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Address: B-dul Revoluției nr. 77, 310130 Arad, Romania, P.O. BOX 2/158 AR
Telephone:+40-257-283010
Fax:+40-257-280070
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Contents

Stability issues in infinite dimensional spaces

Codruţa Stoica* Diana Pătraşcu Borlea†

Abstract

The aim of this paper is to define various concepts of stability for cocycles over non-autonomous dynamical systems, as generalizations of the skew-evolution semiflows, such as exponential stability, (α, β)–exponential stability, and two more general concepts, the (h, k)–stability and the (h, k)–integral stability. Connections between these notions are also given.

Mathematics Subject Classification: 34D05, 93D20

Keywords: Dynamical systems, skew-evolution cocycles, exponential stability, (α, β)–exponential stability, (h, k)–stability, (h, k)–integral stability

1 Preliminaries

In recent years, several concepts of the control theory, as stability, stabilizability, controllability or observability were refined, based on the fact that the dynamical systems which describe processes from engineering, physics, biology or economics are extremely complex and the identification of the proper mathematical models is difficult.

In the qualitative theory of evolution equations, the exponential stability and instability are two of the most important asymptotic properties, approached lately from various perspectives. Recently, other asymptotic behaviors were studied, the dichotomy and the trichotomy, by generalizing the techniques used in the investigation of the stability.

The stability theory puts into discussion the stability of solutions of differential equations and of trajectories of dynamical systems under small perturbations of initial conditions. Many parts of the qualitative theory of differential equations and dynamical systems deal with asymptotic properties of solutions/ trajectories – what happens with the system after a long period of time. If an orbit is well understood, it is natural to ask whether a small change in the initial condition will lead to similar behavior.

The study of the behaviors of the evolution equations by means of associated operator families has allowed to obtain answers to open problems by involving techniques of functional analysis and operator theory.

*`codruta.stoica@uav.ro` Department of Mathematics and Computer Science, "Aurel Vlaicu" University of Arad, Romania

†`dianab268@yahoo.com` Department of Mathematics, West University of Timişoara, Romania

Remarkable results of the stability theory are due to J.L. Daleckii and M.G. Krein (see [1]) and J.L. Massera and J.J. Schäffer (see [2]).

The concept of evolution operators arises naturally from the theory of well-posed non-autonomous Cauchy problems, while the notion of linear skew-product semiflows is involved when considering the linearization along an invariant manifold of a dynamical system generated by a nonlinear differential equation. The skew-evolution semiflows, defined in [3], are generalizations for both evolution operators and skew-product semiflows. The applicability of this notion was emphasized by P. Viet Hai in [6] and [7], and by T. Yue, X.Q. Song, D.Q. Li in [8].

We consider in this paper the case of cocycles over multivalued dynamical systems (see [5]). We define various concepts of stability, characterizations for these notions, as well as connections between them.

2 Definitions. Examples

$X=(X,d_X)$ denotes a complete metric space, $\mathcal{P}(X)$ the set of all nonempty subsets of X, V a Banach space with the dual V^*, $\mathcal{B}(V)$ the space of all bounded linear operators from V into itself. id_X is the identity map on X, I the identity operator on V and $Y=X\times V$. We consider the sets $\mathcal{E}=\{f:\mathbb{R}_+\to[1,\infty)\big|\ \exists\alpha\in\mathbb{R}_+ \text{ such that } f(t)=e^{\alpha t}\}$ and $T=\big\{(t,s)\in\mathbb{R}^2_+,\ t\geq s\big\}$.

Definition 2.1 A multivalued map $\widetilde{u}:T\times X\to\mathcal{P}(X)$ with

(s_1) $\widetilde{u}(t,t,\cdot)=id_X,\ \forall(t,x)\in\mathbb{R}_+\times X$;

(s_2) $\widetilde{u}(t,t_0,x)\subseteq\widetilde{u}(t,s,\widetilde{u}(s,t_0,x)),\forall(t,s),(s,t_0)\in T,x\in X$ is a *generalized multivalued non–autonomous dynamical system* on X.

Remark 2.2 In what follows we will consider the case of a mapping $u:T\times X\to X$ with the properties

(s_1') $u(t,t,x)=x,\ \forall(t,x)\in\mathbb{R}_+\times X$;

(s_2') $u(t,t_0,x)=u(t,s,u(s,t_0,x)),\forall(t,s),(s,t_0)\in T,x\in X$, called the *semiflow* associated to the generalized multivalued non–autonomous dynamical system on X.

Definition 2.3 A mapping $U:T\times X\to\mathcal{B}(V)$ which satisfies

(c_1) $U(t,t,x)=I$, $\forall(t,x)\in\mathbb{R}_+\times X$;

(c_2) $U(t,s,u(s,t_0,x))U(s,t_0,x)=U(t,t_0,x),\forall(t,s),(s,t_0)\in T,x\in X$, is a *skew-evolution cocycle* over u.

Example 2.4 If U is a skew-evolution cocycle over u and $\lambda\in\mathbb{R}$, then $U_\lambda:T\times X\to\mathcal{B}(V)$, $U_\lambda(t,t_0,x)=e^{\lambda(t-t_0)}U(t,t_0,x)$ is the λ–shifted skew-evolution cocycle over u.

Definition 2.5 A skew-evolution cocycle U has *ω–growth* if there exists $\omega:\mathbb{R}_+\to[1,\infty)$, a nondecreasing function, with $\lim\limits_{t\to\infty}\omega(t)=\infty$, that satisfies $\|U(t,t_0,x)v\|\leq\omega(t-s)\,\|U(s,t_0,x)v\|$, for all $(t,s),(s,t_0)\in T$, $(x,v)\in Y$.

Remark 2.6 (i) If a skew-evolution cocycle U has ω–growth, then $U_{-\lambda}$, $\lambda>0$, has also ω–growth.

(ii) The ω–growth is equivalent with the exponential growth (see [4]).

Definition 2.7 A skew-evolution cocycle U is *$*$-strongly measurable* if for every $(t,t_0,x,v^*)\in T\times X\times V^*$ the mapping $s\mapsto \|U(t,s,u(s,t_0,x))^*v^*\|$ is measurable on $[t_0,t]$.

3 Stability types

3.1 Exponential stability

Definition 3.1 A skew-evolution cocycle U is *exponentially stable* if there exist a mapping $N:\mathbb{R}_+\to[1,\infty)$ and a constant $\nu>0$ such that the relation $e^{\nu(t-s)}\|U(t,t_0,x)v\|\le N(s)\|U(s,t_0,x)v\|$ holds for all $(t,s),(s,t_0)\in T$, $(x,v)\in Y$.

Proposition 3.2 *A skew-evolution cocycle U has ω–growth if and only if there exists $\lambda>0$ such that the $-\lambda$-shifted skew-evolution cocycle $U_{-\lambda}$ is exponentially stable.*

Proof. *Necessity.* As U has ω–growth, according to Remark 2.6, there exist $M\ge1$ and $\omega>0$ such that $e^{-\omega(t-s)}\|U(s,t_0,x)v\|\le M\|U(t,t_0,x)v\|$, for all $(t,s),(s,t_0)\in T$, $(x,v)\in Y$. If we consider $\lambda=2\omega>0$ we obtain

$$\|U_{-\lambda}(t,t_0,x)v\|=e^{-\lambda(t-t_0)}\|U(t,t_0,x)v\|\le$$

$$\le Me^{-\lambda(t-t_0)}e^{\omega(t-s)}\|U(s,t_0,x)v\|=$$

$$=Me^{-\lambda(t-t_0)}e^{\omega(t-s)}\|U_{-\lambda}(s,t_0,x)v\|e^{\lambda(s-t_0)}=$$

$$=Me^{-\omega(t-s)}\|U_{-\lambda}(s,t_0,x)v\|,$$

for all $(t,s),(s,t_0)\in T$ and $(x,v)\in Y$, which shows that $U_{-\lambda}$ is exponentially stable.

Sufficiency. As $U_{-\lambda}$ is exponentially stable, there exist $N:\mathbb{R}_+\to[1,\infty)$ and $\nu>0$ such that $e^{\nu(t-s)}\|U(t,t_0,x)v\|\le N(s)\|U(s,t_0,x)v\|$, for all $(t,s),(s,t_0)\in T$, $(x,v)\in Y$. Further, we obtain

$$\|U(t,t_0,x)v\|=e^{\lambda(t-t_0)}\|U_{-\lambda}(t,t_0,x)v\|\le$$

$$\le N(s)e^{\lambda(t-t_0)}e^{-\nu(t-s)}\|U_{-\lambda}(s,t_0,x)v\|=N(s)e^{\lambda(t-s)}e^{-\nu(t-s)}\|U(s,t_0,x)v\|,$$

for all $(t,s),(s,t_0)\in T$ and $(x,v)\in Y$. We denote

$$\gamma=\begin{cases}\lambda-\nu, & \text{if } \lambda>\nu\\ 1, & \text{if } \lambda\le\nu,\end{cases}$$

and we define $\omega:\mathbb{R}_+\to[1,\infty)$ by $\omega(\tau)=N(\tau)e^{\tau}$. Hence, U has ω–growth. □

3.2 (α,β)–exponential stability

Definition 3.3 A skew-evolution cocycle U is *(α,β)–exponentially stable* if there exist some constants $N\ge1$, $\alpha,\beta>0$ such that following relation $e^{\alpha(t-s)}\|U(t,t_0,x)v\|\le Ne^{\beta s}\|U(s,t_0,x)v\|$ holds for all $(t,s),(s,t_0)\in T$, $(x,v)\in Y$.

Proposition 3.4 *A skew-evolution cocycle U with ω–growth is exponentially stable if and only if there exists a constant $\lambda > 0$ such that the $-\lambda$-shifted skew-evolution cocycle $U_{-\lambda}$ is (α, β)–exponentially stable.*

Proof. It is analogous to the proof of Proposition 3.2. □

3.3 (h, k)–stability and (h, k)–integral stability

Definition 3.5 A skew-evolution cocycle U is said to be (h, k)*–stable* if there exist $N \geq 1$ and two continuous mappings $h, k : \mathbb{R}_+ \to \mathbb{R}_+^*$ such that

$$h(t-s)\, \|U(t, t_0, x)v\| \leq Nk(s)\, \|U(s, t_0, x)v\|,$$

for all $(t, s), (s, t_0) \in T$, $(x, v) \in Y$.

Definition 3.6 A skew-evolution cocycle U is (h, k)*–integrally stable* if there exist a constant $D \geq 1$ and two continuous mappings $h, k : \mathbb{R}_+ \to \mathbb{R}_+^*$, where h is nondecreasing with the property $h(s + t) \leq h(s)h(t)$, for all $s, t \in \mathbb{R}_+$, such that the relation

$$\int_s^t h(\tau - s)\, \|U(\tau, t_0, x)v\|\, d\tau \leq Dk(s)\, \|U(s, t_0, x)v\|$$

holds for all $(t, s), (s, t_0) \in T$, $(x, v) \in Y$.

3.4 Connections

3.4.1 (h, k)–stability vs. exponential stability and (α, β)–exponential stability

Remark 3.7 (α, β)–exponentially stable $\underset{\not\Longleftarrow}{\Longrightarrow}$ exponentially stable ([4]).

Remark 3.8 1) If a skew-evolution cocycle U is (h, k)–stable and $h \in \mathcal{E}$, then U is exponentially stable;

2) If a skew-evolution cocycle U is (h, k)–stable and $h, k \in \mathcal{E}$ are given by $t \mapsto e^{\alpha t}$ respectively $t \mapsto Me^{\beta t}$, $M \geq 1$ and $\beta > \alpha$, then U is (α, β)–exponentially stable.

3.4.2 (h, k)–integral stability vs. (h, k)–stability

Theorem 3.9 *A strongly measurable skew-evolution cocycle U with ω–growth is exponentially stable if and only if there exists a constant $\lambda > 0$ such that the $-\lambda$-shifted skew-evolution cocycle $U_{-\lambda}$ is integrally stable.*

Proof. *Necessity.* The existence of a mapping $N : \mathbb{R}_+ \to \mathbb{R}_+^*$ and of a constant $\nu > 0$ such that $\|U(t, t_0, x)v\| \leq N(s)e^{-\nu(t-s)}\, \|U(s, t_0, x)v\|$, for all $(t, s), (s, t_0) \in T$ and $(x, v) \in Y$, is assured by Definition 3.1. We consider $\lambda = \dfrac{\nu}{2} > 0$. We obtain successively

$$\int_{t_0}^{\infty} e^{\frac{\nu}{2}(s-t_0)}\, \|U(s, t_0, x)v\|\, ds \leq N(t_0) \int_{t_0}^{\infty} e^{-\frac{\nu}{2}(s-t_0)}\, \|v\|\, ds =$$

$$= N(t_0) \int_0^{\infty} e^{-\frac{\nu}{2}\tau}\, \|v\|\, d\tau \leq \widetilde{N}(t_0)\, \|v\|,$$

for all $(t_0,x,v)\in\mathbb{R}_+\times Y$, where we have denoted $\widetilde{N}(t_0)=\dfrac{2}{\nu}N(t_0)$, which shows that $U_{-\lambda}$ is integrally stable.

Sufficiency. The $-\lambda$-shifted skew-evolution cocycle $U_{-\lambda}$ with $\lambda>0$ is integrally stable, with ω–growth, hence, it is stable. Then there exists a mapping $\widetilde{M}:\mathbb{R}_+\to\mathbb{R}_+^*$ such that $e^{\alpha(t-t_0)}\left\|U(t,t_0,x)v\right\|\leq\widetilde{M}(t_0)\left\|v\right\|$, for all $(t,t_0)\in T$ and $(x,v)\in Y$, which proves the exponential stability of the skew-evolution cocycle U and ends the proof. □

Theorem 3.10 *A (h,k)–integrally stable skew-evolution cocycle U with ω–growth is (h,k)–stable.*

Proof. According to Definition 2.5 and Remark 2.6, there exist some constants $M\geq 1$ and $\gamma>0$ such that $\left\|U(t,t_0,x)v\right\|\leq Me^{\gamma(t-s)}\left\|U(s,t_0,x)v\right\|$, for all $(t,s),(s,t_0)\in T$, $(x,v)\in Y$.

Let $t\in[s,s+1)$. We have $h(t-s)\left\|U(t,t_0,x)v\right\|\leq Me^{\gamma}h(1)\left\|U(s,t_0,x)v\right\|$, for all $(x,v)\in Y$.

Let now $s\in[t-1,t]$. We obtain successively

$$h(t-s)\left\|U(t,t_0,x)v\right\|=\int_{t-1}^{t}h(t-s)\left\|U(t,t_0,x)v\right\|d\tau\leq$$

$$\leq\int_{t-1}^{t}h(t-\tau)h(\tau-s)\left\|U(t,\tau,u(\tau,t_0,x))U(s,t_0,x)v\right\|d\tau\leq$$

$$\leq Me^{\gamma}h(1)\int_{s}^{t}h(\tau-s)\left\|U(\tau,t_0,x)v\right\|d\tau\leq DMe^{\gamma}h(1)k(s)\left\|U(s,t_0,x)v\right\|,$$

for all $(x,v)\in Y$. Hence, U is (h,k)–stable. □

4 Conclusions

The dynamical systems, used to model processes in computer science, biology, economics, physics, chemistry and many other fields, deal with the analysis of the long-time behavior. One of the central interests in the asymptotic behavior of dynamical systems is to find conditions for their solutions to be stable, unstable or exponential dichotomic. The skew-evolution cocyles generalize the notions of semigroups of operators, evolution families, evolution operators or skew-product semiflows, being suitable to approach the study of the asymptotic properties of the evolution equations from a non-uniform point of view.

The generality of the concepts presented in this paper is given by the fact that in the definitions of the ω–growth, of the (h,k)–stability and (h,k)–integral stability, exponentials are not necessarily involved.

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Luhn prime numbers

Octavian Cira* Florentin Smarandache†

Abstract

The first prime number with the special property that its addition with reversal gives as result a prime number too is 299. The prime numbers with this property will be called *Luhn prime numbers*. In this article we intend to present a performing algorithm for determining the *Luhn prime numbers*. Using the presented algorithm all the 50598 *Luhn prime numbers* have been, for p prime smaller than $2 \cdot 10^7$.

1 Introduction

The number 299 is the smallest prime number that added with his reverse gives as result a prime number, too. As $1151 = 229 + 922$ is prime.

The first that noted this special property the number 229 has, was Norman Luhn (after 9 February 1999), on the *Prime Curios* website [15]. The prime numbers with this property will be later called *Luhn prime numbers*.

In the *Whats Special About This Number?* list [10], a list that contains all the numbers between 1 and 9999; beside the number 229 is mentioned that his most important property is that, adding with reversal the resulting number is prime too.

The *On-Line Encyclopedia of Integer Sequences*, [9, A061783], presents a list 1000 *Luhn prime numbers*. We owe this list to Harry J. Smith, since 28 July 2009. On the same website it is mentioned that Harvey P. Dale published on 27 November 2010 a list that contains 3000 *Luhn prime numbers* and Bruno Berselli published on 5 August 2013 a list that contains 2400 *Luhn prime numbers*.

2 Smarandache's function

The function $\mu : \mathbb{N}^* \to \mathbb{N}^*$, $\mu(n) = m$, where m is the smallest natural number with the property that n divides $m!$ (or $m!$ is a multiple of n) is know in the specialty literature as Smarandache's function, [5, 6, 12]. The values resulting from $n = 1, 2, \ldots, 18$ are: 1, 2, 3, 4, 5, 3, 7, 4, 6, 5, 11, 4, 13, 7, 5, 6, 17, 6. These

*octavian.cira@uav.ro Department of Mathematics and Computer Science, "Aurel Vlaicu" University of Arad, România

†fsmarandache@gmail.com Mathematics & Science Department, University of New Mexico, USA

values were obtained with an algorithm that results from μ's definition. The program using this algorithm cannot be used for $n \geq 19$ because the numbers 19!, 20!, ... are numbers which exceed the 17 decimal digits limit and the classic computing model (without the arbitrary precisions arithmetic [13]) will generate errors due to the way numbers are represented in the computers memory.

3 Kempner's algorithm

Kempner created an algorithm to calculate $\mu(n)$ using classical factorization $n = p_1^{p_1} \cdot p_2^{p_2} \cdots p_s^{p_s}$, prime number and the generalized numeration base $(\alpha_i)_{[p_i]}$, for $i = \overline{1, s}$, [1]. Partial solutions to the algorithm for $\mu(n)$'s calculation have been given earlier by Lucas and Neuberg, [12].

Remark 3.1 *If $n \in \mathbb{N}^*$, n can be decomposed in a product of prime numbers $n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_s^{\alpha_s}$, were p_i are prime numbers so that $p_1 < p_2 < \ldots < p_s$, and $s \geq 1$, thus Kempner's algorithm for calculating the μ function is.*

$$\mu(n) = \max\left\{p_1 \cdot \left(\alpha_{1_{[p_1]}}\right)_{(p_1)},\ p_2 \cdot \left(\alpha_{2_{[p_2]}}\right)_{(p_2)}, \ldots,\ p_s \cdot \left(\alpha_{s_{[p_s]}}\right)_{(p_s)}\right\} ,$$

*where by $\left(\alpha_{[p]}\right)_{(p)}$ we understand that α is "*written*" in the numeration base p (noted $\alpha_{[p]}$) and it is "*read*" in the p numeration base (noted $\beta_{(p)}$, were $\beta = \alpha_{[p]}$),* [6, p. 39].

4 Programs

The list of prime numbers was generated by a program that uses the Sieve of Eratosthenes the linear version of Pritchard, [3], which is the fastest algorithm to generate prime numbers until the limit of L, if $L \leq 10^8$. The list of prime numbers until to $2 \cdot 10^7$ is generated in about 5 seconds. For the limit $L > 10^8$ the fastest algorithm for generating the prime numbers is the Sieve of Atkin, [7].

Program 4.1 *The Program for the Sieve of Eratosthenes, the linear version of Pritchard using minimal memory space is:*

$$CEPbm(L) := \left|\begin{array}{l} \lambda \leftarrow floor\left(\frac{L}{2}\right) \\ for\ k \in 1..\lambda \\ \quad is\_prime_k \leftarrow 1 \\ prime \leftarrow (2\ 3\ 5\ 7)^T \\ i \leftarrow last(prime) + 1 \\ for\ j \in 4, 7..\lambda \\ \quad is\_prime_j \leftarrow 0 \\ k \leftarrow 3 \\ s \leftarrow (prime_{k-1})^2 \end{array}\right.$$

$$
\begin{array}{|l}
t \leftarrow (prime_k)^2 \\
while\ t \leq L \\
\quad \begin{array}{|l}
for\ j \in t, t + 2 \cdot prime_k..L \\
\quad is\_prime_{\frac{j-1}{2}} \leftarrow 0 \\
for\ j \in s + 2, s + 4..t - 2 \\
\quad if\ is\_prime_{\frac{j-1}{2}} = 1 \\
\quad\quad \begin{array}{|l}
prime_i \leftarrow j \\
i \leftarrow i + 1
\end{array} \\
s \leftarrow t \\
k \leftarrow k + 1 \\
t \leftarrow (prime_k)^2
\end{array} \\
for\ j \in s + 2, s + 4..L \\
\quad if\ is\_prime_{\frac{j-1}{2}} = 1 \\
\quad\quad \begin{array}{|l}
prime_i \leftarrow j \\
i \leftarrow i + 1
\end{array} \\
return\ prime
\end{array}
$$

Program 4.2 *The factorization program of a natural number; this program uses the vector p representing prime numbers, generated with the Sieve of Eratosthenes. The Sieve of Eratosthenes is called upon trough the following sequence:*

$$L := 2 \cdot 10^7 \quad t_0 = time(0) \quad p := CEPbm(L) \quad t_1 = time(1)$$

$$(t_1 - t_0)s = 5.064s \quad last(p) = 1270607 \quad p_{last(p)} = 19999999$$

$$
Fa(m) := \begin{array}{|l}
return\ (\text{"}m = \text{"}\ m\ \text{"} > ca\ ultimul\ p^2\text{"})\ if\ m > (p_{last(p)})^2 \\
j \leftarrow 1 \\
k \leftarrow 0 \\
f \leftarrow (1\ 1) \\
while\ m \geq p_j \\
\quad \begin{array}{|l}
if\ \ \mathrm{mod}\ (m, p_j) = 0 \\
\quad \begin{array}{|l}
k \leftarrow k + 1 \\
m \leftarrow \dfrac{m}{p_j}
\end{array} \\
otherwise \\
\quad \begin{array}{|l}
f \leftarrow stack[f, (p_j, k)]\ if\ k > 0 \\
j \leftarrow j + 1 \\
k \leftarrow 0
\end{array}
\end{array} \\
f \leftarrow stack[f, (p_j, k)]\ if\ k > 0 \\
return\ submatrix(f, 2, rows(f), 1, 2)
\end{array}
$$

We presented the Kempner's algorithm using Mathcad programs required for the algorithm.

Program 4.3 *The function counting all the digits in the p base of numeration in which is n.*

$$ncb(n,p) := \left| \begin{array}{l} return\ ceil(\log(n,p))\ if\ n > 1 \\ return\ 1\ otherwise \end{array} \right.$$

Where $ceil(x)$ is a Mathcad function which gives the smallest integer $\geq x$ and $\log(n,p)$ is logarithm in base p from n.

Program 4.4 *The program intended to generate the p generalized base of numeration (noted by Smarandache $[p]$) for a number with m digits.*

$$a(p,m) := \left| \begin{array}{l} for\ i \in 1..m \\ \quad a_i \leftarrow \dfrac{p^i - 1}{p - 1} \\ return\ a \end{array} \right.$$

Program 4.5 *The program intended to generate for the p base of numeration (noted by Smarandache (p)) to write the α number.*

$$b(\alpha,p) := \left| \begin{array}{l} return\ (1)\ if\ p = 1 \\ for\ i \in 1..ncb(\alpha,p) \\ \quad b_i \leftarrow p^{i-1} \\ return\ b \end{array} \right.$$

Program 4.6 *Program that determines the digits of the generalized base of numeration $[p]$ for the number n.*

$$Nbg(n,p) := \left| \begin{array}{l} m \leftarrow ncb(n,p) \\ a \leftarrow a(p,m) \\ return\ (1)\ if\ m{=}0 \\ for\ i \in m..1 \\ \quad \left| \begin{array}{l} c_i \leftarrow trunc\left(\dfrac{n}{a_i}\right) \\ n \leftarrow \mod(n, a_i) \end{array} \right. \\ return\ c \end{array} \right.$$

Where $trunc(x)$ returns the integer part of x by removing the fractional part, and $mod(x,y)$ returns the remainder on dividing x by y (x modulo y).

Program 4.7 *Program for Smarandache's function.*

$$\mu(n) := \left| \begin{array}{l} return\ \text{"}Err.\ n\ nu\ este\ intreg\text{"}\ if\ n \neq trunc(n) \\ return\ \text{"}Err.\ n < 1\text{"}\ if\ n < 1 \\ return\ (1)\ if\ n{=}1 \\ f \leftarrow Fa(n) \\ p \leftarrow f^{\langle 1 \rangle} \\ \alpha \leftarrow f^{\langle 2 \rangle} \\ for\ k = 1..rows(p) \\ \quad \eta_k \leftarrow p_k \cdot Nbg(\alpha_k, p_k) \cdot b(\alpha_k, p_k) \\ return\ \max(\eta) \end{array} \right.$$

This program calls the $Fa(n)$ factorization with prime numbers. The program uses the Smarandache's Remark 3.1 – about the Kempner algorithm. The $\mu.prn$ file generation is done once. The reading of this generated file in Mathcad's documents results in a great time–save.

Program 4.8 *Program with which the file μ.prn is generated*

$$VF\mu(N) := \left| \begin{array}{l} \mu_1 \leftarrow 1 \\ for\ n \in 2..N \\ \quad \mu_n \leftarrow \mu(n) \\ return\ \mu \end{array} \right.$$

This program calls the program 4.7 for calculating the value of the μ function. The sequence of the μ.prn file generation is:

$$t_0 := time(0) \quad WRITEPRN("\mu.prn") := VF\mu(2 \cdot 10^7) \quad t_1 := time(1)$$

$$(t_1 - t_0)sec = "5:17:32.625" hhmmss$$

Smarandache's function is important because it characterizes prime numbers – through the following fundamental property:

Theorem 4.9 *Let be p an integer > 4, than p is prime number if and only if $\mu(p) = p$.*

Proof. See [6, p. 31]. □

Hence, the fixed points of this function are prime numbers (to which is added 4). Due to this property the function is used as primality test.

Program 4.10 *Program for testing μ's primality. This program returns the 0 value if the number is not prime number and the 1 value if the number is a prime. The file μ.prn will be read and it will be assigned to the μ vector.*

$$ORIGIN := 1 \qquad \mu := READPRN(" \ldots \backslash \mu.prn")$$

$$Tp\mu(n) := \left| \begin{array}{l} return\ "Err.\ n < 1\ sau\ n \notin \mathbb{Z}"\ if\ n < 1 \vee n \neq trunc(n) \\ if\ n > 4 \\ \quad \left| \begin{array}{l} return\ 0\ if\ \mu_n \neq n \\ return\ 1\ otherwise \end{array} \right. \\ otherwise \\ \quad \left| \begin{array}{l} return\ 0\ if\ n{=}1 \vee n{=}4 \\ return\ 1\ otherwise \end{array} \right. \end{array} \right.$$

Program 4.11 *Program that provides the reveres of the given m number.*

$$R(m) := \left| \begin{array}{l} n \leftarrow floor(\log(m)) \\ x \leftarrow m \cdot 10^{-n} \\ for\ k \in 1..n \end{array} \right.$$

$$\left|\begin{array}{l} \left|\begin{array}{l} c_k \leftarrow trunc(x) \\ x \leftarrow (x - c_k) \cdot 10 \end{array}\right. \\ c_{n+1} \leftarrow round(x) \\ Rm \leftarrow 0 \\ for\ k \in n+1..2 \\ \quad Rm \leftarrow (Rm + c_k) \cdot 10 \\ return\ Rm + c_1 \end{array}\right.$$

Where $floor(x)$ returns the greatest integer $\leq x$ and $round(x)$ returns x rounded to the nearest integer.

Program 4.12 *Search program for the* Luhn prime numbers*.*

$$PLuhn(L) := \left|\begin{array}{l} n \leftarrow last(p) \\ S \leftarrow (229) \\ k \leftarrow 51 \\ while\ p_k \leq L \\ \quad \left|\begin{array}{l} N \leftarrow R(p_k) + p_k \\ S \leftarrow stack(S, p_k)\ if\ Tp\mu(N) = 1 \\ k \leftarrow k + 1 \end{array}\right. \\ return\ S \end{array}\right.$$

The function $stack(A, B, \ldots)$ is applied for merging matrixes top-down. The number of columns in matrixes should also be the same. The discussed functions could be applied to vectors as well.

Execution of the program $PLuhn$ was made with sequence

$$S := PLuhn(2 \cdot 10^7)$$

The initialization of the S stack is done with the vector that contains the number 229. The variable k has the initial value of 51 because the index of the 229 prime number is 50, so that the search for the *Luhn prime numbers* will begin with $p_{51} = 233$.

5 List of prime numbers Luhn

We present a partial list of the 50598 *Luhn prime numbers* smaller than $2 \cdot 10^7$ (the first 321 and the last 120):
229 239 241 257 269 271 277 281 439 443 463 467 479 499 613 641 653 661 673 677 683 691 811 823 839 863 881 20011 20029 20047 20051 20101 20161 20201 20249 20269 20347 20389 20399 20441 20477 20479 20507 20521 20611 20627 20717 20759 20809 20879 20887 20897 20981 21001 21019 21089 21157 21169 21211 21377 21379 21419 21467 21491 21521 21529 21559 21569 21577 21601 21611 21617 21647 21661 21701 21727 21751 21767 21817 21841 21851 21859 21881 21961 21991 22027 22031 22039 22079 22091 22147 22159 22171 22229 22247 22291 22367 22369 22397 22409 22469 22481 22501 22511 22549 22567

22571 22637 22651 22669 22699 22717 22739 22741 22807 22859 22871 22877 22961 23017 23021 23029 23081 23087 23099 23131 23189 23197 23279 23357 23369 23417 23447 23459 23497 23509 23539 23549 23557 23561 23627 23689 23747 23761 23831 23857 23879 23899 23971 24007 24019 24071 24077 24091 24121 24151 24179 24181 24229 24359 24379 24407 24419 24439 24481 24499 24517 24547 24551 24631 24799 24821 24847 24851 24889 24979 24989 25031 25057 25097 25111 25117 25121 25169 25171 25189 25219 25261 25339 25349 25367 25409 25439 25469 25471 25537 25541 25621 25639 25741 25799 25801 25819 25841 25847 25931 25939 25951 25969 26021 26107 26111 26119 26161 26189 26209 26249 26251 26339 26357 26417 26459 26479 26489 26591 26627 26681 26701 26717 26731 26801 26849 26921 26959 26981 27011 27059 27061 27077 27109 27179 27239 27241 27271 27277 27281 27329 27407 27409 27431 27449 27457 27479 27481 27509 27581 27617 27691 27779 27791 27809 27817 27827 27901 27919 28001 28019 28027 28031 28051 28111 28229 28307 28309 28319 28409 28439 28447 28571 28597 28607 28661 28697 28711 28751 28759 28807 28817 28879 28901 28909 28921 28949 28961 28979 29009 29017 29021 29027 29101 29129 29131 29137 29167 29191 29221 29251 29327 29389 29411 29429 29437 29501 29587 29629 29671 29741 29759 29819 29867 29989 ...
8990143 8990209 8990353 8990441 8990563 8990791 8990843 8990881 8990929 8990981 8991163 8991223 8991371 8991379 8991431 8991529 8991553 8991613 8991743 8991989 8992069 8992091 8992121 8992153 8992189 8992199 8992229 8992259 8992283 8992483 8992493 8992549 8992561 8992631 8992861 8992993 8993071 8993249 8993363 8993401 8993419 8993443 8993489 8993563 8993723 8993749 8993773 8993861 8993921 8993951 8994091 8994109 8994121 8994169 8994299 8994463 8994473 8994563 8994613 8994721 8994731 8994859 8994871 8994943 8995003 8995069 8995111 8995451 8995513 8995751 8995841 8995939 8996041 8996131 8996401 8996521 8996543 8996651 8996681 8996759 8996831 8996833 8996843 8996863 8996903 8997059 8997083 8997101 8997463 8997529 8997553 8997671 8997701 8997871 8997889 8997931 8997943 8997979 8998159 8998261 8998333 8998373 8998411 8998643 8998709 8998813 8998919 8999099 8999161 8999183 8999219 8999311 8999323 8999339 8999383 8999651 8999671 8999761 8999899 8999981

6 Conclusions

The list of all *Luhn prime numbers*, that totalized 50598 numbers, was determined within a time span of 54 seconds, on an Intel processor of 2.20 GHz.

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Connections between some well-known concepts of uniform exponential dichotomy for discrete-time systems in Banach spaces

Mihai-Gabriel Babuţia* Traian Ceauşu †

Abstract

The aim of this paper is to unify our recent works concerning three concepts of uniform exponential dichotomy defined for linear discrete-time systems in Banach spaces: uniform strong exponential dichotomy, uniform exponential dichotomy,uniform weak exponential dichotomy. By defining them in the case of strongly invariant sequences of projections, we point out the connections between them and we also make reference to previous works in this field, by linking them as well with the concepts defined for noninvertible systems.

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Keywords: uniform exponentia dichotomy; uniform strong exponential dichotomy; uniform weak exponential dichotomy; discrete linear system.

1 Introduction

The notion of uniform exponential dichotomy plays a key role in the study of the asymptotic behaviors of discrete-time linear systems in Banach spaces. Several characterizations of such concepts and interesting results were obtained recently, and we point out a selection of them, as well as the references within: [1], [2], [3], [4], [6], [7], [9], [11]. A natural generalization of the dichotomy property, when it cannot completely describe the system's behavior due to the presence of a central manifold, is the property of uniform exponential trichotomy, and several recent developments have been made ([5], [8] and the references therein).

In this paper we define three concepts of uniform exponential dichotomy for discrete-time linear systems in Banach spaces and establish the connections between them. Using the previous concepts and results obtained in [1] and [2], we establish the connections that are present between five concepts of uniform exponential dichotomy, two arising from

*`mbabutia@math.uvt.ro` Department of Mathematics, West University of Timişoara, Romania

†`ceausu@math.uvt.ro` Department of Mathematics, West University of Timişoara, Romania

the cited sources. We also pose two remaining open problems in successfully linking all the five presented concepts.

2 Preliminaries

Let X be a Banach space and $\mathcal{B}(X)$ the Banach algebra of all bounded linear operators on X. The norms on X and on $\mathcal{B}(X)$ will be denoted by $\|\cdot\|$. The identity operator on X is denoted by I. We will denote by $\Delta = \{(m,n) \in \mathbb{N}^2 \;:\; m \geq n\}$.

We consider the linear difference system

$$x_{n+1} = A_n x_n, \tag{A}$$

where $A : \mathbb{N} \to \mathcal{B}(X)$ is a given sequence.

Definition 2.1 *The* **discrete evolution operator** *associated to the system* (A) *is defined, for $(m,n) \in \Delta$, by:*

$$A_m^n = \begin{cases} A_{m-1} \cdot \ldots \cdot A_n, & \text{if } m > n \\ I, & \text{if } m = n \end{cases} \tag{2.1}$$

Remark 2.2 *It is obvious that $A_m^n A_n^p = A_m^p$, for all $(m,n),(n,p) \in \Delta$ and every solution of (A) satisfies $x_m = A_m^n x_n$ for all $(m,n) \in \Delta$.*

Definition 2.3 *An operator valued sequence $P : \mathbb{N} \to \mathcal{B}(X)$ is called a* ***sequence of projections*** *if $P_n^2 = P_n$ for all $n \in \mathbb{N}$.*

If $P : \mathbb{N} \to \mathcal{B}(X)$ is a sequence of projections, then the sequence $Q : \mathbb{N} \to \mathcal{B}(X)$ defined by $Q_n = I - P_n$ is also a sequence of projections, called **the complementary sequence of projections of** P.

Definition 2.4 *A sequence of projections $P : \mathbb{N} \to \mathcal{B}(X)$ is called*

- **invariant** *for the system* (A) *if for all $n \in \mathbb{N}$ we have that $A_n P_n = P_{n+1} A_n$;*
- **strongly invariant** *for the system* (A) *if it is invariant for* (A) *and for all $n \in \mathbb{N}$, the restriction $A_n : Ker\ P_n \to Ker\ P_{n+1}$ is an isomorphism;*
- **bounded** *if there exist $M \geq 1$ such that for all $n \in \mathbb{N}$, $\|P_n\| \leq M$.*

If the sequence of projections $P : \mathbb{N} \to \mathcal{B}(X)$ is invariant for the system (A), then we will say that (A, P) is a **dichotomy pair**.

Remark 2.5 *If the sequence of projections P is strongly invariant for the system (A) then*

(i) for every $n \in \mathbb{N}$ there is an isomorphism B_n from the $Ker\ P_{n+1}$ to $Ker\ P_n$ such that $A_n B_n Q_{n+1} = Q_n$ and $B_n A_n Q_n = Q_n$;

(ii) for all $(m,n) \in \Delta$ there is an isomorphism $B_m^n : Ker\ P_m \to Ker\ P_n$ with $A_m^n B_m^n Q_m = Q_m$ and $B_m^n A_m^n Q_n = Q_n$ for all $(m,n) \in \Delta$.

Throughout the paper, if not stated otherwise, we will consider $P : \mathbb{N} \to \mathcal{B}(X)$ be a sequence of projections which is strongly invariant for the linear discrete-time system (A).

3 Uniform exponential dichotomies

We will proceed by presenting the three main concepts of uniform exponential dichotomy with strongly invariant sequences of projections.

Definition 3.1 *Let (A, P) be a dichotomy pair in which $P : \mathbb{N} \to \mathcal{B}(X)$ is strongly invariant for the discrete system* (A). *We say that the dichotomy pair (A, P) is:*

- **uniformly strongly exponentially dichotomic** *(u.s.e.d) if there exist $N \geq 1$ and $\alpha > 0$ such that for all $(m, n) \in \Delta$,*

$$(used_1) \quad \|A_m^n P_n\| \leq N e^{-\alpha(m-n)}$$
$$(used_2) \quad \|B_m^n Q_m\| \leq N e^{-\alpha(m-n)}.$$

- **uniformly exponentially dichotomic** *(u.e.d) if there exist $N \geq 1$ and $\alpha > 0$ such that for all $(m, n, x) \in \Delta \times X$,*

$$(ued_1) \quad \|A_m^n P_n x\| \leq N e^{-\alpha(m-n)} \|P_n x\|$$
$$(ued_2) \quad \|B_m^n Q_m x\| \leq N e^{-\alpha(m-n)} \|Q_m x\|.$$

- **uniformly weakly exponentially dichotomic** *(u.w.e.d) if there exist $N \geq 1$ and $\alpha > 0$ such that for all $(m, n) \in \Delta$,*

$$(uwed_1) \quad \|A_m^n P_n\| \leq N e^{-\alpha(m-n)} \|P_n\|$$
$$(uwed_2) \quad \|B_m^n f Q_m\| \leq N e^{-\alpha(m-n)} \|Q_m\|.$$

A first set of connections between the above defined concepts is given by the following result.

Proposition 3.2 *The following assertions hold.*

(a) (A, P) is u.s.e.d if and only if there exist $N \geq 1$ and $\alpha > 0$ such that for all $(m, n, x) \in \Delta \times X$ the following conditions hold:

$$(used'_1) \quad \|A_m^n P_n x\| \leq N e^{-\alpha(m-n)} \|x\|$$
$$(used'_2) \quad \|B_m^n Q_m x\| \leq N e^{-\alpha(m-n)} \|x\|.$$

(b) If the dichotomy pair (A, P) is u.s.e.d then (A, P) is u.e.d and $P : \mathbb{N} \to \mathcal{B}(X)$ is bounded.

(c) If the dichotomy pair (A, P) is u.e.d then (A, P) is u.w.e.d.

(d) If $P : \mathbb{N} \to \mathcal{B}(X)$ is bounded and (A, P) is u.w.e.d then (A, P) is u.s.e.d.

Proof. (a) follows straightforward from the definition of the operator norm on $\mathcal{B}(X)$. To prove (b), let $(m, n, x) \in \Delta \times X$ and N, α given by the u.s.e.d property of the pair (A, P). The conclusion follows by considering the vectors $P_n x$ and $Q_m x$ in $(used'_1)$ and $(used'_2)$ respectively. Moreover, by taking $m = n$ in $(used_1)$ we obtain that $\|P_n\| \leq N$.
Assertion (c) follows immediately, by taking the operator norm in (ued_1) and (ued_2). In order to prove (d), consider $(m, n) \in \Delta$ and N, α given by

the u.w.e.d property of (A, P). Consider $M \geq 1$ to be an upper bound for $P : \mathbb{N} \to \mathcal{B}(X)$. From the following estimations

$$\|A_m^n P_n\| \leq Ne^{-\alpha(m-n)}\|P_n\| \leq 2MNe^{-\alpha(m-n)}$$
$$\|B_m^n Q_m\| \leq Ne^{-\alpha(m-n)}\|Q_m\| \leq 2MNe^{-\alpha(m-n)}$$

we obtain that (A, P) is u.s.e.d with constants $2MN$ and α. $\square$

Proposition 3.3 *A dichotomy pair (A, P) is u.e.d if and only if there exist $N, \alpha > 0$ such that for all $(m, n, x) \in \Delta \times X$,*

$$(i) \quad \|A_m^n P_n x\| \leq Ne^{-\alpha(m-n)}\|P_n x\|$$
$$(ii) \quad \|Q_n x\| \leq Ne^{-\alpha(m-n)}\|A_m^n Q_n x\|.$$

Proof. We only have to prove that $(ued_2) \Leftrightarrow (ii)$. Let $(m, n, x) \in \Delta \times X$. For the necessity, we compute $\|Q_n x\| = \|B_m^n Q_m A_m^n Q_n x\| \leq Ne^{-\alpha(m-n)}\|A_m^n Q_n x\|$. The sufficiency follows from $\|B_m^n Q_m x\| = \|Q_n B_m^n Q_m x\| \leq Ne^{-\alpha(m-n)}\|A_m^n Q_n B_m^n Q_m x\| = Ne^{-\alpha(m-n)}\|Q_m x\|$. $\square$

Remark 3.4 *The above proposition shows us that in both cases in which P is invariant and strongly invariant for* (A) *respectively, the dichotomy concepts are equivalent. In the next section we will see that in the case of the strong and weak concepts, this is not the case.*

Remark 3.5 *If a pair (A, P) is u.e.d, it does not necessarily follow that (A, P) is u.s.e.d. The following example points out this fact.*

Example 3.6 *On $X = \mathbb{R}^2$ endowed with the max-norm, consider, for every $n \in \mathbb{N}$, $P_n, A_n : \mathbb{R}^2 \to \mathbb{R}^2$ defined, for $x = (x_1, x_2) \in \mathbb{R}^2$, by*

$$P_n x = (x_1 + \ln(n+1) \cdot x_2, 0), \quad A_n = \frac{1}{e}P_n + eQ_{n+1},$$

where $Q_n = I - P_n$. We have that $P : \mathbb{N} \to \mathcal{B}(X)$ is a sequence of projections which is strongly invariant for the system (A). *The evolution operator associated to* (A) *is given by*

$$A_m^n = e^{n-m}P_n + e^{m-n}Q_m, \quad (m, n) \in \Delta.$$

Having in mind that for all $(m, n, x) \in \Delta \times X$, $\|A_m^n P_n x\| = e^{n-m}\|P_n x\|$ and $\|A_m^n Q_n x\| = e^{m-n}\|Q_m x\| \geq e^{m-n}\|Q_n x\|$ it follows that the pair (A, P) is u.e.d. But (A, P) cannot be u.s.e.d because, from the fact that for all $n \in \mathbb{N}$, $\|P_n(0, 1)\| = \ln(n+1)$, it follows that $\|P_n\| \geq \ln(n+1)$ hence $P : \mathbb{N} \to \mathcal{B}(X)$ is not bounded.

Remark 3.7 *From the above example, in light of (iii) from Proposition 3.2, we deduce that the dichotomy pair (A, P) from Example 3.6 is u.w.e.d but fails to be u.s.e.d.*

Open problem. At this moment we do not know wether the implication "(A, P) is u.w.e.d $\Rightarrow$ (A, P) is u.s.e.d" is generally true or not.

4 Connections with the dichotomy concepts for noninvertible discrete systems

For a more facile reading of the paper, the concepts defined in this paragraph will be denoted with a bar over their abbreviations. Although these concepts can be defined for dichotomy pairs (A, P) without the invertibility condition on the kernels of the projections, we will assume, as in the previous section, that $P : \mathbb{N} \to \mathcal{B}(X)$ is strongly invariant for the system (A), in order to establish some connections between the concepts presented throughout this paper.

Definition 4.1 *We say that the dichotomy pair (A, P) is:*

- $\overline{u.s.e.d}$ *if there exist $N \geq 1$ and $\alpha > 0$ such that for all $(m, n) \in \Delta$*

$$(\overline{used_1}) \quad \|A_m^n P_n\| \leq N e^{-\alpha(m-n)}$$
$$(\overline{used_1}) \quad 1 \leq N e^{-\alpha(m-n)} \|A_m^n Q_n\|$$

- $\overline{u.w.e.d}$ *if there exist $N \geq 1$ and $\alpha > 0$ such that for all $(m, n) \in \Delta$*

$$(\overline{uwed_1}) \quad \|A_m^n P_n\| \leq N e^{-\alpha(m-n)} \|P_n\|$$
$$(\overline{uwed_1}) \quad \|Q_n\| \leq N e^{-\alpha(m-n)} \|A_m^n Q_n\|$$

Remark 4.2 *The above defined concepts, in the continuous case, were studied in [1], and the connections between them can easily be established in a similar manner. Moreover, in the general framework of (h, k)-dichotomies for discrete-time linear systems, the $\overline{u.s.e.d}$ concept is used in [2]. Taking into account the results from these two papers, we state the following result.*

Proposition 4.3 *(a) If (A, P) is u.s.e.d then (A, P) is $\overline{u.s.e.d}$.*

(b) If (A, P) is $\overline{u.s.e.d}$ then (A, P) is $\overline{u.w.e.d}$.

(c) If (A, P) is u.e.d then (A, P) is $\overline{u.w.e.d}$.

(d) The concepts of $\overline{u.s.e.d}$ and u.e.d do not imply each other, existing linear discrete-time systems that satisfy one condition, but not verifying the other.

(e) If (A, P) is $\overline{u.w.e.d}$ it does not result that (A, P) is u.e.d.

(f) If (A, P) is $\overline{u.w.e.d}$ it does not result that (A, P) is $\overline{u.s.e.d}$.

Proof. Assertion (a) follows from the fact tha for all $(m, n) \in \Delta$, $\|Q_n x\| \leq \|B_m^n Q_m\| \cdot \|A_m^n Q_n x\|$. Assertion (b) follows from the fact that if (A, P) is $\overline{u.s.e.d}$ then $1 \leq \max\{\|P_n\|, \|Q_n\|\} \leq N$, for all $n \in \mathbb{N}$. Assertion (c) follows by taking the operator norm in (i) and (ii) in Proposition 3.3. In order to prove assertion (d), we refer to Examples 13 and 14 from [2], for the particular case in which $h_n = e^n = k_n$, $n \in \mathbb{N}$. Finally, in order to prove (e) we refer to Example 4.4. For (f), Example 3.6 gives a dichotomy pair (A, P) which is also $\overline{u.w.e.d}$ but cannot be $\overline{u.s.e.d}$, because $P : \mathbb{N} \to \mathcal{B}(X)$ is not bunded. □

Example 4.4 *We will give an example in the general case of noninvertible linear systems. On* $X = \mathbb{R}^3$ *endowed with the max-norm, consider the system* (A) *described by the operators* $A_n : \mathbb{R}^3 \to \mathbb{R}^3$ *defined, for all* $n \in \mathbb{N}$ *and* $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, *by* $A_n x = (e^{-1}x_1, ex_2, 0)$. *The evolution operator associated to the system* (A) *is given by*

$$A^n_m x = \begin{cases} x & m = n \\ (e^{n-m}x_1, e^{m-n}x_2, 0) & m \geq n+1 \end{cases}, \quad \forall (m, n, x) \in \Delta \times \mathbb{R}^3.$$

Define, for every $n \in \mathbb{N}$ *and* $x \in \mathbb{R}^3$, $P_n x = (x_1, 0, 0)$. *We have that* (A, P) *is a dichotomy pair. Moreover, for all* $(m, n) \in \Delta$, $\|A^n_m P_n\| = e^{-(m-n)}\|P_n\|$ *and* $\|A^n_m Q_n\| = e^{m-n}\|Q_n\|$, *thus* (A, P) *is* $\overline{\textit{u.w.e.d}}$. *By assuming that* (A, P) *is s.e.d, in particular, for* $x = (0, 0, 1)$, *we would obtain that* $0 = Ne^{-\alpha(m-n)}\|A^n_m Q_n x\| \geq \|Q_n x\| = 1$ *which is a contradiction.*

Proposition 4.5 *If* (A, P) *is u.s.e.d then* (A, P) *is* $\overline{\textit{u.s.e.d}}$.

Proof. It is similar to the proof of Proposition 23 from [2]. □

Remark 4.6 *The converse of the preceding proposition does not generally hold, and we refer to Example 24 form [2].*

Open problem. At this moment, we do not have an answer to the implication "(A, P) is u.w.e.d $\Rightarrow$ (A, P) is $\overline{\textit{u.w.e.d}}$".

Remark 4.7 *Regarding the open problem from above, we have a partial answer, which states that if* (A, P) *is* $\overline{\textit{u.w.e.d}}$ *then it does not necessarily imply that* (A, P) *is u.w.e.d, as it can be seen from Example 4.8.*

Example 4.8 *On* $X = \mathbb{R}^3$ *endowed with the max-norm, consider, for* $n, \in \mathbb{N}$, $A_n, P_n : \mathbb{R}^3 \to \mathbb{R}^3$ *defined, for every* $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, *by*

$$A_n x = (\frac{1}{e}x_1, ex_2, e^{-(2n+1)}x_3), \quad P_n x = (x_1, 0, 0).$$

The evolution operator associated to the system (A) *is given by*

$$A^n_m x = (e^{n-m}x_1, e^{m-n}x_2, e^{n^2-m^2}x_3), \quad (m, n, x) \in \Delta \times \mathbb{R}^3.$$

It is easy to check that (A, P) *is a dichotomy pair, that* $P : \mathbb{N} \to \mathcal{B}(X)$ *is strongly invariant for* (A) *and* $\|A^n_m P_n\| = e^{n-m}\|P_n\|$, $\|A^n_m Q_n(0, 1, 1)\| \geq e^{m-n}\|Q_n(0, 1, 1)\|$, *from where* $\|A^n_m Q_n\| \geq e^{m-n}\|Q_n\|$ *hence* (A, P) *is* $\overline{\textit{u.w.e.d}}$. *Assume, by a contradiction, that* (A, P) *is u.w.e.d. Then there exist* $N \geq 1$ *and* $\alpha > 0$ *such that for all* $(m, n) \in \Delta$, *we have that* $\|B^n_m Q_m\| \leq Ne^{-\alpha(m-n)}\|Q_m\|$. *Taking into account that for all* $(m, n, x) \in \Delta \times \mathbb{R}^3$, $\|B^n_m Q_m x\| = \max\{e^{n-m}|x_2|, e^{m^2-n^2}|x_3|\}$, *for* $x_0 = (0, 0, 1)$, *we get that for all* $(m, n) \in \Delta$, $\|B^n_m Q_m x_0\| = e^{m^2-n^2}\|Q_m x_0\|$ *hence* $\|B^n_m Q_m\| \geq e^{m^2-n^2}$. *We finally obtain that*

$$e^{m^2-n^2} \leq \|B^n_m Q_m\| \leq Ne^{-\alpha(m-n)}\|Q_m\| = Ne^{-\alpha(m-n)}, \quad \forall (m, n) \in \Delta,$$

which is a contradiction hence (A, P) *is not u.w.e.d.*

Remark 4.9 *Taking into account the connections between the studied concepts, the results from the cited papers and the open problems stated in this paper, we can emphasize the connections between the presented concepts throughout the following diagram.*

$$\begin{array}{ccccccc}
\boxed{u.s.e.d} & \overset{\nLeftarrow}{\Rightarrow} & \boxed{u.e.d} & \Rightarrow & \boxed{u.w.e.d} & \overset{\nRightarrow}{\Leftarrow} & \boxed{u.s.e.d} \\
\not\Uparrow\Downarrow & & \Updownarrow & & \not\Uparrow & & \not\Uparrow\Downarrow \\
\boxed{\overline{u.s.e.d}} & \overset{\nLeftarrow}{\nRightarrow} & \boxed{u.e.d} & \overset{\nLeftarrow}{\Rightarrow} & \boxed{\overline{u.w.e.d}} & \overset{\nRightarrow}{\Leftarrow} & \boxed{\overline{u.s.e.d}}
\end{array}$$

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Secret communication using cryptography and steganography

Dominic Bucerzan * Crina Ratiu †

Abstract

Cryptography and steganography are two techniques used to provide data confidentiality. Cryptography scrambles the digital data so the new content becomes impossible to understand to unauthorized users. Steganography embeds the digital data in a cover file (usually an image or audio file). Steganography, is a technique that "camouflages" a communication to hide its existence and make it seem "invisible" to unauthorized users. In this paper we present both methods, but we insist on the advantage provided by combining these two methods in order to improve the security of communication over Internet by means of commonly available equipments as PCs, tablets and smart phones.

Mathematics Subject Classification: 94A62

Keywords: Cryptography, Steganography, LSB, Android, SmartSteg

1 Introduction

Nowadays we are witnessing a high level of development in both hardware and software technology. Even though the progress in this fields has occurred with a higher speed than ever before, digital information security problems remain present and become an interdisciplinary issue, having to be constantly optimized, developed and innovated [1].

In figure 1 we present briefly a classification of the security techniques used today to ensure the main characteristics that defines digital information: integrity, confidentiality, availability, authenticity and no-repudiation[8].

In the scientific literature, Simmons, in 1983, proposed the "Prisoners' Problem" to define the digital information security environment [3]. The "Prisoners Problem" describes Alice and Bob who are in jail and wish to establish an escape plan; they may communicate through the warden, Willie who shouldn't suspect and uncover their plan. In this situation the two prisoners must find a way to hide their secret communication in order not to arouse any suspicion from Willie [3]. So Alice and Bob

*dominic@bbcomputer.ro Department of Mathematics and Computer Science, "Aurel Vlaicu" University of Arad, Romania

†ratiu_anina@yahoo.com Babes-Bolyai University of Cluj-Napoca, Romania

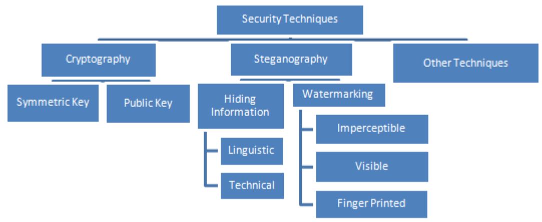

Figure 1: Security techniques [1]

use cryptography combined with steganography to encode and hide their messages.

These two techniques are widely used today to ensure the security of the information transferred through available channels for communication, which are vulnerable to interception. Moreover in recent years these techniques were combined and used together to optimize the security of digital information [1].

2 Cryptography

Cryptography is a technique that scrambles digital data in a way that it becomes unreadable to unauthorized users in the process of storing or transferring it to its intended receiver using today's open channels for communication.

Cryptography is recognized today to be the main component of the security policy of every organization and considered a standard to digital data security [4].

Kirchhoff has stated the principle that stands as a basis for every cryptographic system: the security of an encryption system should rely on the secrecy of the encryption/decryption key and not on the secrecy of the algorithm used [4]. This means that if an unauthorized user knows the algorithm used for encryption, without knowing the key used, is unable to decrypt the secret data. Having this into consideration we may say that the security level of a cryptographic system stands in the length of the key used. The larger the key, the longer is the time needed to uncover it [4].

According to the type of key used, cryptographic systems are classified as [5]:

- Secret / Symmetric key cryptography: the same key is used for both processes encryption and decryption. The sender uses the key to encode the secret data; the receiver uses the same key to decode the data received.
- Public / Asymmetric key cryptography: it uses two key for encrypting and decrypting the data. The sender uses the public key of the receiver (the key is public, known by anyone) to encrypt the data;

the receiver uses his personal secret key (only the receiver knows the secret key) to decode the received data.

The cryptographic methods mentioned are the basis for two popular techniques used today in information security technology, namely:

- Digital signature: ensures the authentication of the sender
- Hash function: ensures the authentication and the integrity of the sent data.

3 Steganography

Steganography is a technique that embeds information in different types of files (usually media files) in such a way that its presence becomes "difficult to notice" [1].

Some countries raised restrictions upon the usage of cryptography; in this scenario steganography became an alternative for confidential communications. Steganography was seen as a solution to this situation because a steganographic message couldn't be detected so its usage couldn't be controlled [1].

By comparison with cryptography which only scrambles the message in a way unreadable for unauthorized users, steganography embeds the data using a cover file in such a way that the changes made to the cover file are not obvious to human means of observation and it can pass through open channels of communication without raising any attention or suspicion.

Steganography is not something new, it is a technique used since antiquity. Nowadays it has evolved and developed to be used with digital technology. Most used cover files for steganography are media files like video, audio and especially image files[7].

Image files are proper to steganography due to their structure, to the fact that the changes made to an image file are difficult to notice with the naked eye. Also, the technology has evolved and it became cheaper, hence digital images are omnipresent in everyday life.

One of the most used methods in digital image steganography is the LSB (Last Significant Bit) technique also known as noise insertion. This technique usually uses as cover files digital images of RGB type [8]. By using the last significant bits of the bytes of the original cover file it embeds the bits of secret information. Here you have an example: consider three pixels of an 24 bits RGB image using nine bytes of memory.

(001001111110100111001000)
(001001111100100011101001)
(001001111100100011101001)

To embed the character A which has the following bit structure 100000001, the original bits schem will change as follows:

(001001111110100 [0] 11001000)
(0010011 [0] 110010001110100 [0])
(110010000010011111101001)

As it can be seen only the value of three bits needs to change in order to embed the desired character. These changes are not visible to the naked eye. In average only a half of the LSB of a digital image needs to be changed to embed secret data.

4 Communication using cryptography and steganography

A method to optimize the secrecy of digital information communication in today's insecure environments like Internet and Mobile Networks is to combine cryptography with steganography. By hiding the previously encoded message, the information that is intended to be secret may be transmitted without attracting attention. Also, this method, offers an alternative to the classical storage of data. A user may store secret data by encoding and embedding them in digital images to protect them from a possible intruder [2].

Both, steganography and cryptography are intended as means to protect the confidentiality of information. However, on their own, they are not perfect and can be decrypted and revealed. This is why most experts would suggest the use of multiple layers of security.

The literature does not recommend replacing steganography with cryptography or cryptography with steganography. It is recommended the simultaneous use of the two techniques. This recommendation derives from the differences between the two techniques and the fact that they can complete each other. Figure 2, presents briefly the different purposes that cryptography and steganography fulfill [5].

DIFFERENCES	
STEGANOGRAPHY	CRYPTOGRAPHY
Invisible communication; confidentiality; privacy	Visible communication
Prevents the discovery of the existence of communication	Prevents the discovery of the content of the communication
Algorithms are still developing and improving	Most algorithms are public
Once detected the message is revealed	Algorithms are still resistant to attacks, expensive computing power needed to crack
Does not alter the structure of secret data	Alter the structure of secret data

Figure 2: Differences between steganography and cryptography [5]

Current trends, both in hardware and software technology led industry towards the development of smaller, faster and high-performance mobile devices, which can support a wide range of features and open source operating systems [2].

A rapid growth in this area is registered by mobile hand-held devices which are popularly called smart gadgets and they include: smart phones, tablets, e-book readers. The Smartphone's life-cycle has evolved drastically in recent years, having a lifetime of approximately 6 months between generations [2].

Among the major issues that occur in this dynamic, ever changing and evolving environment is the fact that almost all platforms have dedicated application. This fact is in contradiction with one of the principal characteristic of digital information, namely availability. In other words, if a sender uses an iPhone to encode and hide some information, the receiver must also use an iPhone to unhide and decode the secret information. And this issue also applies when using a computer [2].

In the case when the sender is restricted to using a computer whereas the receiver has access only to a smart-phone, the secret information is no longer available in this scenario.

A proposed solution to this situation is SmartSteg project that includes a set of applications that use the same algorithm to encode - decode, hide - reveal secret information (almost any kind of digital file) in digital images independent of the type of device used. The idea is shown in figure 3.

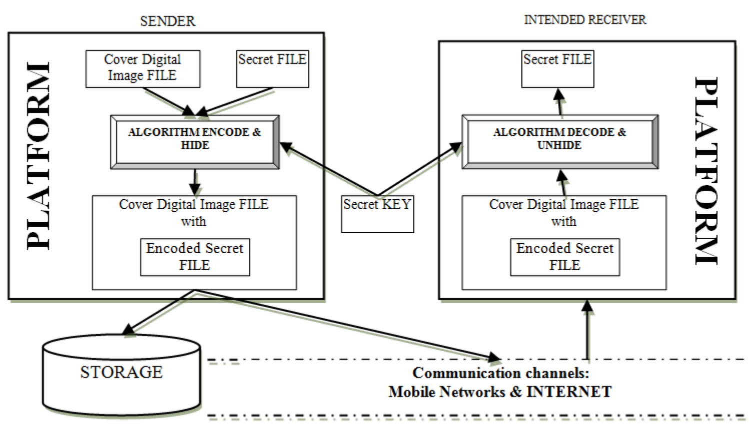

Figure 3: SmartSteg project [9]

At this stage of the project, SmartSteg provides confidential communication between computers that run under Windows Operating System and devices that run under Android. This means that a secret file that is encrypted and encoded with SmartSteg using a computer can be revealed and decrypted with SmartSteg from a device that runs under Android. Figure 4 and 5 show the design of the application chosen for the two versions of SmartSteg [1].

5 Conclusions

Steganography and cryptography serve the same purpose, both systems provide secret communications. However, they differ in the methods of attacking / breaking of the systems [6].

A cryptographic system is considered broken if an attacker manages to decode the secret information. A steganography system is considered broken if an attacker can detect the existence of the hidden information

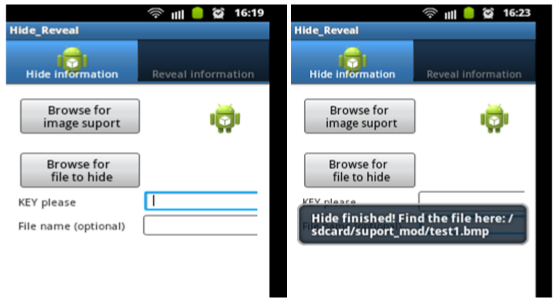

Figure 4: Design for the application that runs under Windows [1]

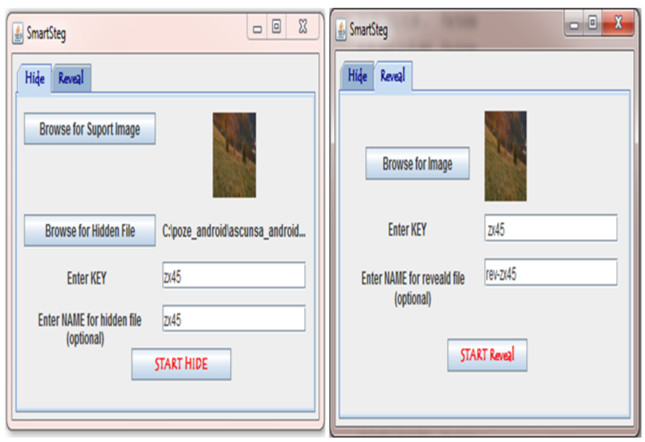

Figure 5: Design for the SmartSteg application running under Android [1]

or read the hidden information. Moreover, the steganography system is considered to have failed even if an attacker only suspects the existence of a file with hidden information or the method used to conceal the secret information even without being able to extract it.

Steganography adds an extra layer of security to cryptography, so combining the two techniques may achieve an efficient solution in terms of systems that ensure confidential communications [6].

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Fuzzy logic applications in power systems

Violeta Chis* Deacu Antheia †

Abstract

Fundamentals of fuzzy logic and an overview of its applications in power systems are presented first. Fuzzy logic based systems with their capability to deal with incomplete information, imprecision, and incorporation of qualitative knowledge have shown great potential for application in electric load forecasting. The paper investigates the application of fuzzy logic (FL) as forecasting tools for predicting short term load forecasting (STLF).

Mathematics Subject Classification: 34D09

Keywords: Fuzzy logic, short term load forecasting (STLF), power systems.

1 Introduction

Electric power systems are large, complex, geographically widely distributed systems and influenced by unexpected events. These facts make it difficult to effectively deal with many power system problems through strict mathematical approaches. Therefore, intelligent techniques such as expert systems, artificial neural networks (ANN), genetic algorithms (GA) and FL have emerged in recent years in power systems as a complement to mathematical approaches and have proved to be effective when properly coupled. As the real world power system problems may neither fit the assumptions of a single technique nor be effectively solved by the strengths and capabilities of a single technique, it is now becoming apparent that the integration of various intelligent techniques is a very important way forward in the next generation of intelligent systems [1].

Uncertainty and imprecision widely exist in engineering problems. The complexity in the world generally arises from uncertainty in the form of ambiguity. The following lists just some examples of such uncertainty and imprecision in power systems:

- Changing power system operation conditions, such as changes in load or generation, and changes in the topology of power systems.
- Various power system constructions, such as untransposed/transposed, shunt compensation, and series compensation, especially the introduction of new techniques - flexible AC transmission systems.

*viochis@uav.ro Department of Mathematics and Computer Science, "Aurel Vlaicu" University of Arad, Romania

†antheia.deacu@upt.ro Universitatea Politehnica din Timisoara

- Many different fault/disturbance conditions, including fault inception, fault location, fault types, and fault path resistance.
- Inaccuracies caused by voltage and current transducers or SCADA measurements/state estimations or noise introduced through electromagnetic interference.
- Uncertainty caused by electricity market or deregulation.
- Imprecise information caused by human beings involved in the planning, management, operation and control of power systems.
- Many concepts in power system problems are fuzzy in nature and the knowledge and reasoning used by human experts to solve these problems is approximate.

2 Fuzyy Logic applications in power systems

Analytical approaches have been used over the years for many power system operation, planning and control problems. However, the mathematical formulations of real-world problems are derived under certain restrictive assumptions and even with these assumptions, the solutions of large scale power systems problems are not trivial. On the other hand, there are many uncertainties in various power systems problems because power systems are large, complex, geographically widely distributed systems and influenced by unexpected events. More recently, the deregulation of power utilities has introduced new issues into the existing problems. These facts make it difficult to effectively deal with many power systems problems through strict mathematical formulations alone. Although a large number of AI techniques have been employed in power systems, fuzzy logic is the only possible answer to a number of challenging problems [1].

For the most complex system where few numerical data exist and only ambiguous or imprecise information may be available, fuzzy reasoning provides a way to understand system behavior by allowing us to interpolate approximately between observed input and output situation. In recent years, the number of publications in the area of fuzzy logic applications to power systems has been growing rapidly [1]. The areas include:

- modelling and control e.g. power system stability control;
- pattern recognition and predication, e.g. power system security assessment, fault diagnosis, load forecasting and power system protection;
- optimisation, e.g. power system planning, unit commitment and economic dispatch etc.

3 Short-term load forecasting

Load forecasting is an important component of power system to establish economical and reliable operations for power stations and their generating units. An accurate load forecasting approach used to predict load demand is essential part of any energy management system [2].

The load forecasting can be divided into three categories: long term, medium term and short term. Long term load forecasting (LTLF) is applicable for system and long term network planning. Mid Term Load Forecasting (MTLF) refers to quarterly, half yearly and yearly load forecasting needs. Short-term load forecasting (STLF) is used to predict load demands up to a week ahead so that the day-to-day operation of a power system can be efficiently planned [3].

There are many techniques that could be employed for loads forecasting like statistical method, linear regression, FL, ANN, GA, expert system, support vector machine, and data mining model.

Electricity load demand is influenced by many factors, such as weather, economic and social activities, and different load. By analysis of only historical load data, it is difficult to obtain accurate load demand for forecasting. The relation between load demand and the independent variables is complex and it is not always possible to fit the load curve using statistical models. The numerical aspects and uncertainties of this problem appear suitable for fuzzy methodologies [4].

4 Case Study

This paper studies the applicability of FL model on STLF. The analysis of fuzzy model was implemented using Matlab Fuzzy Toolbox. There are four basic elements in a fuzzy system (Figure 1) which are:

1. Fuzzification: the process of associating crisp input values with the linguistic terms of corresponding input linguistic variables
2. Fuzzy inference engine: provides the decision making logic of the system The fuzzy inference system is a popular computing framework based on the concept of fuzzy set theory, fuzzy if-then rules, and fuzzy reasoning.
3. Fuzzy rule base: a set of linguistic rules or conditional statements in the form of

   "IF a set of conditions IS satisfied, THEN a set of consequences are inferred"
4. Defuzzification interface: Defuzzifies the fuzzy outputs of the fuzzy inference machine and generates a non-fuzzy (crisp) output which is the actual output of the fuzzy system [5]. The Center of Gravity method (COG) is the most popular defuzzification technique and is widely utilized in actual applications.

Our model forecasts the load for one whole day at a time. Fuzzy logic based model is developed and presented for STLF using real data (real load data of Romanian electric utility). Weather related variation is certainly critical in predicting electricity demand for lead times beyond a day-ahead.

In this case, the inputs are two previous temperature. Temperature is important because demand of load is depending on temperature of the day. Normally when temperature is high, the demand will also high.

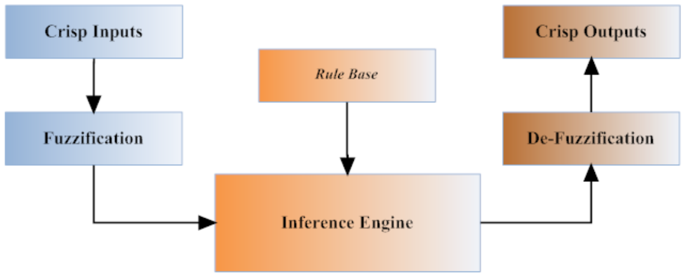

Figure 1: Basic Configuration of Fuzzy System

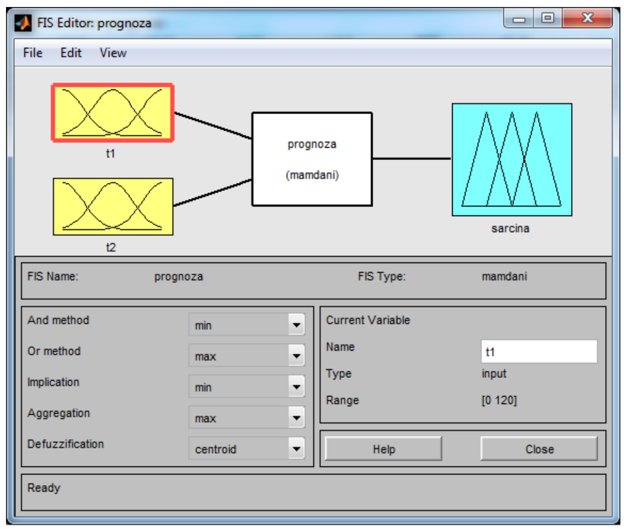

Figure 2: Fuzzy System Structure

There are two most used types of Fuzzy Inference System (FIS): Mamdanis' and Sugenos'. These two types of inference systems vary somewhat in the way the outputs are determined. Mamdani's fuzzy inference method is the most commonly seen fuzzy methodology due to its simple structure of 'min-max' operations. Figure 2 shows the whole structure of fuzzy logic system included input, reasoning rules and also the proposed output.

Construction of membership functions can be based on intuition, experience or probabilistic methods. The two inputs taken for STLF are t1 and t2. As shown in figure (Figure 3) t1 and t2 is divided into five triangular membership functions.

Figure 4 shows forecasted load (output) divided into five triangular membership functions. Fuzzy Rule Base is the heart of the fuzzy system. The heuristic knowledge of the forecasted is stored in terms of "IF-THEN" rules. It sends information to fuzzy inference system, which evaluates

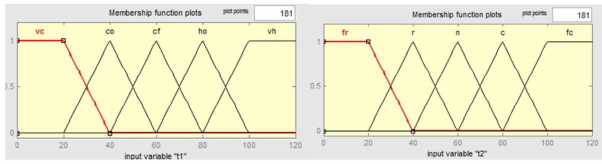

Figure 3: Memberships functions for input

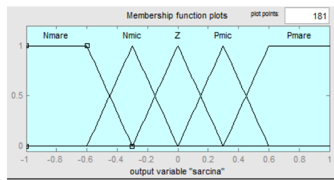

Figure 4: Membership function for forecasted load

the gained information to get the load forecasted output. For the STLF problem, a set of multiple-antecedent fuzzy rules have been established. Some of the rules are prezented in Figure 5 as follows:

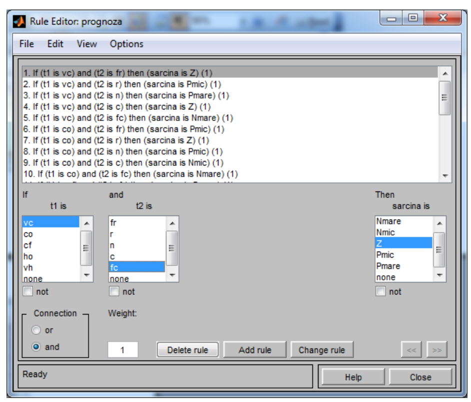

Figure 5: Rule Editor

The Surface Viewer (Figure 6) is used to display the dependency of one of the outputs on any one or two of the inputs - that is, it generates and plots an output surface map for the system.

The Rule Viewer (Figure 7) is based on the fuzzy inference diagram and displays a roadmap of the whole fuzzy inference process.

Results of fuzzy logic based models are compared with the actual demand of electricity for validation. To evaluate the result of fuzzy systems

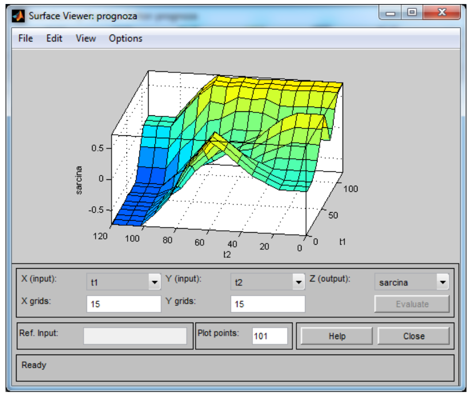

Figure 6: Surface Viewer

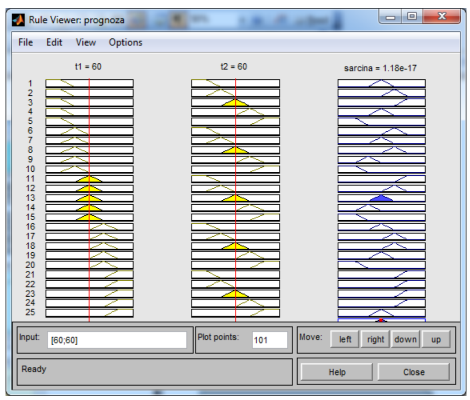

Figure 7: Rule viewer

performance, the following percentage error measure is employed.

$$error = \frac{|actualload - forecastedload|}{actualload} \times 100 \tag{4.1}$$

The predicted data of electrical load for one day is compared with actual load demand and presented graphically in Figure 8.

The performance of the model is evaluated on the basis of statistical indicator; the average error in the forecasted load in comparison with the desired load is 2.59

5 Conclusions

Although a large number of AI techniques have been employed in power systems, fuzzy logic is the only possible answer to a number of challenging problems.

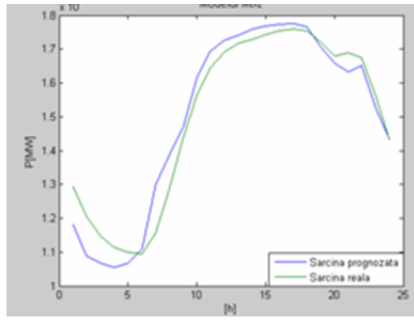

Figure 8: Predicted electrical load in comparison with actual load

The growing number of publications on applications of fuzzy-set-based approaches to power systems clearly demonstrates that fuzzy logic has been, is, and can be used to solve power systems problems.

The fuzzy logic model for the short term electrical load forecasting is developed and presented. The developed model is accurate and effective for short term load forecasting.

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Inverse narcissistic numbers

Octavian Cira*

Abstract

An n-digit number that is the sum of the n^{th} powers of its digits is called an n-narcissistic number. The sum of n numbers n to powers d_k, where $d_k \in \{0, 1, \ldots, b-1\}$, $n, b \in \mathbb{N}$, $1 < n \leq b$ are the digits in base b, is a inverse narcissistic number. In this article determine inverse narcissistic numbers, in bases of numeration $b = 2, 3, \ldots, 16$.

1 Introduction

An n-digit number that is the sum of the n^{th} powers of its digits is called an n-narcissistic number. It is also sometimes known as an Armstrong number, perfect digital invariant [3], or plus perfect number. Hardy [7] wrote, "*There are just four numbers, after unity, which are the sums of the cubes of their digits*: $1^3 + 5^3 + 3^3 = 153$, $3^3 + 7^3 + 0^3 = 370$, $3^3 + 7^3 + 1^3 = 371$, *and* $4^3 + 0^3 + 7^3 = 407$. *These are odd facts, very suitable for puzzle columns and likely to amuse amateurs, but there is nothing in them which appeals to the mathematician*". Narcissistic numbers therefore generalize these "unappealing" numbers to other powers [3, page 164]. In the article [14] was determined Narcissistic numbers in bases of numeration $b = 2$, $b = 3$,... $b = 16$. These numbers are solutions Diophantine equations of order n

$$d_{n-1}^n + d_{n-2}^n + \ldots + d_0^n = d_{n-1}b^{n-1} + d_{n-2}b^{n-2} + \ldots + d_0 \ ,$$

where $d_0, d_1, \ldots, d_{n-1} \in \{0, 1, \ldots, b-1\}$, $n, b \in \mathbb{N}$, $1 < n \leq b$. These numbers narcissistic dealt many authors, among which [1, 2, 4, 5, 6, 8, 9, 10, 11, 12, 13, 18, 15, 16, 17].

The solution the equation Diophantine of the order $\leq b - 1$,

$$n^{d_{n-1}} + n^{d_{n-2}} + \ldots + n^{d_0} = d_{n-1}b^{n-1} + d_{n-2}b^{n-2} + \ldots + d_0 \qquad (1.1)$$

we call the inverse narcissistic number, where $n, b \in \mathbb{N}$, $1 < n \leq b$.

*`octavian.cira@uav.ro` Department of Mathematics and Computer Science, "Aurel Vlaicu" University of Arad, România

2 Inverse Narcissistic numbers

We use the notation $d_{n-1}d_{n-2}\ldots d_{0\,(b)}$ for number written in the base b, $d_{n-1}\cdot b^{n-1}+d_{n-2}\cdot b^{n-2}+\ldots+d_0\cdot b^0$. For numbers written in base $b=10$ we use classical notation.

The largest number of the form $n^{d_{n-1}}+n^{d_{n-2}}+\ldots+n^{d_0}$ in base b is $n\cdot n^{b-1}$. The smallest number of n digits in base b is $b^{n-1}+1$. Let the function $h:\{2,3,\ldots,16\}\times\{2,3,\ldots\}\to\mathbb{R}$

$$h(b,n)=\ln\left(n\cdot n^{b-1}\right)-\ln\left(b^{n-1}+1\right)\ . \tag{2.1}$$

If we have $h(b,n)>0$ then it follows that Diophantine equation (1.1) may have solutions, if $h(b,n)\le 0$ then equation (1.1) has not solutions. To see the discrete values (b,n) for which equation (1.1) may have solutions refer to figure 1.

The solutions with $n=2$ digits and for $b=16$, $b=15$, $\ldots b=2$:

$$\begin{aligned}
2^1+2^4 &= 14_{(14)}=18\ ,\\
2^6+2^2 &= 62_{(11)}=68\ ,\\
2^2+2^4 &= 24_{(8)}=20\ ,\\
2^1+2^3 &= 13_{(7)}=10\ ,\\
2^4+2^4 &= 44_{(7)}=32\ ,\\
2^1+2^2 &= 12_{(4)}=6\ ,\\
2^1+2^0 &= 10_{(3)}=3\ ,\\
2^1+2^1 &= 11_{(3)}=4\ ,\\
2^2+2^2 &= 22_{(3)}=9\ ,
\end{aligned}$$

The time execution of program for $b=16$, $b=15$, $\ldots b=2$ and $n=2$ was $0.25s$, where they analyzed a total of 1360 cases.

The solutions with $n=3$ digits and for $b=16$, $b=15$, $\ldots b=2$:

$$\begin{aligned}
3^1+3^0+3^4 &= 104_{(9)}=85\ ,\\
3^5+3^0+3^0 &= 500_{(7)}=245\ ,\\
3^5+3^2+3^2 &= 522_{(7)}=261\ ,\\
3^1+3^3+3^3 &= 133_{(6)}=57\ ,\\
3^3+3^4+3^3 &= 343_{(6)}=135\ ,\\
3^1+3^1+3^3 &= 113_{(5)}=33\ ,\\
3^4+3^3+3^2 &= 432_{(5)}=117\ ,\\
3^2+3^1+3^3 &= 213_{(4)}=39\ ,
\end{aligned}$$

The time execution of program for $b=16$, $b=15$, $\ldots b=2$ and $n=3$ was $1.684s$, where they analyzed a total of 17000 cases.

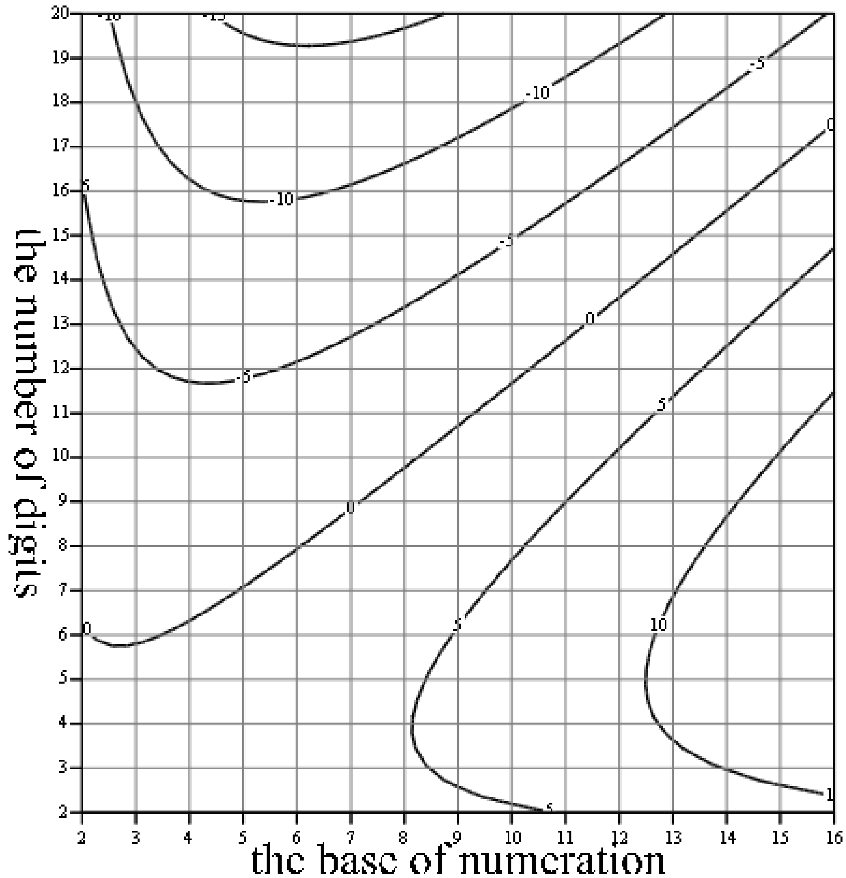

Figure 1: The function h

The solutions with $n = 4$ digits and for $b = 16$, $b = 15$, $\ldots b = 2$:

$$
\begin{aligned}
4^1 + 4^0 + 4^0 + 4^6 &= 1006_{(16)} = 4102 \ , \\
4^4 + 4^6 + 4^2 + 4^4 &= 4624_{(10)} = 4624 \ , \\
4^1 + 4^3 + 4^2 + 4^4 &= 1324_{(6)} = 340 \ , \\
4^1 + 4^0 + 4^1 + 4^0 &= 1010_{(2)} = 10 \ , \\
4^1 + 4^1 + 4^0 + 4^1 &= 1101_{(2)} = 13 \ ,
\end{aligned}
$$

The time execution of program for $b = 16$, $b = 15$, $\ldots b = 2$ and $n = 4$ was $0.593s$, where they analyzed a total of 225352 cases.

The solutions with $n = 5$ digits and for $b = 16$, $b = 15$, $\ldots b = 2$:

$$\begin{aligned}
5^2+5^6+5^3+5^1+5^7 &= 26317_{(14)} = 93905\ ,\\
5^4+5^1+5^2+5^7+5^7 &= 41277_{(14)} = 156905\ ,\\
5^6+5^7+5^7+5^7+5^7 &= 67177_{(14)} = 250005\ ,\\
5^1+5^1+5^3+5^4+5^6 &= 11346_{(11)} = 16385\ ,\\
5^2+5^3+5^1+5^0+5^5 &= 23105_{(6)} = 3281\ ,
\end{aligned}$$

The time execution of program for $b = 16$, $b = 15$, $\ldots b = 2$ and $n = 5$ was $4.898s$, where they analyzed a total of 3103928 cases

The solution with $n = 6$ digits and for $b = 16$, $b = 15$, $\ldots b = 2$:

$$6^5+6^6+6^1+6^4+6^0+6^7 = 561407_{(9)} = 335671\ ,$$

The time execution of program for $b = 16$, $b = 15$, $\ldots b = 2$ and $n = 6$ was $84.037s$, where they analyzed a total of 43912360 cases

The solutions with $n = 7$ digits and for $b = 16$, $b = 15$, $\ldots b = 2$:

$$\begin{aligned}
7^3+7^9+7^6+7^1+7^4+7^5+7^7 &= 3961457_{(15)} = 41314357\ ,\\
7^1+7^5+7^2+7^1+7^3+7^2+7^7 &= 1521327_{(9)} = 840805\ ,\\
7^1+7^5+7^2+7^4+7^1+7^1+7^7 &= 1524117_{(9)} = 842821\ ,\\
7^1+7^5+7^2+7^7+7^4+7^2+7^4 &= 1527424_{(9)} = 845257\ ,\\
7^1+7^2+7^6+7^6+7^6+7^4+7^3 &= 1266643_{(8)} = 355747\ ,
\end{aligned}$$

The time execution of program for $b = 16$, $b = 15$, $\ldots b = 2$ and $n = 7$ was $24 : 53.587$ $mmss$, where they analyzed a total of 633596120 cases

The solutions with $n = 8$ digits and for $b = 16$, $b = 15$, $\ldots b = 2$:

$$\begin{aligned}
8^3+8^9+8^1+8^6+8^0+8^0+8^1+8^6 &= 39160016_{(12)} = 134742546\ ,\\
8^3+8^9+8^1+8^6+8^0+8^0+8^6+8^2 &= 39160062_{(12)} = 134742602\ ,\\
8^7+8^6+8^9+8^4+8^0+8^3+8^6+8^9 &= 76940369_{(12)} = 271061505\ ,\\
8^3+8^5+8^1+8^6+8^1+8^8+8^3+8^5 &= 35161835_{(9)} = 17105936\ ,
\end{aligned}$$

The time execution of program for $b = 16$, $b = 15$, $\ldots b = 2$ and $n = 8$ was $24 : 53.587$ $mmss$, where they analyzed a total of 9280593352 cases

The solutions with $n = 9$ digits and for $b = 14$, $b = 13$, $\ldots b = 2$ not exist, where they analyzed a total of 37253987498 cases.

3 Programs

Program for finding inverse Narcissistic numbers with 2 digits.

$P_2(b) := \big|\ k \leftarrow 0$

$$\left|\begin{array}{l} n \leftarrow 2 \\ \text{for } j \in 0..b-1 \\ \quad D_j \leftarrow n^j \\ \text{for } d_1 \in 1..b-1 \\ \quad \left|\begin{array}{l} N_1 \leftarrow d_1 \cdot b \\ \text{for } d_0 \in 0..b-1 \\ \quad \left|\begin{array}{l} M \leftarrow D_{d_1} + D_{d_0} \\ N \leftarrow N_1 + d_0 \\ \text{if N=M} \\ \quad \left|\begin{array}{l} s_k \leftarrow (d_1\ d_0) \\ k \leftarrow k+1 \end{array}\right. \end{array}\right. \end{array}\right. \\ \text{return s} \end{array}\right.$$

Programs for finding numbers N digits are similar. Program for finding inverse Narcissistic numbers with 3 digits.

$$P_3(b) := \left|\begin{array}{l} k \leftarrow 0 \\ n \leftarrow 3 \\ \text{for } j \in 0..b-1 \\ \quad D_j \leftarrow n^j \\ \text{for } j \in 2..n-1 \\ \quad B_j \leftarrow b^j \\ \text{for } d_2 \in 1..b-1 \\ \quad \left|\begin{array}{l} N_2 \leftarrow d_2 \cdot B_2 \\ \text{for } d_1 \in 0..b-1 \\ \quad \left|\begin{array}{l} N_1 \leftarrow d_1 \cdot b \\ \text{for } d_0 \in 0..b-1 \\ \quad \left|\begin{array}{l} M \leftarrow D_{d_2} + D_{d_1} + D_{d_0} \\ N \leftarrow N_2 + N_1 + d_0 \\ \text{if N=M} \\ \quad \left|\begin{array}{l} s_k \leftarrow (d_2\ d_1\ d_0) \\ k \leftarrow k+1 \end{array}\right. \end{array}\right. \end{array}\right. \end{array}\right. \\ \text{return s} \end{array}\right.$$

Program call is $P_3(9) = [(1\ 0\ 4)]$, where $3^1 + 3^0 + 3^4 = 104_{(9)} = 85$.

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A study on classification algorithms for imbalanced datasets

Carmen Ţerei* Marius Tomescu †

Abstract

Any data set that exhibits an unequal distribution between its classes can be considered imbalanced. Imbalance severely biases the learning process and badly affects the performances of the mining algorithms, because often the classifier has not enough samples of the minority class for proper training. In the current paper we are making a comparative study on some of the most popular classification algorithms for imbalanced datasets.

Mathematics Subject Classification: 68T05

Keywords: decision trees, support vector machines, undersampling, oversampling, cost-sensitive learning

1 Introduction

Technically speaking, any data set that exhibits an unequal distribution between its classes can be considered imbalanced. However, the common understanding in the community is that imbalanced data correspond to data sets exhibiting significant, and in some cases extreme, imbalances. Specifically, this form of imbalance is referred to as a *between-class* imbalance; not uncommon are between-class imbalances on the order of 100:1, 1,000:1, and 10,000:1, where in each case, one class severely outrepresents another[1]. Imbalances of this form are commonly referred to as *intrinsic*, i.e., the imbalance is a direct result of the nature of the dataspace.

However, imbalanced data are not solely restricted to the intrinsic variety. Variable factors such as time and storage also give rise to data sets that are imbalanced. Imbalances of this type are considered *extrinsic*, i.e., the imbalance is not directly related to the nature of the dataspace. Extrinsic imbalances are equally as interesting as their intrinsic counterparts since it may very well occur that the dataspace from which an extrinsic imbalanced data set is attained may not be imbalanced at all.

In addition to intrinsic and extrinsic imbalance, it is important to understand the difference between *relative imbalance* and *imbalance due to*

*carmen_terei@yahoo.com Department of Mathematics and Computer Science, "Aurel Vlaicu" University of Arad, Romania

†tom_uav@yahoo.com Department of Mathematics and Computer Science, "Aurel Vlaicu" University of Arad, Romania

rare instances (or *absolute rarity*). Imbalance due to rare instances is representative in domains where minority class examples are very limited, i.e., where the target concept is rare. In this situation, the lack of representative data will make learning difficult regardless of the between-class imbalance.

This, in fact, is the result of another form of imbalance, a *within-class* imbalance, which concerns itself with the distribution of representative data for subconcepts within a class.

The existence of within-class imbalances is closely intertwined with the problem of *small disjuncts*, which has been shown to greatly depreciate classification performance. The problem of small disjuncts can be understood as follows: a classifier will attempt to learn a concept by creating multiple disjunct rules that describe the main concept. In the case of homogeneous concepts, the classifier will generally create large disjuncts, i.e., rules that cover a large portion (cluster) of examples pertaining to the main concept.

In most cases, the imbalanced class problem is associated to binary classification, but the multi-class problem often occurs and, since there can be several minority classes, it is more difficult to solve [2][3].

Imbalance severely biases the learning process and badly affects the performances of the mining algorithms, because often the classifier has not enough samples of the minority class for proper training. The minority class usually represents the most important concept to be learned, and it is difficult to identify it since it might be associated with exceptional and significant cases [4], or because the data acquisition of these examples is costly [5].

2 Classification algorithms

A large number of approaches have been proposed to deal with the class imbalance problem. These approaches can be categorized into two groups: the *internal* approaches that create new algorithms or modify existing ones to take the class-imbalance problem into consideration [6, 7, 8] and *external* approaches that preprocess the data in order to diminish the effect of their class imbalance.

Furthermore, cost-sensitive learning solutions incorporating both the data (external) and algorithmic level (internal) approaches assume higher misclassification costs for samples in the minority class and seek to minimize the high cost errors [9, 10, 11, 12].

There are more approaches, but they are not subject of the current paper.

3 Internal approaches

1. Decision trees

Decision trees use simple knowledge representation to classify examples into a finite number of classes. In a typical setting, the tree nodes represent the attributes, the edges represent the possible values for a particular attribute, and the leaves are assigned with class labels. Classifying a test sample is straightforward once a decision tree has been constructed. An object is classified by following paths from the root node through the tree to a leaf, taking the edges corresponding to the values of attributes.

C4.5 is a decision tree generating algorithm. It induces classification rules in the form of decision trees from a set of given examples. The decision tree is constructed top-down using the normalized information gain (difference in entropy) that results from choosing an attribute for splitting the data. The attribute with the highest normalized information gain is chosen to make the decision. The C4.5 algorithm then recurs on the smaller sublists.

2. Support vector machines

Support vector machines (SVM) belong to a family of generalized linear models which achieves a classification model based on the linear combination of independent variables. The mapping function in SVM can be either a classification function or a regression function. For classification, nonlinear kernel functions are often used to transform the input data (inherently representing highly complex nonlinear relationships) to a high dimensional feature space in which the input data becomes more separable (i.e., linearly separable) compared to the original input space. Then, the maximum-margin hyperplanes are constructed to optimally separate the classes in the training data. The assumption is that the larger the margin or distance between these hyperplanes the better the generalization performance of the classifier.

4 External approaches

The imbalanced data problem is quite usual in machine learning and data mining applications as it appears in many real-world prediction tasks. However, the techniques and concept of balancing the data prior to model building is relatively new to many information systems researchers. A wide variety of balancing techniques have been applied to data sets in many areas such as medical diagnosis, classifiers for database marketing, property refinance prediction, among others.

Preprocessing imbalanced datasets: resampling techniques

Sampling strategies are often used to overcome the class imbalance problem.

It was empirically proved that applying a preprocessing step in order to balance the class distribution is usually an useful solution [13][14]. Furthermore, the main advantage of these techniques is that they are independent of the underlying classifier.

Resampling techniques can be categorized into three groups or families:

1. Undersampling methods, which create a subset of the original dataset by eliminating instances (usually majority class instances).

2. Oversampling methods, which create a superset of the original dataset by replicating some instances or creating new instances from existing ones.

3. Hybrids methods, which combine both sampling approaches from above.

Synthetic minority over-sampling technique (SMOTE). This heuristic, originally developed by Chawla et al. (2002), generates synthetic minority examples to be added to the original dataset. For each minority example, its k nearest neighbors of the same class are found. Some of these nearest neighbors are then randomly selected according to the over-sampling rate. A new synthetic example is generated along the line between the minority example and every one of its selected nearest neighbors. This process is repeated until the number of examples in all classes is roughly equal to each other. Instead of replicating the existing instances, SMOTE generates new synthetic minority class instances by interpolating between several minority class examples that lie close together. It allows the classifiers to carve broader decision regions, which leads to more coverage of the minority class.

5 Cost-sensitive learning

Instead of changing class distribution, applying cost in decision making is another way to improve the performance of a classifier. Cost-sensitive learning methods try to maximize a loss function associated with a data set. These learning methods are motivated by finding that most real-world applications do not have uniform costs for misclassifications.

Cost-sensitive learning takes into account the variable cost of a misclassification with respect to the different classes. In this case, a cost matrix codifies the penalties $C(i, j)$ of classifying examples of one class i as a different one j.

Weighted LPSVM (wLPSVM) is a solution to the class imbalance problem for LPSVM. It can be seen as a example of the weighting method. The class imbalance problem for LPSVM is mainly caused by the imbalance force of the two-class training error. The idea of wLPSVM is: balancing the training error from two classes to obtain a separating plane learning that is robust to the class imbalance. This training error balancing is achieved by assigning smaller weight to the training errors from the majority class while assigning larger weight to that of the minority class.

The cost-sensitive C4.5 decision tree (C4.5CS) is a method to induce cost-sensitive trees that seeks to minimize the number of high cost errors and, as a consequence of that, leads to minimization of the total misclassification costs in most cases. The method changes the class distribution such that the tree induced is in favor of the class with high weight/cost and is less likely to commit errors with high cost. C4.5CS modifies the weight of an instance proportional to the cost of misclassifying the class to which the instance belonged, leaving the sum of all training instance weights still equal to N. Let $C(j)$ be the cost of misclassifying a class j instance; the weight of a class j instance can be computed as

$$w(j) = C(j)\frac{N}{\sum_i C(i)N_i}$$

such that the sum of all instance weights is $\sum_j w(j)N_j = N$.

6 A comparison of the classification algorithms

A study on several datasets from various fields and different imbalance ratios on which were applied preprocessing methods and cost-sensitive learning algorithms has lead to some useful conclusions.

a. Study on the preprocessing methods

Some of the most representative techniques were compared, developing a ranking according to the performance obtained in each case. The set of methods is composed by the following techniques: SMOTE, SMOTE+ENN, Borderline-SMOTE (Border-SMOTE), ADASYN, Safe-Level-SMOTE (SL-SMOTE), SPIDER2 and DBSMOTE. The interpolations that are computed to generate new synthetic data were made considering the 5-nearest neighbors of minority class instances using the euclidean distance.

The conclusion was that SMOTE+ENN and SMOTE obtained the highest performance, because of some possible reasons. The first one is related to the addition of significant information within the minority class examples by including new synthetic examples. The second reason is that the more sophisticated the technique is, the less general it becomes for the high number of benchmark problems selected for our study.

b. Study on the cost-sensitive learning algorithms

Three different approaches were used, namely the CS-Weighted, MetaCost, and the CostSensitive Classifier (CS-Classifier) from the Weka environment. In the first case, the base classifiers were modified usually by weighting the instances of the dataset to take into account the a priori probabilities, according to the number of samples in each class. In the two latter cases, it was used an input cost-matrix defining $C(+,-) = IR$ and $C(-,+) = 1$.

It has been appreciated that the CS-Weighted approach achieves the highest rank overall. The MetaCost method obtains also a good average for C4.5 and kNN, but it is outperformed by the CS-Classifier when SVM is used.

The good behavior shown by introducing weights to the training examples can be explained by its simplicity, because the algorithm procedure is maintained and is adapted to the imbalanced situation. Therefore, it works similarly to an oversampling approach but without adding new samples and complexity to the problem itself. On the other hand, the MetaCost method follows a similar aim, therefore obtaining high quality results.

7 Conclusions

In this paper were compared some classification algorithms for imbalanced datasets. The approaches which were studied were the datasets preprocessing methods and cost-sensitive learning algorithms. The aim was to emphasize which ones lead to the best data classification results.

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Techniques of evaluating classification algorithms for imbalanced data

Carmen Ţerei* Marius Tomescu †

Abstract

The datasets that exhibit an unequal distribution between their classes can be considered imbalanced. Imbalance severely biases the learning process and badly affects the performances of the mining algorithms. Conventional evaluation practice of using singular assessment criteria, such as the overall accuracy or error rate, does not provide adequate information in the case of imbalanced learning. In the current paper we are presenting some of the metrics which are dedicated for evaluating the data classification for imbalanced datasets.

Mathematics Subject Classification: 68T05

Keywords: imbalanced data, confusion matrix, accuracy, Receiver Operating Characteristic (ROC)

1 Introduction

As the research community continues to develop a greater number of intricate and promising imbalanced learning algorithms, it becomes a must to have standardized evaluation metrics to properly assess the effectiveness of such algorithms.

The measures of the quality of classification are built from a confusion matrix, which records correctly and incorrectly recognized examples for each class. The most used empirical measure, accuracy, does not distinguish between the number of correct labels of different classes, which in the ambit of imbalanced problems may lead to erroneous conclusions. For example a classifier that obtains an accuracy of 90% in a data-set with a 90% of negative instances, might not be accurate if it does not cover correctly any positive class instance.

	Positive prediction	Negative prediction
Positive class	True Positive (TP)	False negative (FN)
Negative class	False Positive (FP)	True negative (TN)

*`carmen_terei@yahoo.com` Department of Mathematics and Computer Science, "Aurel Vlaicu" University of Arad, Romania

†`tom_uav@yahoo.com` Department of Mathematics and Computer Science, "Aurel Vlaicu" University of Arad, Romania

Because of this, instead of using accuracy, more appropriate metrics in this situation are considered. Two common measures, sensitivity and specificity, approximate the probability of the positive (negative) label being true. In other words, they assess the effectiveness of the algorithm on a single class.

2 Numerical metrics

Evaluation measures have a crucial role in classifier analysis and design. Accuracy, Recall, Precision, F-measure, kappa, ACU and some other new proposed measures like Informedness and Markedness are examples of different evaluation measures.

Depending on the problem and the field of application one measure could be more suitable than another. While in the behavioral sciences, Specificity and Sensitivity are commonly used, in the medical sciences, ROC analysis is a standard for evaluation. On the other hand, in the Information Retrieval community and fraud detection, Recall, Precision and F-measure are considered appropriate measures for testing effectiveness.

In a learning design strategy, the best rule for the specific application will be the one that get the optimal performance for the chosen measure. This problem is particularly important in those applications where the instances of a class (the majority one) heavily out- number the instances of the other (the minority) class and it is costly to misclassify samples from the minority class. For example in information retrieval, nontechnical losses in power utilities or medical diagnosis.

For example, in a decision tree the pruning criterion is usually the classification error, which can remove branches related with the minority class. In back- propagation neural networks, the expected gradient vector length is proportional to the class size, and so the gradient vector is dominated by the prevalent class and consequently the weights are determined by this class.

SVMs are thought to be more robust to the class imbalance problem since they use only a few support vectors to calculate region boundaries. However, in a two class problem, the boundaries are determined by the prevalent class, since the algorithm tries to find the largest margin and the minimum error. A different approach is taken in one-class learning, for example one class SVM, where the model is created based on the samples of only one of the classes. In [21] the optimality of one-class SVMs over two-class SVM classifiers is demonstrated for some important imbalanced problems.

In most of the approaches that deal with an imbalanced problem, the idea is to adapt the classifiers that have good Accuracy in balanced domains. A variety of ways of doing this have been proposed: changing class distributions, incorporating costs3 in decision making, and using alternative performance metrics instead of Accuracy in the learning process with the standard algorithms.

The evaluation criteria is a key factor in assessing the classification performance and guiding the classifier modeling. In a two-class problem, the confusion matrix records the results of correctly and incorrectly recognized examples of each class.

Traditionally, the accuracy rate has been the most commonly used empirical measure. However, in the framework of imbalanced datasets, accuracy is no longer a proper measure, since it does not distinguish between the number of correctly classified examples of different classes. Hence, it may lead to erroneous conclusions, i.e., a classifier achieving an accuracy of 90% in a dataset with an IR value of 9 is not accurate if it classifies all examples as negatives.

$$Acc = \frac{TP + TN}{TP + FN + FP + TN}$$

In imbalanced domains, the evaluation of the classifiers' performance must be carried out using specific metrics in order to take into account the class distribution. Concretely, we can obtain four metrics to measure the classification performance of both, positive and negative, classes independently:

- True positive rate: $TP_{rate} = \frac{TP}{TP+FN}$ is the percentage of positive instances correctly classified.
- True negative rate: $TN_{rate} = \frac{TN}{FP+TN}$ is the percentage of negative instances correctly classified.
- False positive rate: $FP_{rate} = \frac{FP}{FP+TN}$ is the percentage of negative instances misclassified.
- False negative rate: $FN_{rate} = \frac{FN}{TP+FN}$ is the percentage of positive instances misclassified.

Other metric of interest to be stressed in this area is the geometric mean of the true rates, which can be defined as:

$$GM = \sqrt{\frac{TP}{TP + FN} \cdot \frac{TN}{FP + TN}}$$

This metric attempts to maximize the accuracy on each of the two classes with a good balance, being a performance metric that correlates both objectives. However, due to this symmetric nature of the distribution of the geometric mean over TPrate (sensitivity) and the TNrate (specificity), it is hard to contrast different models according to their precision on each class.

Another significant performance metric that is commonly used is the F-measure:

$$F_m = \frac{(1 + \beta^2)(PPV \cdot TP_{rate})}{\beta^2 PPV + TP_{rate}}$$

$$PPV = \frac{TP}{TP + TP + FP}$$

According to the previous comments, some authors try to propose several measures for imbalanced domains in order to be able to obtain as much information as possible about the contribution of each class to the final performance and to take into account the IR of the dataset as an indication of its difficulty.

For example, in [10,14] the Adjusted G-mean is proposed. This measure is designed towards obtaining the highest sensitivity (TPrate) without decreasing too much the specificity (TNrate). This fact is measured with respect to the original model, i.e. the original classifier without addressing the class imbalance problem.

$AGM = \dfrac{GM + TN_{rate} \cdot (FP + TN)}{1 + FP + TN}$; If $TP_{rate} > 0$

$AGM = 0$; If $TP_{rate} = 0$

Additionally, in [54] the authors presented a simple performance metric called Dominance, which is aimed to point out the dominance or prevalence relationship between the positive class and the negative class, in the range $[-1, +1]$. Furthermore, it can be used as a visual tool to analyze the behavior of a classifier on a 2-D space from the join perspective of global precision (Y-axis) and dominance (X-axis).

$Dom = TP_{rate} - TN_{rate}$

The same authors, using the previous concept of dominance, Index of Balanced Accuracy (IBA). IBA weights a performance measure, that aims to make it more sensitive for imbalanced domains. The weighting factor favors those results with moderately better classification rates on the minority class. IBA is formulated as follows:

$IBA_{\alpha}(M) = (1 + \alpha \cdot Dom)M$

where $(1 + \alpha \cdot Dom)$ is the weighting factor and M represents a performance metric. The objective is to moderately favor the classification models with higher prediction rate on the minority class (without underestimating the relevance of the majority class) by means of a weighted function of any plain performance evaluation measure.

3 Graphical metrics

A well-known approach to unify these measures and to produce an evaluation criteria is to use the Receiver Operating Characteristic (ROC) graphic. This graphic allows the visualization of the trade-off between the benefits (TPrate) and costs (FPrate), as it evidences that any classifier cannot increase the number of true positives without also increasing the false positives.

ROC curves, graph true positive rates on the y-axis vs. the false positive rates on the x-axis. The resulting curve illustrates the trade-off between detection rate and false alarm rate. The ROC curve illustrates the performance of a classifier across the complete range of possible decision thresholds, and accordingly does not assume any particular misclassification costs or class prior probabilities.

For a single numeric measure, the area under the ROC curve (AUC) is widely used, providing a general idea of the predictive potential of the classifier. A higher AUC is better, as it indicates that the classifier, across the entire possible range of decision thresholds, has a higher true positive rate. The AUC is the performance metric used for this study. Provost and Fawcett give an extensive overview of ROC curves and their potential use for creating optimal classifiers.

The Area Under the ROC Curve (AUC)corresponds to the probability of correctly identifying which one of the two stimuli is noise and which one is signal plus noise. The AUC provides a single measure of a classifier's performance for evaluating which model is better on average. The AUC measure is computed just by obtaining the area of the graphic:

$$AUC = \frac{1 + TP_{rate} + FP_{rate}}{2}$$

According to the authors, the main advantage of this type of methods resides in their ability to depict the trade-offs between evaluation aspects in a multidimensional space rather than reducing these aspects to an arbitrarily chosen (and often biased) single scalar measure. In particular, they present a review of several representation mechanisms emphasizing the best scenario for their use; for example, in imbalanced domains, when we are interested in the positive class, it is recommended the use of precision-recall graphs [36]. Furthermore, the expected cost or profit of each model might be analyzed using cost curves, lift and ROI graphs.

4 Conclusions

In this paper we presented some of the techiques which are adapted for evaluating the quality of the algorithms for the classification of imbalanced datasets. There are several numerical metrics and also some graphical metrics.

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Clusters describing IFRS adoption stage

Mariana Nagy* Burca Valentin ‡
Negruşa Adina †

Abstract

Financial statements are essential on capital investment decision, especially in the new context of a globalized world, increasing international capital market listing, or amplified commercial relation among national economies, and the promising international process of accounting convergence. Our study aim to analyze the closeness of different IFRS adoption strategies, by creating four clusters, based on the level of IFRS adoption referring to consolidated financial statements, individual financial statements and simplified financial statements accounting regulation. The results reveal small differences between the clusters determined, leading to a promising future of the recent IASB projects of continuous improvement of existing standards and for new standards.

Jel classification: M21, M41.

Keywords: *IAS/IFRS, accounting convergence, clustering, enforcement, globalization.*

1 Introduction

Declared objectives of presenting financial position, financial performance and modification in financial position within financial statements has raised vivid debate among the researchers, especially in the past decades characterized by a strong will of reducing international accounting differences that affected seriously decision-making process. Most recent ambitious project of international accounting convergence, conducted by IASB in cooperation with FASB, supported by international professional organizations and local government institutions, has underlined not just the opportunity of improving financial information quality, but also has shown chances of cost reduction at firm-level and country-level as well.

Though, the process of international accounting convergence seems to take a longer time than expected, the signals are positive as nowadays there are more than 100 jurisdictions that decided to adopt IFRS opting for different strategies well known at international level. Once the capital markets have opened on international listing and foreign investments have increased exponentially, especially

*`first.author@uav.ro` PhD prof. "Aurel Vlaicu" University of Arad Department of Mathematics and Computer Science, "Aurel Vlaicu" University of Arad, Romania

†`second.author@uvt.ro` PhD, assoc. prof., "Babeş Bolyai" University, Cluj-Napoca

‡`third.author@uvt.ro` PhD candidate, West University of Timisoara Department of Mathematics, West University of Timişoara, Romania

in the case of the emerging and economies in transition, the incentives offered for IFRS adoption has determined investors and international financial institutions to make pressure for IFRS adoption, against the political factor option (Burca & Cilan, 2013).

Unfortunately, even within countries with similar accounting systems, persist accounting differences caused by various factors describing the environment in all its aspects. The problem of overt and covert options permitted by IFRSs and the ambiguity on defining and utilizing various conceptual terms has led a more relevant financial information than the previous period of local GAAPs, but hasn't solved completely the problem of opportune accounting strategies (Nobes, 2008; Doupnik & Perrera, 2012).

On this article we attempt to make a classification of jurisdictions worldwide considering different strategies of IFRS adoption used, as the adoption described a gradual transition from local GAAP to IFRS. The global positive perception for IFRS adoption is valid, being reiterated by all accounting regulators. But there still persist reluctance on implementing IFRS for all financial statements for most of the jurisdictions. We subscribe to Christensen (2012) opinion who sustain that a cause may be the inconsistencies around benefits and costs of IFRS adoption revealed by numerous recent studies.

2 Literature review

The step towards a unique set of international core standards has been promoted fervently by capital market actors, as the comparability and value relevance of the financial information is essential investment decision. The option for IFRS adoption is recommended in the light of the results of recent studies that envisage a more accurate reporting framework drawn by IFRSs, that improve significantly the quality of financial information (Daske et. al., 2008; Ramana & Sletten, 2009; Chen et. al., 2010; Barth et. al., 2012; Daske et. al., 2013), leading to macroeconomic and microeconomic positive effects (Daske et. al., 2013; Banker et. al., 2014; Biddle et. al., 2013).

Even if there were outlined serious compatibility issues between local accounting system and IFRS provisions, recent studies revealed that fundamental in the transition process to IFRS regulation is the power of will of change. Moreover, the discussion around the overt and covert accounting policy options seems to be more complex, underlining the trade-off between the preparers and the user of the financial information. Essential on this debate is the power of intention when designing the accounting policies (Fields et. al., 2001; Jianu, 2012).

It is less important what financial statement depict a faithful representation of the economic reality, as this can be easily set by the financial system, in the limits of efficiency (Ball & Shivakumar, 2005; Nobes & Parker, 2008). Important is that the mechanism and tools of financial reporting to disclose any doubt of accounting manipulation. Here the role of the institutional framework of a jurisdiction, and complementary a consolidated mechanism of corporate governance, have to bring their contribution to the proper enforcement of the new accounting regulation (Samaresekera et. al., 2012; Christensen et. al., 2013).

Most of negative perception translated in reluctance to IFRS adoption of some firms was mainly determined by the lack of prior studies, or insignificant reporting incentives provided by an emerging capital market. On this context it is essential that accounting standard-setters and local enforcement institutions to pay attention to the role of reporting incentives (especially the market-driven ones), cause the quality of the accounting standards does not necessary traduce into qualitative financial reporting (Burgstahler et. al.2006; Jayaraman & Verdi, 2014).

The option for IFRS standards, but not for US GAAP, is not only a matter of a economic cost-benefits analysis, but also, a matter of politics of accounting standards. Ramanna (2013) has revealed different strategies of harmonization of accounting regulation with IFRS provisions, explaining these strategies by two factors: country's potential influence on IASB decisions and the proximity of the existing accounting system to the ones of the political powers at the IASB. Nobes (2011) identify six ways of IFRS implementation:

⇒ full adoption of IFRS, when the jurisdiction use the most recent versions of IFRSs, without passing the content of the standards though an endorsement process;

⇒ insertion of IFRSs into law, which differs from first way by delaying the date of use of the new or revised standards;

⇒ endorsing IFRS, which involves detailed scrutiny of all IFRS standards, leading many times to several carve-outs, altering the content quality of the original IFRS standard;

⇒ full convergence with IFRS, with expressed intention of full compliance, which imply IFRS insertion into national regulation with several amendments leading to textual changes, several carve-outs, and early adoption banning;

⇒ adapting IFRS, is the case of classic accounting harmonization, when a jurisdiction use IFRSs as a source of regulation;

⇒ allowing IFRS, is the way jurisdictions preserve their power of accounting regulation for a restricted category of companies, as most of jurisdiction do currently when discussing potential IFRS for SMEs adoption.

IFRS are currently used in the consolidated financial statements in IFRS of over 100 jurisdictions .Adopting IFRS takes many forms. While some states choose to implement the full incorporation of IFRS into national legislation (Trinidad and Tobago -1988 , Bosnia - Herzegovina 1995 Vietnam - 2002 Bahamas 2007); or have successfully completed harmonization projects of local accounting standards with the international standards (China - 2006-2010 Algeria , India , Indonesia - 2012) most jurisdictions choose adopting IFRS in its original form , by limiting the applying sphere only to consolidated financial statements of listed companies . But currently, there are also jurisdictions (Cameroon , Congo , Senegal - 2014 , Bolivia , Colombia -2015, US- 2016 , Saudi Arabia - 2017) that did not appeal to the adoption of IFRS (around 15% of jurisdictions worldwide) but many of them have expressed their desire to conduct a future project for implementing IFRS .

3 Methodological research

The aim of the study is to classify different jurisdictions in order to observe a trend on the next future regarding IFRS adoption worldwide. The sample used consist of 57 jurisdiction, which can be split by region as the chart below show.

The information regarding regulation of financial reporting according to IFRS provisions are collected from multiple sources reminding here PWC (2013) study ,,*IFRS adoption by country*", Deloitte website dedicated for IFRS concerns and IASB website relating jurisdictions profiles.

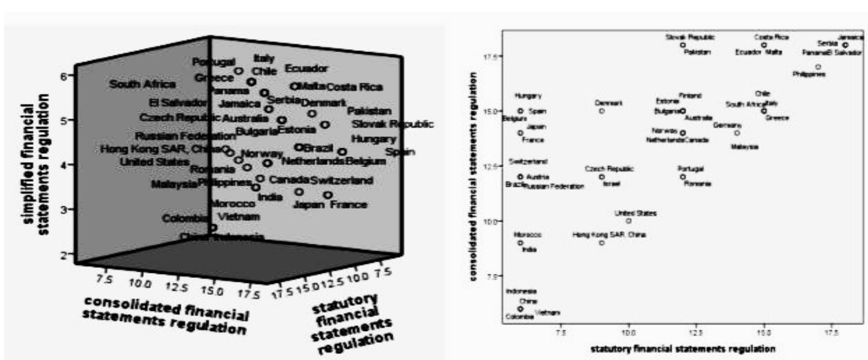
Making abstraction of the endorsement process used in case of some jurisdiction, as is the case of EU countries, the jurisdictions will be classified based on a score variable designed to describe the strategies of IFRS adoption mentioned above. The sample countries are first analyzed based on the treatment for IFRS use for the following groups of companies: are first classified in three groups: listed foreign companies, listed domestic companies and unlisted companies and SMEs. The scor variable is calculated considering different multiples to be used for the type of companies, the level of IFRS use on consolidated and statutory financial statement as well is order to underline the reluctance of most developed jurisdictions to apply IFRS for all financial statements. For the treatment of prohibiting IFRS we consider the value 1, as for the treatment of permitting IFRS we use 2 and for the treatment of mandatory use of IFRS ($value_m$) there is allocated value 3.

The classification is realized using the *k-means cluster analysis* technique which have as variables the scores for each type of financial statement. Leuz (2010) has achieved to classify 49 countries into a small number of clusters considering a suite of factors concerning institutional characteristics, such as legal origin, disclosure requirements rule of law etc. Nanda & Wysocki (2011) study the causal relation between societal trust and firms' voluntary and regulated financial reporting and disclosure quality. Later Nanda & Wysocki (2013) used the same method of clustering, this time to analyze the relation between societal trust and firms' financial transparency, and how firms' external capital demand affects this relation.

4 Discussion and results

In our analysis we will consider the number of necessary clusters to be identified by analyzing graphically the positioning of all jurisdictions on a 3D graph describing all three above mentioned scores to be used (Field, 2005). The basic rationale of our approach is that all jurisdictions included on our sample different more or less on the level IFRS is allowed or even mandatory to use on preparing financial statements. The evidence reveal a gradual IFRS adoption showing that on first stage the consolidated financial statements only are prepared according to IFRS.

Representation by type of financial statements regulation

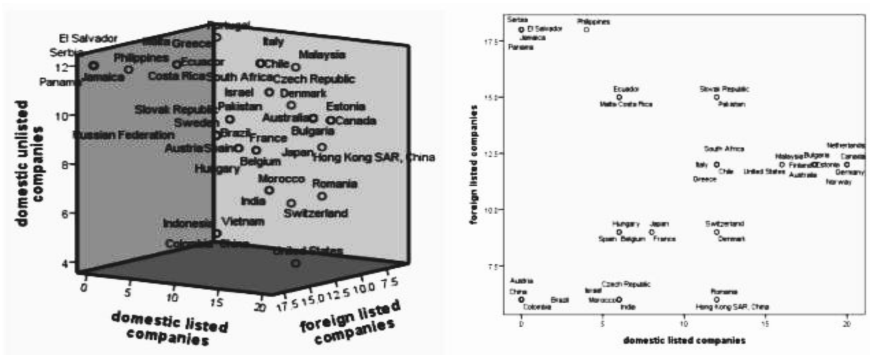


Source: projection with SPSS 20.0

On the second stage, the national regulators decide the extent of IFRS use to preparation of individual financial statements. This step depends strongly on how strong is the relation between accounting policies and fiscal rules, as the continental accounting systems are well known for fiscal economies obtained with the condition to comply with a prescribed formula of accounting registration of different types of economic transactions (Lamb et. al.,1998).

The last stage is supposed to make the transition to IFRS of the small and medium enterprises, too. On this direction IASB has designed a simplified set of accounting standards, meant to respond properly to the lower demand of financial information implied. This way they have published the IFRS for SMEs in order to eliminate the duality of the national accounting system of jurisdictions that have already decided IFRS adoption for listed companies. But, till now there is long debate around the opportunity of this decision. For instance, the EU community has given great attention to this subject, even revising the European directives regulating accounting treatments.

Representation by type of company group



Source: projection with SPSS 20.0

We have represented graphically the scores reflecting level of IFRS adoption by type of financial statements regulation and by type of company analyzed as well. The general conclusion is that jurisdictions expressed their will to adopt IFRS, but just for consolidated financial accounts as they are not used on fiscal purpose, explained also by the negative strong correlation of about -0.288.

Indeed, there is seen on last years a more clear direction towards extent of IFRS use to individual financial statements, too. The statistics reveal a relatively high level of IFRS adoption, but with higher variation within the sample, but the focus for mandatory IFRS adoption still continue to be in case of listed companies. Unlisted companies are rather permitted to use IFRS on statutory annual accounts, but just as a second set of financial statements which lead to higher costs of preparation implied by the reconciliation costs registered. This way, the voluntary adoption is really low, even if the literature confirmed higher benefits of IFRS voluntary adoption than in case of mandatory IFRS adoption.

But this trend can be mainly justified by the pressure exerted by investors and international financial institutions in case of developed countries, as they already have a solid knowledge in financial accounting regulation, with a strong financial education provided by well know universities, and a powerful accounting research tradition. But, jurisdictions perception on IFRS adoption depends especially on the ratio between similarities and differences between the local GAAP and IFRS standards.

Contrary, the underdeveloped economies and economies in transition, the solution of fully IFRS adoption is more reliable as this would lead to significant government cost reduction through the deregulation process. Here potential issues can raise around the quality of training the professional on the IFRS hot topics, such countries being forced more or less to access consulting services provided by accounting and auditing international offices, such as the Big4

The same graph illustrate a persistent visible reluctance of jurisdictions towards IFRSs for SMEs adoption. But this reluctance is not valid only for the political factor, being shown by managers of SMEs as well. In the case of SMEs there is more visible the cultural component of managers behavior regarding financial reporting strategies. If the fiscal factor remain a constant determinant

			Cluster			
Financial statements type	**Company type**	**Treatment**	*1*	*2*	*3*	*4*
Consolidated financial statements	domestic listed companies	prohibited	5.56%	14.29%	5.26%	9.09%
		mandatory	***94.44%***	***57.14%***	***84.21%***	***90.91%***
		permitted	0.00%	28.57%	10.53%	0.00%
	domestic unlisted companies	prohibited	33.33%	42.86%	21.05%	27.27%
		mandatory	22.22%	14.29%	21.05%	27.27%
		permitted	***44.44%***	***42.86%***	***57.89%***	***45.45%***
	foreign listed companies	prohibited	5.56%	14.29%	5.26%	0.00%
		mandatory	***72.22%***	***57.14%***	***57.89%***	***90.91%***
		permitted	22.22%	28.57%	36.84%	9.09%
Satutory financial statements	domestic listed companies	prohibited	22.22%	28.57%	36.84%	***36.36%***
		mandatory	27.78%	***42.86%***	***42.11%***	***36.36%***
		permitted	***50.00%***	28.57%	21.05%	27.27%
	domestic unlisted companies	prohibited	38.89%	***42.86%***	42.11%	***45.45%***
		mandatory	16.67%	14.29%	10.53%	9.09%
		permitted	***44.44%***	***42.86%***	***47.37%***	***45.45%***
	foreign listed companies	prohibited	22.22%	28.57%	36.84%	27.27%
		mandatory	22.22%	***42.86%***	36.84%	***36.36%***
		permitted	***55.56%***	28.57%	26.32%	***36.36%***
SMEs		prohibited	***72.22%***	***71.43%***	***78.95%***	***72.73%***
		mandatory	11.11%	0.00%	15.79%	0.00%
		permitted	16.67%	28.57%	5.26%	27.27%

Source: own projection

of accounting differences, as in the case of individual financial statements, there is evidence of a stronger correlation between score of IFRS for SMEs adoption[1] and level of uncertainty avoidance perceived.

Proceeding to clustering the sample, we have obtained four slightly equal clusters. They are different, especially based on the variations of treatment regarding IFRS adoption for consolidated accounts preparation and treatment regarding IFRS adoption for simplified accounts preparation:

⇒ the first cluster is dominated by the highest level of IFRS adoption, in case of all types of companies and all levels of financial reporting. This can be explained by the emergent capital markets, except the case of New Zeeland which follow the way of IFRS cause of its common law origins;

⇒ the second cluster group jurisdictions which prefer IFRS adoption especially for the foreign listed companies, in order to support potential foreign investor, leading to a growing market capitalization;

⇒ the third cluster is mainly comprised by EU community jurisdictions, which are affected by the IAS regulation (1606/2002) which obliged listed companies to report under the IFRS provisions the consolidated accounts.;

⇒ the forth cluster contains countries that rather permit IFRS use than decide for mandatory adoption in most cases, especially in the case of unlisted companies and SMEs

There are not major differences between the four clusters, which means that all jurisdictions tend to consider IFRS adoption as a viable scenario for future

[1]Even if we refer to IFRS for SMEs adoption as the scenario in case of SMEs, there are cases which still use complete version of IFRSs for SMEs, probably till the debate around the opportunity for IFRS for SMEs adoption will end in favor;

Clustering results

First group	Second group	Third group	Forth group
South Africa	China	Australia	Austria
Chile	Colombia	Bulgaria	Belgium
Costa Rica	Hong Kong	Canada	Brazil
Ecuador	India	Czech Republic	Switzerland
Philippines	Indonesia	Denmark	France
Greece	Morocco	Estonia	Japan
Italy	Vietnam	Finland	Russian Federation
Jamaica	USA	Germany	Spain
Malta		Israel	Sweden
New Zealand		Luxembourg	Hungary
Pakistan		Malaysia	Uruguay
Panama		United Kingdom	
Peru		Norway	
El Salvador		Netherlands	
Serbia		Poland	
Slovak Republic		Portugal	
Thailand		Singapore	
Turkey		***Romania***	
Venezuela			

Source: projection with SPSS 20.0

accounting regulation. The differences consist of the timeframe of the transition process, the date of implementation, the differences of level of local GAAP harmonization with IFRS etc. All the clusters describe one general trend of IFRS adoption which predict a long way to fully IFRS adoption. We tend to even think that this objective is impossible. But the interest for IFRS philosophy and rationale still remain high among the regulators. Thus, even if the process of international accounting convergence will not find a reliable solution for international accounting systems uniformity, the process of international harmonization will continue.

5 Conclusions

It is obvious that IFRS represent the global solution for a new era of an international accounting language. Just that the uncertainty for new of most jurisdictions is visible and the more flexible accounting philosophy promoted by IASB means the professionals rationale to become the center of the entire financial reporting process. Unfortunately, not all the time the preparers of the annual accounts prove to be honest and sincere, often making use of various creative accounting techniques, forced by economic context and shared life values. Adding to this reality the fact that, on an international capital market and a globalized economy, the international financial analyses are of high importance on decision-making, the financial information is disclosed based on different accounting regulations, the quality of financial information decrease drastically.

If the contracting theory can't be solved by specific regulation, just through an open negotiation language establishing the incentives area, the accounting differences were significantly reduced by the international accounting harmonization process, and after by the more ambitious international accounting convergence efforts.

The evidence confirm as a general accepted solution worldwide the IFRS adoption on a gradual process of transition. Unfortunately, the reality reveal just a superficial success of the accounting convergence success as there is long way till all jurisdictions will decide to extend IFRS use not just in purpose of preparing the consolidated accounts. This is because of the reluctance of some players who are still not sure about the impact of IFRS quality on the economy and on the profound reform of the society.

We have revealed that the jurisdiction, even if differ based on the strategies of IFRS adoption, eventually have built a positive perception about IFRS quality and the necessity of international accounting practices comparability.

The complexity of the accounting convergence process explain the long timeframe necessary for full compliance with IFRS at a generalized level. The odds will be in favor of IASB project till the political factor will support IASB work. Otherwise, the limited legitimacy of IASB will not be enough for a proper IFRS implementation. We underline the fact that IFRS accounting quality does not transpose mandatory into qualitative accounting practices. There is necessary a continuous relation of communication between managers, shareholders and stakeholders as well, in order to build the bases of a confident business model which can be financed at a lower cost of capital, registering high rate of returns.

It is obvious that nobody expect that IFRS provisions will solve entire problems raise along the financial reporting supply chain. That is why preparers and users as well have to understand that their contribution to financial disclosures optimization is essential, especially through the voluntary financial reporting component.

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Series representations for random distribution fields

Păstorel Gaşpar*   Lorena Popa †

Abstract

In [3] we started a study of multivariate second order random distribution fields (m.s.o.r.d.f). This note is devoted to ordinary, modular and tensor series representations and some relation between these for m.s.o.r.d.f. Finally a connection of series representation of m.s.o.r.d.f. with the correlation distribution is given.

Mathematics Subject Classification:

Keywords: random distribution field, series representation

1 Introduction

Series and integral representations play an important sole in the study of stochastic processes and random fields. Integral representations appear often in the case of stationary random fields (see [6] and the literature cited there). In [3] the authors extended the study of infinite dimensional random fields to the setting of random distribution fields and stochastic mappings, while [4] is devoted to the stationarity and stationarily cross correlatedness including integral representations.
Series representations for stochastic processes appeared first in [7] and [8] and then in [9] and [1]. This study was continued in [5] and [6]. In this paper we consider the three types of series representations presented in [6, Sec. 4.6] in the extended framework of [3] and [4]. Hence, we may use to a significant extent concepts and notations from [6], [3] and [4].
So $(\Omega, \mathscr{A}, \wp)$ will be a probability space and H a separable Hilbert space, for which $\mathcal{B}(H)$ and $\mathscr{C}_1(H)$ mean the Banach algebra of all continuous linear operators on H, respectively the ideal of trace class operators from $\mathcal{B}(H)$. The space $L^2_0(\wp, H)$ of second order H-valued random variables on Ω of zero mean will be denoted by $\mathscr{H}$. It's elements will be also called *multivariate second order random variables*, or briefly *m.s.o.r.v.* It will be endowed (see [6, Sec. 1.3]) with a normal Hilbert $\mathcal{B}(H)$-module structure,

*`pastorel.gaspar@uav.ro` Department of Mathematics and Computer Science, "Aurel Vlaicu" University of Arad, Romania

†`lorena.popa@yahoo.com` Department of Mathematics and Computer Science, "Aurel Vlaicu" University of Arad, Romania

having a $\mathscr{C}_1(H)$-valued gramian

$$[\mathfrak{h}, \mathfrak{k}]_{\mathscr{H}} = \int_\Omega \mathfrak{h}(\omega) \otimes \overline{\mathfrak{k}(\omega)} d\wp(\omega), \quad \mathfrak{h}, \mathfrak{k} \in \mathscr{H},$$

where $\otimes$ is the tensor product in the sense of Schatten, while $(\mathfrak{h}, \mathfrak{k}) := \mathrm{tr}[\mathfrak{h}, \mathfrak{k}]_{\mathscr{H}}$, $\mathfrak{h}, \mathfrak{k} \in \mathscr{H}$, is the natural scalar product making $\mathscr{H}$ a Hilbert space. As such it can be identified (see also [6, Sec. 2.1]) with the Hilbert space tensor product $H \otimes L_0^2(\wp)$, where $L_0^2(\wp) : = L_0^2(\wp, \mathbb{C})$. The norm on $\mathscr{H}$ will be consequently expressed by

$$\|\mathfrak{h}\|_{\mathscr{H}} : = (\mathfrak{h}, \mathfrak{h})_{\mathscr{H}}^{1/2} = \|[\mathfrak{h}, \mathfrak{h}]_{\mathscr{H}}\|_{\mathscr{C}_1(H)}^{1/2}, \quad \mathfrak{h} \in \mathscr{H}.$$

If we intent to consider the m.s.o. stochastic processes not only as $\mathscr{H}$-valued *functions* on $\mathbb{R}^d$, but more generally as $\mathscr{H}$-valued *distributions* on $\mathbb{R}^d$, then we consider $\mathcal{D}(\mathbb{R}^d) = \mathcal{D}_d$, the space of test functions in the theory of distributions instead of the time parameter set of the field. Such an object will be called *m.s.o. random distribution field (m.s.o.r.d.f.)*.

2 Series representations of m.s.o.r.d.f.

In this section we extend different kinds of series representations from multivariate second order random fields as were given in [6, Sec. 4.6] to the setting treated in [3] and [4] of multivariate second order random distribution fields.

A family of elements $\{\mathfrak{h}_\iota\}_{\iota \in \mathcal{I}}$ of $\mathscr{H}$ of norm 1 is said to be *gramian orthonormal* if $[\mathfrak{h}_\iota, \mathfrak{h}_\iota]^2 = [\mathfrak{h}_\iota, \mathfrak{h}_\iota]$ for each $\iota \in \mathcal{I}$ and $[\mathfrak{h}_\iota, \mathfrak{h}_\kappa] = 0$, for $\iota \neq \kappa \in \mathcal{I}$. A maximal gramian orthonormal family will be called *gramian orthonormal basis*.

The following lemma (see [5]) is also useful in what follows.

Lemma 2.1 *For a subset $\mathcal{H} \subset \mathscr{H}$ of the normal Hilbert module $\mathscr{H}$, we have that the submodule generated by it, $\mathfrak{S}(\mathcal{H})$, is separable if the Hilbert space generated by it, $Sp(\mathcal{H})$, is separable.*

We shall refer here to three such ways to express a multivariate second order random distribution field $U = \{U_\varphi\}_{\varphi \in \mathcal{D}_d}$. Namely we say that

- U admits an *ordinary series representation*, briefly *o.s.r.*, if there exists a sequence of scalar distributions $\{u_j\}_{j \in \mathbb{N}}$ and a sequence $\{\mathfrak{h}_j\}_{j \in \mathbb{N}}$ of elements from $L_0^2[\wp, H] = : \mathscr{H}$ such that

$$U_\varphi(\omega) = \sum_{j=1}^\infty u_j(\varphi)\mathfrak{h}_j(\omega)\,; \quad \omega \in \Omega, \tag{2.1}$$

  which converges in the norm of $\mathscr{H}$ for each $\varphi \in \mathcal{D}_d$;

- U admits a *modular series representation*, briefly *m.s.r.*, if there exists a sequence of $\mathcal{B}(H)$ - valued distributions $\{\mathfrak{u}_j\}_{j \in \mathbb{N}}$ and a sequence of functions $\{\mathfrak{h}_j\}_{j \in \mathbb{N}}$ from $L_0^2[\wp, H] = \mathscr{H}$ such that

$$U_\varphi(\omega) = \sum_{j=1}^\infty \mathfrak{u}_j(\varphi)\mathfrak{h}_j(\omega)\,; \quad \omega \in \Omega, \tag{2.2}$$

  which converges in the norm of $\mathscr{H}$ for each $\varphi \in \mathcal{D}_d$;

- U admits a *tensor series representation*, briefly *t.s.r.*, if for each $\varphi \in \mathcal{D}_d$, U_φ is the sum of a series of elementary tensors from $\mathcal{D}'_d(\mathscr{H})$ written in the form $\mathcal{D}'_d(H) \otimes L^2_0(\wp)$, i.e. there exists a sequence of H-valued distributions $\{\mathbf{u}_j\}_{j\in\mathbb{N}}$ and a sequence of functions $\{f_j\}_{j\in\mathbb{N}}$ from $L^2_0(\wp)$ such that for each $\varphi \in \mathcal{D}_d$

$$U_\varphi(\omega) = \sum_{j=1}^{\infty} \mathbf{u}_j(\varphi) f_j(\omega)\,; \quad \omega \in \Omega, \tag{2.3}$$

also converging in the norm of $\mathscr{H}$.

We notice that it is immediate that a series representation of U of type (2.1), implies a modular series representation, i.e. of type (2.2), if we put $\mathfrak{u}_j(\varphi) = u_j(\varphi) I_H,\ \varphi \in \mathcal{D}_d$.

Proposition 2.2 *Suppose that the m.s.o.r.d.f. U has one of the above three representations, where the systems $\{\mathfrak{h}_j\}_{j\in\mathbb{N}}$ are orthonormal in $\mathscr{H}$ as a Hilbert space in the case of o.s.r., gramian orthonormal in $\mathscr{H}$ as a normal Hilbert $\mathcal{B}(H)$-module in the case of m.s.r. and $\{f_j\}_{j\in\mathbb{N}}$ are orthonormal in $L^2_0(\wp)$ in the case of t.s.r. Then the corresponding distributions forming the systems $\{u_j\}_{j\in\mathbb{N}}$, $\{\mathfrak{u}_j\}_{j\in\mathbb{N}}$, $\{\mathbf{u}_j\}_{j\in\mathbb{N}}$ are respectively uniquely determined.*

Proof. We shall restrain ourselves to the last case. Suppose U has a series representation (2.3) with an orthonormal basis $\{f_j\}_{j\in\mathbb{N}} \subset L^2_0(\wp)$ and $U_\varphi(\omega) = \sum\limits_{j=1}^{\infty} \mathbf{v}_j(\varphi) f_j(\omega)\,;\ \varphi \in \mathcal{D}_d,\ \omega \in \Omega$ is another tensor representation of $U_\varphi, \varphi \in \mathcal{D}_d$. Then for some arbitrary $h \in H$ it holds

$$0 = \left[f_l h, \sum_{j=1}^{\infty} \big(\mathbf{u}_j(\varphi) - \mathbf{v}_j(\varphi)\big) f_j \right] = h \otimes \overline{\big(\mathbf{u}_l(\varphi) - \mathbf{v}_l(\varphi)\big)}\,, \varphi \in \mathcal{D}_d, l \in \mathbb{N}.$$

As h as taken arbitrary it results that $\mathbf{u}_l \equiv \mathbf{v}_l\,, l \in \mathbb{N}$. □ Some characterizations of the series representability of m.s.o.r.d.f. are contained in

Theorem 2.3 *Let $\{U_\varphi\}_{\varphi\in\mathcal{D}_d}$ be a m.s.o.r.d.f. The following characterizations hold:*

(i) U has a representation of type (2.1) iff its vector time domain $\mathscr{H}_{(U)}$ is separable;

(ii) U has a representation of the form (2.2) iff its modular time domain $\mathscr{H}_U$ is separable;

(iii) U has a representation of the form (2.2) iff U has a tensor representation as a series, i.e. of the form (2.3).

Proof. It suffices to show (ii) since (i) can be deduced from (ii) for $H = \mathbb{C}$. Suppose that representation (2.2) holds for U. It results then easy that $\mathscr{H}_U$ is contained in the Hilbert $\mathcal{B}(H)$ - module generated by $\{\mathfrak{h}_j,\ j \in \mathbb{N}\}$, i.e. $\mathscr{H}_U$ is separable. Conversely, if $\mathscr{H}_U$ is separable, there exists a countable set $\{\mathfrak{h}_j,\ j \in \mathbb{N}\}$ in $\mathscr{H}_U$ that forms a gramian basis for $\mathscr{H}_U$, wherefrom for each $\varphi \in \mathcal{D}_d$, the Fourier development in $\mathscr{H}_U$ of U_φ

$$U_\varphi(\omega) = \sum_{j=1}^{\infty} [U_\varphi, \mathfrak{h}_j] \mathfrak{h}_j(\omega), \tag{2.4}$$

infers (2.2) with $\mathfrak{u}_j(\varphi) = [U_\varphi, \mathfrak{h}_j]_{\mathscr{H}}$.

(iii) Now, if (2.2) holds, according to (ii) we have that $\mathscr{H}_U$ is separable. From the operatorial model of the normal Hilbert $\mathcal{B}(H)$ - module (see [6, Corr.7, pp.30]) $\mathscr{H}_U$ can be written as in (2.2) from [3] as the Hilbert space tensor product $H \otimes \overline{G}_U$, hence the measurements space $G_U(\subset L_0^2(\wp))$ is separable. Choosing an orthonormal basis $\{f_j\}$ in G_U and $h \in H$, $\|h\| = 1$, it results (by Proposition 11, pp. 31 [6]) that the family $\{h \otimes f_j\}_{j \in \mathbb{N}}$ (where $(h \otimes f_j)(\omega)$ is identified with the product $f_j h$) is a gramian basis in $H \otimes G_U$. Thus it holds

$$U_\varphi(\omega) = \sum_{j=1}^{\infty} [U_\varphi, f_j h] f_j(\omega) h, \tag{2.5}$$

where $f_j(\omega)h = (h \otimes f_j)(\omega)$. Putting $\mathbf{u}_j(\varphi) = [U_\varphi, f_j h]h,, \quad (\varphi \in \mathcal{D}_d)$ from (2.5) we obtain (2.3).

Conversely, say that U has a representation (2.3), which means that U_φ is the limit of some linear combinations of elementary tensors from the Hilbert space tensor product $Sp\{\mathbf{u}_j(\varphi), j \in \mathbb{N}\} \otimes \overline{Sp\{f_j, j \in \mathbb{N}\}}$.

Now, since for any $\varphi \in \mathcal{D}_d$, we have

$$U_\varphi \in Sp\{\mathbf{u}_j(\varphi), j \in \mathbb{N}\} \otimes \overline{Sp\{f_j, j \in \mathbb{N}\}} \subset H \otimes \overline{Sp\{f_j, j \in \mathbb{N}\}},$$

it results that $\mathscr{H}_U \subset H \otimes \overline{Sp\{f_j, j \in \mathbb{N}\}}$. The second member of the tensor product being separable, it results that $\mathscr{H}_U$ is separable. $\square$

Remark 2.4 *If $(\Omega, \mathscr{A}, \wp)$ is separable, then each $U \in \mathcal{D}'_d(\mathscr{H})$ has all types of series representation.*

Corollary 2.5 *Any stationary m.s.o.r.d.f. U has all types of series representation.*

3 Series representations and the covariance distribution

In the study of a m.s.o.r.d.f. U, its operator and scalar covariance kernels Γ_U and γ_U respectively, as well as it reproducing kernel normal $\mathcal{B}(H)$-module $\mathscr{H}_{\Gamma_U}$ and reproducing kernel Hilbert space G_{Γ_U}, respectively the reproducing kernel Hilbert space K_{γ_U} play important roles. In what follows they will be connected to the series representations of U. First we easily deduce a series representation for the covariance kernels of a m.s.o.r.d.f. U.

Proposition 3.1 *If U is a m.s.o.r.d.f. of scalar covariance γ_U and operator covariance Γ_U, then the following assertion hold:*

(i) if U has a series representation (2.1), where h_j is an orthonormal system, then

$$\gamma_U(\varphi, \psi) = \sum_{j=1}^{\infty} u_j(\varphi)\overline{u_j(\psi)}\,; \quad \varphi,\, \psi \in \mathcal{D}_d; \tag{3.1}$$

(ii) if U has a modular series representation of the form (2.2), where $\{\mathfrak{h}_j\}_{j\in\mathbb{N}}$ is a gramian orthonormal system, then

$$\Gamma_U(\varphi,\psi)=\sum_{j=1}^{\infty}\mathfrak{u}_j(\varphi)[\mathfrak{h}_j,\mathfrak{h}_j]\mathfrak{u}_j(\psi)^*\,;\quad \varphi,\,\psi\in\mathcal{D}_d, \tag{3.2}$$

in the absolute convergence from $\mathscr{C}_1(H)$;

(iii) if U has a tensor series representation of the form (2.3), such that $\{f_j\}$ is an orthonormal system in $L_0^2(\wp)$, then

$$\Gamma_U(\varphi,\psi)=\sum_{j=1}^{\infty}\mathbf{u}_j(\varphi)\otimes\overline{\mathbf{u}_j(\psi)}\,;\quad \varphi,\,\psi\in\mathcal{D}_d, \tag{3.3}$$

holds, where the elementary tensors are understood in their sesquilinear form.

Proof. Direct verification. □ Now, regarding the connections of the series representations of U with the reproducing kernel structures associated to γ_U and Γ_U, we need first

Proposition 3.2 *Let $U=\{U_\varphi\}_{\varphi\in\mathcal{D}_d}$ be a m.s.o.r.d.f. with scalar covariance γ_U and operator covariance Γ_U. K_{γ_U} will be the reproducing kernel Hilbert space associated to γ_U, while $\mathscr{H}_{\Gamma_U}$ the reproducing kernel module associated to Γ_U.*

(i) If $\{h_\iota\}_{\iota\in\mathcal{I}}$ is an orthonormal system in the vector time domain $\mathscr{H}_{(U)}$ and $u_\iota(\varphi)=(h_\iota,U_\varphi)_{\mathscr{H}}$, $\iota\in\mathcal{I}$, $\varphi\in\mathcal{D}_d$, then $\{u_\iota\}_{\iota\in\mathcal{I}}$ form an orthonormal system in K_{γ_U}.

(ii) If $\{\mathfrak{h}_\iota\}_{\iota\in\mathcal{I}}$ form a gramian orthogonal basis for the modular time domain $\mathscr{H}_U$ and $\mathfrak{u}_\iota(\varphi)=[\mathfrak{h}_\iota,U_\varphi]$ for $\iota\in\mathcal{I}$, $\varphi\in\mathcal{D}_d$, then the family $\{\mathfrak{u}_\iota\}_{\iota\in\mathcal{I}}$ forms a gramian basis for $\mathscr{H}_{\Gamma_U}$.

Proof. Since (i) is rather easy, we shall prove only (ii).

Let $\mathscr{H}_0$ be the set of distributions of the form $\sum\limits_{k=1}^{n}a_k\Gamma_U(\varphi_k,\cdot)$, where a_k are from $\mathcal{B}(H)$ and $\varphi_k\in\mathcal{D}_d$. Define an operator $T_0:\mathscr{H}_0\to\mathscr{H}_U$ by

$$T_0\left(\sum_{k=1}^{n}a_k\Gamma(\varphi_k,\cdot)\right)=\sum_{k=1}^{n}a_kU_{\varphi_k}.$$

Thus defined, T_0 commutes with the modular action and preserves the gramian. Since $\mathscr{H}_0$ is dense in $\mathscr{H}_{\Gamma_U}$ and the range of T_0 is dense in $\mathscr{H}_U$, T_0 can be uniquely extended to a gramian unitary operator $T:\mathscr{H}_{\Gamma_U}\to\mathscr{H}_U$. For any $\iota\in\mathcal{I}$, $\mathfrak{h}_\iota$ can be expressed as

$$\mathfrak{h}_\iota=\sum_{k=1}^{\infty}a_{\iota,k}U_{\varphi_{\iota,k}},$$

for $a_{\iota,k}\in\mathcal{B}(H)$ and $\varphi_{\iota,k}\in\mathcal{D}_d$. We thus have

$$\begin{aligned}\mathfrak{u}_\iota(\cdot)=[\mathfrak{h}_\iota,U_\cdot]_{\mathscr{H}}&=\left[\sum_{l=1}^{\infty}a_{\iota,l}U_{\varphi_{\iota,l}},U_\cdot\right]_{\mathscr{H}}\\&=\sum_{l=1}^{\infty}a_{\iota,l}\Gamma(\varphi_{\iota,l},\cdot)=T^{-1}\left(\sum_{l=1}^{\infty}a_{\iota,l}U_{\varphi_{\iota,l}}\right)=T^{-1}\mathfrak{h}_\iota.\end{aligned}$$

As $\{\mathfrak{h}_\iota\}_{\iota\in\mathcal{I}}$ was a gramian basis in $\mathscr{H}_U$, it results that $\{\mathfrak{u}_\iota\}_{\iota\in\mathcal{I}}$ forms a gramian basis in $\mathscr{H}_{\Gamma_U}$. $\square$ Combining Proposition 2.2 with Proposition 3.1 (ii) we get

Corollary 3.3 *Let $U = \{U_\varphi\}_{\varphi\in\mathcal{D}_d}$ be a m.s.o.r.d.f. having the operator correlation $\Gamma = \Gamma_U$. If it admits a modular series representation, i.e. of the form* (2.2)*, where $\{\mathfrak{h}_j\}_{j=1}^\infty$ is a gramian basis of $\mathscr{H}_U$, then the family $\{\mathfrak{u}_j(\cdot)^*\}_{j=1}^\infty$ forms a gramian basis of the reproducing kernel module $\mathscr{H}_\Gamma$.*

The next result establishes a connection to the measurements space.

Theorem 3.4 *Let $U = \{U_\varphi\}_{\varphi\in\mathcal{D}_d}$ be a m.s.o.r.d.f. having the operator covariance $\Gamma = \Gamma_U$. If it admits a tensor series representation, i.e. of the form* (2.3)*, where $\{f_j\}_{j=1}^\infty$ is an orthonormal basis in the measurements space $G_U \subset L_0^2(\wp)$, then $\{\mathbf{u}_j(\cdot)\}_{j=1}^\infty$ form an orthonormal basis in the reproducing kernel Hilbert space G_{Γ_U}.*

Proof. Let $\{f_j\}_{j=1}^\infty$ be an orthonormal basis in G_U. Then $\{f_j\}_{j=1}^\infty$ is an orthonormal set in $L_0^2(\wp)$, which according to Proposition 2.2 means that the coefficients (H-valued distributions) $\mathbf{u}_j$ are uniquely determined. On the other side, we have that for any $h \in H$, $\|h\| = 1$, $\{(h \otimes f_j)(\omega)\}_j = \{f_j(\omega)h\}_j$ form a gramian basis in $H \otimes G_U = \mathscr{H}_U$ (cf.), from which, writing the Fourier development of U_φ with respect to this gramian basis $U_\varphi(\omega) = \sum [U_\varphi, f_j h] f_j h$, which is just (2.3) with

$$\mathbf{u}_j(\varphi) = [U_\varphi, f_j h]h, \qquad \varphi \in \mathcal{D}_d,\ j \geq 1.$$

Since $f_j(\cdot)h = (h \otimes f_j)(\omega) \in \mathscr{H}_U$, for any $j \geq 1$, we have that there exist $\{a_{j,m}\}_{m=1}^\infty \subseteq \mathcal{B}(H)$ and $\{\varphi_{j,m}\}_{m=1}^\infty \subset \mathcal{D}_d$, such that

$$f_j(\cdot)h = \sum_{m=1}^\infty a_{j,m} U_{\varphi_{j,m}}(\cdot), \qquad j \geq 1.$$

Then for any $j \geq 1$ we have that

$$\begin{aligned}\mathbf{u}_j(\varphi) = [U_\varphi, f_j h]h &= \left[U_\varphi, \sum_m a_{j,m} U_{\varphi_{j,m}}\right] h \\ &= \sum_m [U_\varphi, U_{\varphi_{j,m}}] a_{j,m}^* h = \sum_m \Gamma(\varphi, \varphi_{j,m}) a_{j,m}^* h.\end{aligned}$$

Thus having in view the construction of the reproducing kernel Hilbert space G_{Γ_U} we have $\mathbf{u}_j \in G_{\Gamma_U}$, for any $j \geq 1$ and moreover,

$$\begin{aligned}
(\mathbf{u}_j, \mathbf{u}_k)_{G_{\Gamma_U}} &= \left(\sum_m \Gamma_U(\cdot, \varphi_{j,m}) a^*_{j,m} h, \sum_n \Gamma_U(\cdot, \varphi_{k,n}) a^*_{k,n} h \right)_{G_{\Gamma_U}} \\
&= \sum_{m,n} (\Gamma_U(\varphi_{k,n}, \varphi_{j,m}) a^*_{j,m} h, a^*_{k,n} h)_H \\
&= \sum_{m,n} (a_{k,n}[U_{\varphi_{k,n}}, U_{\varphi_{j,m}}] a^*_{j,m} h, h)_H \\
&= \left(\left[\sum_n a_{k,n} U_{\varphi_{k,n}}, \sum_m a_{j,m} U_{\varphi_{j,m}} \right] h, h \right)_H \\
&= ([f_k h, f_j h] h, h)_H \\
&= ((f_k, f_j)_{L^2_0(\wp)} (h \otimes \bar{h}) h, h)_H \\
&= (f_k, f_j)_{L^2_0(\wp)}.
\end{aligned}$$

Hence, $\{\mathbf{u}_k\}_{k=1}^{\infty}$ is an orthonormal set in G_{Γ_U}.
From the Corollary 3.3 we get $\Gamma_U(\varphi, \psi) = \sum_{k=1}^{\infty} \mathbf{u}_k(\varphi) \otimes \overline{\mathbf{u}_k(\psi)}$, for $\varphi, \psi \in \mathcal{D}_d$. From Theorem II.4.24 ([6]) it results that $\{\mathbf{u}_k\}_{k=1}^{\infty}$ is complete in G_{Γ_U} and is thus an orthonormal basis. □

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On quaternions

Moţ Ghiocel* Popa Lorena †

Abstract

In this paper we present a logical and methodical approach of introducing the complex numbers and the quaternions and to link these concepts with the theory of matrices.

Mathematics Subject Classification: 12F05, 97H40

Keywords: algebraic structure, complex number, quaternion, matrix

1 The set of complex numbers

We consider the set $\mathbb{C}$ as the set of all ordered pairs of the real numbers (a, b) and we denote:

$$\mathbb{C} = \mathbb{R} \times \mathbb{R} = \{(a, b) | a, b \in \mathbb{R}\}.$$

Theorem 1.1 *The set* $\mathbb{C}$ *endowed with the operations:*

$$(a, b) + (c, d) = (a + c, b + d)$$
$$(a, b) \cdot (c, d) = (ac - bd, ad + bc)$$

where $(a, b), (c, d) \in \mathbb{C}$*, forms an algebraic structure of commutative field, called the field of complex numbers.*

Theorem 1.2 *The set* $\mathbb{C}$ *endowed with the operations:*

$$(a, b) + (c, d) = (a + c, b + d)$$
$$\alpha \cdot (a, b) = (\alpha a, \alpha b)$$

where $\alpha \in \mathbb{R}$, $(a, b), (c, d) \in \mathbb{C}$*, forms an algebraic structure of linear space, called the space of complex numbers.* (see [3], pp.42)

Remark 1.3 *The field* $\mathbb{C}$ *contains the field of real numbers* $\mathbb{R}$ *as a subfield and it has the element* $i \in \mathbb{C}$ *such that:* $i^2 = -1$, $i = (0, 1)$.

*ghiocel.mot@uav.ro Department of Mathematics and Computer Science, "Aurel Vlaicu" University of Arad, Romania

†lorena.popa@uav.ro Department of Mathematics and Computer Science, "Aurel Vlaicu" University of Arad, Romania

Indeed,

$$i^2 = (0,1) \cdot (0,1) = (0 \cdot 0 - 1 \cdot 1, 0 \cdot 1 + 1 \cdot 0) = (-1,0) = -1_{\mathbb{C}}.$$

Using the notations $1_{\mathbb{R}} = (1,0)$ şi $i = (0,1)$ and the theorems [1.1] and [1.2], we obtain the algebraic form of complex numbers:

$$(a,b) = (a,0) + (0,b) = a + bi.$$

Therefore:

$$\mathbb{C} = \{a + bi | a, b \in \mathbb{R}\}.$$

The following proposition gives us a matrix representation of the complex numbers:

Proposition 1.4 *The set of matrix:*

$$\mathbb{C}_1 = \left\{ M_{a,b} = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \middle| a, b \in \mathbb{R} \right\}$$

with the operations of matrix addition and matrix multiplication, forms an isomorphic field with the field $\mathbb{C}$.

$$M_{a,b} + M_{c,d} = M_{a+c,b+d}$$

$$M_{a,b} \cdot M_{c,d} = M_{ac-bd,ad+bc}.$$

This result is immediately by direct calculation (see [1]).

Remark 1.5 *For any $M_{a,b} \in \mathbb{C}_1$, we have:*

$$M_{a,b} = a \cdot M_{1,0} + b \cdot M_{0,1} = aI_2 + bM_{0,1}.$$

2 The set of the quaternions

The another extension of the set of real numbers is given by the set of quaternions. This set of hypercomplex numbers was introduced by W. Hamilton in 1843.

Theorem 2.1 *The set*

$$\mathbb{R}^4 = \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} = \{(a,b,c,d) | a,b,c,d \in \mathbb{R}\}$$

forms an algebraic structure of noncommutative field with respect to operations:

$$(a_1,b_1,c_1,d_1) + (a_2,b_2,c_2,d_2) = (a_1+a_2, b_1+b_2, c_1+c_2, d_1+d_2)$$

$$\begin{aligned}(a_1,b_1,c_1,d_1) \cdot (a_2,b_2,c_2,d_2) = (&a_1a_2 - b_1b_2 - c_1c_2 - d_1d_2, a_1b_2 + b_1a_2 + \\ &+ c_1d_2 - d_1c_2, a_1c_2 + c_1a_2 + d_1b_2 - b_1d_2, a_1d_2 + d_1a_2 + b_1c_2 - c_1b_2)\end{aligned} \tag{2.1}$$

where (a_1,b_1,c_1,d_1), $(a_2,b_2,c_2,d_2) \in \mathbb{R}^4$. (see [3], pp.25)

Definition 2.2 *The field $\mathbb{R}^4$ which contains the field of real numbers $\mathbb{R}$ as a subfield and which has an imaginary units i, j, $k \in \mathbb{C}$ such that*

$$i^2 = -1, j^2 = -1, k^2 = -1, ij = k = -ji, jk = i = -kj, ki = j = -ik, \tag{2.2}$$

is called the quaternions field and it is denoted by $\mathbb{K}[i, j, k]$ or $\mathbb{K}$.

Theorem 2.3 *The set $\mathbb{K}[i, j, k]$ form an algebraic structures of linear space with the operations:*

$$(a_1, b_1, c_1, d_1) + (a_2, b_2, c_2, d_2) = (a_1 + a_2, b_1 + b_2, c_1 + c_2, d_1 + d_2)$$
$$\alpha \cdot (a, b, c, d) = (\alpha a, \alpha b, \alpha c, \alpha d)$$

where $\alpha \in \mathbb{R}, (a, b, c, d), (a_1, b_1, c_1, d_1), (a_2, b_2, c_2, d_2) \in \mathbb{K}[i, j, k]$. (see [3], pp.44)

Because, the imaginary units are identified with the elements from $K[i, j, k]$: $i = (0, 1, 0, 0)$, $j = (0, 0, 1, 0)$, $k = (0, 0, 0, 1)$, using the theorems [2.1] and [2.3], we have:

$$(a, b, c, d) = (a, 0, 0, 0)+(0, b, 0, 0)+(0, 0, c, 0)+(0, 0, 0, d) = a+bi+cj+dk.$$

Therefore,

$$\mathbb{K} = \{a + bi + cj + dk | a, b, c, d \in \mathbb{R},\ i, j, k \,\text{with prop.[2.2]}\}.$$

To give a matrix representation for the quaternions, similarly with [1.4] for complex numbers, we denote:

$$\mathbb{K}_1[i, j, k] = \left\{ \begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix} \middle| \alpha, \beta \in \mathbb{C} \right\} = \left\{ \begin{pmatrix} a + bi & c + di \\ -(c - di) & a - bi \end{pmatrix} \middle| a, b, c, d \in \mathbb{R} \right\}.$$

Theorem 2.4 *$\mathbb{K}_1[i, j, k]$ with the operations of matrix addition and matrix multiplication, forms a field isomorphic with $\mathbb{K}$.*

Proof. Let us consider the mapping:

$$\varphi : \mathbb{K} \to \mathbb{K}_1,\ \varphi(a + bi + cj + dk) = \begin{pmatrix} a + bi & c + di \\ -(c - di) & a - bi \end{pmatrix}. \tag{2.3}$$

φ is bijective and preserves the operations - it follows immediately by direct calculation from [2.1] and from usual rules of addition and multiplication for matrices.

□

Remark 2.5 *Any element from $\mathbb{K}_1[i, j, k]$ can be written as:*

$$\begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix} = aI_2 + bI + cJ + dK,$$

$$\textit{where } I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, I = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, K = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

Also, if we denote:

$$\mathbb{K}_2[i,j,k] = \left\{ \begin{pmatrix} a & -b & d & -c \\ b & a & -c & -d \\ -d & c & a & -b \\ c & d & b & a \end{pmatrix} \middle| \, a,b,c,d \in \mathbb{R} \right\}$$

we obtain another matrix representation of quaternions, in this case using only a real numbers.

Theorem 2.6 *$\mathbb{K}_2[i,j,k]$ with the operations of matrix addition and matrix multiplications, forms a field isomorphic with $\mathbb{K}$.*

Proof. Let us consider the mapping:

$$\psi : \mathbb{K} \to \mathbb{K}_2, \; \varphi(a+bi+cj+dk) = \begin{pmatrix} a & -b & d & -c \\ b & a & -c & -d \\ -d & c & a & -b \\ c & d & b & a \end{pmatrix}. \tag{2.4}$$

ψ is bijective and preserves the operations - it follows immediately by direct calculation from [2.1] and from usual rules of addition and multiplication of matrices. $\square$

Remark 2.7 *Any element from $\mathbb{K}_2[i,j,k]$ can be written as:*

$$\begin{pmatrix} a & -b & d & -c \\ b & a & -c & -d \\ -d & c & a & -b \\ c & d & b & a \end{pmatrix} = aI_4 + b\boldsymbol{I} + c\boldsymbol{J} + d\boldsymbol{K},$$

where

$$I_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \boldsymbol{I} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix},$$

$$\boldsymbol{J} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \boldsymbol{K} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

Let us mention that W. Hamilton also introduced a vector representation for quaternions: $q = a + bi + cj + dk \in K$, admits the vectorial writing:

$$q = s + \overrightarrow{v} = [s, \overrightarrow{v}], \tag{2.5}$$

where s - real scalar and $\overrightarrow{v} = a\overrightarrow{i} + b\overrightarrow{j} + c\overrightarrow{k}$ - vector from the euclidean space $\mathbb{R}^3$.

From the relation [2.5] we easily deduce that:

- the quaternion's conjugate is: $q = s - \overrightarrow{v} = s - (a\overrightarrow{i} + b\overrightarrow{j} + c\overrightarrow{k})$;
- the norm of a quaternion is: $q = s^2 - (a^a + b^2 + c^2 + d^2)$;

- the quaternions multiplication can be rewritten as:

$$q_1 \cdot q_2 = (s_1 + \overrightarrow{v}_1) \cdot (s_2 + \overrightarrow{v}_2) = s_1 s_2 - \overrightarrow{v}_1 \overrightarrow{v}_2 + s_2 \overrightarrow{v}_1 + s_1 \overrightarrow{v}_2 + \overrightarrow{v}_1 \times \overrightarrow{v}_2 \tag{2.6}$$

Remark 2.8 *The quaternions which satisfy the condition $s = 0$, are called vector - quaternions or ternions and they represent the image of a real vector in quadridimensional space with the base $(1, \overrightarrow{i}, \overrightarrow{j}, \overrightarrow{k})$. Thus, for $q_1 = [0, \overrightarrow{v}_1]$ and $q_2 = [0, \overrightarrow{v}_2]$, [2.6] become:*

$$q_1 \cdot q_2 = -\overrightarrow{v}_1 \overrightarrow{v}_2 + \overrightarrow{v}_1 \times \overrightarrow{v}_2$$

The vectorial representation of quaternions is useful for the representation of rotations, which is actually based on multiplication between a quaternion and a vector.

3 Conclusions

$\mathbb{R} \subseteq \mathbb{C} \subseteq \mathbb{K}$, and $\mathbb{C}$ can be included in $\mathbb{K}$ in three ways:

$$\mathbb{C} \cong \{a + bi | a, b \in \mathbb{R}\}$$
$$\mathbb{C} \cong \{a + cj | a, c \in \mathbb{R}\}$$
$$\mathbb{C} \cong \{a + dk | a, d \in \mathbb{R}\}.$$

But when we extend the set of complex numbers to the set of quaternions, commutativity is lost. We mention also that: $\mathbb{R} \subseteq \mathbb{C} \cong \mathbb{C}_1 \subseteq \mathbb{K}_1$.

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Three concepts of uniform polynomial dichotomy for discrete-time linear systems in Banach spaces

Nicolae Marian Seimeanu* Mihail Megan †

Abstract

This paper presents three concepts of uniform polynomial dichotomy for linear discrete-time systems in Banach spaces. We treat the case in which the sequence of projections is strongly invariant for the discrete system (the evolution operator associated to the systems satisfies a local invertibility property of the kernels of the projections). We give characterizations of the presented concepts and present the connections between them through counterexamples. The main result of this paper is a result of boundedness of the sequence of projections that give a dichotomy property.

Mathematics Subject Classification: 34D09

Keywords: uniform polynomial dichotomy; uniform strong polynomial dichotomy; uniform weak polynomial dichotomy; discrete linear system.

1 Introduction

The property of dichotomy for evolution operators plays a key role in the qualitative study of asymptotics of dynamical systems. Among the behaviors that govern these systems, one that led to many interesting results is the exponential dichotomy property (both in the discrete / continuous and uniform / nonuniform framework) - see for example [1], [5], [13], [16], [17], [19], [20], [21], [22], [23],[24]. More general behaviors can also be considered, for example the (generalized) (h,k)-dichotomies ([2], [3], [7], [10],[14], [15]) and (h,k,μ,ν)-dichotomies ([25]). As a natural "relaxation" of the exponential growth / decay, one can consider polynomial growth rates with (at most) polynomial nonuniformities as asymptotics for dynamical systems. This idea was successfully embraced and very interesting results concerning the polynomial stability, instability and dichotomy were obtained, which vary from characterizations ([11], [12]) to the study of stable manifolds ([6], [7], [8], [9]) and robustness of the polynomial dichotomy property ([4]).

In this paper we define three concepts of uniform polynomial dichotomy

*nicolae.seimeanu77@e-uvt.ro Department of Mathematics, West University of Timişoara, Romania

†megan@math.uvt.ro Department of Mathematics, West University of Timişoara, Romania

for evolution operators with respect to a strongly invariant sequence of projections. We emphasize the connections between them and propose an open problem regarding the weak behavior defined in this paper. Under the assumption of boundedness of the sequence of projections, we prove that the three concepts merge. The main result of our paper gives an upper bound to the dichotomic sequence of projections, under the assumption of uniform polynomial growth of the linear discrete-time system in discussion.

2 Preliminaries

Let X be a Banach space and $\mathcal{B}(X)$ the Banach space of all bounded linear operators on X. The norms on X and on $\mathcal{B}(X)$ will be denoted by $\|\cdot\|$. The identity operator on X is denoted by I. We will denote by $\Delta = \{(m,n) \in \mathbb{N}^2 \ : \ m \geq n\}$.

We consider the linear difference system

$$x_{n+1} = A_n x_n, \tag{A}$$

where $A : \mathbb{N} \to \mathcal{B}(X)$ is a given sequence.

Definition 2.1 *The* **discrete evolution operator** *associated to the system* (A) *is defined, for* $(m,n) \in \Delta$*, by:*

$$A_m^n = \begin{cases} A_{m-1} \cdot \ldots \cdot A_n, & \text{if } m > n \\ I, & \text{if } m = n \end{cases} \tag{2.1}$$

Remark 2.2 *It is obvious that* $A_m^n A_n^p = A_m^p$*, for all* $(m,n),(n,p) \in \Delta$ *and every solution of (A) satisfies* $x_m = A_m^n x_n$ *for all* $(m,n) \in \Delta$.

Definition 2.3 *The discrete linear system* (A) *is said to have a* **uniform polynomial growth** *if there exist* $M, \omega > 0$ *such that*

$$\|A_m^n\| \leq M \left(\frac{m+1}{n+1}\right)^{\omega}, \quad \forall (m,n) \in \Delta.$$

Definition 2.4 *An operator valued sequence* $P : \mathbb{N} \to \mathcal{B}(X)$ *is called a* ***sequence of projections*** *if* $P_n P_n = P_n$ *for all* $n \in \mathbb{N}$.

If $P : \mathbb{N} \to \mathcal{B}(X)$ is a sequence of projections, then the sequence $Q : \mathbb{N} \to \mathcal{B}(X)$ defined by $Q_n = I - P_n$ is also a sequence of projections, called **the complementary sequence of projections of** $P : \mathbb{N} \to \mathcal{B}(X)$.

Definition 2.5 *We say that the sequence of projections* $P : \mathbb{N} \to \mathcal{B}(X)$ *is* **bounded** *if there exist* $M \geq 1$ *such that*

$$\|P_n\| \leq M, \quad \forall n \in \mathbb{N}.$$

3 Uniform polynomial dichotomies

Definition 3.1 *A sequence of projections $P : \mathbb{N} \to \mathcal{B}(X)$ is called*

- **invariant** *for the system* (A) *if for all $n \in \mathbb{N}$ we have that*

$$A_n P_n = P_{n+1} A_n.$$

- **strongly invariant** *for the system* (A) *if it is invariant for* (A) *and for all $n \in \mathbb{N}$, the restriction $A_n : Ker\ P_n \to Ker\ P_{n+1}$ is an isomorphism.*

If the sequence of projections $P : \mathbb{N} \to \mathcal{B}(X)$ is invariant for the system (A), then we will say that (A, P) is a **dichotomy pair**.

Remark 3.2 *If a sequence of projections $P : \mathbb{N} \to \mathcal{B}(X)$ is strongly invariant for* (A) *then it is also invariant for* (A). *The converse is not generally true, as the below example shows it.*

Example 3.3 *Let $X = \mathbb{R}^3$ and define, for every $n \in \mathbb{N}$, define $A_n, P_n : \mathbb{R}^3 \to \mathbb{R}^3$ by*

$$A_n x = \begin{cases} x, & n \geq 1 \\ (x_1, 0, x_3), & n = 0 \end{cases}, \quad P_n x = \begin{cases} (x_1, x_2, 0), & n = 0 \\ (x_1 + x_2, 0, 0), & n \geq 1. \end{cases}$$

for $x = (x_1, x_2, x_3) \in \mathbb{R}^3$. Having in mind that for all $n \in \mathbb{N}$ and $x = (x_1, x_2, x_3) \in \mathbb{R}^3$,

$$A_n P_n x = \begin{cases} A_n(x_1, x_2, 0), & n = 0 \\ A_n(x_1 + x_2, 0, 0), & n \geq 1 \end{cases} = \begin{cases} (x_1, 0, 0), & n = 0, \\ (x_1 + x_2, 0, 0), & n \geq 1 \end{cases}$$

and

$$P_{n+1} A_n x = \begin{cases} P_{n+1} x, & n \geq 1 \\ P_{n+1}(x_1, 0, x_3), & n = 0 \end{cases} = \begin{cases} (x_1, 0, 0), & n = 0 \\ (x_1 + x_2, 0, 0), & n \geq 1 \end{cases}$$

we deduce that (A, P) is a dichotomy pair.

But $P : \mathbb{N} \to \mathcal{B}(X)$ is not strongly invariant for (A). *Assuming the contrary, there exists $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ such that $A_0 x = y$, where $y = (-1, 1, 0) \in Ker\ P_1$. We henceforth get the contradiction $(x_1, 0, x_3) = (-1, 1, 0)$.*

Remark 3.4 *If the sequence of projections P is strongly invariant for the system (A) then*

(i) for every $n \in \mathbb{N}$ there is an isomorphism B_n from the $Ker\ P_{n+1}$ to $Ker\ P_n$ such that $A_n B_n Q_{n+1} = Q_n$ and $B_n A_n Q_n = Q_n$;

(ii) for all $(m, n) \in \Delta$ there is an isomorphism B_m^n from $Ker\ P_m$ to $Ker\ P_n$ with $A_m^n B_m^n Q_m = Q_m$ and $B_m^n A_m^n Q_n = Q_n$ for all $(m, n) \in \Delta$.

Throughout the paper, if not stated otherwise, we will consider $P : \mathbb{N} \to \mathcal{B}(X)$ to be a sequence of projections which is strongly invariant for (A).

Definition 3.5 *We say that the pair (A,P) is* **uniformly strongly polynomially dichotomic** *(u.s.p.d) if there exist $N \geq 1$ and $\alpha > 0$ such that for all $(m,n) \in \Delta$ the following conditions hold:*

$$(uspd_1) \quad (m+1)^{\alpha}\|A_m^n P_n\| \leq N(n+1)^{\alpha};$$

$$(uspd_2) \quad (m+1)^{\alpha}\|B_m^n Q_m\| \leq N(n+1)^{\alpha}.$$

Remark 3.6 *If the pair (A,P) is u.s.p.d with constants $N,\alpha > 0$ given by Definition 3.5, then for all $n \in \mathbb{N}$, $\max\{\|P_n\|,\|Q_n\|\} \leq N$.*

Definition 3.7 *We say that the pair (A,P) is* **uniformly polynomially dichotomic** *(u.p.d) if there exist $N \geq 1$ and $\alpha > 0$ such that for all $(m,n,x) \in \Delta \times X$ the following properties hold:*

$$(upd_1) \quad (m+1)^{\alpha}\|A_m^n P_n x\| \leq N(n+1)^{\alpha}\|P_n x\|;$$

$$(upd_2) \quad (m+1)^{\alpha}\|B_m^n Q_m x\| \leq N(n+1)^{\alpha}\|Q_m x\|.$$

Remark 3.8 *If the pair (A,P) is u.p.d with constants $N,\alpha > 0$ given by Definition 3.7, then (upd_2) is equivalent with the following condition:*

$$(m+1)^{\alpha}\|Q_n x\| \leq N(n+1)^{\alpha}\|A_m^n Q_n x\|, \quad \forall (m,n,x) \in \Delta \times X.$$

A first connection between the above defined concepts is given by

Proposition 3.9 *If the dichotomy pair (A,P) is u.s.p.d then it is also u.p.d.*

Proof. If (A,P) us u.s.p.d, let $N,\alpha > 0$ be such that $(uspd_1)$ and $(uspd_2)$ hold. The conclusion follows from the fact that for all $(m,n,x) \in \Delta \times X$ one has that

$$(m+1)^{\alpha}\|A_m^n P_n x\| \leq (m+1)^{\alpha}\|A_m^n P_n\| \cdot \|P_n x\| \leq N(n+1)^{\alpha}\|P_n x\|$$

$$(m+1)^{\alpha}\|B_m^n Q_m x\| \leq (m+1)^{\alpha}\|B_m^n Q_m\| \cdot \|Q_m x\| \leq N(n+1)^{\alpha}\|Q_m x\|.$$

□

Proposition 3.10 *If $P : \mathbb{N} \to \mathcal{B}(X)$ is bounded and (A,P) is u.p.d then it is also u.s.p.d.*

Proof. Let $M \geq 1$ be an upper-bound for $P : \mathbb{N} \to \mathcal{B}(X)$ and $N,\alpha > 0$ be given by the u.p.d property. Let $x \in X$ and $(m,n) \in \Delta$. It follows that

$$(m+1)^{\alpha}\|A_m^n P_n x\| \leq N(n+1)^{\alpha}\|P_n x\| \leq 2NM(n+1)^{\alpha}\|x\|$$

$$(m+1)^{\alpha}\|B_m^n Q_m x\| \leq N(n+1)^{\alpha}\|Q_m x\| \leq 2NM(n+1)^{\alpha}\|x\|.$$

From the above relations it follows that (A,P) is u.s.p.d with constants $2MN,\alpha > 0$. □

Corollary 3.11 *(A,P) is u.s.p.d if and only if (A,P) is u.p.d and P is bounded.*

Remark 3.12 *The converse implication from the preceding proposition is not generally valid, as the following example shows it.*

Example 3.13 *On* $X = \mathbb{R}^2$ *endowed with* $\|x\| = |x_1|+|x_2|$, *for* $x = (x_1, x_2) \in \mathbb{R}^2$ *consider the sequence of projections* $P : \mathbb{N} \to \mathcal{B}(X)$ *given by* $P_n(x_1, x_2) = (x_1 + nx_2, 0)$, *for* $n \in \mathbb{N}$ *and* $(x_1, x_2) \in \mathbb{R}^2$. *Define, for every* $n \in \mathbb{N}$,

$$A_n = \frac{n+1}{n+2}P_n + \frac{n+2}{n+1}(I - P_{n+1}).$$

The evolution operator associated to the system (A) *is given by*

$$A_m^n = \frac{n+1}{m+1}P_n + \frac{m+1}{n+1}(I - P_m), \quad (m,n) \in \Delta.$$

It can easily be seen that $P : \mathbb{N} \to \mathcal{B}(X)$ *is strongly invariant for* (A), *with* $\|A_m^n P_n x\| = \frac{n+1}{m+1}\|P_n x\|$ *and* $\|B_m^n Q_m x\| = \frac{n+1}{m+1}\|Q_m x\|$, *for all* $(m,n,x) \in \Delta \times \mathbb{R}^2$ *hence* (A, P) *is u.p.d. Having in mind that for all* $n \in \mathbb{N}$, $\|P_n(0,1)\| = \|(n,0)\| = n$, *it follows that* $\sup\limits_{n\in\mathbb{N}} \|P_n\| = +\infty$ *hence* (A, P) *is not u.s.p.d.*

Definition 3.14 *We say that the pair* (A, P) *is* **uniformly weakly polynomially dichotomic** *(u.w.p.d) if there exist constants* $N \geq 1$ *and* $\alpha > 0$ *such that for all* $(m,n) \in \Delta$ *the following hold:*

$$(uwpd_1) \quad (m+1)^\alpha \|A_m^n P_n\| \leq N(n+1)^\alpha \|P_n\|;$$

$$(uwpd_1) \quad (m+1)^\alpha \|B_m^n Q_m\| \leq N(n+1)^\alpha \|Q_m\|.$$

Remark 3.15 *If the dichotomy pair* (A, P) *is u.p.d then it is also u.w.p.d.*

Open problem. At this point we ask wether the implication "u.w.p.d $\Rightarrow$ u.p.d" generally holds or not.

Proposition 3.16 *If* (A, P) *is u.w.p.d and* P *is bouded then* (A, P) *is u.s.p.d.*

Proof. Let $M \geq 1$ be an upper bound for $P : \mathbb{N} \to \mathcal{B}(X)$ and $N, \alpha > 0$ be given by the u.w.p.d property. Then, for all $(m,n,x) \in \Delta \times X$ it follows that

$$(m+1)^\alpha \|A_m^n P_n\| \leq N(n+1)^\alpha \|P_n\| \leq 2MN(n+1)^\alpha$$
$$(m+1)^\alpha \|B_m^n Q_m\| \leq N(n+1)^\alpha \|Q_m\| \leq 2MN(n+1)^\alpha$$

hence (A, P) is u.s.p.d. □

Remark 3.17 *If* (A, P) *is u.w.p.d then it does not necessarily follow that it is also s.p.d, fact pointed out by the following example.*

Example 3.18 *Consider the system given in Example 3.13. Because* (A, P) *is u.p.d it is also u.w.p.d, but* (A, P) *is not u.s.p.d.*

Remark 3.19 *The connections between the concepts of uniform polynomial dichotomies presented in this paper are given by the following diagram.*

$$\begin{array}{ccccc} & & \boxed{u.s.p.d} & & \\ & \not\nearrow\ \swarrow & & \not\nwarrow\ \searrow & \\ \boxed{u.p.d} & & \Longrightarrow & & \boxed{u.w.p.d} \end{array}$$

Remark 3.20 *If the sequence of projections which give the three concepts of dichotomy for the system* (A) *is bounded, then we have that*

$$\boxed{u.s.p.d} \Leftrightarrow \boxed{u.p.d} \Leftrightarrow \boxed{u.w.p.d}$$

The main result of the paper is given by the following

Theorem 3.21 *Consider the system* (A) *having a uniform polynomial growth. If there exists a strongly invariant sequence of projections $P : \mathbb{N} \to \mathcal{B}(X)$ such that the dichotomy pair (A, P) is u.p.d then $P : \mathbb{N} \to \mathcal{B}(X)$ is bounded.*

Proof. Let $M, \omega > 0$ be given by the uniform polynomial growth of (A) and $N, \alpha > 0$ given by the u.p.d property of (A, P). Let $x \in X$, $n \in \mathbb{N}$ be fixed an $m \geq n$. Consider the below estimations:

$$\frac{\|P_n x\| - \|x\|}{N}\left(\frac{m+1}{n+1}\right)^{\alpha} - N\left(\frac{m+1}{n+1}\right)^{-\alpha}\|P_n x\| \leq$$

$$\leq \frac{\|Q_n x\|}{N}\left(\frac{m+1}{n+1}\right)^{\alpha} - N\left(\frac{m+1}{n+1}\right)^{-\alpha}\|P_n x\| \leq$$

$$\leq \|A_m^n P_n x\| - \|A_m^n Q_m x\| \leq \|A_m^n x\| \leq M\left(\frac{m+1}{n+1}\right)^{\omega}\|x\|.$$

It follows that

$$\|P_n x\| \cdot \left(\frac{1}{N} - N\left(\frac{m+1}{n+1}\right)^{-2\alpha}\right) \leq M\left(\frac{m+1}{n+1}\right)^{\omega}\|x\|. \tag{3.1}$$

Take $\lambda_0 > 0$ such that $N\lambda_0^{-2\alpha} < \frac{1}{N}$ and consider $m = [\lambda_0 + 1](n+1) - 1 \geq n$, where $[\cdot]$ is the integer part function. Then (3.1) becomes

$$\|P_n x\|\left(\frac{1}{N} - N[\lambda_0 + 1]^{2\alpha}\right) \leq M([\lambda_0 + 1])^{\omega}\|x\|$$

and by denoting $K = \frac{([\lambda_0+1])^{\omega}}{\frac{1}{N} - N[\lambda_0+1]^{-2\alpha}} > 0$, it follows that

$$\|P_n x\| \leq K\|x\|, \quad \forall (n, x) \in \mathbb{N} \times X.$$

□

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The T-convolution product

L. Sida*   P. Gaspar †

Abstract

In this paper we extend the T-convolution product of periodic distributions to *vector valued* T-periodic distributions.

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1 Introduction

In this paper we will define the T-convolution product for vector valued T - periodic distributions and extend some of the main properties of this product to vector valued T - periodic distributions. Having in mind the convolution product for vector valued distributions introduced by L. Schwartz [9], we will define the T-convolution product for vector valued T - periodic distributions in analogy to the T-convolution product introduced by A. H. Zemanian (see [10], chap. 11) for scalar T-periodic distributions.

As was stated and exemplified by A. H. Zemanian in [10, chp. 11] for the scalar case, since T - periodic distributions do not have compact support, we can't have a proper definition of the convolution product for this distributions (excepting the case when one of this distributions is null). To overcome this A. H. Zemanian introduced a special kind of the convolution product (see [10]), called *the T-convolution product.*

2 Periodic functions and periodic distributions

Definition 2.1 *([10, chp.11, pp.314]) A function $f : \mathbb{R}^d \to \mathbb{C}$ is said to be periodic if there exists $T = (T_1, T_2, ..., T_d) \in \mathbb{R}^d$, $T_i > 0$, such that $(L_T f)(t) = f(t)$, $t \in \mathbb{R}^d$, where L_τ, $\tau \in \mathbb{R}^d$ stands for the translation operator on $\mathbb{R}^d$. T is called a period of f. The set of all periods of f is*

*`lavinia_sd@yahoo.com` Department of Mathematics and Computer Science, "Aurel Vlaicu" University of Arad, Romania

†`pastorel.gaspar@uav.ro` Department of Mathematics and Computer Science, "Aurel Vlaicu" University of Arad, Romania

kT ($kT = (k_1T_1, ..., k_dT_d)$, $k \in \mathbb{Z}^d$). The "smallest" period is called the fundamental period of f.

Definition 2.2 *([10, chp.11, pp. 314]) A function $\theta : \mathbb{R}^d \to \mathbb{C}$ will be called T-periodic test function, if it is periodic of period T and infinitely smooth. The space of all such T-periodic test functions will be denoted by $\mathcal{D}_T(\mathbb{R}^d)$ or $\mathcal{D}_{d,T}$.*

We shall also need the basic well known spaces of test functions from distributions theory (see [1], [8]): $\mathcal{D}(\mathbb{R}^d)$, $\mathcal{S}(\mathbb{R}^d)$, $\mathcal{E}(\mathbb{R}^d)$, $\mathcal{B}(\mathbb{R}^d)$, which we shall briefly denote $\mathcal{D}_d$, $\mathcal{S}_d$, $\mathcal{E}_d$ and $\mathcal{B}_d$.

A function $\xi(x)$ is said to be *unitary* if it is an element of $\mathcal{D}_d$ and there exists $T \in \mathbb{R}^d$, $T_i > 0$ for which

$$\sum_{n\in\mathbb{Z}^d} (L_{nT}\xi)(x) = 1$$

for all $x \in \mathbb{R}^d$ (see [10]). The space of unitary functions will be denoted by $\mathcal{U}_{d,T}$,

Remark 2.3 *For every $\varphi \in \mathcal{D}_d$, the sum $\sum\limits_{n\in\mathbb{Z}^d} (L_{nT}\varphi)(t)$ is finite and, since $L_T(\sum\limits_{n\in\mathbb{Z}^d} (L_{nT}\varphi)) = \sum\limits_{n\in\mathbb{Z}^d} (L_{nT}\varphi)$, it defines a function from $\mathcal{D}_{d,T}$.*

Definition 2.4 *([9, chp. II, § 2]) Let E be a Banach space. Any linear and continuous operator*

$$U : \mathcal{D}_d \to E$$

is an E-valued distribution on $\mathbb{R}^d$.

Definition 2.5 *([9]) A vector valued distribution U is periodic of period $T \in \mathbb{R}^d$, $T_i > 0$ if $L_TU = U$.*

We shall also use the basic spaces from distributions theory, namely

- $\mathcal{D}'(\mathbb{R}^d, E) = \mathcal{D}'_d(E)$ the space of all E-valued distribution on $\mathbb{R}^d$,
- $\mathcal{D}'_T(\mathbb{R}^d, E) = \mathcal{D}'_{d,T}(E)$ the space of E-valued periodic distribution of period $T \in \mathbb{R}^d$, $T_i > 0$,
- $\mathcal{S}'(\mathbb{R}^d, E) = \mathcal{S}'_d(E)$ the space of E-valued tempered distribution,
- $\mathcal{E}'(\mathbb{R}^d, E) = \mathcal{E}'_d(E)$ the space of E-valued distribution with compact support,
- $\mathcal{B}'(\mathbb{R}^d, E) = \mathcal{B}'_d(E)$ the space of E-valued bounded distributions.

Remark 2.6 *The derivative of an E-valued tempered distribution is also an E-valued tempered distribution. The E-valued tempered distributions generalize the bounded (slow-growing) locally integrable E-valued functions; all E-valued distributions with compact support and all square-integrable E-valued functions are E-valued tempered distributions.*

We also recall the inclusion relations (with continuous embedding) between the spaces from distribution theory:

$$\mathcal{D}'_{d,T}(E) \subset \mathcal{B}'_d(E) \subset \mathcal{S}'_d(E) \subset \mathcal{D}'_d(E). \tag{2.1}$$

For locally integrable E - valued function F, we have that, the operator U_F defined by

$$U_F(\varphi) := \int_{\mathbb{R}^d} \varphi(t)F(t)dt,\ \varphi \in \mathcal{D}_d \tag{2.2}$$

is linear and continuous on $\mathcal{D}_d$, hence $U_F \in \mathcal{D}'_d(E)$.

Every E-valued tempered distribution is a derivative of finite order of some continuous E-valued function of polynomial growth (the *structure theorem* for $\mathcal{S}'_d(E)$).

3 The T-convolution product

Let us recall the function from $L^1_{d,T}$, and, by analogy with the case of non-periodic functions from L^1_d, we will quote a theorem for the T-convolution product for scalar T-periodic functions (meaning from $L^1_{d,T}$).

We note first that for a function $f \in L^1_{d,T}$, $T \in \mathbb{R}^d$, $T_i > 0$, for any $a \in \mathbb{R}^d$ the following equality holds (see [6]):

$$\int_{[a,a+T]} f(t)dt = \int_{[0,T]} f(t)dt. \tag{3.1}$$

We enounce now the main result for the T-convolution product on $L^1_{d,T}$.

Theorem 3.1 *Let $f, g \in L^1_{d,T}$. Then for any $s \in [0,T)$, the function $t \to f(t)g(s-t)$ is in $L^1_{d,T}$. The T-convolution $f *^T g$ defined by:*

$$(f *^T g)(s) = \int_{[0,T)} f(t)g(s-t)dt,\ s \in [0,T) \tag{3.2}$$

is also a function from $L^1_{d,T}$ and also

$$\| f *^T g \|_1 \leq \| f \|_1 \cdot \| g \|_1 . \tag{3.3}$$

Thus L^1_T becomes a Banach algebra with the T-convolution for function as multiplication.

The proof of this theorem is based on the reasoning presented in [1] paragraph 2.4.3, p. 129, and for the T-periodicity we have

$$(f *^T g)(t-T) = \int_{[0,T]} f(s)g(t-T-s)dt = \int_{[0,T]} f(s)g(t-s-T)dt \overset{g(t-T)=g(t)}{=}$$

$$\overset{g(t-T)=g(t)}{=} \int_{[0,T]} f(s)g(t-s)dt = (f *^T g)(t).$$

Corollary 3.2 *The convolution product between two functions of which only one is periodic, is a periodic function of same period, i.e. if $f \in \mathcal{D}_{d,T}$ şi $g \in \mathcal{D}_d$, then $f * g \in \mathcal{D}_{d,T}$. (see [6], p. 14)*

Proposition 3.3 *(see [6]) The T-convolution product $f *^T g$ is a continuous application in the meaning of the topology from $L^1_{d,T}$, i.e. for any convergent sequence $(f_n)_{n\geq 1}$ from $L^1_{d,T}$, we have*

$$f_n *^T g \xrightarrow{L^1_T} f *^T g \text{ for any } g \in L^1_T.$$

At the beginning let us recall same known results for the convolution product of scalar distributions.

• The convolution between a distribution u and a function $\varphi \in \mathcal{D}_d$:

$$(u * \varphi)(t) = (u, \varphi_t),\ \varphi \in \mathcal{D}_d,\ t \in \mathbb{R}^d \tag{3.4}$$

where $\varphi_t = L_t\check{\varphi}$.
If $u \in \mathcal{D}'_d$, $\varphi \in \mathcal{D}_d$ then $u * \varphi \in \mathcal{E}_d$, or if $u \in \mathcal{E}'_d$, $\varphi \in \mathcal{E}_d$ then $u * \varphi \in \mathcal{E}_d$, etc. (see [1], p. 212)

• The convolution product between two distributions, one of them with compact support $u \in \mathcal{D}'_d$, $v \in \mathcal{E}'_d$:

$$(u * v, \varphi) = (u, \check{v} * \varphi),\ \varphi \in \mathcal{D}_d. \tag{3.5}$$

If $u \in \mathcal{D}'_d$, $v \in \mathcal{E}'_d$ then $u * v \in \mathcal{D}'_d$, or if $u \in \mathcal{E}'_d$, $v \in \mathcal{E}'_d$ then $u * v \in \mathcal{E}'_d$, etc.(see [1], p. 216)
If $u \in \mathcal{D}'_d$, $v \in \mathcal{E}'_d$ then $u * v \in \mathcal{D}'_d$, or if $u \in \mathcal{E}'_d$, $v \in \mathcal{E}'_d$ then $u * v \in \mathcal{E}'_d$, etc.

• The T-convolution between two T-periodic distributions $u_T, v_T \in (\mathcal{D}_{d,T})'$ (or finite convolution):

$$(u_T *^T v_T, \theta) = v_T(\check{u}_T *^T \theta),\ \theta \in \mathcal{D}_{d,T}, \tag{3.6}$$

It is known that $u_T *^T v_T \in (\mathcal{D}_{d,T})'$ (see [10]).

• The T-convolution between a T-periodic distributions $u_T \in (\mathcal{D}_{d,T})'$ and a test T-periodic function $\theta \in \mathcal{D}_{d,T}$:

$$(u_T *^T \theta)(t) = u_T(L_t\check{\theta}) = (u, \xi\theta_t),\ t \in \mathbb{R}^d,\ \xi \in \mathcal{U}_{d,T}. \tag{3.7}$$

It is clear that $u_T *^T \theta \in \mathcal{D}_{d,T}$ (see [10] p. 326).

For the case of vector valued distributions we consider the locally convex quasi - complete space X. Let $U \in \mathcal{D}'_d(X)$ and $V \in \mathcal{E}'_d(X)$.

Definition 3.4 *The convolution product between two vector valued distributions $U \in \mathcal{D}'_d(X)$, $V \in \mathcal{E}'_d(X)$, is defined as:*

$$(U * V)(\varphi) = V(\check{U} * \varphi),\ \varphi \in \mathcal{D}_d, \tag{3.8}$$

and $U * V \in \mathcal{D}'_d(X)$ (see [9], p. 72).

We mention that, the above definition obviously makes sense if U is a scalar distribution. In this general case the formula (3.8) is meant in the weak sense, i.e.

$$x'(U * V(\varphi)) = V\big[x'(\check{U} * \varphi)\big],\ x' \in X'. \tag{3.9}$$

We recall here some properties of the convolution product between two vector valued distributions:

• The convolution with δ :

$$U * \delta = U, \forall U \in \mathcal{D}'_d(X). \tag{3.10}$$

• The derivation $\boldsymbol{D}^\alpha$ is a convolution with $D^\alpha\delta$, ie

$$\boldsymbol{D}^\alpha U = U * D^\alpha\delta, \, \alpha \in \mathbb{N}. \tag{3.11}$$

• From the convolution with a test function $\varphi \in \mathcal{D}_d$ results the regularization of a vector valued distribution by a scalar function, infinitely differentiable. (see [9], vol. II p. 72)

$$(U * \varphi)(t) = (\boldsymbol{L}_t U)(\check{\varphi}), \, \varphi \in \mathcal{D}_d, \, t \in \mathbb{R}^d.$$

We mention that, the convolution of a vectorial distribution with a scalar function φ că, in particular, if U a temperate distribution, then the convolution have sense for any $\varphi \in \mathcal{S}_d$, and if U is a distribution with compact support then the convolution have sense for any $\varphi \in \mathcal{E}_d$.

In the next definition we extend the notion of T-convolution product, for the case of *vector valued* T-periodic distributions.

Definition 3.5 *The T-convolution product between two vector valued T-periodic distributions $U_T, V_T \in \mathcal{B}(\mathcal{D}_{d,T}, X)$ is:*

$$(U_T *^T V_T)(\theta) = V_T(\check{U}_T *^T \theta), \tag{3.12}$$

for any $\theta \in \mathcal{D}_{d,T}$.

We mention that

$$(\check{U}_T *^T \theta)(t) = (\check{U}_T, \xi L_{-t}\theta) = (U_T, \xi(L_{-t}\theta)^\vee) = (U, \eta(\xi L_{-t}\theta)^\vee),$$

like function of t, is (see [10] Cor. 2.7-2a, p. 74) from $\mathcal{D}'_{d,T}$.

Remark 3.6 *Because $(\check{U}_T, \xi L_{-t}\theta)$ is in $\mathcal{D}_{d,T}$, the right member of the relation (3.12) always exist for any choice of U_T şi V_T from $\mathcal{B}(\mathcal{D}_{d,T}, X)$. This is in contrast with the usual convolution product of two distributions of $\mathcal{D}'_d(X)$, where, to ensure the existence of convolution, it requires to introduction the additional restrictions.*

Remark 3.7 *The definition (3.12) of T-convolution product is correct, since for regular periodic distributions it coincides with the standard definition of the T-convolution of periodic functions.*

In the next theorem we show that, the space of vector valued T-periodic distributions is closed under the T-convolution product.

Theorem 3.8 *The T-convolution product between two vector valued T-periodic distributions $\mathcal{B}(\mathcal{D}_{d,T}, X)$ is still in $\mathcal{B}(\mathcal{D}_{d,T}, X)$. In other words, the space $\mathcal{B}(\mathcal{D}_{d,T}, X)$ is closed under the T-convolution product.*

Proof. We show that (3.12) defines an operator from $\mathcal{B}(\mathcal{D}_{d,T}, X)$.

For the linearity of $U_T *^T V_T$ we consider $\alpha, \beta \in \mathbb{C}$, and $\theta_1, \theta_2 \in \mathcal{D}_{d,T}$. Then from the relation (3.12) it results

$$(U_T *^T V_T)(\alpha\theta_1 + \beta\theta_2) = V_T(\check{U}_T *^T (\alpha\theta_1 + \beta\theta_2)) =$$

$$= V_T(\alpha(\check{U}_T *^T \theta_1) + \beta(\check{U}_T *^T \theta_2)) =$$
$$= \alpha V_T(\check{U}_T *^T \theta_1) + \beta V_T(\check{U}_T *^T \theta_2) =$$
$$= \alpha(U_T *^T V_T)(\theta_1) + \beta(U_T *^T V_T)(\theta_2).$$

For the continuity $U_T *^T V_T$ we consider the sequence $\{\theta_k\}_{k=1}^{\infty} \to \theta$ in $\mathcal{D}_{d,T}$. Now

$$(U_T *^T V_T)(\theta_k) = (V \times \check{U}, \eta(t)\xi(s)\theta_k(t+s)) \to (V \times \check{U}, \eta(t)\xi(s)\theta(t+s)) =$$
$$= (U_T *^T V_T)(\theta),$$

because if $\{\theta_k\}_{k=1}^{\infty} \to \theta$ in $\mathcal{D}_{d,T}$ then $\{\eta\xi\theta_k\}_{k=1}^{\infty} \to \eta\xi\theta$ în $\mathcal{D}_d$ as well.

So, $U_T *^T V_T \in \mathcal{B}(\mathcal{D}_{d,T}, X)$, where it results that $U_T *^T V_T = W_T$ where $W \in \mathcal{D}'_{d,T}(X)$, i.e. W is T-periodic. □

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Applications of graph theory in chemistry

Şandru Andrea* Palcu Adrian †

Abstract

Chemical Graph theory is used to model physical properties of molecules called alkanes. Indices based on the graphical structure of the alkanes are defined and used to model both the boiling point and melting point of the molecules.

Mathematics Subject Classification: 94C15, 97K30

Keywords: Chemical Graph theory, alkanes, boiling point

1 Introduction

This article has arisen because the curriculum of computer science states that the examples used in teaching will be mainly drawn from the curriculum Mathematics and Natural Sciences. Because in class XI studying concept graph and his applications, we thought it would be a good example linking the concepts learned in chemistry with concepts learned in computer science. Chemical graph theory is a branch of mathematics which combines graph theory and chemistry. Graph theory is used to mathematically model molecules in order to gain insight into the physical properties of these chemical compounds. Some physical properties, such as the boiling point, are related to the geometric structure of the compound. This is especially true in the case of chemical compounds known as alkanes. Alkanes are organic compounds exclusively composed of carbon and hydrogen atoms. One example of an alkane is ethane, shown in Figure 1.

*sandru.andrea@uav.ro Department of Mathematics and Computer Science, "Aurel Vlaicu" University of Arad, Romania

†palcu.adrian@uav.ro DDepartment of Mathematics and Computer Science, "Aurel Vlaicu" University of Arad, Romania

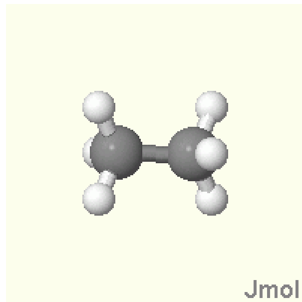

Figure 1:

Each carbon atom has four chemical bonds and each hydrogen atom has one chemical bond. Therefore, the hydrogen atoms can be removed without losing information about the molecule. The resulting representation of ethane is the carbon tree shown in Figure 2.

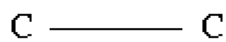

Figure 2:

This carbon tree can be represented as a graph by replacing the carbon atoms with vertices. Chemical bonds are then represented as an edge in the graph. Figure 3 shows the graphical representation of ethane composed of two vertices connected by a single edge.

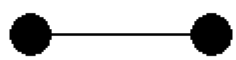

Figure 3:

2 Main results

The structure of an alkane determines its physical properties. Physical properties of alkanes can be modeled using topological indices. Some of these indices are well known outside of the chemical and mathematical communities such as the relative molecular mass (Mr) of a compound. For alkanes, the relative molecular mass is a function of the number of carbon atoms, denoted by n, and is given by $Mr(n) = 12.01115n + 1.00797(2n + 2)$ atomic mass units (amu). Using this formula, you can determine that the relative molecular mass of ethane in Figure 3 is 30.0701amu.
The boiling point of alkanes is determined by the geometric structure of the alkane. Boiling points are a measure of the forces of attraction between like molecules. For essentially nonpolar compounds such as alkanes, these forces are London dispersion forces due to instantaneous dipole-induced

dipole attractions. Dispersion forces are very short range forces which increase with the number of electrons which is proportional to the relative molecular mass for the alkanes. The alkane boiling point should depend on the relative molecular mass and on how well the molecules pack together, which is related to the geometry of the molecule. The dependence on the geometry is complex, but the boiling point should decrease in a general way as the compactness of the molecule increases if the relative molecular mass stays the same. Balaban noted that for the same relative molecular mass, the boiling point decreased with increased branching.

We examine the structures and boiling points of octane and 2,2,4-trimethylpentane to illustrate this result. Both are composed of 8 carbon atoms so they have the same molecular weight. A 3D representation is given in Figure 4 and the graphical representation is given in Figure 5.

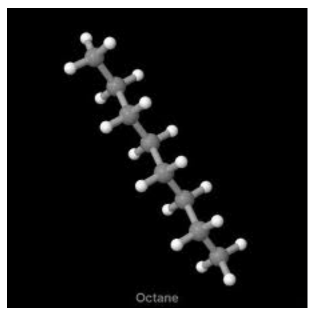

Figure 4:

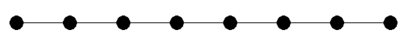

Figure 5:

2,2,4-trimethylpentane is a more compact alkane and is sometimes called isooctane. A 3D representation is given in Figure 6 and the graphical representation is given in Figure 7.

From the above discussion, we expect the boiling point of 2,2,4-trimethylpentane to be lower than that of octane. This is indeed the case. The boiling point of octane is 398.7 K while the boiling point of 2,2,4-trimethylpentane is 372.4 K. It is possible to model the boiling point of families of alkanes having similar geometric structure using molecular weight as the only index in the model. In modeling the alkanes in general, more topological indices are needed to reduce the error in the model. Some examples include the Hosoya index, the Wiener number, the Wiener path numbers, the Mean Wiener index, and the Methyl index.

The Hosoya index (denoted Z) is the sum of the coefficients of the simple matching polynomial for a graph. This is equivalent to the number of matchings a graph contains plus 1 to account for the matching consisting of no edges. A matching of a graph G is a (possibly empty) set of edges

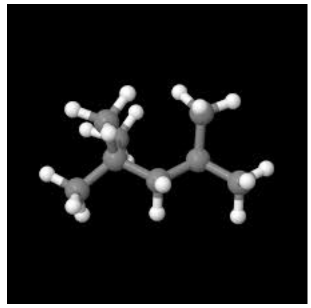

Figure 6:

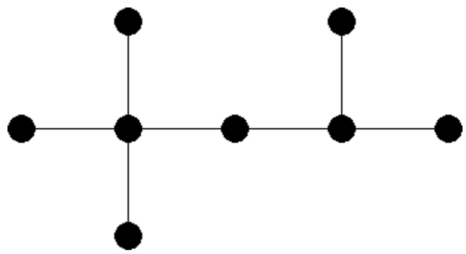

Figure 7:

of G in which no two edges share a common vertex. The set of edges in a matching are said to be independent. An algorithm for computing the simple matching polynomial of a graph is given by Farrell. The Hosoya index for ethane is 2 since it contains only one edge yielding one matching with zero edges and one matching with one edge. Using this algorithm we determine the Hosoya index for 2,2,4-trimethylpentane is 19. You can verify this index by carefully inspecting all the sets of independent edges. There is 1 way to choose zero edges, there are 7 ways to choose only one edge in the matching, 11 ways to choose two edges in the matching, and there is no way to choose three or more edges for a matching. This gives, $1 + 7 + 11 = 19$ simple matchings of 2,2,4-trimethylpentane verifying the Hosoya index for this alkane.
The Wiener number (denoted W) is the sum of the distances between all pairs of vertices in a graph. It can be computed by adding the entries in the upper (or lower) triangular part of the distance matrix of a graph. Ethane has a Wiener number of 1 since it has only one pair of vertices separated by an edge. For 2,2,4-trimethylpentane, we use the distance matrix in Figure 9 computed from the labeled graph in Figure 8. The Wiener number of 2,2,4-trimethylpentane is therefore 66.

The Wiener path numbers (denoted ${}^1P$, ${}^2P$, ${}^3P$, ${}^4P$, ...) are defined by ${}^iP$ which is the number of pairs of vertices in the graph separated by i edges. ${}^iP$ can be computed using the distance matrix of a graph and counting the number of times i appears in the upper triangular part of

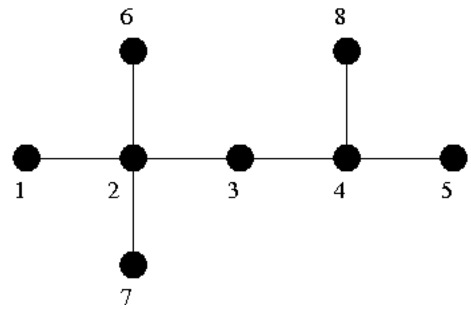

Figure 8:

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 2 & 2 & 4 \\ 1 & 0 & 1 & 2 & 3 & 1 & 1 & 3 \\ 2 & 1 & 0 & 1 & 2 & 2 & 2 & 2 \\ 3 & 2 & 1 & 0 & 1 & 3 & 3 & 1 \\ 4 & 3 & 2 & 1 & 0 & 4 & 4 & 2 \\ 2 & 1 & 2 & 3 & 4 & 0 & 2 & 4 \\ 2 & 1 & 2 & 3 & 4 & 2 & 0 & 4 \\ 4 & 3 & 2 & 1 & 2 & 4 & 4 & 0 \end{pmatrix}$$

Figure 9:

the matrix. Using the distance matrix for 2,2,4-trimethylpentane given in Figure 9, we find that ${}^1P = 7$, ${}^2P = 10$, ${}^3P = 5$, and ${}^4P = 6$. The Mean Wiener index (denoted W) is the average of the distances between all pairs of vertices in a graph. We previously showed the Wiener number for 2,2,4-trimethylpentane is 66. Using this, we calculate that W = 66/28 = 2.35714. The methyl index was introduced to help graphically represent the branching of the alkanes. The methyl index (denoted Mth) is defined to be the number of degree one vertices which are adjacent to a vertex of degree three or greater. For example, the methyl index for 2,2,4-trimethylpentane is 5. The five methyl edges as seen in Figure 8 are (1, 2), (2, 6), (2, 7), (4, 5), and (4, 8). All of the indices described can be used to construct models of the various physical properties of alkanes. Several of these indices were used to model the boiling points of alkanes having six to twelve carbon atoms. The normal alkanes with thirteen through twenty-two carbons were also included to facilitate the prediction of test data having thirteen to twenty-two carbons. The total number of alkanes modeled was 187. One such model is $f({}^1P,{}^2P,...,{}^6P, Mth, Z) = 847.41474 + 221.61698({}^1P)^{0.49420} 1182.20853({}^2P)^{0.03689} + 0.00125({}^3P)^{3.39724} -$
$-3.02445({}^4P)^{0.93751} - 2.16070({}^5P)^{1.01631} - 0.56366({}^6P)^{1.38233} -$
$-2.10575 Mth^{0.5695} - 9.61075 Z^{0.19907}$
The coefficient of determination for this model is 0.997068 and the standard deviation is 2.1 degrees (C). Table 1 gives the number and percentage of alkanes with the specified absolute boiling point deviations given by this

model.

BP dev.	alkanes	% alkanes
$0 - 1^0$	84	44.9
$1 - 2^0$	48	25.7
$2 - 4^0$	46	24.6
$4 - 6^0$	5	2.7
$6 - 9^0$	3	1.6
$> 9^0$	1	0.5

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