Exact quasiclassical asymptotic beyond Maslov canonical operator and quantum jumps nature

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Keywords: quantum jumps, quantum measurements, quantum averages, limiting quantum trajectory, Schrodinger equation, Stochastic quantum jump equation, Colombeau solution, Feynman path integral, Maslov canonical operator, Feynman-Colombeau propagator.

I.Introduction

A number of experiments on trapped single ions or atoms have been performed in recent years [1,2,3,4]. Monitoring the intensity of scattered laser light off of such systems has shown abrupt changes that have been cited as evidence of "quantum jumps" between states of the scattered ion or atom . The existence of such jumps was required by Bohr in his theory of the atom. Bohr's quantum jumps between atomic states [5] were the first form of quantum dynamics to be postulated. He assumed that an atom remained in an atomic eigenstate until it made an instantaneous jump to another state with the emission or absorption of a photon. Since these jumps do not appear to occur in solutions of the Schrodinger equation, something similar to Bohr's idea has been added as an extra postulate in modern quantum mechanics.
Stochastic quantum jump equations [6], [7],[8]were introduced as a tool for simulating the dynamics of a dissipative system with a large Hilbert space and their links with quantum measurement the or were also noted [9],[10],[11],[12],[13].This measurement interpretation is generally known as quantum trajectory theory [14].By adding filter cavities as part of the apparatus, even the quantum jumps in the dressed state model can be interpreted as approximations to measurement-induced jumps [15].
The question arises whether an explanation of these jumps can be found to result from an Colombeau solution [16]-[18]$(\Psi_\varepsilon(x,t;\hbar))_\varepsilon$ of the Schrödinger equation alone without additional postulates. We found exact quasi-classical asymptotic of the quantum averages

$$(\langle i,t,x_0;\hbar,\varepsilon\rangle)_\varepsilon=\left(\int x_i|\Psi_\varepsilon(x,t;\hbar)|^2dx\right)_\varepsilon,\varepsilon\in(0,1],x\in\mathbb{R}^d,i=1,\dots,d,\qquad(1.1)$$

i.e. we found the limiting quantum average (limiting quantum trajectory) such that [18]

$$(\lim_{\hbar\to0}\langle i,t,x_0;\hbar,\varepsilon\rangle)_\varepsilon=\left(\lim_{\hbar\to0}\int x_i|\Psi_\varepsilon(x,t;\hbar)|^2dx\right)_\varepsilon,\varepsilon\in(0,1],x\in\mathbb{R}^d,i=1,\dots,d.\qquad(1.2)$$

The physical interpretation of these asymptotics given below, shows that the answer is "yes."

II. Colombeau solutions of the Schrödinger equation and corresponding path integral representation

Let **H** be a complex infinite dimensional separable Hilbert space, with inner product $<\cdot,\cdot>$ and norm $|\cdot|$.
Let us consider Schrödinger equation:

$$-i\hbar\left(\frac{\partial\Psi(t)}{\partial t}\right)+\widehat{H}(t)\Psi(t)=0,\Psi(0)=\Psi_0(x),(2.1)$$

$H(t) = -\left(\frac{\hbar^2}{2m}\right)\Delta + V(x,t)$.(2.2)

Here operator $H(t): \mathbb{R} \times \mathbf{H} \to \mathbf{H}$ is essentially self-adjoint, $\widehat{H}(t)$is the closure of $H(t)$.
Theorem 2.1. [19],[20].Assume that:(1) $\Psi_0(x) \in L_2(\mathbb{R}^d)$,(2) $V(x,t)$ is continuous and $\sup_{x\in\mathbb{R}^d, t\in[0,T]} |V(x,t)|) < +\infty$.Then corresponding solution of the Schrödinger equation (2.1)-(2.2) exist and can be represented via formulae

$$\Psi(t,x) = \lim_{n\to\infty}\left(\frac{nm}{4\pi it\hbar}\right)^{d(n+1)/2} \int_{\mathbb{R}^d} \dots \int_{\mathbb{R}^d} dx_0 dx_1 \dots dx_n \Psi_0(x_0) \exp\left[\frac{i}{\hbar} S(x_0, x_1, \dots, x_n, x_{n+1}; t)\right],(2.3)$$

where we have set $x_{n+1} = x$and

$$S(x_0, x_1, \dots, x_n, x_{n+1}; t) = \sum_{i=1}^{n}\left[\frac{m}{4}\frac{|x_{i+1}-x_i|^2}{(t/n)^2} - \mathrm{V}(x_{i+1}, t_i)\right],(2.4)$$

where $t_i = \frac{it}{n}$. Let $q_n(t)$be a trajectory; that is, a function from$[0,t]$to$\mathbb{R}^d$ with $q_n(0) = x_0$and set$q_n(t_i) = x_i, i = 1, \dots, n+1$.We rewrite Eq.(2.3) for a future application symbolically for short of the following form

$$\Psi(t,x) = \lim_{n\to\infty} \int_{q_n(t)=x} D[q_n(t)] \Psi_0(q_n(0)) \exp\left[\frac{i}{\hbar} S(q_n(t), x; t)\right],(2.5)$$

where we have set (i) $S(q_n(t), x; t) = S(x_0, x_1, \dots, x_n, x_{n+1}; t)$ and (ii)$D[q_n(t)]$that is, a

$$D[q_n(t)] = \left(\frac{nm}{4\pi it\hbar}\right)^{d(n+1)/2} \prod_{j=0}^{n} dx_j.(2.6)$$

Trotter and Kato well known classical results give a precise meaning to the Feynman integral when the potential $V(x,t)$is sufficiently regular [18]-[19]. However if potential $V(x,t)$ is a non-regular this is well known problem to represent solution of the Schrödinger equation (2.1)-(2.2) via formulae (2.3), see [19]. We avoided this difficulty using contemporary Colombeau framework [16]-[18]. Using replacement $x_i \to \frac{x_i}{1+\varepsilon^{2k}|x|^{2k}}, \varepsilon \in (0,1], k \geq 1$we obtain from potential $V(x,t)$regularized potential $V_\varepsilon(x,t), \varepsilon \in (0,1]$, such that $V_{\varepsilon=0}(x,t) = V(x,t)$and

(i) $\left(V_\varepsilon(x,t)\right)_\varepsilon \in G(\mathbb{R}^d)$,

(ii) $\sup_{x\in\mathbb{R}^d, t\in[0,T]} |V_\varepsilon(x,t)|) < +\infty, \varepsilon \in (0,1]$.(2.7)

Here $G(\mathbb{R}^d)$ is Colombeau algebra of Colombeau generalized functions [16]-[18].

Finally we obtain regularized Schrödinger equation of Colombeau form [16]-[18]:

$$-i\hbar\left(\frac{\partial\Psi_\varepsilon(t)}{\partial t}\right)_\varepsilon + \left(\widehat{H}_\varepsilon(t)\Psi_\varepsilon(t)\right)_\varepsilon = 0, \left(\Psi_\varepsilon(0)\right)_\varepsilon = \Psi_0(x),(2.8)$$

$$H_\varepsilon(t) = -\left(\frac{\hbar^2}{2m}\right)\Delta + V_\varepsilon(x,t).(2.9)$$

Using the inequality (2.7) Theorem 2.1 asserts again that corresponding solution of the Schrödinger equation (2.8)-(2.9)exist and can be represented via formulae [18]:

$$\left(\Psi_\varepsilon(t,x)\right)_\varepsilon =$$

$$\left(\lim_{n\to\infty}\left(\frac{nm}{4\pi it\hbar}\right)^{d(n+1)/2} \int_{\mathbb{R}^d} \dots \int_{\mathbb{R}^d} dx_0 dx_1 \dots dx_n \Psi_0(x_0) \exp\left[\frac{i}{\hbar} S_\varepsilon(x_0, x_1, \dots, x_n, x_{n+1}; t)\right]\right)_\varepsilon (2.10)$$

where we have set $x_{n+1} = x$ and

$$S_\varepsilon(x_0, x_1, \dots, x_n, x_{n+1}; t) = \sum_{i=1}^{n} \left[\frac{m}{4} \frac{|x_{i+1} - x_i|^2}{(t/n)^2} - V_\varepsilon(x_{i+1}, t_i) \right], (2.11)$$

where we have set $t_i = \frac{it}{n}$.

We rewrite Eq.(2.10) for a future application symbolically of the following form

$$\left(\Psi_\varepsilon(t,x)\right)_\varepsilon = \left(\lim_{n\to\infty} \int_{q_n(t)=x} D[q_n(t)] \Psi_0(q_n(0)) \exp\left[\frac{i}{\hbar} S_\varepsilon(q_n(t), x; t)\right]\right)_\varepsilon, \quad (2.12)$$

or of the following form

$$\left(\Psi_\varepsilon(t,x)\right)_\varepsilon = \left(\lim_{n\to\infty} \int_{q(t)=x} D_{\mathbf{n}}[q(t)] \Psi_0(q_n(0)) \exp\left[\frac{i}{\hbar} S_\varepsilon(\dot q(t), q(t), x; t)\right]\right)_\varepsilon. \quad (2.13)$$

For the limit in RHS of (2.12) and (2.13) we will be used canonical path integral notation

$$\left(\Psi_\varepsilon(t,x)\right)_\varepsilon = \left(\int_{q(t)=x} D[q(t)] \Psi_0(q(0)) \exp\left[\frac{i}{\hbar} S_\varepsilon(\dot q(t), q(t))\right]\right)_\varepsilon, (2.14)$$

where$S_\varepsilon(\dot q(t), q(t)) = \int_0^t \left[\frac{m}{4} \dot q^2(s) - V_\varepsilon(q(s), s)\right] ds.$

Substitution $n = 8k + 7$ into RHS of the Eq.(2.10) gives

$$\left(\Psi_\varepsilon(t,x)\right)_\varepsilon =$$

$$\left(\lim_{k\to\infty} \left(\frac{(8k+7)m}{4\pi t\hbar}\right)^{d(4k+4)} \int_{\mathbb{R}^d} \dots \int_{\mathbb{R}^d} dx_0 dx_1 \dots dx_{8k+7}\, \Psi_0(x_0) \exp\left[\frac{i}{\hbar} S_\varepsilon(x_0, x_1, \dots, x_{8k+7}, x_{8k+8}; t)\right]\right)_\varepsilon. (2.15)$$

We rewrite Eq.(2.15) for a future application symbolically of the following form

$$\left(\Psi_\varepsilon(t,x)\right)_\varepsilon = \left(\lim_{n\to\infty} \int_{q_n(t)=x} D^+[q_n(t)] \Psi_0(q_n(0)) \exp\left[\frac{i}{\hbar} S_\varepsilon(q_n(t), x; t)\right]\right)_\varepsilon, \quad (2.16)$$

or of the following form

$$\left(\Psi_\varepsilon(t,x)\right)_\varepsilon = \left(\lim_{n\to\infty} \int_{q(t)=x} D_n^+[q(t)] \Psi_0(q_n(0)) \exp\left[\frac{i}{\hbar} S_\varepsilon(\dot q(t), q(t), x; t)\right]\right)_\varepsilon. \quad (2.17)$$

For the limit in RHS of (2.16) and (2.17) we will be used following path integral notation

$$\left(\Psi_\varepsilon(t,x)\right)_\varepsilon = \left(\int_{q(t)=x} D^+[q(t)] \Psi_0(q(0)) \exp\left[\frac{i}{\hbar} S_\varepsilon(\dot q(t), q(t))\right]\right)_\varepsilon. (2.14)$$

III. Exact quasiclassical asymptotic beyond Maslov canonical operator.

Theorem. Let us consider Cauchy problem (2.8) with initial condition $\Psi_0(x)$ given via formula

$$\Psi_0(x) = \frac{\eta^{d/4}}{(2\pi)^{d/4} \hbar^{d/4}} \exp\left[\frac{\eta (x - x_0)^2}{2\hbar}\right], (3.1)$$

where $0 < \hbar \ll \eta \ll 1$ and $x^2 = \langle x, x \rangle$.

(1) We assume now that: (i) $\left(V_\varepsilon(x,t)\right)_\varepsilon \in G(\mathbb{R}^d)$, (ii)$V_{\varepsilon=0}(x,t) = V(x,t): \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$ and (iii)$\forall t \in \mathbb{R}_+$function$V(x,t)$ is a polynomial on variable $x = (x_1, \dots, x_d)$, i.e.

$V(x,t)=\sum_{\|\alpha\|\leq m} g_{\alpha}(t)x^{\alpha}, \alpha=(i_1,\dots,i_d), x^{\alpha}=x_1^{i_1}\times\dots\times x_d^{i_d}, \|\alpha\|=\sum_{r=1}^{d} i_r$(3.2)

(2) Let $u(\tau,t,\lambda,x,y)=\left(u_1(\tau,t,\lambda,x,y),\dots,u_d(\tau,t,\lambda,x,y)\right)$be the solution of the boundary problem

$\frac{\partial^2 u^{\mathrm{T}}(\tau,t,\lambda,x,y)}{\partial\tau^2}=\mathrm{Hess}[V(\lambda,\tau)]u^{\mathrm{T}}(\tau,t,\lambda,x,y)+[V'(\lambda,\tau)]^{\mathrm{T}}$,(3.3)

$u(0,t,\lambda,x,y)=y, u(t,t,\lambda,x,y)=x$.(3.4)

Here $\lambda=(\lambda_1,\dots,\lambda_d)\in\mathbb{R}^d, u^{\mathrm{T}}(\tau,t,\lambda,x,y)=\left(u_1(\tau,t,\lambda,x,y),\dots,u_d(\tau,t,\lambda,x,y)\right)^{\mathrm{T}}$,

$V'(\lambda,\tau)=\left(\left[\frac{\partial V(x,t)}{\partial x_1}\right]_{x=\lambda},\dots,\left[\frac{\partial V(x,t)}{\partial x_d}\right]_{x=\lambda}\right)$and $\mathrm{Hess}[V(\lambda,\tau)]=\left[\frac{\partial^2 V(x,t)}{\partial x_i\partial x_j}\right]_{x=\lambda}$ (3.5)

(3) Let$\mathrm{S}(t,\lambda,x,y)$ be the *master action* given via formula

$\mathrm{S}(t,\lambda,x,y)=\int_0^t L(\dot{u}(\tau,t,\lambda,x,y),u(\tau,t,\lambda,x,y),\tau)\,d\tau$,(3.6)

where *master Lagrangian* $L(\dot{u},u,\tau)$ are

$L(\dot{u},u,\tau)=\frac{m}{2}\dot{u}^2(\tau,t,\lambda,x,y)-\hat{V}(u(\tau,t,\lambda,x,y),\tau), \dot{u}=\left(\frac{\partial u_1}{\partial\tau},\dots,\frac{\partial u_d}{\partial\tau}\right), \dot{u}^2=\langle\dot{u},\dot{u}\rangle$,(3.7)

$\hat{V}\left((\tau,t,\lambda,x,y),\tau\right)=u(\tau,t,\lambda,x,y)\mathrm{Hess}[V(\lambda,\tau)]u^{\mathrm{T}}(\tau,t,\lambda,x,y)+V'(\lambda,\tau)u^{\mathrm{T}}(\tau,t,\lambda,x,y)$. (3.8)

Let $y_{cr}=y_{cr}(t,\lambda,x)\in\mathbb{R}^d$ be solution of the linear system of the algebraic equations

$\left[\frac{\partial\mathrm{S}(t,\lambda,x,y)}{\partial y_i}\right]_{y=y_{cr}}=0, i=1,\dots,d$ (3.9)

(4) Let $\hat{x}=\hat{x}(t,\lambda,x_0)\in\mathbb{R}^d$ be solution of the linear system of the algebraic equations

$y_{cr}(t,\lambda,\hat{x})+\lambda-x_0=0$. (3.10)

Assume that: for a given values of the parameters $t,\lambda,x_0$the point $\hat{x}=\hat{x}(t,\lambda,x_0)$is not a focal point on a corresponding trajectory is given by corresponding solution of the boundary problem (3.3).Then for the limiting quantum average given via formulae (1.1) the inequalities is satisfied:

$$\underline{\lim}_{\substack{\hbar\to 0\\ \varepsilon\to 0}}\left|\langle i,t,x_0;\hbar\rangle-\lambda_i\right|\leq$$

$\leq 2\left[\left|\det S_{y_{cr}y_{cr}}\left(t,\lambda,\hat{x}(t,\lambda,x_0),y_{cr}\left(t,\lambda,\hat{x}(t,\lambda,x_0)\right)\right)\right|\right]^{-1}\left|\hat{x}_i(t,\lambda,x_0)\right|, i=1,\dots,d$. (3.11)

Thus one can to calculate the limiting quantum trajectory corresponding to potential$V(x,t)$by using *transcendental master equation*

$\hat{x}_i(t,\lambda,x_0)=0, i=1,\dots,d$.(3.12)

Proof. From inequality (A.15) and Theorem A1, using inequalities (A.53.a) and (A.53.b) we obtain

$\underline{\lim_{\varepsilon\to 0}\lim_{\sigma\to 0}}_{\hbar\to 0}\left|\langle\hat{x}_i,T;\sigma,l,\lambda,\varepsilon\rangle-\lambda_i\right|\leq\lim_{\hbar\to 0}\left[\mathcal{R}_1(T,\lambda)+\mathcal{R}_2(T,\lambda)\right], i=1,\dots,d$, (3.13)

where

$$\mathcal{R}_1(T,\lambda) = \int dx \left\{\int_{q(T)=x} D^+[q(t)]\Psi\big(q(0)\big)[|q_i(T)|]^{\frac{1}{2}} \cos\left[\frac{1}{\hbar}\boldsymbol{S}_1(\dot{q},q,\lambda,T)\right]\right\}^2, \quad (3.14)$$

$$\mathcal{R}_2(T,\lambda) = \int dx \left\{\int_{q(T)=x} D^+[q(t)]\Psi\big(q(0)\big)[|q_i(T)|]^{\frac{1}{2}} \sin\left[\frac{1}{\hbar}\boldsymbol{S}_1(\dot{q},q,\lambda,T)\right]\right\}^2. \quad (3.15)$$

We note that

$$\mathcal{R}_1(T,\lambda) = \int dx \left[\breve{\mathcal{R}}_1(x,T,\lambda)\right]^2, \ \ \mathcal{R}_2(T,\lambda) = \int dx \left[\breve{\mathcal{R}}_2(x,T,\lambda)\right]^2, \quad (3.16)$$

where

$$\breve{\mathcal{R}}_1(x,T,\lambda) = \int_{q(T)=x} D^+[q(t)]\Psi\big(q(0)\big)[|q_i(T)|]^{\frac{1}{2}} \cos\left[\frac{1}{\hbar}\boldsymbol{S}_1(\dot{q},q,\lambda,T)\right] =$$

$$= \int dy \int_{\substack{q(T)=x \\ q(0)=y}} D^+[q(t)]\Psi\big(q(0)\big)[|q_i(T)|]^{\frac{1}{2}} \cos\left[\frac{1}{\hbar}S_1(\dot{q},q,\lambda,T)\right] = \int dy \breve{\mathcal{R}}_1(x,y,T,\lambda), \quad (3.17)$$

$$\breve{\mathcal{R}}_1(x,y,T,\lambda) = \int_{\substack{q(T)=x \\ q(0)=y}} D^+[q(t)]\Psi\big(q(0)\big)[|q_i(T)|]^{\frac{1}{2}} \cos\left[\frac{1}{\hbar}\boldsymbol{S}_1(\dot{q},q,\lambda,T)\right] \quad (3.18)$$

and

$$\breve{\mathcal{R}}_2(x,T,\lambda) = \int_{q(T)=x} D^+[q(t)]\Psi\big(q(0)\big)[|q_i(T)|]^{\frac{1}{2}} \sin\left[\frac{1}{\hbar}\boldsymbol{S}_1(\dot{q},q,\lambda,T)\right] =$$

$$= \int dy \int_{\substack{q(T)=x \\ q(0)=y}} D^+[q(t)]\Psi\big(q(0)\big)[|q_i(T)|]^{\frac{1}{2}} \sin\left[\frac{1}{\hbar}S_1(\dot{q},q,\lambda,T)\right] = \int dy \breve{\mathcal{R}}_2(x,y,T,\lambda), \quad (3.19)$$

$$\breve{\mathcal{R}}_2(x,y,T,\lambda) = \int_{\substack{q(T)=x \\ q(0)=y}} D^+[q(t)]\Psi\big(q(0)\big)[|q_i(T)|]^{\frac{1}{2}} \sin\left[\frac{1}{\hbar}\boldsymbol{S}_1(\dot{q},q,\lambda,T)\right]. \quad (3.20)$$

From Eq.(3.18) one obtain

$$\breve{\mathcal{R}}_1(x,y,T,\lambda) = \frac{1}{2}\left[\breve{\mathcal{R}}_{1,1}(x,y,T,\lambda) + \breve{\mathcal{R}}_{1,2}(x,y,T,\lambda)\right], \quad (3.21)$$

where

$$\breve{\mathcal{R}}_{1,1}(x,y,T,\lambda) = \int_{\substack{q(T)=x \\ q(0)=y}} D^+[q(t)]\Psi\big(q(0)\big)[|q_i(T)|]^{\frac{1}{2}} \exp\left[\frac{i}{\hbar}\boldsymbol{S}_1(\dot{q},q,\lambda,T)\right], \quad (3.22)$$

$$\breve{\mathcal{R}}_{1,2}(x,y,T,\lambda) = \int_{\substack{q(T)=x \\ q(0)=y}} D^+[q(t)]\Psi\big(q(0)\big)[|q_i(T)|]^{\frac{1}{2}} \exp\left[-\frac{i}{\hbar}\boldsymbol{S}_1(\dot{q},q,\lambda,T)\right]. \quad (3.23)$$

Let us calculate now path integral $\breve{\mathcal{R}}_{1,1}(x,y,T,\lambda)$ and path integral $\breve{\mathcal{R}}_{1,2}(x,y,T,\lambda)$, using stationary phase approximation. From Eq.(A.23) follows directly that action $\boldsymbol{S}_1(\dot{q},q,\lambda,T)$ coincide with *master action* $S(t,\lambda,x,y)$ is given via formulae (3.6)-(3.8) and therefore from Eq.(3.22) and Eq.(3.23) one obtain

$$\breve{\mathcal{R}}_{1,1}(x,y,T,\lambda) = \int_{\substack{q(T)=x \\ q(0)=y}} D^+[q(t)]\Psi\big(q(0)\big)[|q_i(T)|]^{\frac{1}{2}} \exp\left[\frac{i}{\hbar}\boldsymbol{S}_1(\dot{q},q,\lambda,T)\right] =$$

$$= \breve{\mathcal{R}}_{1,1}(x,y,T,\lambda) = [|x_i|]^{\frac{1}{2}}\,\Psi(y)\exp\left[\frac{i}{\hbar}S(t,\lambda,x,y)\right] \quad (3.24)$$

and

$$\breve{\mathcal{R}}_{1,2}(x,y,T,\lambda) = \int_{\substack{q(T)=x \\ q(0)=y}} D^+[q(t)]\Psi\big(q(0)\big)[|q_i(T)|]^{\frac{1}{2}} \exp\left[\frac{i}{\hbar}\boldsymbol{S}_1(\dot{q},q,\lambda,T)\right] =$$

$$= \breve{\mathcal{R}}_{1,2}(x,y,T,\lambda) = [|x_i|]^{\frac{1}{2}}\Psi(y)\exp\left[-\frac{i}{\hbar}S(t,\lambda,x,y)\right]. \quad (3.25)$$

From Eq.(3.17) and Eq.(3.24) we obtain

$$\breve{\mathcal{R}}_1(x,T,\lambda) = \int dy \breve{\mathcal{R}}_1(x,y,T,\lambda). \quad (3.26)$$

Substitution Eq.(3.25) into Eq.(3.26) gives

$$\breve{\mathcal{R}}_{1,1}(x,T,\lambda) = [|x_i|]^{\frac{1}{2}} \int dy\, \Psi(y)\exp\left[\frac{i}{\hbar}S(t,\lambda,x,y)\right]. \quad (3.26)$$

Similarly one obtain

$$\breve{\mathcal{R}}_{1,2}(x,T,\lambda) = [|x_i|]^{\frac{1}{2}} \int dy\, \Psi(y)\exp\left[-\frac{i}{\hbar}S(t,\lambda,x,y)\right]. \quad (3.27)$$

Let us calculate now integral $\breve{\mathcal{R}}_{1,1}(x,T,\lambda)$ and integral $\breve{\mathcal{R}}_{1,2}(x,T,\lambda)$ using stationary phase approximation. Let $y_{cr} = y_{cr}(t,\lambda,x) \in \mathbb{R}^d$ be the stationary point of master action $S(t,\lambda,x,y)$ and therefore Eq.(3.9) is satisfied. Having applied stationary phase approximation one obtain

$$\breve{\mathcal{R}}_{1,1}(x,y_{cr}(t,\lambda,x),T,\lambda) =$$

$$\left[\left|\det S_{y_{cr}y_{cr}}(t,\lambda,x,y_{cr}(t,\lambda,x))\right|\right]^{-\frac{1}{2}}[|x_i|]^{\frac{1}{2}}\Psi(y_{cr}(t,\lambda,x))\exp\left[\frac{i}{\hbar}S(t,\lambda,x,y_{cr}(t,\lambda,x))\right], \quad (3.28)$$

$$\breve{\mathcal{R}}_{1,2}(x,y_{cr}(t,\lambda,x),T,\lambda) =$$

$$\left[\left|\det S_{y_{cr}y_{cr}}(t,\lambda,x,y_{cr}(t,\lambda,x))\right|\right]^{-\frac{1}{2}}[|x_i|]^{\frac{1}{2}}\Psi(y_{cr}(t,\lambda,x))\exp\left[-\frac{i}{\hbar}S(t,\lambda,x,y_{cr}(t,\lambda,x))\right]. \quad (3.29)$$

Substitution Eq.(3.28)-Eq.(3.29) into Eq.(3.21) gives

$$\breve{\mathcal{R}}_1(x,y_{cr}(t,\lambda,x),T,\lambda) = \frac{1}{2}\left[\breve{\mathcal{R}}_{1,1}(x,y_{cr}(t,\lambda,x),T,\lambda) + \breve{\mathcal{R}}_{1,2}(x,y_{cr}(t,\lambda,x),T,\lambda)\right] =$$

$$= \left[\left|\det S_{y_{cr}y_{cr}}(t,\lambda,x,y_{cr}(t,\lambda,x))\right|\right]^{-\frac{1}{2}}[|x_i|]^{\frac{1}{2}}\Psi(y_{cr}(t,\lambda,x)) \times$$

$$\times\left\{\exp\left[\frac{i}{\hbar}S(t,\lambda,x,y_{cr}(t,\lambda,x))\right] + \exp\left[-\frac{i}{\hbar}S(t,\lambda,x,y_{cr}(t,\lambda,x))\right]\right\} =$$

$$= \left[\left|\det S_{y_{cr}y_{cr}}(t,\lambda,x,y_{cr}(t,\lambda,x))\right|\right]^{-\frac{1}{2}}[|x_i|]^{\frac{1}{2}}\Psi(y_{cr}(t,\lambda,x))\cos\left[\frac{1}{\hbar}S(t,\lambda,x,y_{cr}(t,\lambda,x))\right]. \quad (3.30)$$

Substitution Eq.(3.30) into Eq.(3.16) gives

$$\mathcal{R}_1(T,\lambda) = \int dx \int dx \left[\breve{\mathcal{R}}_1(x,T,\lambda)\right]^2 =$$

$$= \int dx \left[\left|\det S_{y_{cr}y_{cr}}(t,\lambda,x,y_{cr}(t,\lambda,x))\right|\right]^{-1}|x_i|\Psi^2(y_{cr}(t,\lambda,x))\cos^2\left[\frac{1}{\hbar}S(t,\lambda,x,y_{cr}(t,\lambda,x))\right]. \quad (3.31)$$

Similarly one obtain

$$\mathcal{R}_2(T,\lambda) = \int dx \int dx \left[\breve{\mathcal{R}}_2(x,T,\lambda)\right]^2 =$$

$$= \int dx \left[\left|\det S_{y_{cr}y_{cr}}(t,\lambda,x,y_{cr}(t,\lambda,x))\right|\right]^{-1}|x_i|\Psi^2(y_{cr}(t,\lambda,x))\sin^2\left[\frac{1}{\hbar}S(t,\lambda,x,y_{cr}(t,\lambda,x))\right]. \quad (3.32)$$

Therefore

$$\mathcal{R}(T,\lambda) = \mathcal{R}_1(T,\lambda) + \mathcal{R}_2(T,\lambda) =$$

$$= 2\int dx\,\left[\left|\det S_{y_{cr}y_{cr}}\big(t,\lambda,x,y_{cr}(t,\lambda,x)\big)\right|\right]^{-1}|x_i|\Psi^2\big(y_{cr}(t,\lambda,x)\big). \qquad (3.33)$$

Substitution Eq.(3.1) into Eq.(3.33) gives

$$\mathcal{R}(T,\lambda) = 2\frac{\eta^{d/2}}{(2\pi)^{d/2}\hbar^{d/2}}\int dx\,\left[\left|\det S_{y_{cr}y_{cr}}\big(t,\lambda,x,y_{cr}(t,\lambda,x)\big)\right|\right]^{-1}|x_i|\exp\left[\frac{\eta(y_{cr}(t,\lambda,x)-x_0)^2}{\hbar}\right]. \qquad (3.34)$$

Let us calculate now integral (3.34) using Laplace's approximation. It is easy to see that corresponding stationary point $\hat{x} = \hat{x}(t,\lambda,x_0) \in \mathbb{R}^d$ is the solution of the linear system of the algebraic equations (3.10). Therefore finally we obtain

$$\mathcal{R}(T,\lambda) = 2|\hat{x}_i(t,\lambda,x_0)|\left[\left|\det S_{y_{cr}y_{cr}}\Big(t,\lambda,\hat{x}(t,\lambda,x_0),y_{cr}\big(t,\lambda,\hat{x}(t,\lambda,x_0)\big)\Big)\right|\right]^{-1} + O(\hbar^d),$$

$$i = 1,\ldots,d. \qquad (3.35)$$

Substitution Eq.(3.35) into inequality (3.13) gives the inequality (3.11). The inequality (3.11) completed the proof.

IV. Quantum anharmonic oscillator with a cubic potential.

Let us consider quantum anharmonic oscillator with potential

$V(x) = \frac{m\omega^2}{2}x^2 - ax^3 + bx, x \in \mathbb{R}, a, b > 0$(4.1)

supplemented by an additive sinusoidal driving. Thus

$$V(x,t) = \frac{m\omega^2}{2}x^2 - ax^3 + bx - [A\sin(\Omega t)]x. \qquad (4.2)$$

The corresponding master Lagrangian given by (3.7), are

$$L(\dot{u},u,\tau) = \left(\frac{m}{2}\right)\dot{u}^2 - m\left(\left(\frac{\omega^2}{2}\right) + \left(\frac{3a\lambda}{m}\right)\right)u^2 - (m\omega^2\lambda + 3a\lambda^2 - b - A\sin(\Omega t))u.(4.3)$$

We assume now that: $\frac{\omega^2}{2} + \frac{3a\lambda}{m} \geq 0$ and rewrite (4.3) of the form

$$L(\dot{u},u,\tau) = (m/2)\dot{u}^2 - (m\varpi^2\lambda/2)u^2 + g(\lambda,t)u, \qquad (4.4)$$

where $\varpi(\lambda) = \sqrt{2\left|\frac{\omega^2}{2} + \frac{3a\lambda}{m}\right|}$ and $g(\lambda,t) = -[m\omega^2\lambda + 3a\lambda^2 - b - A\sin(\Omega\cdot t)]$.

The corresponding master action$S(t,\lambda,x,y)$given by(3.6), are

$$S(t,\lambda,x,y) = \frac{m\varpi}{2\sin\varpi t}[(\cos\varpi t)(y^2 + x^2) - 2xy + \frac{2x}{m\varpi}\int_0^t g(\lambda,\tau)\sin(\varpi\tau)\,d\tau +$$

$$+\frac{2y}{m\varpi}\int_0^t g(\lambda,\tau)\sin(\varpi(t-\tau))\,d\tau - \frac{2}{m^2\varpi^2}\int_0^t\int_0^\tau g(\lambda,\tau)g(\lambda,s)\sin\varpi(t-\tau)\sin(\varpi s)\,ds\,d\tau].(4.5)$$

The linear system of the algebraic equations (3.9) are

$$\frac{\partial S(t,\lambda,x,y)}{\partial y} = 2y\cos\varpi t - 2x + \frac{2}{m\varpi}\int_0^t g(\lambda,t)\sin((\varpi(t-\tau))d\tau = 0. \qquad (4.6)$$

Therefore

$$y_{cr}(t,\lambda,x)=\frac{x}{\cos\varpi t}-\frac{1}{m\varpi\cos\varpi t}\int_0^t g(\lambda,t)\sin((\varpi(t-\tau))d\tau \quad (4.7)$$

The linear system of the algebraic equations (3.10) are

$$\frac{x}{\cos\varpi t}-\frac{1}{m\varpi\cos\varpi t}\int_0^t g(\lambda,t)\sin((\varpi(t-\tau))d\tau+\lambda-x_0=0. \qquad (4.8)$$

Therefore the solution of the linear system of the algebraic equations (3.10) are

$$\hat{x}(t,\lambda,x_0)=\frac{1}{m\varpi}\int_0^t g(\lambda,t)\sin((\varpi(t-\tau))d\tau+(\lambda-x_0)\cos\varpi t. \qquad (4.9)$$

Transcendental master equation (3.11) are

$$\frac{1}{m\varpi}\int_0^t g(\lambda,t)\sin((\varpi(t-\tau))d\tau+(\lambda-x_0)\cos\varpi t=0 \qquad (4.10)$$

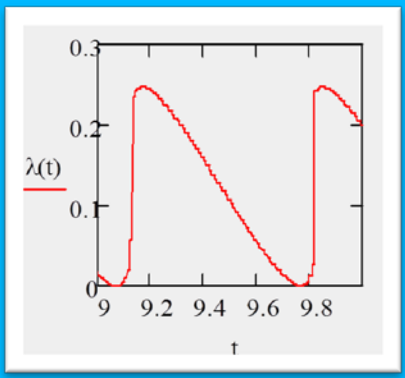
Finally from Eq.(4.10) one obtain

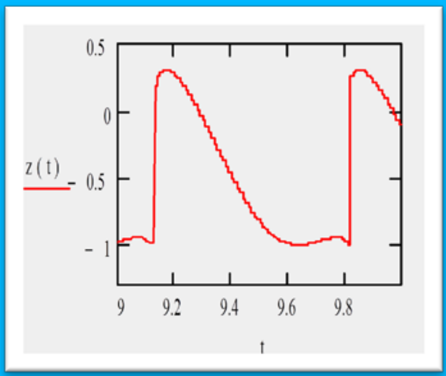
$$d(\lambda)\left(\frac{\cos(\varpi t)}{\varpi}-\frac{1}{\varpi}\right)+\frac{A(\varpi\sin(\Omega t)-\Omega\sin(\varpi t))}{\varpi^2-\Omega^2}-(\lambda-x_0)m\varpi\cos(\varpi t)=0, \quad (4.11)$$

where $d(\lambda)=m\omega^2\lambda+3a\lambda^2-b.$

Numerical Examples.

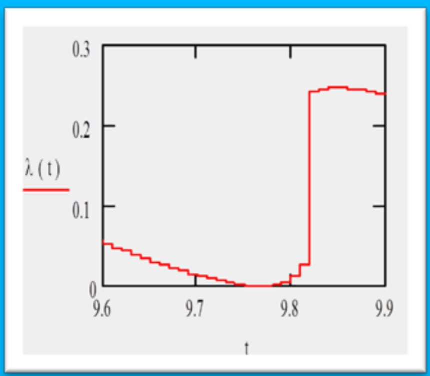
Example1. $x_0=0, m=1, \Omega=0, \omega=9, a=3, b=10, A=0.$

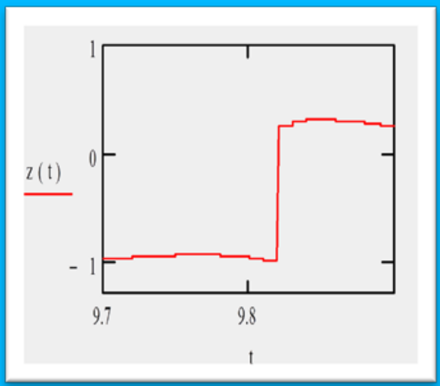




Pic.1. Function $\lambda(t)$

Pic.2. Function $z(t)$





Appendix

Let us consider now regularized Feynman-Colombeau propagator$\left(K_{\varepsilon}(x,T|y,0)\right)_{\varepsilon}$
given by Feynman path integral:

$$\widetilde{K}_{\varepsilon}(x,T|y,0;\sigma,l) = \int_{\substack{q(T)=x\\ q(0)=y}} D^{+}[q(t)]\exp\left[-\frac{1}{\hbar}\boldsymbol{S_1}(q,T;\sigma,l)\right]\exp\left[-\frac{1}{\hbar}\boldsymbol{S_2}(q(T),\lambda)\right]\exp\left[\frac{i}{\hbar}\boldsymbol{S}_{\varepsilon}(\dot{q},q,T)\right], \quad (A.1)$$

where $\hbar \in (0,1]$,

$$\boldsymbol{S}_1(q,T;\sigma,l) = \int_0^T dt[\{[q(t)-\lambda]^2;\sigma,l\}], \quad (A.2)$$

$$\boldsymbol{S}_2(q(T),\lambda) = [q(T)-\lambda]^2,\ \ \lambda = (\lambda_1,\dots,\lambda_d) \in \mathbb{R}^d, \quad (A.3)$$

$$\boldsymbol{S}_{\varepsilon}(\dot{q},q,T) = \int_0^T L(\dot{q}(t),q(t),t)\,dt,\ \ L(\dot{q}(t),q(t),t) = \frac{m}{2}\dot{q}^2(t) - V_{\varepsilon}(q(t),t), V_{\varepsilon=0}(x,t) = V(x,t), \quad (A.4)$$

$$V(x,t) = g_1(t)x + g_2(t)x^2 + g_3(t)x^3 + \cdots + g_{\alpha}(t)x^{\alpha}, \quad (A.5)$$

$$\alpha = (i_1,\dots,i_d), x^{\alpha} = x_1^{i_1} \times \dots \times x_d^{i_d}, \|\alpha\| = \sum_{r=1}^{d} i_r,$$

$$V_{\varepsilon}(q(t),t) = V(q_{\varepsilon}(t),t),\ \ q_{\varepsilon}(t) = \left(q_{1,\varepsilon}(t),\dots,q_{d,\varepsilon}(t)\right), \quad (A.6)$$

$$q_{i,\varepsilon}(t) = \frac{q_i(t)}{1+\varepsilon^{2k}|q(t)|^{2k}}, \varepsilon \in (0,1], k \geq 1. \quad (A.7)$$

Here: (1)$\sigma \in (0,1], \hbar \ll \sigma$and (2) for each path $q(t)$ such that $q(t) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi t}{T}\right) + u(t,T,y,x)$, $u(0,T,y,x) = y, u(T,T,y,x) = x$,where $u(t,T,y,x)$ is a given function, operator $\{p(t);\sigma,l\}$ are

$$\{q(t);\sigma,l\} = \sum_{n=1}^{l} \sigma\, a_n \sin\left(\frac{n\pi t}{T}\right) + \sum_{n=l+1}^{\infty} a_n \sin\left(\frac{n\pi t}{T}\right). \quad (A.8)$$

(3) $D^{+}[q(t)]$is a positive Feynman "measure".

Therefore regularized Colombeau solution of the Schrödinger equation corresponding to regularized propagator (A.1) are

$$\left(\Psi_{\varepsilon}(T,x;\sigma,l,\lambda)\right)_{\varepsilon} = \left(\int_{-\infty}^{\infty} dy \Psi(y) \widetilde{K}_{\varepsilon}(x,T|y,0;\sigma,l)\right)_{\varepsilon} =$$

$$\left(\int\limits_{q(T)=x} D^{+}[q(t)]\Psi\big(q(0)\big)\exp\left[-\frac{1}{\hbar}\boldsymbol{S}(q,T;\sigma,l,\lambda)\right]\exp\left[\frac{i}{\hbar}\boldsymbol{S}_{\varepsilon}(\dot{q},q,T)\right]\right)_{\varepsilon} =$$

$$= \left(\int dy \int_{\substack{q(T)=x\\ q(0)=y}} D^{+}[q(t)]\Psi\big(q(0)\big)\exp\left[-\tfrac{1}{\hbar}\boldsymbol{S}(q,T;\sigma,l,\lambda)\right]\exp\left[\tfrac{i}{\hbar}\boldsymbol{S}_{\varepsilon}(\dot{q},q,T)\right]\right)_{\varepsilon}. \quad (A.9)$$

Here

$$\boldsymbol{S}(q,T;\sigma,l,\lambda) = \boldsymbol{S}_1(q,T;\sigma,l) + \boldsymbol{S}_2(q(T),\lambda). \quad (A.10)$$

Let us consider now regularized quantum average

$$(\langle \hat{x}_i, T;\sigma,l,\lambda,\varepsilon\rangle)_{\varepsilon} = \left(\int_{-\infty}^{\infty} dx x_i\, |\Psi_{\varepsilon}(T,x;\sigma,l,\lambda)|^2\right)_{\varepsilon}. \quad (A.11)$$

From (A.5) and (A.11) one obtain

$$(|\hat{x}_i, T;\sigma,l,\lambda,\varepsilon|)_{\varepsilon} \leq \left(\int dx \left\{\int_{q(T)=x} D^{+}[q(t)]\Psi\big(q(0)\big)[|q_i(T)|]^{1/2}\exp\left[-\tfrac{1}{\hbar}\boldsymbol{S}(q,T;\sigma,l,\lambda)\right]\cos\left[\tfrac{1}{\hbar}\boldsymbol{S}_{\varepsilon}(\dot{q},q,T)\right]\right\}^2\right)_{\varepsilon} +$$

$$+\left(\int dx \left\{\int_{q(T)=x} D^{+}[q(t)]\Psi\big(q(0)\big)[|q_i(T)|]^{1/2}\exp\left[-\tfrac{1}{\hbar}\boldsymbol{S}(q,T;\sigma,l,\lambda)\right]\sin\left[\tfrac{1}{\hbar}\boldsymbol{S}_{\varepsilon}(\dot{q},q,T)\right]\right\}^2\right)_{\varepsilon} \quad (A.13)$$

From Eq.(A.5)-(A.13) one obtain

$$|\langle \hat{x}_i, T;\sigma,l,\lambda,\varepsilon\rangle - \lambda_i| = \left|\langle \hat{x}_i, T;\sigma,l,\lambda,\varepsilon\rangle - \lambda_i \int_{-\infty}^{\infty} dx |\Psi_{\varepsilon}(T,x;\sigma,l,\lambda)|^2\right| =$$

$$= \left|\int_{-\infty}^{\infty} dx[x_i-\lambda_i]\, |\Psi_{\varepsilon}(T,x;\sigma,l,\lambda)|^2\right| \leq \int_{-\infty}^{\infty} dx|x_i-\lambda_i|\, |\Psi_{\varepsilon}(T,x;\sigma,l,\lambda)|^2 =$$

$$= \int dx \left\{\int_{q(T)=x} D^{+}[q(t)]\Psi\big(q(0)\big)[|q_i(T)-\lambda_i|]^{1/2}\exp\left[-\tfrac{1}{\hbar}\boldsymbol{S}(q,T;\sigma,l,\lambda)\right]\cos\left[\tfrac{1}{\hbar}\boldsymbol{S}_{\varepsilon}(\dot{q},q,T)\right]\right\}^2 +$$

$$+ \int dx \left\{\int_{q(T)=x} D^{+}[q(t)]\Psi\big(q(0)\big)[|q_i(T)-\lambda_i|]^{1/2}\exp\left[-\tfrac{1}{\hbar}\boldsymbol{S}(q,T;\sigma,l,\lambda)\right]\sin\left[\tfrac{1}{\hbar}\boldsymbol{S}_{\varepsilon}(\dot{q},q,T)\right]\right\}^2 \quad (A.14)$$

Using replacement $q_i(t) - \lambda_i \coloneqq q_i(t), i = 1,\dots,d$ into RHS of the Eq.(A.9) one obtain

$$|\langle \hat{x}_i, T;\sigma,l,\lambda,\varepsilon\rangle - \lambda_i| \leq$$

$$\leq \int dx \left\{\int_{q(T)=x} D^{+}[q(t)]\Psi\big(q(0)\big)[|q_i(T)|]^{1/2}\exp\left[-\tfrac{1}{\hbar}\boldsymbol{S}(q,T;\sigma,l)\right]\cos\left[\tfrac{1}{\hbar}\boldsymbol{S}_{\varepsilon}(\dot{q},q+\lambda,T)\right]\right\}^2 +$$

$$+ \int dx \left\{\int_{q(T)=x} D^{+}[q(t)]\Psi\big(q(0)\big)[|q_i(T)|]^{1/2}\exp\left[-\tfrac{1}{\hbar}\boldsymbol{S}(q,T;\sigma,l)\right]\sin\left[\tfrac{1}{\hbar}\boldsymbol{S}_{\varepsilon}(\dot{q},q+\lambda,T)\right]\right\}^2 =$$

$$= \int dx\,[I_1^2(x,T;\sigma,l,\lambda,\varepsilon)] + \int dx\,[I_2^2(x,T;\sigma,l,\lambda,\varepsilon)]. \quad (A.15)$$

Here

$$\boldsymbol{S}(q,T;\sigma,l) = \boldsymbol{S}_1(q,T;\sigma,l) + \boldsymbol{S}_2\big(q(T)\big), \boldsymbol{S}_1(q,T;\sigma,l) = \int_0^T dt[\{[q(t)]^2;\sigma,l\}],$$

$$\boldsymbol{S}_2\big(q(T)\big) = [q(T)]^2, \lambda \in \mathbb{R}^d \quad (A.16)$$

and

$$I_1(x,T;\sigma,l,\lambda,\varepsilon) =$$

$$= \int_{q(T)=x} D^+[q(t)]\Psi\big(q(0)\big)[|q_i(T)|]^{1/2}\exp\left[-\frac{1}{\hbar}\boldsymbol{S}(q,T;\sigma,l)\right]\cos\left[\frac{1}{\hbar}\boldsymbol{S}_\varepsilon(\dot{q},q+\lambda,T)\right] \qquad (A.17)$$

$$I_2(x,T;\sigma,l,\lambda,\varepsilon) =$$

$$= \int_{q(T)=x} D^+[q(t)]\Psi\big(q(0)\big)[|q_i(T)|]^{1/2}\exp\left[-\frac{1}{\hbar}\boldsymbol{S}(q,T;\sigma,l)\right]\sin\left[\frac{1}{\hbar}\boldsymbol{S}_\varepsilon(\dot{q},q+\lambda,T)\right]. \qquad (A.18)$$

Let us rewrite a function $V_\varepsilon(q(t)+\lambda,t)$ in the following equivalent form:

$$V_\varepsilon(q(t)+\lambda,t) = V_{\varepsilon,0}(q(t),t,\lambda) + V_{\varepsilon,1}(q(t),t,\lambda), \qquad (A.19)$$

$$V_{\varepsilon,0}(q(t),t,\lambda) = a_{\varepsilon,1}(q(t),t,\lambda)q(t) + a_{\varepsilon,2}(q(t),t,\lambda)q^2(t), \qquad (A.20)$$

$$V_{\varepsilon,1}(q(t),t,\lambda) = a_{\varepsilon,3}(q(t),t,\lambda)q^3(t) + \cdots + a_{\varepsilon,\alpha}(q(t),t,\lambda)q^\alpha(t), \qquad (A.21)$$

where $a_{\varepsilon=0,1}(q(t),t,\lambda) = c_1(t,\lambda), a_{\varepsilon=0,2}(q(t),t,\lambda) = c_2(t,\lambda), \dots, a_{\varepsilon=0,\alpha}(q(t),t,\lambda) = c_\alpha(t,\lambda)$.
Let us evaluate now path integral $I_1(T;\sigma,l,\lambda)$ given via Eq.(A.17). Substitution Eq.(A.19) into RHS of the Eq.(A.17) gives

$$I_1(x,T;\sigma,l,\lambda,\varepsilon) = I_1^{(1)}(x,T;\sigma,l,\lambda,\varepsilon) + I_1^{(2)}(x,T;\sigma,l,\lambda,\varepsilon) =$$

$$\int_{q(T)=x} D^+[q(t)]\Psi\big(q(0)\big)[|q_i(T)|]^{\frac{1}{2}}\exp\left[-\frac{1}{\hbar}\boldsymbol{S}(q,T;\sigma,l)\right]\cos\left[\frac{1}{\hbar}\boldsymbol{S}_{\varepsilon,1}(\dot{q},q+\lambda,T)\right]\cos\left[\frac{1}{\hbar}\boldsymbol{S}_{\varepsilon,2}(q+\lambda,T)\right] +$$

$$+ \int_{q(T)=x} D^+[q(t)]\Psi\big(q(0)\big)[|q_i(T)|]^{1/2}\exp\left[-\frac{1}{\hbar}\boldsymbol{S}(q,T;\sigma,l)\right]\sin\left[\frac{1}{\hbar}\boldsymbol{S}_{\varepsilon,1}(\dot{q},q,\lambda,T)\right]\sin\left[-\frac{1}{\hbar}\boldsymbol{S}_{\varepsilon,2}(q,\lambda,T)\right], \qquad (A.22.a)$$

$$I_2(x,T;\sigma,l,\lambda,\varepsilon) = I_2^{(1)}(x,T;\sigma,l,\lambda,\varepsilon) + I_2^{(2)}(x,T;\sigma,l,\lambda,\varepsilon) =$$

$$\int_{q(T)=x} D^+[q(t)]\Psi\big(q(0)\big)[|q_i(T)|]^{\frac{1}{2}}\exp\left[-\frac{1}{\hbar}\boldsymbol{S}(q,T;\sigma,l)\right]\cos\left[\frac{1}{\hbar}\boldsymbol{S}_{\varepsilon,1}(\dot{q},q+\lambda,T)\right]\sin\left[\frac{1}{\hbar}\boldsymbol{S}_{\varepsilon,2}(q+\lambda,T)\right] +$$

$$+ \int_{q(T)=x} D^+[q(t)]\Psi\big(q(0)\big)[|q_i(T)|]^{1/2}\exp\left[-\frac{1}{\hbar}\boldsymbol{S}(q,T;\sigma,l)\right]\sin\left[\frac{1}{\hbar}\boldsymbol{S}_{\varepsilon,1}(\dot{q},q,\lambda,T)\right]\cos\left[-\frac{1}{\hbar}\boldsymbol{S}_{\varepsilon,2}(q,\lambda,T)\right], \qquad (A.22.b)$$

where

$$\boldsymbol{S}_{\varepsilon,1}(\dot{q},q,\lambda,T) = \int_0^T L_\varepsilon(\dot{q}(t),q(t),t,\lambda)\,dt, \quad L_\varepsilon(\dot{q}(t),q(t),t,\lambda) = \frac{m}{2}\dot{q}^2(t) - V_{\varepsilon,0}(q(t),t,\lambda), \qquad (A.23)$$

$$\boldsymbol{S}_{\varepsilon,2}(q,\lambda,T) = \int_0^T V_{\varepsilon,1}(q(t),t,\lambda)\,dt, \qquad (A.24)$$

$$I_1^{(1)}(x,T;\sigma,l,\lambda,\varepsilon) =$$

$$\int_{q(T)=x} D^+[q(t)]\Psi\big(q(0)\big)[|q_i(T)|]^{\frac{1}{2}}\exp\left[-\frac{1}{\hbar}\boldsymbol{S}(q,T;\sigma,l)\right]\cos\left[\frac{1}{\hbar}\boldsymbol{S}_{\varepsilon,1}(\dot{q},q,\lambda,T)\right]\cos\left[\frac{1}{\hbar}\boldsymbol{S}_{\varepsilon,2}(q,\lambda,T)\right], \qquad (A.25.a)$$

$$I_2^{(1)}(x,T;\sigma,l,\lambda,\varepsilon) =$$

$$\int_{q(T)=x} D^+[q(t)]\Psi\big(q(0)\big)[|q_i(T)|]^{\frac{1}{2}}\exp\left[-\frac{1}{\hbar}\boldsymbol{S}(q,T;\sigma,l)\right]\cos\left[\frac{1}{\hbar}\boldsymbol{S}_{\varepsilon,1}(\dot{q},q,\lambda,T)\right]\sin\left[\frac{1}{\hbar}\boldsymbol{S}_{\varepsilon,2}(q,\lambda,T)\right], \qquad (A.25.b)$$

$$I_1^{(2)}(x,T;\sigma,l,\lambda,\varepsilon) =$$

$$\int_{q(T)=x} D^{+}[q(t)]\Psi(q(0))[|q_i(T)|]^{1/2}\exp\left[-\frac{1}{\hbar}\boldsymbol{S}(q,T;\sigma,l)\right]\sin\left[\frac{1}{\hbar}\boldsymbol{S}_{\varepsilon,1}(\dot{q},q,\lambda,T)\right]\sin\left[-\frac{1}{\hbar}\boldsymbol{S}_{\varepsilon,2}(q,\lambda,T)\right]. \quad \text{(A.26.a)}$$

$$I_2^{(2)}(x,T;\sigma,l,\lambda,\varepsilon) =$$

$$\int_{q(T)=x} D^{+}[q(t)]\Psi(q(0))[|q_i(T)|]^{1/2}\exp\left[-\frac{1}{\hbar}\boldsymbol{S}(q,T;\sigma,l)\right]\sin\left[\frac{1}{\hbar}\boldsymbol{S}_{\varepsilon,1}(\dot{q},q,\lambda,T)\right]\cos\left[-\frac{1}{\hbar}\boldsymbol{S}_{\varepsilon,2}(q,\lambda,T)\right]. \quad \text{(A.26.b)}$$

Let us evaluate now n-dimensional path integral $I_{1,n}^{(1)}(x,T;\sigma,l,\lambda,\varepsilon)$:

$$I_{1,n}^{(1)}(x,T;\sigma,l,\lambda,\varepsilon) =$$

$$= \int_{q(T)=x} D_n^{+}[q(t)]\Psi(q(0))[|q_i(T)|]^{\frac{1}{2}}\exp\left[-\frac{1}{\hbar}\boldsymbol{S}(q,T;\sigma,l)\right]\cos\left[\frac{1}{\hbar}\boldsymbol{S}_{\varepsilon,1}(\dot{q},q,\lambda,T)\right]\cos\left[\frac{1}{\hbar}\boldsymbol{S}_{\varepsilon,2}(q,\lambda,T)\right] =$$

$$= \int_{q(T)=x} D_n^{+}[q(t)]\Psi(q(0))[|q_i(T)|]^{\frac{1}{2}}\exp\left[-\frac{1}{\hbar}\boldsymbol{S}(q,T;\sigma,l)\right]\left\{\cos\left[\frac{1}{\hbar}S_{\varepsilon,1}(\dot{q},q,\lambda,T)\right]+1\right\}\cos\left[\frac{1}{\hbar}\boldsymbol{S}_{\varepsilon,2}(q,\lambda,T)\right] -$$

$$- \int_{q(T)=x} D_n^{+}[q(t)]\Psi(q(0))[|q_i(T)|]^{\frac{1}{2}}\exp\left[-\frac{1}{\hbar}\boldsymbol{S}(q,T;\sigma,l)\right]\cos\left[\frac{1}{\hbar}\boldsymbol{S}_{\varepsilon,2}(q,\lambda,T)\right]. \quad \text{(A.27)}$$

From Eq.(A.27) one obtain the inequality

$$\left|I_{1,n}^{(1)}(x,T;\sigma,l,\lambda,\varepsilon)\right| \leq \left|\int \quad D_n^{+}[q(t)]\Psi(q(0))[|q_i(T)|]^{\frac{1}{2}}\exp\left[-\frac{1}{\hbar}\boldsymbol{S}(q,T;\sigma,l)\right]\left\{\cos\left[\frac{1}{\hbar}\boldsymbol{S}_{\varepsilon,1}(\dot{q},q,\lambda,T)\right]+1\right\}\right| -$$

$$- \int_{q(T)=x} D_n^{+}[q(t)]\Psi(q(0))[|q_i(T)|]^{\frac{1}{2}}\exp\left[-\frac{1}{\hbar}\boldsymbol{S}(q,T;\sigma,l)\right]\cos\left[\frac{1}{\hbar}\boldsymbol{S}_{\varepsilon,2}(q,\lambda,T)\right] =$$

$$= \left|\int_{q(T)=x} D_n^{+}[q(t)]\Psi(q(0))[|q_i(T)|]^{\frac{1}{2}}\exp\left[-\frac{1}{\hbar}\boldsymbol{S}(q,T;\sigma,l)\right]\cos\left[\frac{1}{\hbar}\boldsymbol{S}_{\varepsilon,1}(\dot{q},q,\lambda,T)\right]\right| +$$

$$+ \int\limits_{q(T)=x} D_n^{+}[q(t)]\Psi(q(0))[|q_i(T)|]^{\frac{1}{2}}\exp\left[-\frac{1}{\hbar}\boldsymbol{S}(q,T;\sigma,l)\right] -$$

$$- \int_{q(T)=x} D_n^{+}[q(t)]\Psi(q(0))[|q_i(T)|]^{\frac{1}{2}}\exp\left[-\frac{1}{\hbar}\boldsymbol{S}(q,T;\sigma,l)\right]\cos\left[\frac{1}{\hbar}\boldsymbol{S}_{\varepsilon,2}(q,\lambda,T)\right]. \quad \text{(A.28)}$$

From Inq.(A.28) one obtain the inequality

$$\left|I_{1,n}^{(1)}(x,T;\sigma,l,\lambda,\varepsilon)\right| \leq \left|\int_{q(T)=x} D_n^{+}[q(t)]\Psi(q(0))[|q_i(T)|]^{\frac{1}{2}}\exp\left[-\frac{1}{\hbar}\boldsymbol{S}(q,T;\sigma,l)\right]\cos\left[\frac{1}{\hbar}\boldsymbol{S}_{\varepsilon,1}(\dot{q},q,\lambda,T)\right]\right| -$$

$$- \sum_{i=1}^{\infty}\frac{(-1)^i\hbar^{-2i}}{(2i)!}\int_{q(T)=x} D_n^{+}[q(t)]\Psi(q(0))[|q_i(T)|]^{\frac{1}{2}}\left[\boldsymbol{S}_{\varepsilon,2}(q,\lambda,T)\right]^{2i}\exp\left[-\frac{1}{\hbar}\boldsymbol{S}(q,T;\sigma,l)\right] =$$

$$= \left|\mathbf{J}_{1,n}^{(1)}(x,T;\sigma,l,\lambda,\varepsilon,\hbar)\right| - \sum_{i=1}^{\infty}\frac{(-1)^i\hbar^{-2i}}{(2i)!}\mathcal{R}_{\varepsilon}^{(i)}(x,T;\sigma,l,n), \quad \text{(A.29)}$$

where

$$\mathbf{J}_{1,n}^{(1)}(x,T;\sigma,l,\lambda,\varepsilon,\hbar) =$$

$$= \int_{q(T)=x} D_n^{+}[q(t)]\Psi(q(0))[|q_i(T)|]^{\frac{1}{2}}\exp\left[-\frac{1}{\hbar}\boldsymbol{S}(q,T;\sigma,l)\right]\cos\left[\frac{1}{\hbar}\boldsymbol{S}_{\varepsilon,1}(\dot{q},q,\lambda,T)\right], \quad \text{(A.30)}$$

$$\mathcal{R}_{\varepsilon}^{(i)}(x,T;\sigma,l,n) = \int_{q(T)=x} D_n^{+}[q(t)]\Psi(q(0))[|q_i(T)|]^{\frac{1}{2}}\left[\boldsymbol{S}_{\varepsilon,2}(q,\lambda,T)\right]^{2i}\exp\left[-\frac{1}{\hbar}\boldsymbol{S}(q,T;\sigma,l)\right]. \quad \text{(A.31)}$$

Using replacement $q_i(t) \coloneqq \hbar^{\frac{1}{2}}q_i(t), t \in [0,T], i = 1,\ldots,d$ into RHS of the Eq.(A.31) one obtain

$$\mathcal{R}_{\varepsilon}^{(i)}(x,T;\sigma,l,n) = \hbar^{1/4}\int_{q(T)=\frac{x}{\sqrt{\hbar}}} \breve{D}_n^{+}[q(t)]\Psi\left(\hbar^{\frac{1}{2}}q(0)\right)[|q_i(T)|]^{\frac{1}{2}}\left[\boldsymbol{S}_{\varepsilon,2}\left(\hbar^{1/2}q,\lambda,T\right)\right]^{2i}\exp\left[-\frac{1}{\hbar}\boldsymbol{S}\left(\hbar^{1/2}q,T;\sigma,l\right)\right] =$$

$$\hbar^{1/4}\hbar^{i/2}\int dy \int_{\substack{q(T)=\frac{x}{\sqrt{\hbar}}\\ q(0)=\frac{y}{\sqrt{\hbar}}}} \breve{D}_n^{+}[q(t)]\breve{\Psi}\left(q(0)\right)[|q_i(T)|]^{\frac{1}{2}}\left[\widehat{\boldsymbol{S}}_{\varepsilon,2}(q,\lambda,T,\hbar)\right]^{2i}\exp[-\boldsymbol{S}(q,T;\sigma,l)] =$$

$$= \hbar^{1/4}\hbar^{i/2}\widehat{\mathcal{R}}_{\varepsilon}^{(i)}(x,T;\sigma,l,n), \qquad \text{(A.32)}$$

where

$$\breve{D}_n^{+}[q(t)] = D_n^{+}\left[\hbar^{\frac{1}{2}}q(t)\right], t\in[0,T],\ \ \breve{\Psi}\left(q(0)\right) = \frac{\eta^{d/4}}{(2\pi)^{d/4}\hbar^{d/4}}\exp\left[\frac{\eta q^2(0)}{2}\right],$$ see Eq.(3.1) and

$$\widehat{\boldsymbol{S}}_{\varepsilon,2}(q,\lambda,T,\hbar) = \int_0^T \widehat{V}_{\varepsilon,1}(q(t),t,\lambda,\hbar)\,dt, \qquad \text{(A.33)}$$

$$\widehat{V}_{\varepsilon,1}(q(t),t,\lambda,\hbar) = a_{\varepsilon,3}(q(t),t,\lambda)q^3(t) + \cdots + \hbar^{\frac{\alpha-3}{2}}a_{\varepsilon,\alpha}(q(t),t,\lambda)q^{\alpha}(t). \qquad \text{(A.34)}$$

$$\widehat{\mathcal{R}}_{\varepsilon}^{(i)}(x,T;\sigma,l,n) = \int dy \int_{\substack{q(T)=\frac{x}{\sqrt{\hbar}}\\ q(0)=\frac{y}{\sqrt{\hbar}}}} \breve{D}_n^{+}[q(t)]\breve{\Psi}\left(q(0)\right)[|q_i(T)|]^{\frac{1}{2}}\left[\widehat{\boldsymbol{S}}_{\varepsilon,2}(q,\lambda,T,\hbar)\right]^{2i}\exp[-\boldsymbol{S}(q,T;\sigma,l)]. \qquad \text{(A.35)}$$

From (A.29)-(A.35) one obtain

$$\left|I_{1,n}^{(1)}(x,T;\sigma,l,\lambda,\varepsilon)\right| \leq \left|\mathbf{J}_{1,n}^{(1)}(x,T;\sigma,l,\lambda,\varepsilon,\hbar)\right| - \hbar^{\frac{1}{4}}\sum_{i=1}^{\infty}\frac{(-1)^i\hbar^i}{(2i)!}\widehat{\mathcal{R}}_{\varepsilon}^{(i)}(x,T;\sigma,l,n) =$$

$$\leq \left|\mathbf{J}_{1,n}^{(1)}(x,T;\sigma,l,\lambda,\varepsilon,\hbar)\right| - \hbar^{\frac{1}{4}}\Xi_{\varepsilon}(x,T;\sigma,l,n), \ \text{ where} \qquad \text{(A.36)}$$

$$\Xi_{\varepsilon}(x,T;\sigma,l,\hbar,n) = \sum_{i=1}^{\infty}\frac{(-1)^i\hbar^i}{(2i)!}\widehat{\mathcal{R}}_{\varepsilon}^{(i)}(x,T;\sigma,l,n). \qquad \text{(A.37)}$$

Proposition A.1. [21]-[23] Let $\{s_{n,m}\}_{n,m=1}^{n,m=\infty}$ be a double sequence $s:\mathbb{N}\times\mathbb{N}\to\mathbb{C}$. Let $\lim_{n,m\to\infty}s_{n,m} = a$. Then the iterated limit: $\lim_{n\to\infty}\left(\lim_{m\to\infty}s_{n,m}\right)$ exist and equal to a if and only if $\lim_{m\to\infty}s_{n,m}$ exists for each $n\in\mathbb{N}$.

Proposition A.2. Let $I_1^{(1)}(x,T;\sigma,l,\lambda,\varepsilon,\hbar) = I_1^{(1)}(x,T;\sigma,l,\lambda,\varepsilon)$, where $I_1^{(1)}(x,T;\sigma,l,\lambda,\varepsilon)$ is given via Eq.(A.25) and let $I_1^{(2)}(x,T;\sigma,l,\lambda,\varepsilon,\hbar) = I_1^{(2)}(x,T;\sigma,l,\lambda,\varepsilon)$, where $I_1^{(2)}(x,T;\sigma,l,\lambda,\varepsilon)$ is given via Eq.(A.26). Then $I_2^{(2)}(x,T;\sigma,l,\lambda,\varepsilon) =$

(1) $$\underline{\lim}_{\substack{\varepsilon\to\mathbf{0}\\ \sigma\to 0}}\underline{\lim}_{\hbar\to 0}\int dx\left[I_1^{(1)}(x,T;\sigma,l,\lambda,\varepsilon)\right]^2 \leq$$

$$\leq \lim_{\hbar\to 0}\int dx\left\{\int_{q(T)=x} D^{+}[q(t)]\Psi\left(q(0)\right)[|q_i(T)|]^{\frac{1}{2}}\cos\left[\frac{1}{\hbar}\boldsymbol{S}_1(\dot{q},q,\lambda,T)\right]\right\}^2,$$

(2) $$\underline{\lim}_{\substack{\varepsilon\to\mathbf{0}\\ \sigma\to 0}}\underline{\lim}_{\hbar\to 0}\int dx\left[I_1^{(2)}(x,T;\sigma,l,\lambda,\varepsilon)\right]^2 = 0,$$

(3) $$\underline{\lim}_{\substack{\varepsilon\to\mathbf{0}\\ \sigma\to 0}}\underline{\lim}_{\hbar\to 0}\int dx\left[I_1^{(1)}(x,T;\sigma,l,\lambda,\varepsilon)I_1^{(2)}(x,T;\sigma,l,\lambda,\varepsilon)\right] = 0,$$

(4) $$\underline{\lim}_{\substack{\varepsilon\to\mathbf{0}\\ \sigma\to 0}}\underline{\lim}_{\hbar\to 0}\int dx\left[I_2^{(2)}(x,T;\sigma,l,\lambda,\varepsilon)\right]^2 \leq$$

$$\leq \lim_{\hbar\to 0}\int dx\left\{\int_{q(T)=x} D^{+}[q(t)]\Psi\left(q(0)\right)[|q_i(T)|]^{\frac{1}{2}}\sin\left[\frac{1}{\hbar}\boldsymbol{S}_1(\dot{q},q,\lambda,T)\right]\right\}^2,$$

(5) $$\underline{\lim}_{\substack{\varepsilon\to 0\\ \sigma\to 0}}\underline{\lim}_{\hbar\to 0}\int dx\left[I_2^{(1)}(x,T;\sigma,l,\lambda,\varepsilon)\right]^2=0,$$

(6) $$\underline{\lim}_{\substack{\varepsilon\to 0\\ \sigma\to 0}}\underline{\lim}_{\hbar\to 0}\int dx\left[I_2^{(2)}(x,T;\sigma,l,\lambda,\varepsilon)I_2^{(1)}(x,T;\sigma,l,\lambda,\varepsilon)\right]=0.$$

Here

$$\boldsymbol{S}_1(\dot q,q,\lambda,T)=\boldsymbol{S}_{\varepsilon=0,1}(\dot q,q,\lambda,T)=\int_0^T L_{\varepsilon=0}(\dot q(t),q(t),t,\lambda)\,dt,$$

$$L_{\varepsilon=0}(\dot q(t),q(t),t,\lambda)=\frac{m}{2}\dot q^2(t)-V_{\varepsilon=0,0}(q(t),t,\lambda).$$

Proof (I) Let us to choose an sequence $\{\hbar_m\}_{m=1}^{\infty}$ such that

(i) $\lim_{m\to\infty}\hbar_m=0$ and

(ii) $\lim_{m,n\to\infty}\int dx\left\{\Xi_\varepsilon^{(m)}(x,T;\sigma,l,\hbar_m,n)\right\}^2=\lim_{m,n\to\infty}\int dx\left\{\sum_{i=1}^{m}\frac{(-1)^i\hbar_m^i}{(2i)!}\hat{\mathcal{R}}_\varepsilon^{(i)}(x,T;\sigma,l,n)\right\}^2=0.$

We note that from (ii) follows that: perturbative expansion

$$\int dx\{\Xi_\varepsilon(x,T;\sigma,l,\hbar_m,n)\}^2=\hbar_m^{1/4}\int dx\left\{\sum_{i=1}^{\infty}\frac{(-1)^i\hbar_m^i}{(2i)!}\hat{\mathcal{R}}_\varepsilon^{(i)}(x,T;\sigma,l,n)\right\}^2$$

vanishes in the limit $m,n\to\infty$. From (A.36) and Schwarz's inequality using Proposition A.1, one obtain

$$\underline{\lim}_{m,n\to\infty}\int dx\left[I_{1,n}^{(1)}(x,T;\sigma,l,\lambda,\varepsilon,\hbar_m)\right]^2\leq\underline{\lim}_{m,n\to\infty}\int dx\left\{\left|\mathbf{J}_{1,n}^{(1)}(x,T;\sigma,l,\lambda,\varepsilon,\hbar_m)\right|-\hbar_m^{1/4}\Xi_\varepsilon(x,T;\sigma,l,\hbar,n)\right\}^2$$

$$\leq\overline{\lim}_{m,n\to\infty}\int dx\left\{\mathbf{J}_{1,n}^{(1)}(x,T;\sigma,l,\lambda,\varepsilon,\hbar_m)\right\}^2+$$

$$+\underline{\lim}_{m,n\to\infty}\left\{2\hbar_m^{1/4}\sqrt{\left[\int dx\left\{\mathbf{J}_{1,n}^{(1)}(x,T;\sigma,l,\lambda,\varepsilon,\hbar_m)\right\}^2\int dx\{\Xi_\varepsilon(x,T;\sigma,l,\hbar_m,n)\}^2\right]}+\int dx\{\Xi_\varepsilon(x,T;\sigma,l,\hbar_m,n)\}^2\right\}$$

$$=\lim_{m,n\to\infty}\int dx\left\{\mathbf{J}_{1,n}^{(1)}(x,T;\sigma,l,\lambda,\varepsilon,\hbar_m)\right\}^2=\lim_{\hbar\to 0}\lim_{n\to\infty}\int dx\left\{\boldsymbol{J}_{1,n}^{(1)}(x,T;\sigma,l,\lambda,\varepsilon,\hbar)\right\}^2=$$

$$=\lim_{\hbar\to 0}\int dx\left\{\int_{q(T)=x}D^+[q(t)]\Psi\big(q(0)\big)[|q_i(T)|]^{\frac{1}{2}}\cos\left[\frac{1}{\hbar}\boldsymbol{S}_1(\dot q,q,\lambda,T)\right]\right\}^2.\qquad\text{(A.38)}$$

Let us to choose now an subsequence $\{\hbar_{m_k}\}_{m_k=1}^{\infty}$ such that the limit:

$\lim_{k,n\to\infty}\int dx\left[I_{1,n}^{(1)}(x,T;\sigma,l,\lambda,\varepsilon,\hbar_{m_k})\right]^2$ exist and

$$\lim_{k,n\to\infty}\int dx\left[I_{1,n}^{(1)}(x,T;\sigma,l,\lambda,\varepsilon,\hbar_{m_k})\right]^2=\underline{\lim}_{m,n\to\infty}\int dx\left[I_{1,n}^{(1)}(x,T;\sigma,l,\lambda,\varepsilon,\hbar_m)\right]^2\qquad\text{(A.39)}$$

From (A.39) and Proposition A.1 one obtain

$$\lim_{k,n\to\infty}\int dx\left[I_{1,n}^{(1)}(x,T;\sigma,l,\lambda,\varepsilon,\hbar_{m_k})\right]^2=\lim_{k\to\infty}\left\{\lim_{n\to\infty}\int dx\left[I_{1,n}^{(1)}(x,T;\sigma,l,\lambda,\varepsilon,\hbar_{m_k})\right]^2\right\}.\qquad\text{(A.40)}$$

From (A.39), (A.40) and (A.38) one obtain

$$\underline{\lim}_{\substack{\varepsilon\to 0\\ \sigma\to 0}}\underline{\lim}_{\hbar\to 0}\int dx\left[I_1^{(1)}(x,T;\sigma,l,\lambda,\varepsilon)\right]^2\leq$$

$$\leq\underline{\lim}_{\substack{\varepsilon\to 0\\ \sigma\to 0}}\lim_{k\to\infty}\left\{\lim_{n\to\infty}\int dx\left[I_{1,n}^{(1)}(x,T;\sigma,l,\lambda,\varepsilon,\hbar_{m_k})\right]^2\right\}=$$

$$=\underline{\lim}_{\substack{\varepsilon\to 0\\ \sigma\to 0}}\lim_{k\to\infty}\int dx\left[I_1^{(1)}(x,T;\sigma,l,\lambda,\varepsilon,\hbar_{m_k})\right]^2\leq$$

$$\leq\lim_{\hbar\to 0}\int dx\left\{\int_{q(T)=x}D^+[q(t)]\Psi\big(q(0)\big)[|q_i(T)|]^{\frac{1}{2}}\cos\left[\frac{1}{\hbar}\boldsymbol{S}_1(\dot q,q,\lambda,T)\right]\right\}^2.\qquad\text{(A.41)}$$

The inequality (A.41) completed the proof of the statement (1).

(**II**) Let us estimate now n-dimensional path integral

$$I_{1,n}^{(2)}(x,T;\sigma,l,\lambda,\varepsilon) =$$

$$\int_{q(T)=x} D_n^+[q(t)]\Psi\big(q(0)\big)[|q_i(T)|]^{1/2}\exp\left[-\frac{1}{\hbar}\boldsymbol{S}(q,T;\sigma,l)\right]\sin\left[\frac{1}{\hbar}\boldsymbol{S}_{\varepsilon,1}(\dot{q},q,\lambda,T)\right]\sin\left[-\frac{1}{\hbar}\boldsymbol{S}_{\varepsilon,2}(q,\lambda,T)\right]. \quad \text{(A.42)}$$

From Eq. (A.42) one obtain the inequality

$$\left|I_{1,n}^{(2)}(x,T;\sigma,l,\lambda,\varepsilon)\right| \leq$$

$$\leq \int_{q(T)=x} D_n^+[q(t)]\Psi\big(q(0)\big)[|q_i(T)|]^{1/2}\exp\left[-\frac{1}{\hbar}\boldsymbol{S}(q,T;\sigma,l)\right]\left|\sin\left[\frac{1}{\hbar}\boldsymbol{S}_{\varepsilon,2}(q,\lambda,T)\right]\right| \leq$$

$$\leq \sum_{i=0}^{\infty}\frac{\hbar^{-(2i+1)}}{(2i+1)!}\int_{q(T)=x} D_n^+[q(t)]\Psi\big(q(0)\big)[|q_i(T)|]^{\frac{1}{2}}\left[\left|\boldsymbol{S}_{\varepsilon,2}(q,\lambda,T)\right|\right]^{(2i+1)}\exp\left[-\frac{1}{\hbar}\boldsymbol{S}(q,T;\sigma,l)\right] =$$

$$= \sum_{i=0}^{\infty}\frac{\hbar^{-(2i+1)}}{(2i+1)!}\wp_{\varepsilon}^{(i)}(x,T;\sigma,l,n) \quad \text{(A.43)}$$

where

$$\wp_{\varepsilon}^{(i)}(x,T;\sigma,l,n) = \int_{q(T)=x} D_n^+[q(t)]\Psi\big(q(0)\big)[|q_i(T)|]^{\frac{1}{2}}\left[\left|\boldsymbol{S}_{\varepsilon,2}(q,\lambda,T)\right|\right]^{(2i+1)}\exp\left[-\frac{1}{\hbar}\boldsymbol{S}(q,T;\sigma,l)\right]. \quad \text{(A.44)}$$

Using replacement $q_i(t) \coloneqq \hbar^{\frac{1}{2}}q_i(t), t \in [0,T], i = 1,\ldots,d$ into RHS of the Eq.(A.44) one obtain

$$\wp_{\varepsilon}^{(i)}(x,T;\sigma,l,n) =$$

$$\hbar^{1/4}\int_{q(T)=\frac{x}{\sqrt{\hbar}}}\breve{D}_n^+[q(t)]\Psi\left(\hbar^{\frac{1}{2}}q(0)\right)[|q_i(T)|]^{\frac{1}{2}}\left[\left|\boldsymbol{S}_{\varepsilon,2}(q,\lambda,T)\right|\right]^{(2i+1)}\exp\left[-\frac{1}{\hbar}\boldsymbol{S}\big(\hbar^{1/2}q,T;\sigma,l\big)\right] =$$

$$\hbar^{1/4}\hbar^{i/2}\int dy \int_{\substack{q(T)=\frac{x}{\sqrt{\hbar}}\\ q(0)=\frac{y}{\sqrt{\hbar}}}}\breve{D}_n^+[q(t)]\breve{\Psi}\big(q(0)\big)[|q_i(T)|]^{\frac{1}{2}}\left[\left|\widehat{\boldsymbol{S}}_{\varepsilon,2}(q,\lambda,T)\right|\right]^{(2i+1)}\exp[-\boldsymbol{S}(q,T;\sigma,l)] =$$

$$= \hbar^{1/4}\hbar^{i/2}\widehat{\wp}_{\varepsilon}^{(i)}(x,T;\sigma,l,n), \quad \text{(A.45)}$$

where

$\breve{D}_n^+[q(t)] = D_n^+\left[\hbar^{\frac{1}{2}}q(t)\right], t \in [0,T],\ \breve{\Psi}\big(q(0)\big) = \frac{\eta^{d/4}}{(2\pi)^{d/4}\hbar^{d/4}}\exp\left[\frac{\eta q^2(0)}{2}\right]$, see Eq.(3.1) and

$$\widehat{\boldsymbol{S}}_{\varepsilon,2}(q,\lambda,T,\hbar) = \int_0^T \widehat{V}_{\varepsilon,1}(q(t),t,\lambda,\hbar)\,dt, \quad \text{(A.46)}$$

$$\widehat{V}_{\varepsilon,1}(q(t),t,\lambda,\hbar) = a_{\varepsilon,3}(q(t),t,\lambda)q^3(t) + \cdots + \hbar^{\frac{\alpha-3}{2}}a_{\varepsilon,\alpha}(q(t),t,\lambda)q^{\alpha}(t). \quad \text{(A.47)}$$

$$\widehat{\wp}_{\varepsilon}^{(i)}(x,T;\sigma,l,n) = \int dy \int_{\substack{q(T)=\frac{x}{\sqrt{\hbar}}\\ q(0)=\frac{y}{\sqrt{\hbar}}}}\breve{D}_n^+[q(t)]\breve{\Psi}\big(q(0)\big)[|q_i(T)|]^{\frac{1}{2}}\left[\left|\widehat{\boldsymbol{S}}_{\varepsilon,2}(q,\lambda,T)\right|\right]^{(2i+1)}\exp[-\boldsymbol{S}(q,T;\sigma,l)]. \quad \text{(A.48)}$$

From (A.43)-(A.48) one obtain

$$\left|I_{1,n}^{(2)}(x,T;\sigma,l,\lambda,\varepsilon)\right| \leq \hbar^{\frac{1}{4}} \sum_{i=0}^{\infty} \frac{\hbar^{2(i+1)}}{(2i+1)!} \widehat{\wp}_{\varepsilon}^{(i)}(x,T;\sigma,l,n) = \mathbf{\Theta}_{\varepsilon}(x,T;\sigma,l,\hbar,n). \quad \text{(A.49)}$$

Let us to choose an sequence $\{\hbar_m\}_{m=1}^{\infty}$ such that

(i) $\lim_{m\to\infty}\hbar_m = 0$ and

(ii) $\lim_{m,n\to\infty} \int dx \left\{\mathbf{\Theta}_{\varepsilon}^{(m)}(x,T;\sigma,l,\hbar_m,n)\right\}^{2} = \lim_{m,n\to\infty} \int dx \left\{\sum_{i=0}^{m} \frac{\hbar_m^{2(i+1)}}{(2i+1)!} \widehat{\wp}_{\varepsilon}^{(i)}(x,T;\sigma,l,n)\right\}^{2} = 0.$

We note that from (ii) follows that: perturbative expansion

$$\int dx \{\mathbf{\Theta}_{\boldsymbol{\varepsilon}}(x,T;\sigma,l,\hbar_m,n)\}^2 = \hbar_m^{1/4} \int dx \left\{\sum_{i=0}^{\infty} \frac{\hbar^{2(i+1)}}{(2i+1)!} \widehat{\wp}_{\varepsilon}^{(i)}(x,T;\sigma,l,n)\right\}^{2}$$

vanishes in the limit $m,n \to \infty$. From (A.49) one obtain

$$\underline{\lim}_{m,n\to\infty} \int dx \left[I_{1,n}^{(2)}(x,T;\sigma,l,\lambda,\varepsilon,\hbar_m)\right]^{2} \leq \underline{\lim}_{m,n\to\infty} \int dx \{\mathbf{\Theta}_{\boldsymbol{\varepsilon}}(x,T;\sigma,l,\hbar_m,n)\}^2. \quad \text{(A.50)}$$

Let us to choose now an subsequence $\{\hbar_{m_k}\}_{m_k=1}^{\infty}$ such that the limit:

$\lim_{k,n\to\infty} \int dx \left[I_{1,n}^{(2)}(x,T;\sigma,l,\lambda,\varepsilon,\hbar_{m_k})\right]^{2}$ exist and

$$\lim_{k,n\to\infty} \int dx \left[I_{1,n}^{(2)}(x,T;\sigma,l,\lambda,\varepsilon,\hbar_{m_k})\right]^{2} = \underline{\lim}_{m,n\to\infty} \int dx \left[I_{1,n}^{(2)}(x,T;\sigma,l,\lambda,\varepsilon,\hbar_m)\right]^{2} \quad \text{(A.51)}$$

From (A.51) and Proposition A.1 one obtain

$$\lim_{k,n\to\infty} \int dx \left[I_{1,n}^{(2)}(x,T;\sigma,l,\lambda,\varepsilon,\hbar_{m_k})\right]^{2} = \lim_{k\to\infty} \left\{\lim_{n\to\infty} \int dx \left[I_{1,n}^{(2)}(x,T;\sigma,l,\lambda,\varepsilon,\hbar_{m_k})\right]^{2}\right\}. \quad \text{(A.52)}$$

From (A.50), (A.51) and (A.52) one obtain

$$\underline{\lim}_{\substack{\boldsymbol{\varepsilon}\to\mathbf{0}\\ \sigma\to 0}} \underline{\lim}_{\hbar\to 0} \int dx \left[I_{1}^{(2)}(x,T;\sigma,l,\lambda,\varepsilon)\right]^{2} \leq$$

$$\leq \underline{\lim}_{\substack{\boldsymbol{\varepsilon}\to\mathbf{0}\\ \sigma\to 0}} \lim_{k\to\infty} \left\{\lim_{n\to\infty} \int dx \left[I_{1,n}^{(2)}(x,T;\sigma,l,\lambda,\varepsilon,\hbar_{m_k})\right]^{2}\right\} =$$

$$= \underline{\lim}_{\substack{\boldsymbol{\varepsilon}\to\mathbf{0}\\ \sigma\to 0}} \lim_{k\to\infty} \int dx \left[I_{1}^{(2)}(x,T;\sigma,l,\lambda,\varepsilon,\hbar_{m_k})\right]^{2} = 0.$$

Proof of the statements (3)-(6) is similarly to the proof of the statements (1)-(2).

Theorem A.1. Let $I_1(x,T;\sigma,l,\lambda,\varepsilon,\hbar) = I_1(x,T;\sigma,l,\lambda,\varepsilon), I_2(x,T;\sigma,l,\lambda,\varepsilon,\hbar) = I_2(x,T;\sigma,l,\lambda,\varepsilon)$, where $I_1(x,T;\sigma,l,\lambda,\varepsilon)$ is given via Eq.(A.22a)- Eq.(A.22b). Then

$$\underline{\lim}_{\substack{\varepsilon\to\mathbf{0}\\ \sigma\to 0}} \underline{\lim}_{\hbar\to 0} \int dx \left[I_1^2(x,T;\sigma,l,\lambda,\varepsilon)\right] \leq$$

$$\leq \lim_{\hbar\to 0} \int dx \left\{\int_{q(T)=x} D^{+}[q(t)]\Psi\big(q(0)\big)[|q_i(T)|]^{\frac{1}{2}} \cos\left[\frac{1}{\hbar}\boldsymbol{S}_1(\dot{q},q,\lambda,T)\right]\right\}^{2}, \quad \text{(A.53.a)}$$

$$\underline{\lim}_{\substack{\varepsilon\to\mathbf{0}\\ \sigma\to 0}} \underline{\lim}_{\hbar\to 0} \int dx \left[I_2^2(x,T;\sigma,l,\lambda,\varepsilon)\right] \leq$$

$$\leq \lim_{\hbar\to 0} \int dx \left\{\int_{q(T)=x} D^{+}[q(t)]\Psi\big(q(0)\big)[|q_i(T)|]^{\frac{1}{2}} \sin\left[\frac{1}{\hbar}\boldsymbol{S}_1(\dot{q},q,\lambda,T)\right]\right\}^{2}, \quad \text{(A.53.b)}$$

Here

$$\boldsymbol{S}_{\mathbf{1}}(\dot{q},q,\lambda,T) = \boldsymbol{S}_{\varepsilon=0,1}(\dot{q},q,\lambda,T) = \int_0^T L_{\varepsilon=0}(\dot{q}(t),q(t),t,\lambda)\,dt, \quad \text{(A.54)}$$

$$L_{\varepsilon=0}(\dot{q}(t),q(t),t,\lambda) = \frac{m}{2}\dot{q}^2(t) - V_{\varepsilon=0,0}(q(t),t,\lambda). \quad \text{(A.55)}$$

Proof . We remain that

$$I_1(x,T;\sigma,l,\lambda,\varepsilon,\hbar) = I_1^{(1)}(x,T;\sigma,l,\lambda,\varepsilon,\hbar) + I_1^{(2)}(x,T;\sigma,l,\lambda,\varepsilon,\hbar). \quad \text{(A.56)}$$

From Eq.(A.56) we obtain

$$\int dx\,[I_1^2(x,T;\sigma,l,\lambda,\varepsilon,\hbar)] \leq \int dx\left[I_1^{(1)}(x,T;\sigma,l,\lambda,\varepsilon,\hbar)\right]^2 +$$

$$\int dx\left[I_1^{(2)}(x,T;\sigma,l,\lambda,\varepsilon,\hbar)\right]^2 + 2\int dx\left[\left|I_1^{(1)}(x,T;\sigma,l,\lambda,\varepsilon)I_1^{(2)}(x,T;\sigma,l,\lambda,\varepsilon,\hbar)\right|\right] =$$

$$= \int dx\left[I_1^{(1)}(x,T;\sigma,l,\lambda,\varepsilon,\hbar)\right]^2 + \int dx\left[I_1^{(2)}(x,T;\sigma,l,\lambda,\varepsilon,\hbar)\right]^2 +$$

$$+2\sqrt{\int dx\left[I_1^{(1)}(x,T;\sigma,l,\lambda,\varepsilon,\hbar)\right]^2 \int dx\left[I_1^{(2)}(x,T;\sigma,l,\lambda,\varepsilon,\hbar)\right]^2}. \quad \text{(A.57)}$$

Let us to choose now an sequences $\{\hbar_m\}_{m=1}^{\infty}, \{\varepsilon_k\}_{k=1}^{\infty}, \{\sigma_l\}_{l=1}^{\infty}$ such that:

(i) $\lim_{m\to\infty}\hbar_m = 0,\ \lim_{k\to\infty}\varepsilon_k = 0,\ \lim_{l\to\infty}\sigma_l = 0$

(ii)
$$\lim_{\substack{k\to\infty\\ l\to\infty}}\lim_{m\to\infty}\int dx\left[I_1^{(2)}(x,T;\sigma_l,l,\lambda,\varepsilon_k,\hbar_m)\right]^2 =$$

$$=\underline{\lim}_{\substack{k\to\infty\\ l\to\infty}}\underline{\lim}_{m\to\infty}\int dx\left[I_1^{(2)}(x,T;\sigma_l,l,\lambda,\varepsilon_k,\hbar_m)\right]^2 = 0, \quad \text{(A.58)}$$

(iii)
$$\underline{\lim}_{\substack{k\to\infty\\ l\to\infty}}\underline{\lim}_{m\to\infty}\int dx\left[I_1^{(1)}(x,T;\sigma_l,l,\lambda,\varepsilon_k,\hbar_m)\right]^2 \leq$$

$$\leq \lim_{\hbar\to 0}\int dx\left\{\int_{q(T)=x} D^+[q(t)]\Psi\big(q(0)\big)[|q_i(T)|]^{\frac{1}{2}}\cos\left[\frac{1}{\hbar}\boldsymbol{S}_1(\dot{q},q,\lambda,T)\right]\right\}^2. \quad \text{(A.59)}$$

Therefore from inequality (A.57), Eq.(A.58) and inequality (A.59) we obtain

$$\underline{\lim}_{\substack{\varepsilon\to 0\\ \sigma\to 0}}\underline{\lim}_{\hbar\to 0}\int dx\,[I_1^2(x,T;\sigma,l,\lambda,\varepsilon,\hbar)] \leq \underline{\lim}_{\substack{k\to\infty\\ l\to\infty}}\underline{\lim}_{m\to\infty}\int dx\,[I_1^2(x,T;\sigma_l,l,\lambda,\varepsilon_k,\hbar_m)] \leq$$

$$\leq \underline{\lim}_{\substack{k\to\infty\\ l\to\infty}}\underline{\lim}_{m\to\infty}\int dx\left[I_1^{(1)}(x,T;\sigma_l,l,\lambda,\varepsilon_k,\hbar_m)\right]^2 + \lim_{\substack{k\to\infty\\ l\to\infty}}\lim_{m\to\infty}\int dx\left[I_1^{(2)}(,T;\sigma_l,l,\lambda,\varepsilon_k,\hbar_m)\right]^2 +$$

$$+2\lim_{\substack{k\to\infty\\ l\to\infty}}\lim_{m\to\infty}\sqrt{\int dx\left[I_1^{(1)}(,T;\sigma_l,l,\lambda,\varepsilon_k,\hbar_m)\right]^2 \int dx\left[I_1^{(2)}(,T;\sigma_l,l,\lambda,\varepsilon_k,\hbar_m)\right]^2} =$$

$$\underline{\lim}_{\substack{k\to\infty\\ l\to\infty}}\underline{\lim}_{m\to\infty}\int dx\left[I_1^{(1)}(x,T;\sigma_l,l,\lambda,\varepsilon_k,\hbar_m)\right]^2 =$$

$$= \lim_{\hbar\to 0}\int dx\left\{\int_{q(T)=x} D^+[q(t)]\Psi\big(q(0)\big)[|q_i(T)|]^{\frac{1}{2}}\cos\left[\frac{1}{\hbar}\boldsymbol{S}_1(\dot{q},q,\lambda,T)\right]\right\}^2. \quad \text{(A.60)}$$

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