The Associahedra and Permutohedra Yet Again

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Abstract

The associahedra and permutohedra polytopes are redefined as subsets of discrete simplices, associated respectively to the commutative and noncommutative word monoid.

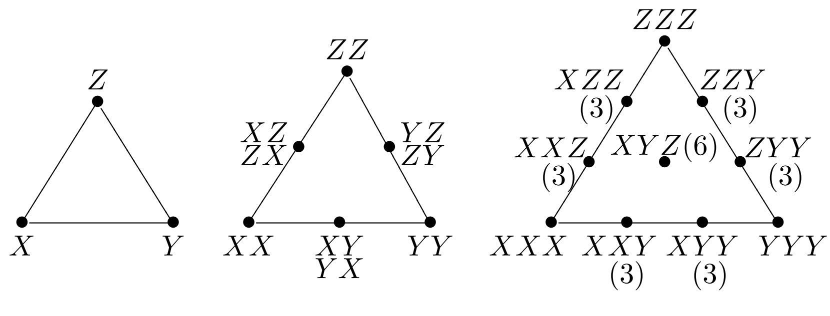
Although ubiquitous in quantum mathematics, the associahedra and permutohedra polytopes do not really have a definition that is natural for applications in number theory. We would like a definition that uses a direct algebraic condition. In [1] Postnikov defines these polytopes by selecting a certain set of indices associated to integral coordinates for generalised simplices. We reinterpret this definition in the simplest possible terms, so that the polytopes are defined by a set of divisors for a certain ordinal.

The next section introduces the commutative and noncommutative word monoids, that define canonical discrete simplices, and then the polytopes are defined as special subsets of these gadgets.

1 The Combinatorial Word Monoid

A word is a noncommutative monomial in a given alphabet of letters. So XY is distinguished from YX. Let n denote the number of letters in the alphabet and l the length of a given word. For example, $XXYX$ has length 4 in the $n = 2$ alphabet. The set $\mathbf{NC}(n, l)$ contains all length l words in an alphabet with n letters.

Forgetting the noncommutativity, words are reduced to ordinary monomials in n variables. The set $\mathbf{C}(n, l)$ denotes commutative monomials of length l in n variables. For example, at $n = 3$ the length 1, 2 and 3 words sit on triangular simplices



(1)

where the number of noncommutative words at each point is given. Blowing up to $\mathbf{NC}(n,l)$, one obtains a cubic array of points.

The ordinals n and l cover all possible words for alphabets of any length. A simplex is canonically coordinatised in the integer lattice $\mathbb{Z}^n$ (we dispense with brackets and commas) as follows. Since the monomials in a diagram are of homogeneous degree, the sum of degrees over the letters in the word is always a constant. As an example, the length two words in three letters specify six vectors in $\mathbb{Z}^3$ that correspond to the degrees of X, Y and Z in each monomial. These are 200, 020, 002, 110, 101 and 011, where 200 stands for XX.

Canonical coordinates can be used to turn the noncommutative words in $\mathbf{NC}(n,l)$ into ordinary monomials in a new set of variables, called *divisors*. The coordinates now specify the powers of n variables, which we call p_1, p_2, $\cdots$ p_n. For example, at $n \geq 2$ the words 000, 001, 011 and 111, along with their permutations, give the monomials 1, $\{p_1, p_2, p_3\}$, $\{p_1p_2, p_1p_3, p_2p_3\}$ and $p_1p_2p_3$ respectively. Thus the ith digit in the string of numbers denotes the power of p_i in the divisor of $n = p_1p_2p_3$. Then $\mathbf{NC}(2,3)$ is just a list of all divisors for $n = p_1p_2p_3$ in $\mathbb{N}$.

We now focus on the diagonal case $n = l$. These are ordinals of the form

$$1, \;\; p_1p_2, \;\; {p_1}^2{p_2}^2{p_3}^2, \;\; {p_1}^3{p_2}^3{p_3}^3{p_4}^3, \;\; \cdots \tag{2}$$

At $n = 3$ the set $\mathbf{C}(3,3)$ is also labeled with Young diagrams, as in figure 1. Divisors extend the commutative monomial points to all of $\mathbf{NC}(3,3)$ by filling in the little boxes of the Young diagram according to the following rules. The numbers denote the variable index: i for p_i. Every right moving list of numbers is strictly increasing. For instance, a row of two boxes can only contain the sequences 12, 13 or 23. After the first row, every subsequent row is filled with numbers from the first row. And that's it.

These Young diagrams define a cell decomposition for the Grassmannian over a finite field $\mathbb{F}$ [2]. For example, the figure $\mathbf{C}(3,3)$ denotes the space $\mathrm{Gr}(3,5)$ of three dimensional planes in $\mathbb{F}^5$. The space $\mathrm{Gr}(2,4)$ is given by six points on a triangle, and this is a finite version of Minkowski space. The prime variables p_i extend the parameter q in the Gaussian polynomials that count the number of points in $\mathrm{Gr}(l, n+l-1)$ for a finite field with q elements. For example, the polynomial for $\mathrm{Gr}(2,4)$, namely [2]

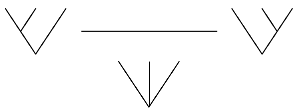
$$1 + q + 2q^2 + q^3 + q^4, \tag{3}$$

is a sum over the six Young diagrams, with p_1 and p_2 identified to define q. The term $2q^2$ corresponds to the two terms of degree two, ${p_i}^2$ and p_1p_2. In this way, the noncommutative space $\mathbf{NC}(n,l)$ extends the prime power spaces to allow for multiple factors p_i.

The projective space $\mathrm{Gr}(1,n)$ is then a simple simplex at $l = 1$. Over the complex number field, the projective space $\mathbb{CP}^{n-1}$ is usually described by assigning a torus to each point of a simplex, with tori degenerating to circles on the edge of a triangle and then to points at the vertices, and so on. The canonical coordinates give the shape of a torus at each point inside a simplex, so that the projective space is homogeneous.

2 Associahedra and Permutohedra with Divisors

By definition [3][4], the nth *associahedron* A_n has its vertex set specified by the rooted planar binary trees on $n+1$ leaves, with node levels not distinguished. So A_1 is a single point, the unique tree with two leaves, and A_2 is an edge between two vertices, as shown.



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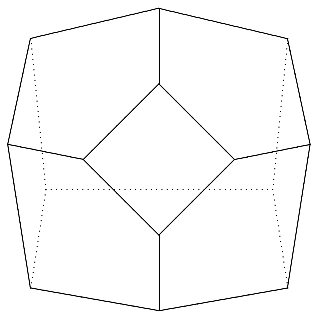
Each edge, in any A_n, represents a flip through a single node on the tree. A_3 is the pentagon and A_4 is a 14 vertex polytope in dimension 3.



(5)

Insisting instead on distinguished node levels within a planar tree, the trees encode permutations. This extends A_n to the *permutohedron* S_n, the polytope in dimension $n-1$ given by all permutations of the $\mathbb{Z}^n$ coordinate $(1, 2, \cdots, n)$. The polytope A_{n-1} can also be mixed with S_n to form the *permutoassociahedron* K_n [5]. For example, K_2 is a 12-gon in the plane, obtained by expanding each vertex of the hexagon S_3 by an edge A_2.

Now we define these polytopes as subsets of simplices from the word monoid, along the diagonal $n = l$. The result follows essentially from [1]. The associahedra A_n live in the commutative triangular simplices, while the S_n are subsets of the noncommutative monoid cube.

Definition 2.1 The *associahedron* polytope A_n is the restriction to the triangular simplex $\mathbf{C}(n, n)$ of all divisors of a factor from the central point of $\mathbf{C}(n, n)$.

Example 2.2 The pentagon A_3 is the set of divisors of ${p_1}^2 p_2$, where the words p_1 and p_2 are identified at a single point on $\mathbf{C}(3, 3)$, which is the point XXY in the diagram (1).

Definition 2.3 The *permutohedron* polytope S_n is the restriction to non-commutative words $\mathbf{NC}(n,n)$ of all divisors of a factor from the central point of $\mathbf{C}(n,n)$.

Example 2.4 The hexagon S_3 splits p_1 and p_2 for the pentagon above.

Example 2.5 The 14 vertices of A_4 are the commutative points of the divisor set of $n = {p_1}^3{p_2}^2p_3$, which lists the 24 points on S_4. The 120 vertex polytope K_4 is built from divisor pairs (D_a, D_b), where D_a is in S_4 and D_b comes from the complementary A_3 of the number $p_2{p_3}^2$.

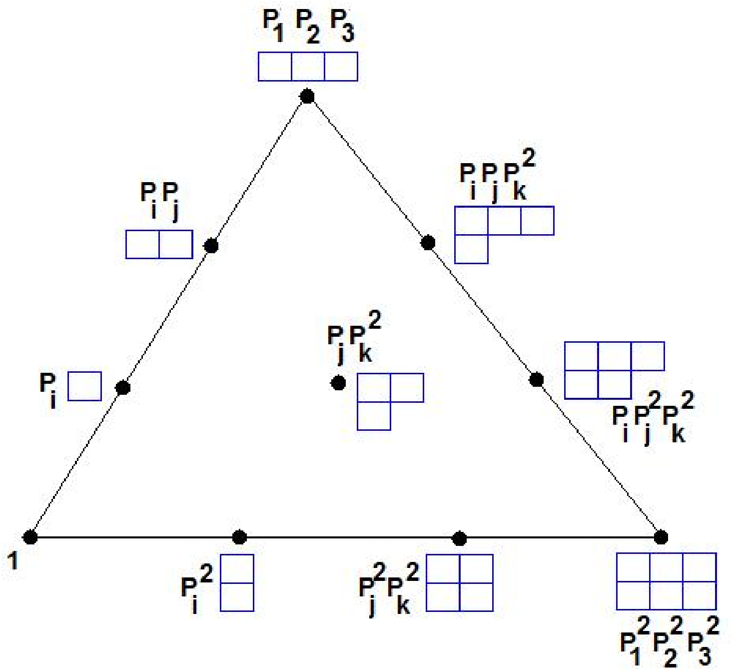


Figure 1: $\mathbf{C}(3,3)$ with Young diagrams

References

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