Maxwell Equations and Total Internal Reflection

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The phenomenon of total internal reflection is considered in terms of exact solutions of Maxwell equations. Matching of plane and evanescent waves at the interface is completed. It is shown that amplitude of the reflected wave cannot be obtained from the matching alone. Since it can differ from that of incident wave due to possible energy loss which may occur in the evanescent wave zone if there is another layer of optically dense medium, this loss is to be specified via amplitude of the reflected wave. Besides, reflected wave potential has phase shift which also depends on this specification.

1 Introduction

Rigorous mathematical descriptions of reflection, refraction and transmission of plane waves are found in numerous monographs on classical electrodynamics and optics [1, 2]. The phenomenon of total internal reflection is somewhat more complicated because it contains waves of two distinct kinds, namely, ordinary plane waves and evanescent waves, which have different properties. Usually, one follows the lines of quantum mechanics in which it suffices to add imaginary part to one of components of the wave vector. This trick works when dealing with scalar fields, but its legality in electrodynamics was to be verified. It turns out that Maxwell equations have no solutions composed this way. It will be shown below that no other waves but ordinary plane and evanescent ones exist and there is no freedom of adding an imaginary part to wave vector of an electromagnetic wave. This fact tells us that there is no continuous passage from plane to evanescent waves and the phenomenon of total internal reflection is to be studied without using the trick used in quantum mechanics.

The main task to be completed when describing the phenomenon of total internal reflection is matching strengths and inductions of the field on the interface. In this work the matching procedure is completed under an additional assumption that reflected wave has an unknown phase shift which can be obtained from the entire analysis. This assumption adds one unknown that broadens opportunities of matching the fields at the interface and after all, allows us to complete the procedure of matching. The result obtained shows that, first, amplitude of the reflected wave cannot be obtained from this procedure because the general solution admits possible energy drain by another layer of an optically dense medium which can be placed behind the first one. In other words, amplitude of the reflected wave is equal to that of the incident wave only if it is know that the less optically dense medium fills the half-space, otherwise it must be specified with account of the energy loss. It turns out that the phase shift and amplitude of the reflected wave are linked via a certain relation.

This work is organized as follows. We start with a special representation of source-free Maxwell equations and boundary conditions in terms of exterior calculus and their solutions which describe plane and evanescent waves. This analysis is necessary because it shows that there is no continuous passage from plane to evanescent waves which is known in quantum mechanics and which is used as a straightforward description of tunneling effect. Then, we derive relations for amplitudes from the boundary conditions and solve them as algebraic ones. The result obtained has the form of relation for amplitudes and phase shift.

2 Exterior calculus and source-free Maxwell equations

Our approach consists in the following. Strengths E and H and inductions Δ and B (Δ stands for electric induction) constitute a complex 2-form Ψ

$$\Psi = \mathrm{d}t \wedge (E + \imath H) + (B - \imath \Delta), \tag{1}$$

where t is Lorentzian time and

$$\Delta = \epsilon \cdot {}^*E, \quad B = \mu \cdot {}^*H. \tag{2}$$

Here the asterisk conjugation is 3-dimensional (otherwise one of signs changes). According to source-free Maxwell equations, the 2-form Ψ is closed, consequently, there exists such a complex 1-form A that

$$\mathrm{d}A = \Psi \tag{3}$$

and the equations (2) play the key role in this mathematical construction.

The equation

$$\mathrm{d}\Psi = 0$$

which, by construction, is an identity, contains boundary conditions at an interface. Indeed, if, say ϵ and μ behave as step functions with jump at the plane $z = 0$:

$$\epsilon = \begin{cases} \epsilon_+, & z > 0 \\ \epsilon_-, & z < 0 \end{cases} \tag{4}$$

this equation requires that xy-components of the 2-forms B and Δ remain continuous and jump of the corresponding strengths does not make sense. On the other hand, the same equation requires that tangent components of the strengths are continuous. Denote $E_\pm$, $H_\pm$, $B_\pm$ and $\Delta_\pm$ these forms on the boundary with sign specifying the side as that of the coordinate z. Then, the boundary conditions take the form

$$\begin{aligned} &(E_+)_t = (E_-)_t, \quad &&(H_+)_t = (H_-)_t, \\ &(B_+)_{xy} = (B_-)_{xy}, \quad &&(\Delta_+)_{xy} = (\Delta_-)_{xy}, \quad z = 0 \end{aligned} \tag{5}$$

where the subscript t stands for tangential components.

3 Plane waves

In case of plane wave the t-component of the 1-form A can be omitted:

$$A = f\,\mathrm{d}x + g\,\mathrm{d}y + h\,\mathrm{d}z, \quad A = A_1 + \imath A_2 \tag{6}$$

where A_1 and A_2 are real-valued 1-forms specified by their components f_1, g_1 h_1 and f_2, g_2 h_2 correspondingly. Taking its exterior derivative and substituting the result into the equations (1-6) yields a system of six equation for six unknowns. This system consists of two subsystems of three equations each. The subsystems are independent on each other, one of them has the form

$$\epsilon\frac{\partial g_1}{\partial t} = \frac{\partial h_2}{\partial x} - \frac{\partial f_2}{\partial z}, \qquad (7)$$
$$\mu\frac{\partial f_2}{\partial t} = -\frac{\partial g_1}{\partial z}, \quad \mu\frac{\partial h_2}{\partial t} = \frac{\partial g_1}{\partial x}$$

and another one is

$$\epsilon\frac{\partial f_1}{\partial t} = \frac{\partial g_2}{\partial z}, \quad \epsilon\frac{\partial h_1}{\partial t} = -\frac{\partial g_2}{\partial x}$$
$$\mu\frac{\partial g_2}{\partial t} = \frac{\partial f_1}{\partial z} - \frac{\partial h_1}{\partial x}.$$

Solutions of these equations represent plane electromagnetic waves of two possible polarizations.

When describing plane waves it is convenient to introduce the phase

$$\phi = \omega t - px - qz$$

as an auxiliary function. Evidently, functions like

$$f = a\cos\phi, \quad g = \cos\phi, \quad h = b\cos\phi,$$

turn both subsystems into that of algebraic ones, namely

$$\epsilon\omega = aq - bp, \quad a\mu\omega = q, \quad b\mu\omega = -p$$

for the first and

$$\mu\omega = bp - aq, \quad a\epsilon\omega = -q, \quad b\epsilon\omega = p$$

for the second subsystem. Both of them yield the well-known dispersion equation

$$\epsilon\mu\omega^2 = p^2 + q^2 \qquad (8)$$

and provide amplitudes a and b explicitly as soon as components of the wave vector p and q are chosen in accord with the equation (8). The 1-form A will be represented as

$$A = \frac{\imath q}{\mu\omega}\cos\phi\,\mathrm{d}x + \cos\phi\,\mathrm{d}y - \frac{\imath p}{\mu\omega}\cos\phi\,\mathrm{d}z \qquad (9)$$

for one polarization and

$$A = -\frac{q}{\epsilon\omega}\cos\phi\,\mathrm{d}x + \imath\cos\phi\,\mathrm{d}y + \frac{p}{\epsilon\omega}\cos\phi\,\mathrm{d}z \tag{10}$$

for another.

Now, calculate the corresponding strengths and inductions. In case of plane waves (9) and (10) we have

$$\begin{aligned}
E =&\ -\omega\sin(\omega t - px - qz)\,\mathrm{d}y, \\
H =&\ -\frac{q}{\mu}\sin(\omega t - px - qz)\,\mathrm{d}x + \frac{p}{\mu}\sin(\omega t - px - qz)\,\mathrm{d}z, \\
B =&\ p\sin(\omega t - px - qz)\,\mathrm{d}x\wedge\mathrm{d}y - q\sin(\omega t - px - qz)\,\mathrm{d}y\wedge\mathrm{d}z, \\
\Delta =&\ -\omega\epsilon\sin(\omega t - px - qz)\,\mathrm{d}z\wedge\mathrm{d}x
\end{aligned} \tag{11}$$

and

$$\begin{aligned}
E =&\ \frac{q}{\epsilon}\sin(\omega t - px - qz)\,\mathrm{d}x - \frac{p}{\epsilon}\sin(\omega t - px - qz)\,\mathrm{d}z, \\
H =&\ -\omega\sin(\omega t - px - qz)\,\mathrm{d}y, \\
B =&\ -\omega\mu\sin(\omega t - px - qz)\,\mathrm{d}z\wedge\mathrm{d}x, \\
\Delta =&\ q\sin(\omega t - px - qz)\,\mathrm{d}y\wedge\mathrm{d}z - p\sin(\omega t - px - qz)\,\mathrm{d}x\wedge\mathrm{d}y
\end{aligned}$$

correspondingly.

4 Evanescent wave

One more pair of solutions of the equations (7) and (8) is obtained below. Consider a more general case with components to be found factorized as $f = F(z)\sin(\omega t - px)$ and $f = F(z)\cos(\omega t - px)$. It will be shown below that both of them are needed. The amplitudes have the form

$$\begin{aligned}
f &= F(z)\sin(\omega t - px), \\
g = G(z)\cos(\omega t - px), &\qquad h = H(z)\cos(\omega t - px)
\end{aligned} \tag{12}$$

or

$$\begin{aligned}
f &= F(z)\cos(\omega t - px), \\
g = G(z)\sin(\omega t - px), &\qquad h = H(z)\sin(\omega t - px).
\end{aligned} \tag{13}$$

We consider only waves with polarization specified by f_2, g_1 and h_2 (E tangent to the interface), therefore, polarization subscripts are unnecessary. Substituting this representation into the equations (7) turns them into ordinary differential equations

$$\begin{aligned} -\epsilon\omega G = &\quad H_2 p - F_2' \\ \mu\omega F = &\quad -G', \quad -\mu\omega H = pG \end{aligned} \tag{14}$$

for the functions (12) and similar for (13). For the earlier, exclusion of the unknown H and denoting

$$q^2 = p^2 - \omega^2\epsilon\mu$$

reduces them to the following pair of equations:

$$\mu\omega F' = -q^2 G, \quad G' = -\mu\omega F$$

and for the latter,

$$\mu\omega F' = -q^2 G, \quad G' = \mu\omega F.$$

Solution of the equations (14) is trivial and so for another triplet of components and finally we have the following pair of 1-form A:

$$\begin{aligned} A = &\quad \frac{\imath q}{\mu\omega} e^{-qz} \sin(\omega t - px)\,\mathrm{d}x + e^{-qz}\cos(\omega t - px)\,\mathrm{d}y - \\ &- \frac{\imath p}{\mu\omega} e^{-qz}\cos(\omega t - px)\,\mathrm{d}z \end{aligned}$$

and similar for another opportunity. It will be seen below why both of them are important. These solutions describe waves which propagate in the x direction and extinct in the z direction.

5 On matching plane and evanescent waves

Expressions for strengths and inductions given in [2] and [1] look quite complicated. We obtain much simpler form

$$\begin{aligned} E = &\quad a\omega e^{-qz}\sin(\omega t - px)\,\mathrm{d}y, \\ H = &\quad -\frac{aq}{\mu} e^{-qz}\cos(\omega t - px)\,\mathrm{d}x - \frac{ap}{\mu} e^{-qz}\sin(\omega t - px)\,\mathrm{d}z, \\ B = &\quad -ape^{-qz}\sin(\omega t - px)\,\mathrm{d}x\wedge\mathrm{d}y - aqe^{-qz}\cos(\omega t - px)\,\mathrm{d}y\wedge\mathrm{d}z, \\ \Delta = &\quad a\omega\epsilon e^{-qz}\sin(\omega t - px)\,\mathrm{d}z\wedge\mathrm{d}x \end{aligned} \tag{15}$$

and, for another opportunity,

$$\begin{aligned}
E =&\ \ a\omega e^{-qz}\cos(\omega t - px)\,\mathrm{d}y, \\
H =&\ \ \frac{aq}{\mu}e^{-qz}\sin(\omega t - px)\,\mathrm{d}x - \frac{ap}{\mu}e^{-qz}\cos(\omega t - px)\,\mathrm{d}z, \\
B =&\ \ -ape^{-qz}\cos(\omega t - px)\,\mathrm{d}x\wedge\mathrm{d}y + aqe^{-qz}\sin(\omega t - px)\,\mathrm{d}y\wedge\mathrm{d}z, \\
\Delta =&\ \ a\omega\epsilon e^{-qz}\cos(\omega t - px)\,\mathrm{d}z\wedge\mathrm{d}x
\end{aligned} \tag{16}$$

which is also important. Note that no other solutions but purely plane waves and purely evanescent waves, exist. In other words, there are no solutions containing an oscillating factor which descends exponentially as if the wave vector has imaginary part.

Consider a plane wave propagating in a more optically dense medium which fill the half-space $z < 0$, towards the interface. For simplicity, we consider only one polarization with electric strength parallel to the interface. The corresponding field has the form (9) multiplied by an amplitude a_i. Reflected wave has amplitude a_r and wave vector with opposite z-component. So, strengths and inductions of the field of incident is given by the equations (11):

$$\begin{aligned}
E =&\ \ -a_i\omega\sin(\omega t - px - q_-z)\,\mathrm{d}y, \\
H =&\ \ -a_i\frac{q_-}{\mu}\sin(\omega t - px - q_-z)\,\mathrm{d}x + \frac{a_ip}{\mu}\sin(\omega t - px - q_-z)\,\mathrm{d}z, \\
B =&\ \ a_ip\sin(\omega t - px - q_-z)\,\mathrm{d}x\wedge\mathrm{d}y - \\
&\ \ -a_iq\sin(\omega t - px - q_-z)\,\mathrm{d}y\wedge\mathrm{d}z, \\
\Delta =&\ \ -a_i\omega\epsilon\sin(\omega t - px - q_-z)\,\mathrm{d}z\wedge\mathrm{d}x.
\end{aligned} \tag{17}$$

As for the reflected wave, we assume that it has a phase shift α, therefore its field has the form

$$\begin{aligned}
E =&\ \ -a_r\omega\sin(\omega t - px + q_-z + \alpha)\,\mathrm{d}y, \\
H =&\ \ \frac{a_rq_-}{\mu}\sin(\omega t - px + q_-z + \alpha)\,\mathrm{d}x + \frac{a_rp}{\mu}\sin(\omega t - px + q_-z + \alpha)\,\mathrm{d}z, \\
B =&\ \ a_rp\sin(\omega t - px + q_-z + \alpha)\,\mathrm{d}x\wedge\mathrm{d}y + \\
&\ \ +a_rq_-\sin(\omega t - px + q_-z + \alpha)\,\mathrm{d}y\wedge\mathrm{d}z, \\
\Delta =&\ \ -a_r\omega\epsilon\sin(\omega t - px - q_-z + \alpha)\,\mathrm{d}z\wedge\mathrm{d}x
\end{aligned} \tag{18}$$

where sign of one of components of the magnetic strength changed together with that of q_-.

The total strengths and inductions in the $z < 0$ half-space is sum of the fields given by the equations (17) and (18). Now, substitute these expressions and those from the equations (11) into the boundary conditions (5). For this end, first we collect components of strengths and inductions for the sum of incident and reflected waves at $z = 0$. Employing the sine and cosine summation theorems, obtain:

$$
\begin{aligned}
(E_i + E_r)_y = &\quad -(a_i + a_r \cos\alpha)\omega \sin(\omega t - px) - a_r \omega \sin\alpha \cos(\omega t - px), \\
(H_i + H_r)_x = &\quad \frac{(-a_i + a_r \cos\alpha) q_+}{\mu_+} \sin(\omega t - px) + \frac{a_r q_+}{\mu_+} \sin\alpha \cos(\omega t - px), \\
(B_i + B_r)_{xy} = &\quad (a_i + a_r \cos\alpha) p \sin(\omega t - px) + a_r p \sin\alpha \cos(\omega t - px)
\end{aligned}
$$

and xy-component of Δ is zero for both plane and evanescent waves. Here the subscripts $+$ and $-$ stand for the sign of the z coordinate. The same for evanescent waves has the form

$$
\begin{aligned}
(E_t)_y = &\quad [a_1 \sin(\omega t - px) + a_2 \cos(\omega t - px)]\omega \\
(H_t)_x = &\quad -\frac{q_-}{\mu_-}[-a_1 \cos(\omega t - px) + a_2 \sin(\omega t - px)], \\
(B_t)_{xy} = &\quad -p[a_1 \sin(\omega t - px) + a_2 \cos(\omega t - px)]\omega.
\end{aligned}
$$

The coefficients a_i and a_r are arbitrary (see below) and two others are amplitudes to be fit. It is seen that matching B_{xy} gives the same equations as matching E_y, therefore, third lines of both systems above are extra. All the rest yields the following quartet of equations

$$
\begin{aligned}
a_1 &= -(a_i + a_r \cos\alpha), & a_2 &= -a_r \sin\alpha, \\
\frac{q_-}{\mu_-} a_1 &= \frac{q_+}{\mu_+} a_r \sin\alpha, & \frac{q_-}{\mu_-} a_2 &= -\frac{q_+}{\mu_+} (a_i - a_r \cos\alpha).
\end{aligned}
$$

Two of them determine the unknowns a_1 and a_2 which can now be excluded so that it remains to solve the system of two equations

$$
Q a_r \sin\alpha = -(a_i + a_r \cos\alpha), \quad Q(a_i - a_r \cos\alpha) = -a_r \sin\alpha
$$

where

$$
Q = \frac{\mu_- q_+}{\mu_+ q_-}.
$$

Solution of this system yields the phase shift α:

$$
\cos\alpha = \frac{1 - Q^2}{1 + Q^2} \frac{a_i}{a_r}.
$$

It was expected that solution should give both phase shift and amplitudes of the evanescent waves. It is not so because boundary conditions used do not specify possible energy drain by another interface which can well be placed behind this one as in experiments on tunneling. If there is an air gap between two optically dense media, evanescent wave in this gap has the same form, but the phase shift and amplitudes are different so that amplitude of the reflected wave is not equal to that of the incident one. If energy loss is known, this amplitude must be found and substituted to the equations just considered that gives another values for two other amplitudes.

6 Conclusion

The procedure of matching the fields in case of total internal reflection differs from that of ordinary refraction. First, reflected wave has a phase shift which depends on possible presence other interface(s) behind that under consideration as in the experiments of the work [3]. If an experiment, there is an air gap between optically dense media, the evanescent wave loses energy that diminishes amplitude of the reflected wave and changes its phase. Taking all this into account allows one to solve the problem of matching the fields. In any case, the amplitude of reflected wave is not known *a priori* and needs to be found by matching the fields at the second interface, if any. If there is no energy loss of this kind, amplitude of the reflected wave is to be put equal to that of the incident wave that simplifies the final result.

References

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