonian of electrons of graphene can be expressed as $H = \hbar v_F(\sigma_1 k_1 + \sigma_2 k_2) + m\sigma_3$ (v_F; the Fermi velocity, m; a band gap) [15]. A similar Hamiltonian is also found in the dynamics of NMR (nuclear magnetic resonance). In that case, the Hamiltonian is given as $H = m(e^{i\omega t}\sigma_+ + e^{-i\omega t}\sigma_-) + M\sigma_3$ ($me^{i\omega t}$: rotational magnetic field, M: an external magnetic field which causes a Zeeman splitting), and a transformation to a rotational coordinates is frequently performed: $g^{-1}Hg$, $g \in SO(3)$.

The model of collective neutrino oscillation of a dense neutrino gas discussed by Raffelt is [23]

$$H = \sum_{\omega} \omega \mathbf{B} \cdot \mathbf{P}(\omega) + \frac{\mu}{2}\mathbf{P} \cdot \mathbf{P}. \tag{76}$$

Here, ω is a frequency, $\mathbf{B}$ is an external "magnetic" field, and the three-dimensional vector $\mathbf{P}$ depicts the polarization of neutrino system in the flavor space. If we fix the length of $\mathbf{P}$, the dynamics of this model gives an oscillation/precession of an angular momentum and thus $\mathbf{P}$ must obey the algebra of $SO(3)$, and $\mathbf{P}\cdot\mathbf{P}$ is a Casimir element of $SO(3)$. On the other hand, the Galilei group E_3 is defined by

$$[P_i, P_j] = 0, \quad [L_i, L_j] = i\epsilon_{ijk}L_k, \quad [L_i, P_j] = i\epsilon_{ijk}P_k, \tag{77}$$

and its second-order Casimir elements are

$$\mathcal{C}_1 \; = \; \mathbf{P}\cdot\mathbf{P}, \quad \mathcal{C}_2 = \mathbf{L}\cdot\mathbf{P}, \tag{78}$$

where, $\mathbf{P} = (P_1, P_2, P_3)$ and $\mathbf{L} = (L_1, L_2, L_3)$. Now, the Lie group is E_3 and Lie($SO(3)$) is embedded as a closed subalgebra of Lie(E_3): Thus, the Hamiltonian of collective oscillation will be converted into a linear combination of the Casimir elements $\alpha\mathcal{C}_1 + \beta\mathcal{C}_2 = \alpha\mathbf{P}\cdot\mathbf{P} + \beta\mathbf{L}\cdot\mathbf{P}$ via a transformation from an axial-vector (angular momentum) to a vector (momentum).

An example of the Lagrangian of conformal mechanics is found in literature [20].

$$L \; = \; \frac{1}{2}g_{ij}\frac{dq_i}{dt}\frac{dq_j}{dt} + A_i\frac{dq_i}{dt} - V(q). \tag{79}$$

Here, A_i is an external magnetic field, and $V(q)$ is a scalar potential. Immediately we recognize that the Lagrangian is re-expressed in the following form:

$$L = \mathbf{q}\cdot\mathbf{q} + \mathbf{A}\cdot\mathbf{q} - V(\mathbf{q}). \tag{80}$$

Here, $\mathbf{q}$ and $\mathbf{A}$ are vector notations. The similarity of the conformal mechanics and neutrino collective oscillation model is now obvious.

The Euler top is a famous example for the classical Nambu mechanics [14]. It is given as follows:

$$H_1 \; = \; \frac{1}{2}\Big(\frac{L_1^2}{I_1^2} + \frac{L_2^2}{I_2^2} + \frac{L_3^2}{I_3^2}\Big), \tag{81}$$

$$H_1 \; = \; \frac{1}{2}\Big(L_1^2 + L_2^2 + L_3^2\Big), \tag{82}$$

($(L_1, L_2, L_3) \in \mathrm{Lie}SO(3)$). Here, H_1 is the usual Hamiltonian of an Euler top, while H_2 is a Casimir element (total angular momentum) of Lie$SO(3)$. Both of the Hamiltonians are expressed as

$$H_1 = g^{(1)}_{ij}(g^{-1}dg)_i(g^{-1}dg)_j, \quad H_2 = g^{(2)}_{ij}(g^{-1}dg)_i(g^{-1}dg)_j, \tag{83}$$

while the Nambu triplet in this case is

$$((g^{-1}dg)_1, (g^{-1}dg)_2, (g^{-1}dg)_3) \tag{84}$$

Thus, the quantum mechanical Nambu-Heisenberg equation is

$$i\frac{d}{dt}(g^{-1}dg)_k \;=\; [(g^{-1}dg)_k, H_1, H_2], \quad (i,j,k=1,2,3). \tag{85}$$

Again, the similarity between the neutrino oscillation model and the Euler top is apparent.

The formalism of Nambu mechanics utilizes two Hamiltonians of conserved quantities, namely constants of motion. One is the usual Hamilton function, while one can use the second-order Casimir element for the second Hamiltonian. The Casimir element can be regarded as a constraint for a time evolution of the dynamical system. This fact just indicates us that the framework of Nambu mechanics strongly correlates with Lie algebras and groups. A phenomenon of neutrino oscillation which takes place in the three-flavor neutrino sector, gives the conservation of total amplitude of wavefunctions as a constraint:

$$|\psi_e|^2 + |\psi_\mu|^2 + |\psi_\tau|^2 = 1 \quad (\text{const.}). \tag{86}$$

Namely, this equation defines a unit sphere S^2. With respect to the vector/spin model of collective neutrino oscillation of Raffelt discussed above, this conserved quantity can be regarded as a Casimir invariant of $SO(3)$. Therefore, again we argue it is possible to employ the framework of Nambu mechanics to describe the (collective) neutrino oscillation. Moreover, it was shown in literature that the Reffelt-type model can be converted into the exactly solvable BCS pairing model (so-called Gaudin model) [21]. Hence, the framework of Nambu mechanics could be extended to a system of fermionic oscillators, in which a (iso)spin vector on the flavor space is defined by a bilinear form of fermionic operators. This may also relate with a current algebra of hadron physics. If the neutrino system of oscillations is a conservative system, $|\psi_e|^2 + |\psi_\mu|^2 + |\psi_\tau|^2$ is conserved, while it is not a conserved quantity if any component of neutrinos (e, μ, or τ) is dissipative. Of course, the eigenvalues of Jacobian,

$$\frac{\partial(\tilde{\psi}_e, \tilde{\psi}_\mu, \tilde{\psi}_\tau)}{\partial(\psi_e, \psi_\mu, \psi_\tau)} \tag{87}$$

can distinguish whether the system is dissipative or not, chaotic or not. It should be noticed that those considerations given here can also be applied to the phenomenon of neutral kaon oscillation.

Let us introduce three variables $\mathbf{X} = (X, Y, Z)$, with $X^2 + Y^2 + Z^2 = 1$ for a system of neutrino oscillation, and parametrize them as follows:

$$X = r\cos\theta(t)\cos\phi(t), \quad Y = r\cos\theta(t)\sin\phi(t), \quad Z = r\sin\theta(t). \tag{88}$$

Then we obtain a correlation function via the Wiener-Khinchin relation

$$\begin{aligned} S(\omega) &\propto \lim_{T\to\infty}\frac{1}{T}\left|\int_0^T dte^{i\omega t}\mathbf{X}(t)\right|^2 \\ &= \int_{-\infty}^{+\infty} dte^{i\omega t}\lim_{\tau\to\infty}\frac{1}{\tau}\int_0^\tau d\tau'\mathbf{X}(\tau')\mathbf{X}(\tau'+t). \end{aligned} \tag{89}$$

If we assume a precession of $\theta(t) = \theta_0$=const., $\phi(t) = \phi_0 t$, ϕ_0=const., we find all of the correlation functions of X, Y, and Z vanishes after an elementary manipulation. Since the Wiener-Khinchin relation is frequently used to study an autocorrelation function of a dynamical system, it is interesting for us to employ the relation to study the neutrino oscillation combined with the three-dimensional quantum Nambu mechanics. Until now, only the periodic orbits are found in several experiments of neutrino oscillation, though it is possible to find a (quantum) chaotic behavior in an oscillation.

5 Summary

In this paper, we have summarized the aspect of dynamical systems in the classical Nambu mechanics in Sec. 2. Our conclusion is that there is no essential difference, no essential difficulty to apply the framework of dynamical systems in the classical Nambu mechanics. In Sec. 3, we have investigated the quantization condition of three-dimensional quantum Nambu mechanics, especially from the viewpoint of Lie algebras/groups. Usually, such an attempt is said that it seems difficult to study a Lie-algebra/group structure in the framework of Nambu mechanics, since the Nambu-Poisson algebra is defined not by a binary, but by a ternary operation (in a three-dimensional case). We have observed that we can easily overcome such a difficulty by employing the notions of symplectic homogeneous spaces and quaternionic algebra, at least in the three-dimensional case. Some geometric aspects of the three-dimensional quantum Nambu bracket has been found by us, and we have met an example of a realization of unification of algebra and geometry. We also have discussed the fact that the ordinary quantum theory of Heisenberg by the Heisenberg's quantization condition is just a special case, just an example of the quantization condition of three-dimensional Nambu bracket: We have mentioned that this fact might give us a framework to find/construct/invent several variants of quantum theories where they are not defined on the Heisenberg algebra but are embedded in the quantum Nambu brackets. What we should do in the next step is to make our insight more concrete one, and also to investigate some geometric realizations of those algebras by employing the framework of quaternionic geometry or hyper-Kähler and quaternionic Kähler geometry. For example, a quantum theory of gravity might be constructed by a consistent manner by using a framework of new quantum theory. Since a new quantum theory might give us a new paradigm of quantum information/communication theory, we can consider a possibility that a quantum nature of gravity might have an essential difficulty to detect it by an apparatus constructed on the ordinary Heisenberg quantum theory. Another possible interpretation for us is that our insight/speculation of a generalization of the Heisenberg's quantum theory might approach toward a region/domain of "beyond quantum physics". For a systematic generalization/extension of the Heisenberg's quantum theory, one might also consider to employ the algebra of octonions. Finally, in Sec. 4, we have listed some physical systems, especially for describing neutrino oscillation phenomena, and have emphasized that their Lie-algebraic structures have some familiality with that of the quantum Nambu mechanics.

(This paper was submitted to ArXiV at 24 Mar. 2014, and removed at 3 Feb. 2015.)

References

[1] A. Yu. Alekseev and A. Z. Malkin, *Symplectic Structures Associated to Lie-Poisson Groups*, Commun. Math. Phys. Vol.162, 147-173 (1994).

[2] V. Arnol'd, *Mathematical Methods of Classical Mechanics* (GTM 60, Springer, 1978).

[3] M. Axenides and E. Floratos, *Strange Attractors in Dissipative Nambu Mechanics: Classical and Quantum Aspects*, arXiv:0910.3881.

[4] A. Ballesteros, A. Blasco, and F. Musso, *Classification of Real Three-Dimensional Poisson-Lie Groups*, arXiv:1202.2077.

[5] J. Bardeen, L. N. Cooper, and J. R. Schrieffer, Phys. Rev. Vol.108, 1175 (1957).

[6] F. Bayen, M. Flato, C. Frnsdal, A. Lichnerowicz, D. Sternheimer, *Deformation Theory and Quantization I, II.* Ann. Phys. Vol.111, 61-110, 111-151 (1978).

[7] N. Boumuki, *On Symplectic Homogeneous Spaces and Adjoint Orbits*, RIMS proceedings (2005).

[8] B.-Y. Chu, *Symplectic Homogeneous Spaces*, Trans. Am. Math. Soc. Vol.197, 145-159 (1974).

[9] T. Curtright and C. K. Zachos, *Classical and Quantum Nambu Mechanics*, Phys. Rev. D68, 085001 (2003).

[10] J.-P. Eckmann and D. Ruelle, *Ergodic Theory of Chaos and Strange Attracters*, Rev. Mod. Phys. Vol.57, 617-656 (1985).

[11] M. Kontsevich, *Deformation Quantization of Poisson Manifolds*, arXiv:q-alg/9709040.

[12] Z. Maki, M. Nakagawa, and S. Sakata, *Remarks on the Unified Model of Elementary Particles*, Prog. Theor. Phys. Vol.28, 870-880 (1962).

[13] Y. Nambu, Phys. Rev. Vol.117, 648 (1960).

[14] Y. Nambu, *Generalized Hamiltonian Mechanics*, Phys. Rev. D7, 2405-2412 (1973).

[15] T. Ohsaku, *Relativistic Model of Two-Band Superconductivity in (2+1)-Dimensions*, Int. J. Mod. Phys. B18, 1771-1794 (2004).

[16] T. Ohsaku, *Deformation Quantization and Quaternions*, math-ph/0609031. (9 pages.)

[17] T. Ohsaku, *Dynamical Mass Generations and Collective Excitations in the (Supersymmetric-)Nambu−Jona-Lasinio Model and a Gauge Theory with Left-Right-Asymmetric Majorana Mass Terms* , arXiv:0802.1286. (98 pages.)

[18] T. Ohsaku, *The Anomalous Nambu-Goldstone Theorem in Relativistic/Nonrelativistic Quantum Field Theory*, arXiv:1312.0916, (71 pages). (The Part I of this paper.)

[19] T. Ohsaku, *The Anomalous Nambu-Goldstone Theorem in Relativistic/Nonrelativistic Quantum Field Theory. II*, (24 pages). (submitted to arXiv.org, 25 Feb. 2014, and removed at 3 Feb. 2015.)

[20] G. Papadopoulos, *Conformal and Superconformal Mechanics*, (Feb. 2000), available from internet.

[21] Y. Pehlivan, A. B. Balantekin, T. Kajino, T. Yoshida, *Invariants of Collective Neutrino Oscillations*, arXiv:1105.1182.

[22] B. Pontecorvo, *Mesonium and Anti-Mesonium*, Sov. Phys. JETP, Vol.6, 429 (1957).

[23] G. G. Raffelt, *N-Mode Coherence in Collective Neutrino Oscillations*, arXiv:1103.2891.

[24] L. Takhtajan, *On Foundation of the Generalized Nambu Mechanics*, Commun. Math. Phys. Vol.160, 295-315 (1994).

[25] A. Weil. *Sur certains groupes d'operateur unitaires*, Acta Math. Vol.111, 143-211 (1964).

[26] H. Yoshida, *Siegel's Modular Forms and the Arithmetic of Quadratic Forms*, Invent. Math. Vol.60, 193-248 (1980).