Remarks Around Lorentz Transformation

Arm Boris Nima
arm.boris@gmail.com

Abstract
After diagonalizing the Lorentz Matrix,
we find the frame where the Dirac
equation is one derivation and
we calculate the 'speed' of
the Schwarschild metric

Introduction

The Lorentz transformation is the only transformation which leaves Maxwell equations invariants. As a matter of fact, the matrix representation of the Lorentz belongs to what we call the Poincare group. However, the Lorentz matrix is often express in the Cartesian coordinates x, y, z, t. Furthermore, I wondered what are the eigenvalues and the eigenvectors of the Lorentz matrix.

After diagonalizing the Lorentz matrix, we see that the first coordinates of its eigenvectors are $x+ct$ and $x-ct$ and that the eigenvalues are inverse each others. In that case the idea came to me to express some well-known physics equation in the coordinates $x+ct, x-ct$. But the only physic equation I found where I could do it was the Dirac equation. Also, there was some derivation in the Dirac equation and I searched which frame I have to derivate by to obtain the derivations in the Dirac equation. The answer of this question is not $x+ct$ but

$$U(x,t) \ = \ c\, t\mathbb{1}_4 \ + \sum_{j=1}^{3} \alpha_i \, x_i \tag{0.1}$$

where the α_i are related with the Pauli matrices. So the derivation by the 'metric' $U(x,t)$ gives the Dirac equation

$$i\,\hbar \ \frac{\partial \psi(x,t)}{\partial U} \ = \ m\, c\, \alpha_0 \ \psi(x,t) \tag{0.2}$$

with $\alpha_0 = \gamma_0$.

Moreover, I remarked that, if we decompose the well known Schwarschild metric, we have a matrix with two eigenvalues inverse each others, which remind us the diagonalization of the Lorentz matrix. Then we decide to calculate the corresponidng speed of the Schwarschild metric which gives us

$$\frac{v}{c} \simeq -\frac{2GM}{c^2 r} \tag{0.3}$$

In the first part, we calculate the eigenvalues and the eigenvectors of the Lorentz matrix expressed with hyperbolic functions. We recall the definition of $O(1,1)$ and the Poincare group and we give the diagonalized matrix in function of the speed v of the translation of the frame. In the second part, we use the coordinates $x+ct$ found in the eigenvectors to express the Dirac equation with a derivation by a frame function of the Pauli matrices. In the third part, we find the corresponding speed of the Schwarzschild metric in decomposing it and comparing it with the diagonalization of the Lorentz matrix.

1 The Poincare Group

We recall the definition of the Lorentz transformation in a direction x (taken on Wikipedia)

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \underbrace{\begin{pmatrix} \cosh(\alpha) & -\sinh(\alpha) & & \\ -\sinh(\alpha) & \cosh(\alpha) & & \\ & & 1 & \\ & & & 1 \end{pmatrix}}_{L} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} \tag{1.4}$$

with $\cosh(\alpha) = \gamma$, $\sinh(\alpha) = \beta\gamma$ and $\gamma = \frac{1}{\sqrt{1-\beta^2}}$, $\beta = \frac{v}{c}$. We often say that L belongs to the Poincare group.

Now we study the matrix

$$g \ = \ \begin{pmatrix} \cosh(\alpha) & -\sinh(\alpha) \\ -\sinh(\alpha) & \cosh(\alpha) \end{pmatrix} \in O(1,1) \tag{1.5}$$

where $O(1,1)$ has been defined by [1] as

$$O(1,1) \ = \ \Big\{ g \in GL(2,\mathbb{R}) \ \ | \ \ g\, J_1\, {}^t g \ = \ J_1 \Big\} \ = \ \Big\{ \forall \alpha \in \mathbb{R} \ \ | \ \ g \ = \exp\Big(\alpha \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}\Big) \Big\} \tag{1.6}$$

with

$$J_1 \ = \ \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \tag{1.7}$$

You can check that

$$\begin{pmatrix} \cosh(\alpha) & -\sinh(\alpha) \\ -\sinh(\alpha) & \cosh(\alpha) \end{pmatrix} \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}^t \begin{pmatrix} \cosh(\alpha) & -\sinh(\alpha) \\ -\sinh(\alpha) & \cosh(\alpha) \end{pmatrix} = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \tag{1.8}$$

Now we calculate the eigenvalues of g

$$\det(g - \lambda Id_2) \ = \ \begin{vmatrix} \cosh(\alpha) - \lambda & -\sinh(\alpha) \\ -\sinh(\alpha) & \cosh(\alpha) - \lambda \end{vmatrix} \ = \ 0 \tag{1.9}$$

So we have two eigenvalues $\lambda_\pm$ given by

$$\lambda_\pm \ = \ e^{\pm\alpha} \tag{1.10}$$

and the corresponding eigenvectors

$$\begin{aligned} \ker \begin{pmatrix} -\sinh(\alpha) & -\sinh(\alpha) \\ -\sinh(\alpha) & -\sinh(\alpha) \end{pmatrix} \ &= \ \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\ \ker \begin{pmatrix} \sinh(\alpha) & -\sinh(\alpha) \\ -\sinh(\alpha) & \sinh(\alpha) \end{pmatrix} \ &= \ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \end{aligned} \tag{1.11}$$

So we can diagonalize the matrix L :

$$\begin{pmatrix} \cosh(\alpha) & -\sinh(\alpha) \\ -\sinh(\alpha) & \cosh(\alpha) \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} e^{\alpha} & \\ & e^{-\alpha} \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

and the equation (1.4)

$$\begin{pmatrix} ct' - x' \\ ct' + x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} e^{\alpha} & & & \\ & e^{-\alpha} & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} ct - x \\ ct + x \\ y \\ z \end{pmatrix} \tag{1.12}$$

Because $\cosh(\alpha) = \gamma, \sinh(\alpha) = \beta\gamma$, we can write :

$$\begin{pmatrix} ct' - x' \\ ct' + x' \\ y' \\ z' \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{1+\beta}{\sqrt{1-\beta^2}} & & & \\ & \frac{1-\beta}{\sqrt{1-\beta^2}} & & \\ & & 1 & \\ & & & 1 \end{pmatrix}}_{L_0} \begin{pmatrix} ct - x \\ ct + x \\ y \\ z \end{pmatrix} \tag{1.13}$$

where $\beta = \frac{v}{c}$.

Thus wee have seen in (1.13) that the coordinates $x + ct$ and $x - ct$ are very special coordinates because these are the coordinates where Lorentz transformation matrices are diagonales. Now we try to write the Dirac equation in these 'kind' of coordinates.

2 The Dirac Equation

We know the Dirac equation under the form (cf. Wikipedia)

$$\left(i\,\hbar\,\gamma^{\mu}\,\partial_{\mu} - m\,c\right)\,\psi(x) \;=\; 0 \tag{2.14}$$

But the explicit form is given by

$$i\,\hbar\,\frac{\partial\psi(x,t)}{\partial t} \;=\; \left(m\,c^2\,\alpha_0 \;-\; i\,\hbar\,c\,\sum_{j=1}^{3}\alpha_j\,\frac{\partial}{\partial x_j}\right)\psi(x,t) \tag{2.15}$$

where

$$\alpha_0 \;=\; \begin{pmatrix} \mathbb{1}_2 & \\ & -\mathbb{1}_2 \end{pmatrix} \tag{2.16}$$

and

$$\alpha_j \;=\; \begin{pmatrix} & \sigma_j \\ \sigma_j & \end{pmatrix} \tag{2.17}$$

for each Pauli matrices

$$\sigma_1 = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} & -i \\ i & \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \tag{2.18}$$

The matrices γ^j in (2.14) are defined by

$$\gamma^0 = \alpha_0 \qquad \gamma^j = \alpha_0\, \alpha_j \tag{2.19}$$

Now we can write (2.15) in the form

$$i\,\hbar\, c \left(\frac{\mathbb{1}_4}{c} \frac{\partial}{\partial t} + \sum_{j=1}^{3} \alpha_j \frac{\partial}{\partial x_j} \right) \psi(x,t) = m\, c^2\, \alpha_0\, \psi(x,t) \tag{2.20}$$

Because $\mathbb{1} = \mathbb{1}^{-1}$ and $\alpha_j = \alpha_j^{-1}$, we can rewrite (2.20) as

$$i\,\hbar \left(\frac{\mathbb{1}_4^{-1}}{c} \frac{\partial}{\partial t} + \sum_{j=1}^{3} \alpha_j^{-1} \frac{\partial}{\partial x_j} \right) \psi(x,t) = m\, c\, \alpha_0\, \psi(x,t) \tag{2.21}$$

For a general function $U(x^1, x^2, x^3, t) \equiv U(x,t)$, we have

$$\frac{\partial}{\partial U(x,t)} = \frac{\partial t}{\partial U(x,t)} \frac{\partial}{\partial t} + \frac{\partial x_1}{\partial U(x,t)} \frac{\partial}{\partial x_1} + \frac{\partial x_2}{\partial U(x,t)} \frac{\partial}{\partial x_2} + \frac{\partial x_3}{\partial U(x,t)} \frac{\partial}{\partial x_3} \tag{2.22}$$

If we take the frame $U(x,t) = c\, t \mathbb{1}_4 + \sum_{j=1}^{3} \alpha_i\, x_i$, we have

$$\frac{\partial}{\partial U(x,t)} = \frac{\mathbb{1}_4^{-1}}{c} \frac{\partial}{\partial t} + \alpha_1^{-1} \frac{\partial}{\partial x_1} + \alpha_2^{-1} \frac{\partial}{\partial x_2} + \alpha_3^{-1} \frac{\partial}{\partial x_3} \tag{2.23}$$

which is the derivation in (2.21). Finally we can express the derivate in (2.21) as

$$i\,\hbar\, \frac{\partial \psi(x,t)}{\partial U} = m\, c\, \alpha_0\, \psi(x,t) \tag{2.24}$$

Then we can see $U(x,t)$ as a 'metric' given by

$$U(x,t) = \begin{pmatrix} ct & 0 & x_3 & x_1 - i x_2 \\ 0 & ct & x_1 + i x_2 & -x_3 \\ x_3 & x_1 - i x_2 & ct & 0 \\ x_1 + i x_2 & -x_3 & 0 & ct \end{pmatrix} \tag{2.25}$$

3 Lorentz transformation in the Schwarschild metric ?

We consider the Schwarschild metric

$$g_{\mu\nu} = \begin{pmatrix} 1-\frac{2GM}{c^2r} & & & \\ & -\left(1-\frac{2GM}{c^2r}\right)^{-1} & & \\ & & -r^2 & \\ & & & -r^2\sin^2\theta \end{pmatrix} \tag{3.26}$$

Now what I will do is really speculative. We "imagine" that the Schwarzschild metric is the product of a Lorentz transformation and the Minskowskian polar metric

$$g_{\mu\nu} = \begin{pmatrix} 1-\frac{2GM}{c^2r} & & & \\ & \left(1-\frac{2GM}{c^2r}\right)^{-1} & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & r^2 & \\ & & & r^2\sin^2\theta \end{pmatrix} \tag{3.27}$$

Now we compare the first matrix of (3.27) with (1.13) and we identify

$$\begin{aligned} \frac{1+\beta}{\sqrt{1-\beta^2}} &= 1-\frac{2GM}{c^2r} \\ (1+\beta)\left(1+\frac{\beta^2}{2}+...\right) &= 1-\frac{2GM}{c^2r} \\ 1+\beta &\simeq 1-\frac{2GM}{c^2r} \\ \frac{v}{c} &\simeq -\frac{2GM}{c^2r} \end{aligned} \tag{3.28}$$

The expression (3.28) has to be compared with the liberation speed of a black hole gravitation

$$v_{lib} = \sqrt{\frac{2GM}{c^2r}} \tag{3.29}$$

Références

[1] Daniel Bump, Lie Groups p34