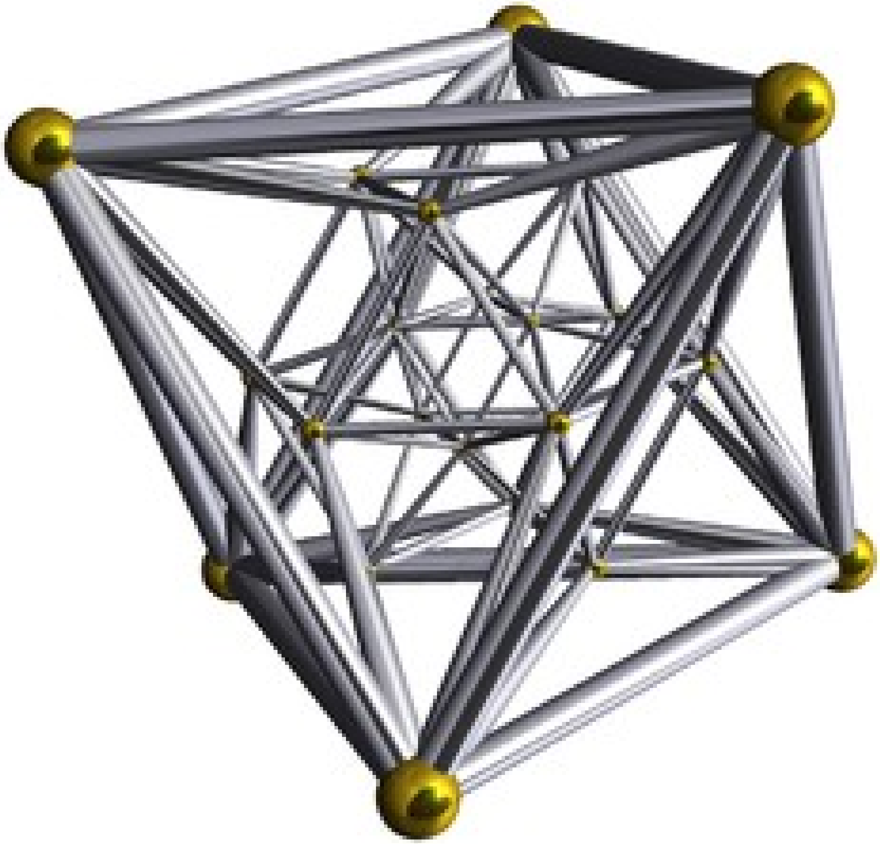
Geography of Time in Qi Men Dun Jia

By John Frederick Sweeney



Abstract

One cause for the failure of western science is the assumption that Time is constant, or the reliance on such concepts as “Space – Time.” In contrast, the ancient divination method known as Qi Men Dun Jia views time as a physical entity. The QMDJ model, based on actual real physical processes, thus uses this temporal model to make accurate predictions in the real world. This paper explains the temporal mechanism, based on Hurwitz Quaternions, by which this is done. Thus Hurwitz Lattices and the 24 – Cell may describe the Geography of Time.

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Introduction

As a teacher of Qi Men Dun Jia, I explain to my students that there exists something called the Geography of Time. This concept is quite difficult for western students to grasp, since the western world assumes that Time is a constant, much as light, and so one can therefore make simple equations about the world, such as $E = MC^2$.

Geoffrey Dixon has written much about Clifford Algebras and specifically about the number 24, which he views as special and thus worthy of extra attention and an essay. While never quite finding the grail of his quest, Dixon reveals a good deal about why 24 is so important, given its relation to the 24 Hurwitz Quaternions. The section “The 24 Problem” is copied directly from his 7 Stones website, albeit from what he describes as the most obscure page of such.

This is so, since Dixon reaches the heart of the problem, perhaps unknowingly. The problem consists of breaking down the 24 Seasons or the 24 Hurwitz Quaternions into triplets, which can then be further broken down into 60 periods of 120 minutes each. Perhaps intuitively, Dixon provides a complete breakdown of the triplets, which apparently took several years to complete.

In another paper, Dixon describes how Octonions replaced Complex Numbers in Physics:

(the work of G¨ursey at Yale University during the 1970s was the inspiration for all of my work - and that of many others - applying the octonion algebra to physics)). The work of G¨ursey (and G¨unaydin) was inspired by the work of von Neumann, Jordan and Wigner [8], who investigated an expansion of quantum theory from a foundation on C to one on O. They linked quantum observability with algebraic associativity, and unobservability with nonassociativity, thinking along these lines being forced by the nonassociativity of O. (I do not know the details of their work, but the notion that nonassociativity could be associated with things unseen, and unseeable, partly motivated this work.) The quantum notion of unobservable is more restrictive than the notion of unseeable being used here. In particular, quarks are unseeable, but they are detectable, and they supply the paradigm - albeit not well defined - of what is meant by unseeable.

In fact, the QMDJ Model is founded upon Octonions, precisely because Octonions may be used to “see” the unsee – able.” This is where QMDJ derives its divinational power. QMDJ is able to foresee events before they take place, for example, to detect disease in its gestation stage before the disease actually manifests in the human body. Unfortunately, this boon to medical science is completely misunderstood by most of humanity at present.

In Vedic Physics, particular numbers taken on huge importance because they are key to basic natural processes in our combinatorial Universe. If one takes Base 60 Math as a fundamental standard, then 24 fits in 2.5 times. Half of 24 is 12, and Chinese medicine theory contains 12 meridians – 6 Yang and 6 Yin. These 12 meridians, added to two vessels, control all the functions in the human body, and form a network through which Qi flows in a 24 – hour cycle. Base 60 forms a fundamental standard in that it is directly related to Pisano (Fibonacci) Periodicity, the natural limits to growth.

This paper introduces Qi Men Dun Jia and some of its basic features, including the Clifford Clock as an isomorph of the Eight Trigrams, an articulation of the 3 x 3 Magic Square by Frank “Tony” Smith; John Baez on the Clifford Clock and related spaces; and the 24 Seasons of China’s traditional calendar and their use in QMDJ, which is the key feature under scrutiny here. Then, a discussion of 24 by Geoffrey Dixon, sections on Hurwitz Quaternions and finally a conclusion, based on the work of a few mathematicians, about where the math leads.

Time is not what you think it to be. Time has a shape, a geometry, and it is possible for us to know and study that geometry, just as the ancients studied the Geography of Time with Qi Men Dun Jia.

Qi Men Dun Jia

In 2003 I bought a copy of *The I Ching and Chinese Medicine* by Yang Li, a professor of medicine in Beijing. Consisting primarily of her lecture notes, the tome lays out the esoteric and metaphysical aspects of Chinese medicine. When I showed it to my 80 + year old doctor of Chinese medicine in Nan Ning, Guang Xi, he found it surprising that China's government had allowed publication of such hallowed national secrets.

Alas, the book was lost on younger generations, who have been taught to believe, wrongly, that their traditional culture consists of superstition. What the Chinese dispose of in their rush to embrace the dying American Dream, the author of this paper gladly accepts. Yang Li's book contains brief descriptions of Qi Men Dun Jia and Da Liu Ren, the most advanced forms of divination known to the Chinese. I had identified the target of my decades – long effort to master Chinese.

To underscore the point, an octogenarian Da Liu Ren master presented me with copies of his ten books, in the hope that I would translate them into English. At least then, he believed, someone in the world would continue the study of Da Liu Ren. There remained little chance of that in China in 2007.

As an undergraduate at U.C. Berkeley, in late August 1984 I stood in between Durant Hall and Dwinnelle Hall, trying to decide whether to study Japanese or Chinese. I had studied with Chalmers Johnson, the reknowned Japan specialist, and Japan was very much in vogue in the early 1980's, before its Bubble Economy burst in 1989. Johnson had stated that it is best to learn languages before the age of 25, since the human brain begins to solidify after that age. I admired Johnson and wished to continue to study with him, and enjoyed Japanese culture far more than Chinese.

Yet I reasoned that Durant Hall and ancient Chinese held many more ancient secrets, that Japanese simply could not hold. Thus I decided to learn Chinese, with the understanding that at some point I would access the ancient secrets the old language held, perhaps secrets unknown even to contemporary Chinese.

In the late 1990's, surrounded by all of the "Feng Shui" experts in Southern California, who knew just a few words in Chinese and practiced their art to earn small fortunes, I decided to investigate the idea of becoming a Feng Shui master. In downtown Los Angeles, in the Chinatown there, I bought a couple of books on Chinese metaphysics and brought them home to study. Perusal of these books made it clear that true mastery of the subject would require moving to Taiwan or Hong Kong where more books would be readily available, and then years of study.

Discovering QMDJ in 2003 presented an avenue of exploration that would not only meet my Feng Shui challenge, but lead to other roads as well, since QMDJ is a multi – purpose divination tool. Eventually I was to extend this tool to a level which the Chinese have never understood, proof that the Chinese did not invent QMDJ, but merely preserved it as a cast – off from the superior civilizations of Vedic India and ancient Egypt. Thus began my QMDJ journey.

Within a short time I found the Clifford Clock on Frank "Tony" Smith's website, and began thinking of how to devise a mathematical foundation for QMDJ. I knew that QMDJ provided accurate predictions – the very basis of theory. Yet I didn't know how to prove how QMDJ did so, and few westerners would believe in such a system without a scientific explanation. I invited Smith to join me, but he was otherwise occupied and declined my invitation, although he said that his monumental website is open to the public as a well from which one may freely draw.

For this reason, I had to learn mathematical physics, primarily from Smith's website. This led to my outsider status, since I obviously had not learned within academia, and Smith had been forced out of academia by Cornell University and the professors who described him as a "crack – pot." Even John Baez could not restrain himself from making exceptions about Smith's work, which deals with the Tarot Card system. While shameful to employed academics in western universities, Smith's website is profoundly earth – shattering in the ground work it has spaded for future, more enlightened generations.

My problem boiled down to this: I knew that QMDJ made accurate predictions. A.J. Gregor had taught at UC Berkeley that a model which can make accurate predictions is a theory in scientific terms. The problem was how to explain, in scientific terms, how QMDJ is capable of making predictions. Vedic Physics and our combinatorial Universe holds the key to this problem, and Cayley Algebras, the work of Smith, Baez, Dixon and others in RCHOS helps to explain how this all takes place.

The final linch pin fell into place when the idea of Dixon's musings about the number 24 clicked with the 24 seasons of China's traditional calendar. I had known and written about the Hurwitz Quaternions in 2013, and understood them as key to the formation of matter and directly related to the Exceptional Lie Algebras, yet it took until the end of 2014 when the realization occurred that the 24 Hurwitz Quaternions could be applied to Time, as well as matter.

In so doing, one would create a model of time which mirrored that of QMDJ. From this mathematical construct, QMDJ derives its divinatory power. That is to say that by dividing the year along the lines one might divide Hurwitz Quaternions, one may predict events in a uniform and systematic fashion. This is how QMDJ is able to make accurate predictions – as a reasonable mirror of natural systems.

This paper provides the necessary parts and explains how it all fits together to create the QMDJ Predictive Model.

Base 60

Sexagesimal (**base 60**) is a numeral system with sixty as its base. It originated with the ancient Sumerians in the 3rd millennium BC, it was passed down to the ancient Babylonians, and it is still used — in a modified form — for measuring time, angles, and geographic coordinates.

The number 60, a superior highly composite number, has twelve factors, namely {1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60}, of which 2, 3, and 5 are prime numbers. With so many factors, many fractions involving sexagesimal numbers are simplified. For example, one hour can be divided evenly into sections of 30 minutes, 20 minutes, 15 minutes, 12 minutes, 10 minutes, 6 minutes, 5 minutes, 4 minutes, 3 minutes, 2 minutes, and 1 minute. 60 is the smallest number that is divisible by every number from 1 to 6; that is, it is the lowest common multiple of 1, 2, 3, 4, 5, and 6.

Cycle of Sixty

The combination of 10 Heavenly Stems and 12 Earth Branches leads to the 60 Jia Zi, or the Cycle of Sixty, another basic component of Chinese metaphysics.

The Cycle of Sixty

甲子	甲戌	甲申	甲午	甲辰	甲寅
乙丑	乙亥	乙酉	乙未	乙巳	乙卯
丙寅	丙子	丙戌	丙申	丙午	丙辰
丁卯	丁丑	丁亥	丁酉	丁未	丁巳
戊辰	戊寅	戊子	戊戌	戊申	戊午
己巳	己卯	己丑	己亥	己酉	己未
庚午	庚辰	庚寅	庚子	庚戌	庚申
幸未	幸巳	幸卯	幸丑	幸亥	幸酉
壬申	壬午	壬辰	壬寅	壬子	壬戌
癸酉	癸未	癸巳	癸卯	癸丑	癸亥

Notice the six Jia Heavenly Stems atop each column. This is where QMDJ derives its name. Each Jia begins a new 12 – part mini – cycle. When used to keep track of time, this cycle runs for five days / 60 hours. This cycle may represent 1/5 of a 300 – year cycle, two months, or five days, and even one minute when applied to seconds.

Cycle of Sixty in English

Jia Zi	Jia Xu	Jia Shen	Jia Wu	Jia Chen	Jia Yin
Yi Chou	Yi Hai	Yi You	Yi Wei	Yi Si	Yi Mao
Bing Yin	Bing Zi	Bing Xu	Bing Shen	Bing Wu	Bing Chen
Ding Mao	Ding Chou	Ding Hai	Ding You	Ding Wei	Ding Si
Wu Chen	Wu Yin	Wu Zi	Wu Xu	Wu Shen	Wu Wu
Ji Si	Ji Mao	Ji Chou	Ji Hai	Ji You	Ji Wei
Geng Wu	Geng Chen	Geng Yin	Geng Zi	Geng Xu	Geng Shen
Xing Wei	Xing Si	Xing Mao	Xing Chou	Xing Hai	Xing You
1Ren Shen	Ren Wu	Ren Chen	Ren Yin	Ren Zi	Ren Xu
Gui You	Gui Wei	Gui Si	Gui Mao	Gui Chou	Gui Hai

Alternatively, one might substitute the values from Wiseman & Ye:

S1/B1	S1/B11	S1/B9	S1/B7	S1/B5	S1/B3
S2/B2	S2/B12	S2/B10	S2/B8	S2/B6	S2/B4
S3/B3	S3/B1	S3/B11	S3/B9	S3/B7	S3/B5
S4/B4	S4/B2	S4/B12	S4/B10	S4/B8	S4/B6
S5/B5	S5/B3	S5/B1	S5/B11	S5/B9	S5/B7
S6/B6	S6/B4	S6/B2	S6/B12	S6/B10	S6/B8
S7/B7	S7/B5	S7/B3	S7/B1	S7/B11	S7//B9
S8/B8	S8/B6	S8/B4	S8/B2	S8/12	S8/B10
S9/B9	S9/B7	S9/B5	S9/B3	S9/B1	S9/B11
S10/B10	S10/B8	S10/B6	S10/B4	S10/B2	S10/B12

8. Heavenly Stems & Earthly Branches

The Heavenly Stems		The Earthly Branches	
甲 *jiă*	S1	子 *zĭ*	B1
乙 *yĭ*	S2	丑 *chŏu*	B2
丙 *bĭng*	S3	寅 *yín*	B3
丁 *dīng*	S4	卯 *măo*	B4
戊 *wù*	S5	辰 *chén*	B5
己 *jĭ*	S6	巳 *sì*	B6
庚 *gēng*	S7	午 *wŭ*	B7
辛 *xīn*	S8	未 *wèi*	B8
壬 *rén*	S9	申 *shēn*	B9
癸 *guĭ*	S10	酉 *yŏu*	B10
		戌 *xū*	B11
		亥 *hài*	B12

Nine Palaces

From the River Diagram and the Luo Book, we have received a Magic Square. The magic square consists of nine consecutive numbers arranged in nine square boxes (3 x 3). The sum of the numerals in any direction always adds up to fifteen, and five is always placed at the center of the Magic Square. In Qi Men Dun Jia, we call the nine square boxes the Nine Palaces. The Nine Palaces may be arranged in the 3x3 square format, or may be arranged in concentric circles.

Xun Palace 4 Wood Southeast Eldest Female	Li Palace 9 Fire South Mid daughter	Kun Palace Two Earth Southwest Mother
Zhen Palace 3 Wood East Eldest Son	Centre Palace 5 " rides" Palace 2 Earth	Dui Palace 7 Metal West Youngest Daughter
Gen Palace 8 Earth Northeast Youngest Son	Kan Palace 1 Water North Middle Son	Qian Palace 6 Metal Northwest Father

Nine Palaces can be overlaid onto the system of twenty-four seasons and Eight Hexagrams described below. Each palace matches one hexagram, and these match one direction. Taken together, these match three seasons of the traditional lunar calendar.

Frank "Tony" Smith's website contains the following introduction:

Chinese cosmology begins with the undivided Tai Chi,
then separating into Yin-Yang, ... :

Let o represent the undivided Tai Chi, a scalar point of origin:

	o	

Then add 4 vector directions of Physical Spacetime:
1, i, j, k of the quaternions

to get the 5 Elements:

	i	
j	o	1
	k	

Then add 4 vector directions of Internal Symmetry Space:
E, I, J, K of the octonions,
which are the basis for the D4-D5-E6-E7 physics model,
to get 9 directions:

J	i	
j	o	1
K	k	E

The 10th direction is Yin-Yang reflection
of the 8 vector directions 1, i, j, k, E, I, J, K.
Now, identify the 3x3 square with the Magic Square

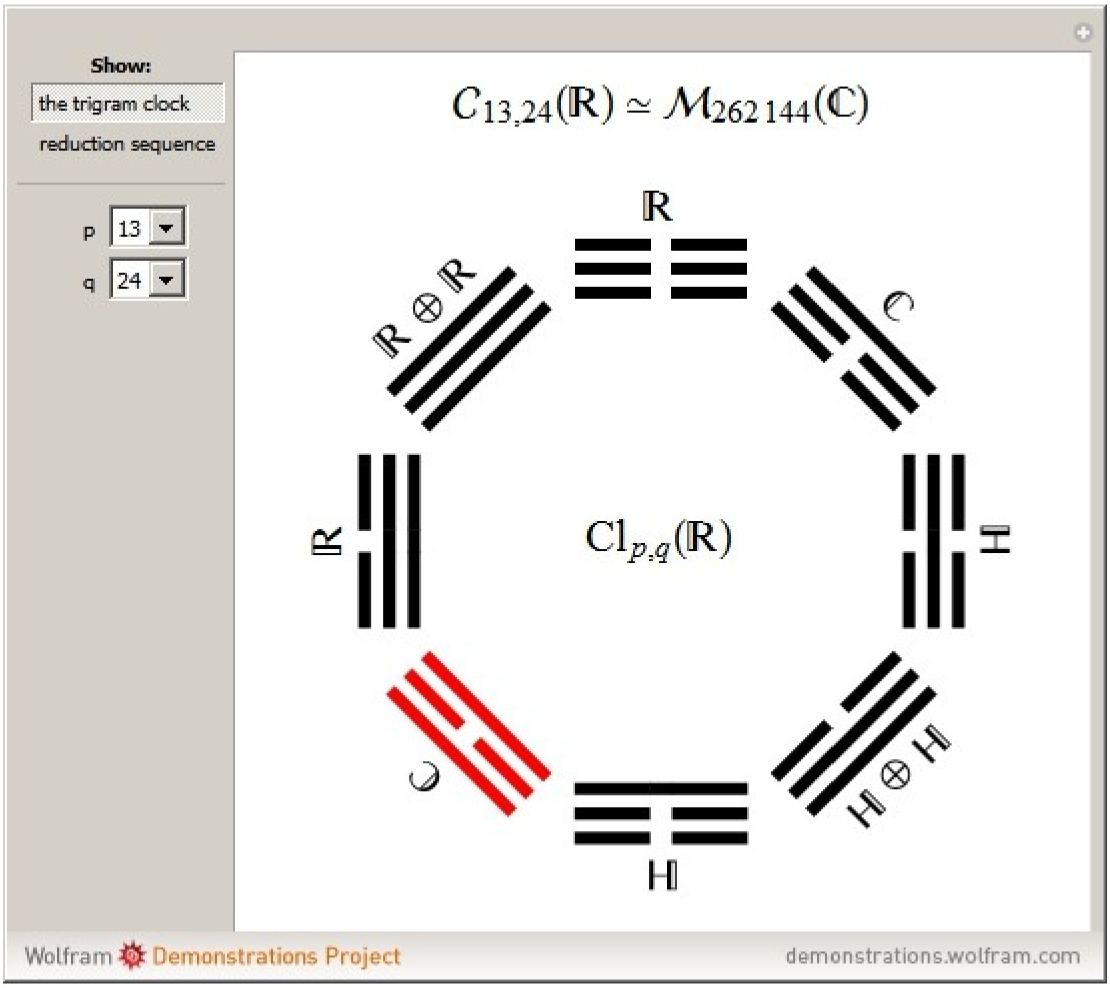
4	9	2
3	5	7
8	1	6

whose central number, 5, is also
central in the sequence 1,2,3,4, 5, 6,7,8,9
which sequence corresponds
to the octonions 1,i,j,k, 0, E,I,J,K
whose total number for each line is 15,
the dimension of the largest Hopf fibration
and the dimension of the imaginary sedenions.
If you take into account the direction in which you add each
of the 8 ways, and add all directed ways together
you get a total of 16x15 = 240
which is the number of vertices of a Witting polytope.
The total of all 9 numbers of the Magic Square is 45,
the dimension of the D5 Lie algebra Spin(10)
that is used in the D4-D5-E6-E7 physics model
in which the D4 Spin(8) subgroup of Spin(10) corresponds

to 28 bi - vector gauge bosons and the 16-dimensional homogeneous space
Spin(10) / Spin(8)xU(1) corresponds to an 8-dimensional complex domain
whose Shilov boundary is RP1 x S7
corresponding to an 8-dimensional spacetime.

As can be observed, Smith links Magic Squares with Quarternions, Octonions (with the Fano Plane and its Golden Ratio associations implicit) and Sedenions, as well as with the Hopf (Bloch) Fibrations, the Witting Polytope (and its Golden Ratio associations implicit), the Exceptional Lie Algebra E8 (and its Golden Ratio associations implicit), 28 bivector gauge bosons, the 16-dimensional homogeneous space Spin(10) / Spin(8)xU(1) and finally to an 8-dimensional spacetime.

Clifford Clock and Clifford Spaces



Wolfram Demo

We denote the real Clifford algebra of signature (p, q)as $\mathrm{Cl}_{p,q}(\mathbb{R}) = \mathrm{Cl}(p, q)$. It is most simply described as the free algebra over the reals constructed from the $n = p + q$basis elements $e_i$subject to the relations

$e_i e_j + e_j e_i = 0$ for $i \neq j$,

$e_i e_i = -1$for $i = 1, \ldots, p$,

$e_i e_i = +1$for $i = p + 1, \ldots, p + q$.

Note that this sign convention is the same as that used in *MathWorld*, but opposite that used in *Wikipedia*.

It is easy to see that as a vector space, the algebra has the basis

$1, e_i, e_{i_1} e_{i_2} (i_1 < i_2), \ldots, e_{i_1} \cdots e_{i_{n-1}} (i_1 < i_2 < \cdots < i_{n-1}), e_1 e_2 \ldots e_n$

and is thus of real dimension 2^n.

The classification of Clifford algebras is the process of realizing the above relations using matrices and matrix multiplication. The classification is tractable due to certain recurrence relations between the Clifford algebras, namely

(1): $C(p+1, q+1) \cong C(p, q) \otimes C(1, 1)$,

(2): $C(p+2, q) \cong C(q, p) \otimes C(2, 0)$,

(3): $C(p, q+2) \cong C(q, p) \otimes C(0, 2)$.

Isomorphism (1) allows an arbitrary Clifford algebra to be reduced to a product of $|p-q|$lots of $C(1, 1)$and either $C(p-q, 0)$or $C(0, q-p)$. Then isomorphisms $(2)$and (3)can reduce the latter factor to a product of powers of $C(2, 0)$and powers of $C(0, 2)$, and a remainder of $C(0, 0)$, $C(1, 0)$, or $C(0, 1)$. So we need the matrix realization of the remaining six, low-dimensional Clifford algebras.

Define the $n \times n$matrix algebra $\mathcal{M}_n(\mathbb{A})$where $\mathbb{A}$is either the reals ($\mathbb{R}$), complexes ($\mathbb{C}$), or quaternions ($\mathbb{H}$) and note that $\mathcal{M}_1(\mathbb{A}) \cong \mathbb{A}$. Remember that both the complex and quaternionic numbers can themselves be realized as real matrix algebras of dimension 2 and 4 respectively.

Direct calculation gives the results

$C(0, 0) \cong \mathbb{R}$, $C(1, 0) \cong \mathbb{C}$, $C(0, 1) \cong \mathbb{R} \oplus \mathbb{R}$, $C(1, 1) \cong \mathcal{M}_2(\mathbb{R})$, $C(2, 0) \cong \mathbb{H}$, $C(0, 2) \cong \mathcal{M}_2(\mathbb{R})$.

To simplify and complete the matrix realization we use the following isomorphisms between the various products of matrix algebras

$\mathcal{M}_n(\mathbb{R}) \otimes \mathcal{M}_m(\mathbb{R}) \cong \mathcal{M}_{nm}(\mathbb{R})$, $\mathbb{C} \otimes \mathcal{M}_n(\mathbb{R}) \cong \mathcal{M}_n(\mathbb{C})$, $\mathbb{H} \otimes \mathcal{M}_n(\mathbb{R}) \cong \mathcal{M}_n(\mathbb{H})$, $\mathbb{C} \otimes \mathbb{H} \cong \mathcal{M}_2(\mathbb{R})$, $\mathbb{H} \otimes \mathbb{H} \cong \mathcal{M}_4(\mathbb{R})$.

The eightfold periodicity displayed in the clock is a consequence of (2) and (3), from which we can derive the relations

$C(p+4, q) \cong C(p, q+4)$and $C(p+8, q) \cong C(p, q+8) \cong C(p, q) \otimes \mathcal{M}_{16}(\mathbb{R})$.

This eightfold periodicity was discovered by Cartan in 1908, but is often given the name Bott periodicity due to related periodicities found by Bott in the late 1950s in the study of the homotopy groups of the classical groups.

The trigrams in the Yì Jīng (I Ching) can be associated with the obvious binary sequence where - - (associated with yin) is given the value $0$and — (associated with yang) is valued at 1. Then a binary digit is constructed with the smallest position at the top. Our clock follows that binary sequence and for the Clifford Algebra $C(p, q)$it points to the position $p - q \bmod 8$. The actual order normally given to trigrams does not match the binary correspondence given above.

In the Yi Jing each trigram is given a value in a range of areas (direction, family, body part, animal, etc.) and then combined into pairs call hexagrams. The meaning of a hexagram is inherited from its component trigrams. The lower trigram is called the inner and is linked to the personal aspects, while the upper trigram is called the outer and linked to the external aspects. Thus a large range of characteristics and situations can be encoded using these hexagrams.

This Demonstration was inspired by a graphic found on a page by Tony Smith (www.valdostamuseum.org).

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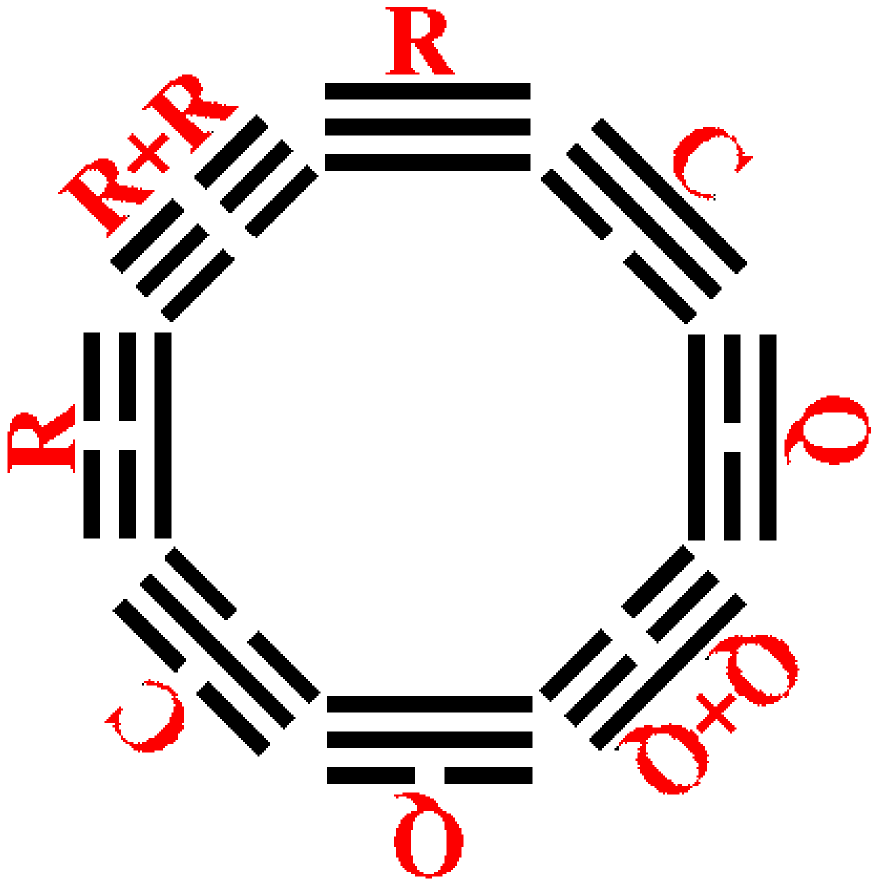
(John Baez)



I'm not sure who first worked out all the Clifford algebras - perhaps it was Cartan - but the interesting fact is that they follow a periodic pattern. If we use C_n to stand for the Clifford algebra generated by n anti - commuting square roots of -1, they go like this:

C_0 R
C_1 C
C_2 H
C_3 H + H
C_4 H(2)
C_5 C(4)
C_6 R(8)
C_7 R(8) + R(8)
where:

- R(n) means n x n real matrices,
- C(n) means n x n complex matrices, and
- H(n) means n x n quaternionic matrices.

All these become algebras with the usual addition and multiplication of matrices. Finally, if A is an algebra, A + A consists of pairs of guys in A, with pairwise addition and multiplication.

What happens next? Well, from then on things sort of "repeat" with period 8: C_{n+8} consists of 16 x 16 matrices whose entries lie in C_n!

So, you can remember all the Clifford algebras with the help of this eight-hour clock:

0

R

7 1

R+R C

6 R H 2

C H+H

5 3

H

4

To use this clock, you have to remember to use matrices of the right size to get C_n to have dimension 2^n. So, when I write "R + R" next to the "7" on the clock, I don't mean C_7 is really R + R.

To get C_7, you have to take R + R and beef it up until it becomes an algebra of dimension $2^7 = 128$. You do this by taking R(8) + R(8), since this has dimension 8 x 8 + 8 x 8 = 128.

Similarly, to get C_{10}, you note that 10 is 2 modulo 8, so you look at "2"

on the clock and see "H" next to it, meaning the quaternions.

But to get C_{10},

you have to take H and beef it up until it becomes an algebra of dimension $2^{10} = 1024$.

You do this by taking H(16), since this has dimension 4 x 16 x 16 = 1024.

This "beefing up" process is actually quite interesting. For any associative algebra A, the algebra A(n) consisting of n x n matrices with entries in A is a lot like A itself. The reason is that they have equivalent categories of representations!

To see what I mean by this, remember that a "representation" of an algebra is a way for its elements to act as linear transformations of some vector space. For example, R(n) acts as linear transformations of R^n by matrix multiplication, so we say R(n) has a representation on R^n. More generally, for any algebra A, the algebra A(n) has a representation on A^n.

More generally still, if we have any representation of A on a vector space V, we get a representation of A(n) on V^n. It's less obvious, but true, that *every* representation of A(n) comes from a representation of A this way.

In short, just as n x n matrices with entries in A form an algebra A(n) that's a beefed-up version of A itself, every representation of A(n) is a beefed-up version of some representation of A.

Even better, the same sort of thing is true for maps between representations of A(n). This is what we mean by saying that A(n) and A have equivalent categories of representations. If you just look at the categories of representations of these two algebras as abstract categories, there's no way to tell them apart! We say two algebras are "Morita equivalent" when this happens.

It's fun to study Morita equivalence classes of algebras - say algebras over the real numbers, for example. The tensor product of algebras gives us a way to multiply these classes. If we just consider the invertible classes, we get a *group*. This is called the "Brauer group" of the real numbers.

The Brauer group of the real numbers is just Z/2, consisting of the classes [R] and [H]. These correspond to the top and bottom of the Clifford clock! Part of the reason is that

```
H tensor H = R(4)
```

so when we take Morita equivalence classes we get

```
[H] x [H] = [R]
```

But, you may wonder where the complex numbers went! Alas, the Morita equivalence class [C] isn't invertible, so it doesn't live in the Brauer group. In fact, we have this little multiplication table for tensor product of algebras:

```
          tensor       R       C       H
                    ----------------------
            R      |   R       C       H
                   |
            C      |   C      C+C     C(2)
                   |
            H      |   H      C(2)    R(4)
```

Anyone with an algebraic bone in their body should spend an afternoon figuring out how this works! But I won't explain it now.

Instead, I'll just note that the complex numbers are very aggressive and infectious - tensor anything with a C in it and you get more C's. That's because they're a field in their own right - and that's why they don't live in the Brauer group of the real numbers.

They do, however, live in the *super-Brauer* group of the real numbers, which is Z/8 - the Clifford clock itself!

But before I explain that, I want to show you what the categories of representations of the Clifford algebras look like:

1 **split Real vector spaces**	**0** **Real vector spaces**	**7** **Complex vector spaces**
6 **Real vector spaces**		**2** **Quaternionic vector**
5 **Complex vector spaces**	**4** **Quaternionic vector spaces**	**3** **split Quaternionic vector spaces**

China’s 24-Season Traditional Lunar Calendar

Chinese		English	Period (approx.)	Ju Number
1. Dong Zhi	冬至	Winter Solstice	22 Dec - 4 January	**1-7-4**
2. Xiao Han	小寒	Little Cold	5 Jan - 19 January	**2-8-5**
3. Da Han	大寒	Great Cold	20 January – 3 February	**3-9-6**
4. Li Chun	立春	Spring starts	4 Feb - 18 February	**8-5-2**
5. Yu Shui	雨水	Rain Water	19 Feb - 4 March	**9-6-3**
6. Jing Zhe	惊蛰	Movement of insects	5 Mar - 20 March	**4-7-4**
7. Chun Fen	春分	Spring Equinox	21 Mar - 4 April	**3-9-6**
8. Qing Ming	晴明	Clear Brilliance	5 Apr - 19 April	**4-1-7**
9. Gu Yu	谷雨	Grain Rain	20 Apr - 4 May	**5-2-8**
10. Li Xia	立夏	Summer starts	5 May - 20 May	**4-1-7**
11. Xiao Man	小满	Surfeit	21 May - 5 June	**5-2-8**
12. Mang Zhong	芒种	Grain in ear	6 Jun - 20 June	**6-3-9**
13. Xia Zhi	**夏至**	**Summer solstice**	**21 Jun - 6 July**	**9-3-6**
14. Xiao Shu	小署	Little Heat	7 July - 22 July	**8-2-5**
15. Da Shu	大署	Great Heat	23 July - 6 August	**7-1-4**
16. Li Qiu	立秋	Autumn starts	7 Aug - 22 August	**2-5-8**
17. Chu Shu	初署	Heat Ends	23 Aug - 6 September	**1-4-7**
18. Bai Lu	白露	White Dew	7 Sep - 21 September	**9-3-6**
19. Qiu Fen	秋分	Autumn Equinox	22 Sep - 7 October	**7-1-4**
20. Han Lu	寒露	Cold Dew	8 Oct - 22 October	**6-9-3**
21. Shuang Jiang	霜降	First Frost	23 Oct - 6 November	**5-8-2**
22. Li Dong	立冬	Winter starts	7 Nov -21 November	**6-9-3**
23. Xiao Xue	小雪	Little Snow	22 Nov - 6 December	**5-8-2**
24. Da Xue	大雪	Great Snow	7 Dec - 21 December	**4-7-1**

The chart above gives the 24 seasons in the left column, their names in Pin Yin and in Chinese characters. The second column gives their translated names. The third column gives their periods, *roughly* adjusted to the western calendar. Westerners often expect exact correspondence, but this is NOT the case.

The right hand column is the most interesting, as it provides the Ju numbers for each season. Each Chinese season lasts 15 days, and these are broken into thirds of five days each, or 60 hours. The thirty double hours which comprise these 60 hours correspond to a Situation, or a specific arrangement of six levels (dimensions) of QMDJ symbols, which include palaces, gates, stars and spirits. When the season begins, then the first Situation begins to cover the initial 120 – minute period.

For example, the Winter Solstice season begins with Ju No. 1 of Yang Dun, and its Situation No. 1. Once the 60 double hours have run their course, then the next Ju will be number 7, followed by No. 4. The remainder of the year follows the same pattern, although the Chinese often add an intercalary month to compensate for calendrical inequalities.

QMDJ divides the year into Yin and Yang binary form between the solstices, with Yang Dun from Winter Solstice to Summer Solstice, or the six – month period when Qi (and daylight from the Sun) increases steadily.

This QMDJ temporal scheme may give inspiration to researchers to follow the pattern established here; divide the 24 Hurwitz Quaternions into their corresponding triplets, then divide these further based on Pisano (Fibonacci) Periodicity.

The 24 Problem

From Geoffrey Dixon

There is only one nontrivial example of the sum of consecutive squares from 1 to some integer n
equalling a perfect square, and that occurs when n = 24.

$1^2 + 2^2 + 3^2 + ... + 24^2 = 70^2$.

This is related to the existence of the Leech lattice, $\mathbf{\Lambda}_{24}$, in $\mathbf{R}^{24}$. The Leech lattice can be represented
as a triplet of octonions.

Question: can the integers from 1 to 24 be be divided into 3 sets of 8
such that the sum of the squares of each octet is a perfect square?
Well, first we need triplets of positive integers whose squares sum to 70^2. There are five such triplets:

2	36	60
12	20	66
12	34	60
20	30	60
30	36	52

Of these there is only one that allows the set of integers from 1 to 24 to be split into 3 subsets of 8
such that the sum of the squares of the integers in each subset is a perfect square. In particular, it only
works for the triplet (30, 36, 52), and in this instance there are many such triplets of octet subsets.

One such is:

1	2	3	4		6	7									16							23		→ 30
				5			8	9	10	11			14	15							22			→ 36
											12	13				17	18	19	20	21			24	→ 52

At this point I don't know how many others there are, but there are very many.
Curiosity: The order of the inner shell of the Leech lattice is 196560.
Note that: 196560/(30×36×52) = 7/2 = 3.5.

(This makes for half a Fano Plane or half of the Octonions, but Dixon apparently fails to note that here – author's note).

(Since this is all numerology, let me add that $(7/2)^6 = 1838.265625$,
which is close to the ratio of the proton or neutron mass to the electron mass.)
(2004.02.24: More anon.)

It has now been determined that there are $961 = 31^2$ ways of splitting the set of integers from 1 to 24 into 3 octets such that the sum of the squares of the integers in the octets are 30^2, 36^2 and 52^2.
These are listed below. For each octet summing to 52^2 there is an associated list of octets summing to 30^2.
The associated octets summing to 36^2 are easily found from these.
[1, 8, 17, 19, 20, 22, 23, 24] sum^2 to 52^2

At present there is no reason to assume that $961 = 31^2$ and $31 = 24 + 7 = 2^5 - 1$ is at all significant.
(2004.03.01: More anon.)

The null vector u = [1,1,1,1,1,1,1,1,1 | 3] in the Lorentzian space $\mathbf{R}^{9,1}$ plays a special role in a derivation
of the $\mathbf{E}_8$ lattice. For any given octet x = [x1, ..., x8], with $x1^2 + ... + x8^2 = k^2$, define the $\mathbf{R}^{9,1}$ null vector
n(x) = [0, x1, ..., x8 | k].
Then the inner product of u and n(x) is
<u, n(x)> = x1 + ... + x8 - 3k.

Of the 361 octet triples (x,y,z) above there are 72 that satisfy the following interesting property:
<u, n(x)> = <u, n(y)> = <u, n(z)> = -18.
Therefore, if
$x1^2 + ... + x8^2 = 30^2$ and $y1^2 + ... + y8^2 = 36^2$ and $z1^2 + ... + z8^2 = 52^2$,
then
x1 + ... + x8 = 72 and y1 + ... + y8 = 90 and z1 + ... + z8 = 138.

If we define
p(x) = [18, x1, ..., x8 | k], then
<u, p(x)> = <u, p(y)> = <u, p(z)> = 0.

The 72 triples are indicated in the list below:

[7, 9, 12, 20, 21, 22, 23, 24: sum 138] sum^2 52^2
------- [1, 3, 4, 8, 10, 14, 15, 17: sum 72] sum^2 30^2
------- [1, 3, 4, 8, 11, 13, 14, 18: sum 72] sum^2 30^2
------- [1, 3, 5, 8, 10, 11, 16, 18: sum 72] sum^2 30^2
------- [2, 3, 4, 5, 13, 14, 15, 16: sum 72] sum^2 30^2
------- [2, 3, 4, 6, 11, 13, 16, 17: sum 72] sum^2 30^2
------- [2, 3, 4, 8, 10, 11, 15, 19: sum 72] sum^2 30^2
------- [2, 3, 5, 6, 10, 13, 14, 19: sum 72] sum^2 30^2
------- [2, 4, 5, 6, 8, 13, 15, 19: sum 72] sum^2 30^2
------- [3, 4, 5, 6, 8, 10, 17, 19: sum 72] sum^2 30^2
9_SUBTOTAL = 72
(2004.03.09: More anon.)

In 2004 I published a paper in JMP in which I motivated a new kind of spinor space
by pointing out the how exceptional were the dimensions 2, 8 and 24 (at least as they
pertain to lattice theory). This other triple of numbers (30, 36, 52) has intrigued me since
I first started playing with the idea of breaking the sequence (1,2,...,24) into
three octets, such that the sum of the squares of each octet is also a perfect square.
Interestingly:
30 = 28 + 2;
36 = 28 + 8;
52 = 28 + 24.
The chances that you have got this far in this most obscure of my pages are remote.
The chances that if you have these sums might spark some thoughts that give this
coincidence meaning are even remoter ... vanishingly remote, in fact.
It's the kind of coincidence that makes me go, "What the ...?", but from which
I find it nearly impossible to divert my gaze.

An alternative:
30 = 26 + 4;
36 = 26 + 10;
52 = 26 + 26.
Magic square, anyone?
(2007.12.04: More anon.)

Regarding the numbers 72, 90, 138 looked at above.

72 = 3*24

90 = 3*30

138 = 3*46

$(24 \pm 6)^2 + (30 \pm 6)^2 + (46 \pm 6)^2 = (60 \pm 10)^2 = (24 + 30 + 46)(6 \pm 1)^2$

Yeh, well, cool ... I think. I mean, how can that not be cool?

(2010.01.07: More anon.)

Hurwitz Quaternions

2.2 The ring of quaternions

let $\mathcal{H}$ be the ring of *Hurwitz integers* generated over $\mathbb{Z}$ by the 24 unit quaternions $\pm 1, \pm i, \pm j, \pm k$, and $(\pm 1 \pm i \pm j \pm k)/2$. The ring $\mathcal{H}$ consists of the elements $(a+bi+cj+dk)/2$, where a, b, c, d are integers all congruent modulo 2, with the standard multiplication rules $i^2 = j^2 = k^2 = ijk = -1$. When tensored with $\mathbb{R}$ we get the skew field of quaternions called $\mathbb{H}$. The conjugate of $q = a + bi + cj + dk$ is $\bar{q} = a - bi - cj - dk$. The *real part* of q is $\operatorname{Re}(q) = a$ and the *imaginary part* is $\operatorname{Im}(q) = q - \operatorname{Re}(q)$. The *norm* of q is $|q|^2 = \bar{q}.q$.

We let $\alpha = (1+i+j+k)/2$ and $p = (1-i)$. In the constructions of the lattices given below, the number $p = 1 - i$ plays the role of $\sqrt{-3}$ in [3]. p generates a two sided ideal $\mathfrak{p}$ in $\mathcal{H}$. We have $i \equiv j \equiv k \equiv 1 \bmod \mathfrak{p}$. (Observe that $(1-i)\alpha = 1 + j$, so $j \equiv 1 \bmod \mathfrak{p}$). The group $\mathcal{H}/\mathfrak{p}\mathcal{H}$ is $\mathbb{F}_4$, generated by $0, 1, \alpha, \bar{\alpha}$.

The multiplicative group of units $\mathcal{H}^*$ is $2 \cdot A_4$. The quotient $\mathcal{H}^*/\{\pm 1\}$ has four Sylow three subgroups generated by $\alpha, i\alpha, j\alpha, k\alpha$. The permutation representation of $\mathcal{H}^*/\{\pm 1\}$ on the Sylow 3-subgroups identifies it with the alternating group A_4.

We also identify the quaternions $a + bi$ with the complex numbers. So any quaternion can be written as $z_1 + z_2 j$ for complex numbers z_1 and z_2. The multiplication is defined by $j^2 = -1$ and $jz = \bar{z}j$. The complex conjugation becomes conjugation by the element j.

When interpreted as the quaternions, the F_4 root lattice (which is integral span of the vertices of the 24-cell) is closed under multiplication and is therefore a ring. This is the ring of Hurwitz integral quaternions. The vertices of the 24-cell form the group of units (i.e. the group of invertible elements) in the Hurwitz quaternion ring (this group is also known as the binary tetrahedral group). The vertices of the 24-cell are precisely the 24 Hurwitz quaternions with norm squared 1, and the vertices of the dual 24-cell are those with norm squared 2. The D_4 root lattice is the dual of the F_4 and is given by the subring of Hurwitz quaternions with even norm squared.

Vertices of other convex regular 4-polytopes also form multiplicative groups of quaternions, but few of them generate a root lattice.

In mathematics, a **Hurwitz quaternion** (or **Hurwitz integer**) is a quaternion whose components are *either* all integers *or* all half-integers (halves of an odd integer; a mixture of integers and half-integers is not allowed). The set of all Hurwitz quaternions is

$$H = \left\{a + bi + cj + dk \in \mathbb{H} \mid a,b,c,d \in \mathbb{Z} \text{ or } a,b,c,d \in \mathbb{Z} + \tfrac{1}{2}\right\}.$$

H is closed under quaternion multiplication and addition, which makes it a subring of the ring of all quaternions **H**.

A **Lipschitz quaternion** (or **Lipschitz integer**) is a quaternion whose components are all integers. The set of all Lipschitz quaternions

$$L = \{a + bi + cj + dk \in \mathbb{H} \mid a,b,c,d \in \mathbb{Z}\}$$

forms a subring of the Hurwitz quaternions H.

Hurwitz's theorem is a theorem of Adolf Hurwitz (1859 - 1919), published posthumously in 1923, on finite-dimensional unital real non-associative algebras endowed with a positive-definite quadratic form. The theorem states that if the quadratic form defines a homomorphism into the positive real numbers on the non-zero part of the algebra, then the algebra must be isomorphic to the real numbers, the complex numbers, the quaternions or the octonions. The non-associative algebras occurring are called **Hurwitz algebras** or **composition algebras**.

In mathematics, a **composition algebra** A over a field K is a not necessarily associative algebra over K together with a nondegenerate quadratic form N which satisfies

$$N(xy) = N(x)N(y)$$

for all x and y in A. Unital composition algebras are called **Hurwitz algebras.**[1] If the ground field K is the field of real numbers and N is positive-definite, then A is called an Euclidean Hurwitz algebra.

The quadratic form N is often referred to as a *norm* on A. Composition algebras are also called **normed algebras**: these should not be confused with associative normed algebras, which include Banach algebras, although three associative Euclidean Hurwitz algebras **R**, **C**, and **H** in fact *are* Banach algebras.

Every unital composition algebra over a field K can be obtained by repeated application of the Cayley - Dickson construction starting from K (if the characteristic of K is different from 2) or a 2-dimensional composition subalgebra (if char(K) = 2). The possible dimensions of a composition algebra are 1, 2, 4, and 8.[2]

- 1-dimensional composition algebras only exist when char(K) $\neq$ 2.
- Composition algebras of dimension 1 and 2 are commutative and associative.
- Composition algebras of dimension 2 are either quadratic field extensions of K or isomorphic to $K \oplus K$.
- Composition algebras of dimension 4 are called quaternion algebras. They are associative but not commutative.
- Composition algebras of dimension 8 are called octonion algebras. They are neither associative nor commutative.

The case char(K) $\neq$ 2

Scalar product

If K has characteristic not equal to 2, then a bilinear form $(a, b) = 1/2[N(a + b) - N(a) - N(b)]$ is associated with the quadratic form N.

Involution in Hurwitz algebras

Assuming A has a multiplicative unity, define involution and right and left multiplication operators by

$$\bar{a} = -a + 2(a,1)1, \quad L(a)b = ab, \quad R(a)b = ba.$$

Evidently   is an involution and preserves the quadratic form. The overline notation stresses the fact that complex and quaternion conjugation are partial cases of it. These operators have the following properties:

- The involution is an antiautomorphism, i.e. $a \ b = b \ a$
- $a \ a = N(a) \ 1 = a \ a$
- $L(a) = L(a)^*$, $R(a) = R(a)^*$, where * denotes the adjoint operator with respect to the form (,)
- $\mathrm{Re}(a \ b) = \mathrm{Re}(b \ a)$ where $\mathrm{Re}\ x = (x + x)/2 = (x, 1)$

- $\mathrm{Re}((a\ \ b)\ \ c) = \mathrm{Re}(a\ \ (b\ \ c))$
- $L(a^2) = L(a)^2$, $R(a^2) = R(a)^2$, so that A is an alternating algebra

These properties are proved starting from polarized version of the identity $(a\ b,\ \ a\ b) = (a,\ \ a)(b,\ \ b)$:

$$2(a,b)(c,d) = (ac,bd) + (ad,bc).$$

Setting $b = 1$ or $d = 1$ yields $L(a) = L(a)*$ and $R(c) = R(c)*$. Hence $\mathrm{Re}(a\ b) = (a\ b,\ 1) = (a,\ \ b) = (b\ a,\ 1) = \mathrm{Re}(b\ a)$. Similarly $(a\ b,\ \ c) = (a\ b,\ \ c) = (b,\ \ a\ c) = (1,\ \ b\ (a\ c)) = (1,\ \ (b\ a)\ \ c) = (b\ a,\ \ c)$. Hence $\mathrm{Re}(a\ b)c = ((a\ b)c,\ 1) = (a\ b,\ \ c) = (a,\ \ c\ b) = (a(b\ c),\ 1) = \mathrm{Re}(a(b\ c))$. By the polarized identity $N(a)\ (c,\ \ d) = (a\ c,\ \ a\ d) = (a\ a\ c,\ \ d)$ so $L(a)\ \mathrm{L}(a) = N(a)$. Applied to 1 this gives $a\ a = N(a)$. Replacing a by a gives the other identity. Substituting the formula for a in $L(a)\ \ L(a) = L(a\ a)$ gives $L(a)^2 = L(a^2)$.

Para-Hurwitz algebra

Another operation ∗ may be defined in a Hurwitz algebra as

$$x * y = x\ y$$

The algebra $(A,\ *)$ is a composition algebra not generally unital, known as a **para-Hurwitz algebra**.[3] In dimensions 4 and 8 these are **para-quaternion**[4] and **para-octonion** algebras.[5]

A para-Hurwitz algebra satisfies[6]

$$(x * y) * x = x * (y * x) = \langle x|x\rangle y\ .$$

Conversely, an algebra with a non-degenerate symmetric bilinear form satisfying this equation is either a para-Hurwitz algebra or an eight-dimensional pseudo-octonion algebra.[7] Similarly, a flexible algebra satisfying

$$\langle xy|xy\rangle = \langle x|x\rangle\langle y|y\rangle$$

is either a Hurwitz algebra, a para-Hurwitz algebra or an eight-dimensional pseudo-octonion algebra.[7]

Euclidean Hurwitz algebras

Main article: Hurwitz's theorem (composition algebras)

If the underlying coefficient field of a Hurwitz algebra is the reals and q is positive-definite, so that $(a, b) = 1/2[q(a + b) - q(a) - q(b)]$ is an inner product, then A is called a **Euclidean Hurwitz algebra**. The Euclidean Hurwitz algebras are precisely the real numbers, the complex numbers, the quaternions and the octonions.[8]

Instances and usage

When the field K is taken to be complex numbers **C**, then the four composition algebras over **C** are **C** itself, the direct sum **C** ⊕ **C** known first as tessarines (1848), the 2×2 complex matrix ring M(2, **C**), and the complex octonions **CO**.

Matrix ring M(2, **C**) has long been an object of interest, first as biquaternions by Hamilton (1853), later in the isomorphic matrix form, and especially as Pauli algebra. Complex octonions have been used in a model of angular momentum.[9]

The squaring function $N(x) = x^2$ on the real number field forms the primordial composition algebra. When the field K is taken to be real numbers **R**, then there are just six other real composition algebras.[10] In two, four, and eight dimensions there are both a "split algebra" and a "division algebra": complex numbers and split-complex numbers, quaternions and split-quaternions, octonions and split-octonions.

The problem has an equivalent formulation in terms of quadratic forms $q(x)$, composability requiring the existence of a bilinear "composition" $z(x, y)$ such that $q(x)\ q(y) = q(z\ (x, y))$. Subsequent proofs have used the Cayley–Dickson construction.

Although neither commutative nor associative, composition algebras have the special property of being alternative algebras, i.e. left and right multiplication preserves squares, a weakened version of associativity. The theory has subsequently been generalized to arbitrary quadratic forms and arbitrary fields.[1]

Hurwitz's theorem implies that multiplicative formulas for sums of squares can only occur in 1, 2, 4 and 8 dimensions, a result originally proved by Hurwitz in 1898. It is a special case of the Hurwitz problem, solved also in Radon (1922).

Subsequent proofs of the restrictions on the dimension have been given by Eckmann (1943) using the representation theory of finite groups and by Lee (1948) and Chevalley (1954) using Clifford algebras. Hurwitz's theorem has been applied in algebraic topology to problems on vector fields on spheres and the homotopy groups of the classical groups[2] and in quantum mechanics to the classification of simple Jordan algebras.[3]

It is routine to check that the real numbers **R**, the complex numbers **C** and the quaternions **H** are examples of associative Euclidean Hurwitz algebras with their standard norms and involutions. There are moreover natural inclusions **R** ⊂ **C** ⊂ **H**.

Analysing such an inclusion leads to the Cayley–Dickson construction, formalized by A.A. Albert. Let A be a Euclidean Hurwitz algebra and B a proper unital subalgebra, so a Euclidean Hurwitz algebra in its own right. Pick a unit vector j in A orthogonal to B. Since $(j, 1) = 0$, it follows that $j^* = -j$ and hence $j^2 = -1$. Let C be subalgebra generated by B and j. It is unital and is again a Euclidean Hurwitz algebra. It satisfies the following **Cayley–Dickson multiplication laws**:

$$C = B \oplus Bj,\ (a+bj)^* = a^* - bj,\ (a+bj)(c+dj) = (ac - d^*b) + (bc^* + da)j.$$

To check this note that B and $B\,j$ are orthogonal, since j is orthogonal to B. If a is in B, then $j\,a = a^*\,j$, since by orthogonal $0 = 2\,(j, a^*) = j\,a - a^*\,j$. The formula for the involution follows. To show that $B \oplus B\,j$ is closed under multiplication note that $Bj = j\,B$. Since $B\,j$ is orthogonal to 1, $(b\,j)^* = -b\,j$.

- $b(c\,j) = (c\,b)j$ since $(b, j) = 0$ so that, for x in A, $(b(c\,j), x) = (b(j\,x), j(c\,j) = -(b(j\,x), c^*) = -(c\,b, (j\,x)^*) = -((c\,b)j, x^*) = ((c\,b)j, x)$.
- $(j\,c)b = j(b\,c)$ taking adjoints above.
- $(b\,j)(c\,j) = -c^*\,b$ since $(b, c\,j) = 0$, so that, for x in A, $((b\,j)(c\,j), x) = -((c\,j)x^*, b\,j) = (b\,x^*, (c\,j)j) = -(c^*\,b, x)$.

Imposing the multiplicativity of the norm on *C* for *a* + *b* *j* and *c* + *d* *j* gives:

$$(\|a\|^2 + \|b\|^2)(\|c\|^2 + \|d\|^2) = \|ac - d^*b\|^2 + \|bc^* + da\|^2,$$

which leads to

$$(ac, d^*b) = (bc^*, da).$$

Hence *d*(*a* *c*) = (*d* *a*)*c*, so that *B must be associative*.

This analysis applies to the inclusion of **R** in **C** and **C** in **H**. Taking **O** = **H** ⊕ **H** with the product and inner product above gives a non - commutative non - associative algebra generated by **J** = (0, 1). This recovers the usual definition of the octonions or Cayley numbers. If *A* is a Euclidean algebra, it must contain **R**. If it is strictly larger then **R**, the argument above shows that it contains **C**. If it is larger than **C**, it contains **H**. If it is larger still, it must contain **O**. But there the process must stop, because **O** is not associative. In fact **H** is not commutative and *a*(*b* *j*) = (*b* *a*) *j* ≠ (*a* *b*)*j* in **O**.[5]

Theorem. The only Euclidean Hurwitz algebras are the real numbers, the complex numbers, the quaternions and the octonions.

Hurwitz quaternion order is a specific order in a quaternion algebra over a suitable number field. The order is of particular importance in Riemann surface theory, in connection with surfaces with maximal symmetry, namely the Hurwitz surfaces.[1] The Hurwitz quaternion order was studied in 1967 by Goro Shimura,[2] but first explicitly described by Noam Elkies in 1998.[3] For an alternative use of the term, see Hurwitz quaternion (both usages are current in the literature).

Definition

Let K be the maximal real subfield of $\mathbb{Q}(\rho)$ where ρ is a 7th-primitive root of unity. The ring of integers of K is $\mathbb{Z}[\eta]$, where the element $\eta = \rho + \bar{\rho}$ can be identified with the positive real $2\cos(\frac{2\pi}{7})$. Let D be the quaternion algebra, or symbol algebra

$$D := (\eta, \eta)_K,$$

so that $i^2 = j^2 = \eta$ and $ij = -ji$ in D.

Also let $\tau = 1 + \eta + \eta^2$ and $j' = \frac{1}{2}(1 + \eta i + \tau j)$. Let

$$\mathcal{Q}_{\mathrm{Hur}} = \mathbb{Z}[\eta][i, j, j'].$$

Then $\mathcal{Q}_{\mathrm{Hur}}$ is a maximal order of D, described explicitly by Noam Elkies.[4]

Hurwitz Number

A number with a continued fraction whose terms are the values of one or more polynomials evaluated on consecutive integers and then interleaved. This property is preserved by Möbius transformations (Gosper 1972, p. 44).

Hurwitz Lattices

A general reference for lattices is [7]. An $\mathcal{H}$-*lattice* is a free finitely generated right $\mathcal{H}$-module with an $\mathcal{H}$-valued bilinear form $\langle\ ,\ \rangle$ satisfying $\overline{\langle x,y\rangle} = \langle y,x\rangle$, $\langle x, y\alpha\rangle = \langle x,y\rangle\alpha$ and $\langle x\alpha, y\rangle = \bar{\alpha}\langle x,y\rangle$, for all x, y in the lattice and α in $\mathcal{H}$. In this article, by a lattice we shall mean an $\mathcal{H}$-lattice, unless otherwise stated. Definite lattices will usually be negative definite. The standard negative definite lattice $\mathcal{H}^n$ has the inner product $\langle x,y\rangle = -\bar{x}_1 y_1 - \cdots - \bar{x}_n y_n$. The indefinite lattice given by $\mathcal{H} \oplus \mathcal{H}$ with inner product $\langle (x,y),(x',y')\rangle = (\bar{x},\bar{y})\left(\begin{smallmatrix} 0 & \bar{p} \\ p & 0\end{smallmatrix}\right)\left(\begin{smallmatrix} x' \\ y'\end{smallmatrix}\right)$ is denoted by H. We call H the *hyperbolic cell*.

Let x be a vector in a $\mathcal{H}$-lattice K. The norm $|x|^2 = \langle x,x\rangle$ is a rational integer. The $\mathbb{Z}$-module K with the quadratic form given by the norm will be called the *real form* of K. For example, the real form of the lattice $\mathcal{H}$ (with the norm multiplied by a factor of 2) is the D_4 root lattice.

The E_8 root lattice can be defined as a sub-lattice of $\mathcal{H}^2$ as $E_8 = \{(x_1,x_2) | x_1 \equiv x_2 \bmod \mathfrak{p}\}$. It has minimal norm -2 and the underlying $\mathbb{Z}$-lattice is the usual E_8 root lattice.

The Leech lattice Λ can be defined as a 6 dimensional negative definite $\mathcal{H}$-lattice with minimal norm -4 whose real form is the usual real Leech lattice. The automorphism group of this lattice was studied by Wilson in [11]. We quote the facts we need from there. Let $\omega = (-1+i+j+k)/2$. The lattice Λ consists of all vectors $(v_\infty, v_0, v_1, v_2, v_3, v_4)$ in $\mathcal{H}^6$ such that $v_2 \equiv v_3 \equiv v_4 \bmod \mathfrak{p}$, $(v_1+v_4)\bar{\omega} + (v_2+v_3)\omega \equiv (v_0+v_1)\omega + (v_2+v_4)\bar{\omega} \equiv 0 \bmod 2$, and $-v_\infty(i+j+k) + v_0 + v_1 + v_2 + v_3 + v_4 \equiv 0 \bmod 2+2i$. The inner product we use is $-1/2$ of the one used in [11], so that the following basis vectors have norm -4. We use the following $\mathcal{H}$-basis for the lattice Λ given in [11] for some computations.

$$bb[1] = [2+2i, 0, 0, 0, 0, 0]$$
$$bb[2] = [2, 2, 0, 0, 0, 0]$$
$$bb[3] = [0, 2, 2, 0, 0, 0]$$
$$bb[4] = [i+j+k, 1, 1, 1, 1, 1]$$
$$bb[5] = [0, 0, 1+k, 1+j, 1+j, 1+k]$$
$$bb[6] = [0, 1+j, 1+j, 1+k, 0, 1+k]$$

Let L be the Lorentzian lattice $L = \Lambda \oplus H \cong 3E_8 \oplus H$. The real form of this lattice is $II_{4,28}$. E_8, Λ, H and L each satisfy $L'p = L$, where L' is the *dual lattice* of L defined by $L' = \{x \in L \otimes \mathbb{H} : \langle x,y\rangle \in \mathcal{H} \forall y \in L\}$.

The lattice of Hurwitz quaternions

The (arithmetic, or field) norm of a Hurwitz quaternion, given by $a^2 + b^2 + c^2 + d^2$, is always an integer. By a theorem of Lagrange every nonnegative integer can be written as a sum of at most four squares. Thus, every nonnegative integer is the norm of some Lipschitz (or Hurwitz) quaternion. More precisely, the number $c(n)$ of Hurwitz quaternions of given positive norm n is 24 times the sum of the odd divisors of n. The generating function of the numbers $c(n)$ is given by the level 2 weight 2 modular form

$$2E_2(2\tau) - E_2(\tau) = \sum_n c(n)q^n = 1 + 24q + 24q^2 + 96q^3 + 24q^4 + 144q^5 + \cdots$$

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where

$$q = e^{2\pi i\tau}$$

and

$$E_2(\tau) = 1 - 24\sum_n \sigma_1(n)q^n$$

is the weight 2 level 1 Eisenstein series (which is a quasimodular form) and $\sigma_1(n)$ is the sum of the divisors of n.

From Stephanie Vance

2. Special Properties of Hurwitz lattices

A Hurwitz lattice is a lattice Λ in $\mathbb{H}^m$ that has the extra algebraic structure of a Hurwitz module. In particular, $\Lambda = \mathcal{H}\Lambda = \{\lambda u : \lambda \in \mathcal{H} \text{ and } u \in \Lambda\}$ where $\mathcal{H} = \mathbb{Z}[i, j, \frac{1+i+j+k}{2}]$ is the ring of Hurwitz integers, a maximal order in the rational positive definite quaternion algebra $\left(\frac{-1,-1}{\mathbb{Q}}\right) = \{a+bi+cj+dk \in \mathbb{H} : a, b, c, d \in \mathbb{Q}\}$. As an $\mathcal{H}$-module, every Hurwitz lattice Λ in $\mathbb{H}^m$ is necessarily free and is generated by a quaternionic basis for $\mathbb{H}^m$. (This is due to the fact that $\mathcal{H}$ has class number one, a special property not satisfied by maximal orders in general.) Furthermore we prove below (see Theorem 2.2) that Λ also contains, but is not necessarily generated by, a special quaternionic basis for $\mathbb{H}^m$ with prescribed lengths $\min_1(\Lambda), \ldots, \min_m(\Lambda)$ defined as follows. (See [Ma] and [Re] for additional properties of properties of Hurwitz lattices, and more generally lattices over maximal orders in positive definite $\mathbb{Q}$-algebras.)

Note that g satisfies the hypothesis of Theorem 2.3 (g is bounded since f is bounded and vanishes off a compact set) and hence there exists a Hurwitz lattice Λ in $\mathbb{H}^m$ with $\det(\Lambda) = 1$ such that

$$
\begin{aligned}
\sum_{\substack{u\in\Lambda \\ u\neq 0}} g(u) &< \int_{\mathbb{H}^m} g(z)\ dz + \frac{\varepsilon}{2} \\
&= \int_{\mathbb{H}^m \setminus \delta B_n} \sum_{k=1}^{\infty} \mu(k) f(kz)\ dz + M\delta^{4m} V_{4m} + \frac{\varepsilon}{2} \\
&< \sum_{k=1}^{\infty} \frac{\mu(k)}{k^{4m}} \int_{\mathbb{H}^m \setminus \delta B_n} f(z)\ dz + \varepsilon \\
&\leq \frac{1}{\zeta(4m)} \int_{\mathbb{H}^m} f(z)\ dz + \varepsilon.
\end{aligned}
$$

Now for the set of lattice vectors $A = \{u \in \Lambda : ||u|| \geq \delta\}$, let A' denote the vectors in A which cannot be written as a positive integer multiple of another vector contained in A. (Note that $\{u \in \Lambda' : ||u|| \geq \delta\} \subseteq A'$.) Using the properties of the Möbius function, the fact that f is non-negative and $g(z) \geq f(z)$ whenever $||z|| < \delta$, we obtain

$$\begin{aligned}
\sum_{u\in\Lambda\setminus\{0\}} g(u) &= \sum_{\substack{u\in\Lambda \\ 0<||u||<\delta}} g(u) + \sum_{u\in A'}\sum_{s=1}^{\infty} g(su) \\
&\geq \sum_{\substack{u\in\Lambda \\ 0<||u||<\delta}} f(u) + \sum_{u\in A'}\sum_{s=1}^{\infty}\sum_{k=1}^{\infty} \mu(k) f(ksu) \\
&\geq \sum_{\substack{u\in\Lambda \\ 0<||u||<\delta}} f(u) + \sum_{u\in A'}\sum_{t=1}^{\infty}\sum_{k|t} \mu(k) f(tu) \\
&= \sum_{\substack{u\in\Lambda \\ 0<||u||<\delta}} f(u) + \sum_{u\in A'} f(u) \\
&\geq \sum_{u\in\Lambda'} f(u).
\end{aligned}$$

4. Remarks

In addition to the lower bound (1.1) proven in the previous section, Corollary 2.4 can also be used to obtain a lower bound for the optimal density of more general Hurwitz packings in $\mathbb{H}^m$ consisting of copies of an $\mathcal{H}^\times$-invariant convex body S translated by the vectors of a Hurwitz lattice and such that the interiors of the copies of S are disjoint. (The term $\mathcal{H}^\times$-invariant convex body is used here to describe a compact convex subset of $\mathbb{H}^m$ with non-empty interior and that is invariant under scalar multiplication by the Hurwitz integer unit group $\mathcal{H}^\times$. Note that an $\mathcal{H}^\times$-invariant convex body in $\mathbb{H}^m$ is the Hurwitz analogue of a 0-symmetric convex body in $\mathbb{R}^n$, i.e., a convex body invariant under multiplication by the units $\{-1, 1\}$ in $\mathbb{Z}$.) This type of lower bound is similar to the lower bound given in the Minkowski-Hlawka Theorem for 0-symmetric convex bodies and is proved using similar techniques; see the proof of the Minkowski-Hlawka theorem in [GL] or [Z]. The extra assumption that the convex body be $\mathcal{H}^\times$-invariant allows one to improve on the lower bound in the Minkowski-Hlawka theorem by a factor of 12.

Theorem 4.1. *Let S be an $\mathcal{H}^\times$-invariant convex body in $\mathbb{H}^m$. If $m \geq 2$ then there exists a Hurwitz lattice packing of translated copies of S such that the density of the packing is at least*

$$\frac{3\zeta(4m)}{2^{4m-3}}.$$

Proof. Without loss of generality assume that $\mathrm{vol}(S) = 24\zeta(4m)$ (otherwise replace S by a dilate having volume $24\zeta(4m)$) and for $\varepsilon > 0$, choose $\delta \in (0,1)$ such that $\mathrm{vol}(\delta S) = (24 - \varepsilon)\zeta(4m)$. Let ρ denote the characteristic function of δS and let Λ be a Hurwitz lattice in $\mathbb{H}^m$ obtained by applying Corollary 2.4 with this ε and ρ so that $\det(\Lambda) = 1$ and

$$\sum_{u \in \Lambda'} \rho(u) < \frac{1}{\zeta(4m)} \int_{\mathbb{H}^m} \rho(z)\ dz + \varepsilon = 24.$$

Observe that if δS contains any non-zero vectors of Λ' then it must contain at least 24 since every Hurwitz lattice has at least 24 minimal vectors and δS is $\mathcal{H}^\times$-invariant (the unit group $\mathbb{H}^\times$ has size 24). In particular, $\sum_{u \in \Lambda'} \rho(u)$ is either zero or greater than or equal to 24. Thus the above inequality implies that $\sum_{u \in \Lambda'} \rho(u) = 0$ and hence Λ intersects δS only at the origin.

Now since $\varepsilon > 0$ is arbitrary, as a consequence of Mahler's compactness theorem, there exists a Hurwitz lattice with determinant one in $\mathbb{H}^m$ that intersects S only at the origin. Such a Hurwitz lattice can be used to obtain a packing of translates of $\frac{1}{2}S$ and the density of this packing is equal to $\frac{3\zeta(4m)}{2^{4m-3}}$. $\square$

Conclusion

The QMDJ model is based on mathematical constructs, including the Clifford Clock, their Spinors, the 3 x 3 Magic Square and the Hurwitz Quaternions. A forthcoming paper by the author will show the fundamental importance of the ten Heavenly Stems and twelve Earth Branches in the combinatorial mathematics of our Universe, and these form an additional mathematical construct for QMDJ. The predictive mechanism derives from Bott Periodicity, Pisano Periodicity (Fibonacci Periodicity) and from the nature of the 24 Hurwitz Quaternions, which describe the geography of time, at least from a contemporary western perspective.

The QMDJ model divides one year into 24 seasons, akin to the 24 Hurwitz Quaternions. These 24 seasons are then divided into two sets of 12 seasons each, dividing the year into a Yin half and a Yang half, evenly at the solstices. Yin Dun begins at Summer Solstice, the period during Yin decline in the Northern Hemisphere, while Yang Dun begins at Winter Solstice, the period during which Yang Qi gradually builds in the Northern Hemisphere.

Next, each season is divided by thirds again, into a numbered “Ju” or five – day period, with each season matched with three of the Ju. These five – day periods consist of 60 hours, a complete run of the 60 Jia Zi double – hour cycle.

In terms of mathematical physics, this breaking down of seasons into 3 Ju matches the Pythagorean Triplets which form the 24 Hurwitz Quaternions.

Thus, the 60 Jia Zi double – hour cycle conforms with the natural Pisano (Fibonacci) Periodicity, which forms the natural limits to growth. In this way, the QMDJ Time Model treats Time itself as if it were a physical entity, thus matching the natural growth cycle with this temporal cycle. Since the model is based on how nature works in reality, the QMDJ Time Model is able to form accurate predictions of real – time events. Bott Periodicity and Pisano (Fibonacci) Periodicity extend the applicability of the model to most events on Earth.

If Time forms around Hurwitz Quaternions, then the shape and form of Time may appear like the Hurwitz Lattice, as described above by Stephanie Vance.

Recent Scholarship

Theorem 4.18 *Let $q \in \mathbb{A}$ be a non-unit Hurwitz quaternion and $q = q_1 \cdots q_n = q'_1 \cdots q'_n$ be two factorizations into irreducible factors. Then $q_1 \cdots q_n$ and $q'_1 \cdots q'_n$ are related by a series of unit-migrations, meta-commutations, and re-combinations, i.e. one is obtained from the other by applying a series of the above operations.*

Proof. First we exhibit the following diagram, where Φ (the two Φ's are in general distinct, but we use the same letter for simplicity) means a series of steps consisting of meta-commutations leading to the the standard models $q_1^{(1)} \cdots q_n^{(1)}$ and $q_1^{(2)} \cdots q_n^{(2)}$, where the prime factors are ordered.

$$\begin{tikzcd}
q_1 \cdots q_n \arrow[r, "?"] \arrow[d, "\Phi"'] & q'_1 \cdots q'_n \arrow[d, "\Phi"'] \\
q_1^{(1)} \cdots q_n^{(1)} \arrow[r, "\Psi"] & q_1^{(2)} \cdots q_n^{(2)}
\end{tikzcd}$$

We need to specify the process of Ψ: It consists of two stages. First by the Corollary 4.15, there exists a unit-migration such that the two associated factorizations of q in blocks (of factors whose norms are of the form $p_i^{n_i}$, where the rational primes p_i and p_j are distinct if $i \neq j$) coincide. Therefore as the second stage of Ψ it remains to show that any two factorizations of a p-pure quaternion are related by the two series of processes, namely unit-migrations and recombinations, which we will do in the following proposition. Observe that all the processes are invertible, hence the first factorization reaches the second one by composing Φ, Ψ and Φ^{-1}.

Example. Consider the two factorizations (A) and (B) of $9 - 3i - 15j$:

$$\left(\frac{-1+i+3j-k}{2}\right) \cdot (2+i) \cdot \left(\frac{-1+3i+j+k}{2}\right) \cdot (2+i+j+k) \qquad \text{(A)}$$

and

$$(1+i+j) \cdot (1-i-j) \cdot \left(\frac{-3+i-3j+k}{2}\right) \cdot \left(\frac{1-i+5j+k}{2}\right) \qquad \text{(B).}$$

Starting from (A), we describe a process to reach (B).

Step 1. Meta-commutation of the middle two factors:

$$\left(\frac{-1+i+3j-k}{2}\right)\cdot(2+i)\cdot\left(\frac{-1+3i+j+k}{2}\right)\cdot(2+i+j+k)$$
$$=\left(\frac{-1+i+3j-k}{2}\right)\cdot\left(\frac{-1+i-j-3k}{2}\right)\cdot(j-2k)\cdot(2+i+j+k).$$

Step 2. Unit-migration with $\epsilon_1 = 1, \epsilon_2 = -i$, and $\epsilon_3 = \frac{1-i+j+k}{2}$:

$$\left(\frac{-1+i+3j-k}{2}\right)\epsilon_1^{-1}\cdot\epsilon_1\left(\frac{-1+i-j-3k}{2}\right)\epsilon_2^{-1}\cdot\epsilon_2(j-2k)\epsilon_3^{-1}\cdot\epsilon_3(2+i+j+k)$$

$$=\left(\frac{-1+i+3j-k}{2}\right)\cdot\left(\frac{-1-i-3j+k}{2}\right)\cdot\left(\frac{-3+i-3j+k}{2}\right)\cdot\left(\frac{1-i+5j+k}{2}\right)$$

Step 3. Recombination $\left(\frac{-1+i+3j-k}{2}\right)\cdot\left(\frac{-1-i-3j+k}{2}\right)=(1+i+j)(1-i-j)=3$:

$$\left(\frac{-1+i+3j-k}{2}\right)\cdot\left(\frac{-1-i-3j+k}{2}\right)\cdot\left(\frac{-3+i-3j+k}{2}\right)\cdot\left(\frac{1-i+5j+k}{2}\right)$$

$$=(1+i+j)\cdot(1-i-j)\cdot\left(\frac{-3+i-3j+k}{2}\right)\cdot\left(\frac{1-i+5j+k}{2}\right).$$

The above suggests a mathematical method for factoring the Hurwitz Quaternions. Compare with Geoffrey Dixon on his 7 Stones website.

However, if we want to look at all possible prime factorizations of x, then we must allow for changes in the order of the primes $p_1, \ldots, p_n$. If P and Q are primes lying over distinct rational primes p and q, then PQ has a unique (up to unit migration) factorization $Q'P'$ modeled on qp. This process of switching two adjacent primes in the model was named *metacommutation* by Conway and Smith in their account of unique factorization for Hurwitz integers (see Chapter 5 of [CS]). They prove that the prime factorization of an arbitrary nonzero Hurwitz integer is unique up to metacommutation, unit migration, and *recombination*: the process of replacing PP^σ with $\tilde{P}\tilde{P}^\sigma$, where P and $\tilde{P}$ are primes of the same norm. Another exposition of this theorem can be found in [CP].

Conway and Smith comment that the metacommutation problem of determining Q' and P' given P and Q does not seem to be addressed in the literature [CS, p. 61]. Their proof that Q' and P' exist yields an efficient method for computing them via the Euclidean algorithm, but it provides little insight into the properties of the metacommutation map.

These authors go a step beyond Conway and Smith to factor the Hurwitz Quaternions.

Theorem 6.1. *Let p and q be distinct primes, with p odd. The sign of the permutation induced on the $p+1$ Hurwitz primes of norm p (up to left multiplication by units) through metacommutation by a prime of norm q is $\left(\frac{q}{p}\right)$.*

Finally, if the present author has followed the logic correctly, Hurwitz Quaternions lead to Shimura Curves, examples of which are:

Let F be a totally real number field and D a quaternion division algebra over F. The multiplicative group $D^{\times}$ gives rise to a canonical Shimura variety. Its dimension d is the number of infinite places over which D splits. In particular, if $d = 1$ (for example, if $F = \mathbf{Q}$ and $D \otimes \mathbf{R} \cong M_2(\mathbf{R})$), fixing a sufficiently small arithmetic subgroup of $D^{\times}$, one gets a Shimura curve, and curves arising from this construction are already compact (i.e. projective).

Some examples of Shimura curves with explicitly known equations are given by the Hurwitz curves of low genus:

- Klein quartic (genus 3)
- Macbeath surface (genus 7)
- First Hurwitz triplet (genus 14)

and by the Fermat curve of degree 7.[1]

Other examples of Shimura varieties include Picard modular surfaces and Hilbert–Blumenthal varieties.

These scholars have realized that much can be learned from studying the various factors of the Hurwitz Quaternions, which form an isomorphic relationship with the 24 Seasons of China's traditional calendar, as well as with the binary Ying and Yan Dun halves of the year, and the 60 double hour periods of five days which form the temporal scheme of QMDJ. Time may share the geography of such Shimura curves as the Klein Quartic and the First Hurwitz Triplet.

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Laplacian growth in a channel and Hurwitz numbers

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December 2012

IMPROVED SPHERE PACKING LOWER BOUNDS FROM HURWITZ LATTICES

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Reflection group of the quaternionic Lorentzian Leech lattice

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Factorization of Hurwitz Quaternions

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METACOMMUTATION OF HURWITZ PRIMES

HENRY COHN AND ABHINAV KUMAR

Wiseman & Ye, A Practical Dictionary of Chinese Medicine, 2000, Beijing.

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Dedication

So let us dedicate ourselves to what the Greeks wrote so long ago:
to tame the savageness of man and make gentle the life of this world.

Robert Francis Kennedy