The Wave Medium, the Electron, and the Proton - Part 2

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Summary

Resting wave functions for the electron and the proton are presented. The proposed resting wave function of the proton is $-6\pi^5$ multiplied by the resting wave function of the electron. Therefore, the hypothesis presented in Part 1 is not rejected. An unexpected consequence of this work is the ability to calculate the size of the proton by using Equation 13.4. The resulting value for the diameter is 1.668 x 10^{-15} meter. This is within the accepted measured range of the proton diameter at 1.755(102) x 10^{-15} meter. Another unexpected consequence of this work is the ability to associate vacuum energy with the wave function of the electron by using Equation 22.1.

A function based upon the exponential of a generic quaternion is used as the basis for the solution of the wave equations. This produces 3 general forms of wave functions.

The **Discussion** is broken into two portions. The portion subtitled **General Ideas** can be largely understood using little mathematics. The portion subtitled **Rigorous Analysis** will require competence with quaternions and with partial differential equations.

Preface

The author is neither a mathematician nor a physicist. None the less, there is a significant amount of mathematics presented. The required mathematics includes quaternions and partial differential equations. The written text can be largely understood without the Appendices. It should be viewed as a mixture of math and art.

Discussion

In Part 1 of this effort[1], the author presented evidence that our rest frame is moving with respect to a scalar field. As a brief restatement of what was presented, the author argued that the true ratio between the rest mass of the proton and the rest mass of the electron is $6\pi^5$ and that the observed deviation from this value is the result of motion of our rest frame with respect to a scalar field. The value of this motion was stated to be 0.006136 c based upon the Lorentz Transform.

An implication of this motion through the aether is that the electrons that we perceive as being in our frame of reference are actually stationary and simply rise up from the aether as required by the wave equation. The key to that argument is the belief that there are two solutions to the wave equation such that their ratio at rest is $6\pi^5$. It is therefore the objective of Part 2 of this work to produce these solutions or to show how they are related. This is not a theoretical discussion of wave mechanics. The objective is to produce a pair of wave functions. This should be viewed as the mathematical equivalent of art.

The author[2,3] has previously produced vector solutions to the spherical wave equation. These were based upon previous work by Wolff[4]. The key to those solutions was the fact that sin(x)/x has a finite value at x = 0. Therefore, versions of Euler's Equation were combined in such a way as to eliminate the cosine terms, thereby leaving only the sine terms. This effort will follow that strategy.

It will be shown below that possible resting wave functions for the electron and the proton are as follows:

Electron:

$$(\boldsymbol{\Psi}_{\mathbf{E}} - \psi_0) = \pm\frac{1}{r}\sin(\alpha r)\, e^{-\mathbf{i}\alpha ct}$$

Proton:

$$(\boldsymbol{\Psi}_{\mathbf{P}} - \psi_0) = \pi(\mathbf{i} + \mathbf{j} + \mathbf{k})(\psi_i + \psi_j + \psi_k)$$

Where

$$\psi_i = \pm\frac{1}{r}\left(e^{q_0+\mathbf{i}\alpha r} - e^{q_0-\mathbf{i}\alpha r}\right)e^{-\mathbf{i}\alpha ct} = \pm\frac{2\mathbf{i}}{r}e^{q_0}\sin(\alpha r)\, e^{-\mathbf{i}\alpha ct}$$

$$\psi_j = \pm\frac{1}{r}\left(e^{q_0+\mathbf{j}\alpha r} - e^{q_0-\mathbf{j}\alpha r}\right)e^{-\mathbf{i}\alpha ct} = \pm\frac{2\mathbf{j}}{r}e^{q_0}\sin(\alpha r)\, e^{-\mathbf{i}\alpha ct}$$

$$\psi_k = \pm\frac{1}{r}\left(e^{q_0+\mathbf{k}\alpha r} - e^{q_0-\mathbf{k}\alpha r}\right)e^{-\mathbf{i}\alpha ct} = \pm\frac{2\mathbf{k}}{r}e^{q_0}\sin(\alpha r)\, e^{-\mathbf{i}\alpha ct}$$

$$\alpha = \frac{2m_E c}{\left(\frac{h}{2\pi}\right)} ; \beta = \frac{\pi m_E c}{\left(\frac{h}{2\pi}\right)\ln\pi} ; \frac{\alpha}{\beta} = \frac{2\ln(\pi)}{\pi} ; q_0 = 4\ln(\pi) = 2\pi\frac{\alpha}{\beta}$$

The ratio between the proposed proton and electron wave functions is $-6\pi^5$ as required by Part 1.

General Ideas:

There was a Figure 1 presented in Part 1 of this work and included here. During the discussion of that figure, it was noted that a necessary condition for this argument to be true is that mass must be proportional to vertical distance in that figure. It was also speculated that the value of the wave function was related to this same distance. If the speculation is true then it follows that the value of the wave function is related to mass. This is not a new concept since the Schrödinger Equation expresses a form of it. The simplest such relationship is linear. This is expressed mathematically as a differential as follows:

Equation 1:

$$dm = \gamma \, d\psi$$

In Equation 1, γ is a presently unknown constant of proportionality. Negative values here should not be interpreted as anti-matter or negative matter. Equation 1 is necessary to ensure that the ratio between two wave functions is equal to the ratio of their masses. The value of γ will be determined when the electron is discussed in the **Rigorous Analysis** section.

The horizontal distance in that figure could be a true physical distance although that was never explicitly stated.

The general idea here is to produce a wave function - actually two wave functions - whose ratio is $6\pi^5$. There must be some rationalization for the structure of the functions and they both must satisfy the wave equations. The easiest way for both to satisfy the wave equation is for both functions to be multiples of a single function that satisfies it. Also, the functions should have similar structures and if quaternions are involved then one function might be advanced by 2π with respect to the other.

The first task is to determine a method of scaling a vector based upon it's angle of rotation. Each 90 degrees of rotation must increase the length of the vector by a factor of π. The author proposes the following relationship:

Equation 2:

$$f = \pi^{\frac{2}{\pi}\theta}$$

At θ = 0, f = 1. At θ = π/2, f = π. At θ = π, $f = \pi^2$. At θ = 3π/2, $f = \pi^3$. At θ = 2π, $f = \pi^4$. These are precisely the values needed to satisfy the quaternion multiplication proposed in Part 1. It might be possible to include a value s to represent spin and a value n with discrete values (n = 0, 1, 2, 3) to produce a generalized solution to the wave equation, but the author's objective here is limited to producing $6\pi^5$.

The next task is to determine a method of producing the value 6. Euler's Equation is one of the foundations used to solve differential equations. It is almost magic. Euler's Equation is also a quaternion that performs a rotation about the axis in the exponential. The author will begin with the following simplified solution:

Equation 3:

$$\psi = e^{+\mathbf{i}\theta_i} - e^{-\mathbf{i}\theta_i} = 2\mathbf{i}\sin(\theta_i)$$

Here, the bold faced **i** indicates a vector of unit length in the direction of the **i** axis. The variable θ_i represents the angle of rotation around the associated axis. There are similar functions for **j** and **k**. By multiplying each of these three simplified solutions by Equation 2 and then adding them together, the following is obtained:

Equation 4:

$$\psi = 2\left(\mathbf{i}\,\pi^{\frac{2}{\pi}\theta_i}\sin(\theta_i) + \mathbf{j}\,\pi^{\frac{2}{\pi}\theta_j}\sin(\theta_j) + \mathbf{k}\pi^{\frac{2}{\pi}\theta_k}\sin(\theta_k)\right)$$

The next step is to multiply Equation 4 by (**i** + **j** + **k**) and to substitute a generic θ for each angle of rotation. Doing so produces the following:

Equation 5:

$$\psi = 2(\mathbf{i}+\mathbf{j}+\mathbf{k})(\mathbf{i}+\mathbf{j}+\mathbf{k})\,\pi^{\frac{2}{\pi}\theta}\sin(\theta) = -6\pi^{\frac{2}{\pi}\theta}\sin(\theta)$$

With relatively little effort, the author has produced a factor of $6\pi^4$ at $\theta = 2\pi$. The minus sign is an unexpected bonus. The remaining factor of π is more difficult to rationalize.

The author was stumped at this point for quite some time. It was tempting to simply arbitrarily include the remaining factor of π but that would risk missing something fundamental. The author believes that the problem is that there is another version of Equation 2 that is associated with rotation in the opposite direction. Equation 2 creates an expanding spiral in the counter-clockwise direction. The choice of rotational direction was arbitrary. It is equally valid to rotate in the clock-wise direction. If CW rotation is subtracted from the CCW rotation, the result is a line segment with ends on the two spirals. It is then reasonable to multiply by π to produce a circle from the line segment. Equation 6 incorporates these revisions. It seems that the exponential π is operating like a quaternion. Therefore, the author included an **i** as part of the exponential. Taking the difference between the two also ensures that the result includes only sine functions since that is a requirement of the spherical wave solutions.

Equation 6:

$$\psi = -6\pi\left(\pi^{+\mathbf{i}\frac{2}{\pi}\theta_i} - \pi^{-\mathbf{i}\frac{2}{\pi}\theta_i}\right)\sin(\theta_i)$$

Equation 6 is getting pretty close to a possible solution. The $\sin(\theta_i)$ could be the wave function for the electron. The exponentials need revision since there is scaling as well as rotation. This is perhaps done as follows:

Equation 7:

$$\psi = -6\pi\left(\pi^{\frac{2}{\pi}(\theta_i+\mathbf{i}\theta_i)} - \pi^{\frac{2}{\pi}(\theta_i-\mathbf{i}\theta_i)}\right)\sin(\theta_i) = -6\pi\pi^{\frac{2\theta_i}{\pi}}\left(\pi^{+\mathbf{i}\frac{2}{\pi}\theta_i} - \pi^{-\mathbf{i}\frac{2}{\pi}\theta_i}\right)\sin(\theta_i)$$

If the $\sin(\theta_i)$ combined with the terms in the parentheses represents the electron, then the problem is essentially solved at this point provided the proton wave function is 2π ahead of the electron wave function. Of course, it must also be confirmed that this satisfies the various wave equations.

The author proposes that the correct generalized form of the wave function is as follows:

Equation 8:

$$\mathbf{\Psi} = \psi_0 + \psi_i\mathbf{i} + \psi_j\mathbf{j} + \psi_k\mathbf{k}$$

In Equation 8, $\mathbf{\Psi}$ is a quaternion and **i**, **j**, and **k** are unit vectors. A motivation for writing the wave function in this form is to present the scalar field as a datum. Therefore, the wave functions of various particles should be written as $\mathbf{\Psi} - \psi_0$.

Going back to Equation 7, the next step is to substitute the wave functions for the proton and the electron.

Equation 9:

$$(\mathbf{\Psi}_P - \psi_0) = -6\pi\pi^{\frac{2\theta_i}{\pi}}(\mathbf{\Psi}_E - \psi_0)$$

In Equation 9, the wave function for the electron is as follows:

Equation 10:

$$(\mathbf{\Psi}_E - \psi_0) = \left(\pi^{+\mathbf{i}\frac{2}{\pi}\theta_i} - \pi^{-\mathbf{i}\frac{2}{\pi}\theta_i}\right)\sin(\theta_i)$$

Equation 10 is useful for the present but will likely be revised with a 1/r term when the spherical wave equation is solved. Also, the $\sin(\theta_i)$ might not be needed because the complex exponentials of π will produce a factor of $2\mathbf{i}\sin(\theta_i)$, and the extra $\sin(\theta_i)$ might prevent ψ from satisfying the wave equations.

The ratio between these two wave functions is $-6\pi^5$ at 2π.

In consideration of the extra sine term in Equation 10, let us revisit the above argument. Suppose that we begin with the following function for **i**:

Equation 11:

$$F_i = \pi \left[\pi^{\frac{2}{\pi}(2\pi + \mathbf{i}\theta_i)} - \pi^{\frac{2}{\pi}(2\pi - \mathbf{i}\theta_i)} \right] = 2\pi\pi^4 \mathbf{i} \sin\left(\frac{2}{\pi}\theta_i \ln(\pi)\right)$$

In Equation 11, the author has combined all of the lessons learned from Equations 2 - 7. Next, define similar functions for **j** and **k**. Then add the three functions together, multiply by (**i** + **j** + **k**), and simplify by using a generic θ. Hint: regarding Eq 11, π = exp(ln(π)).

Equation 12:

$$(\boldsymbol{\Psi}_P - \psi_0) = -6\pi^5 \sin\left(\frac{2}{\pi}\theta \ln \pi\right)$$

Equation 13:

$$(\boldsymbol{\Psi}_E - \psi_0) = \sin\left(\frac{2}{\pi}\theta \ln(\pi)\right)$$

If Equation 13 can be shown to satisfy the wave equations, then the author will have shown that there are two solutions to the wave equations such that their ratio is $-6\pi^5$. By inference, one solution might be the electron while the other solution might be the proton.

Appendix A confirms that Equation 13 satisfies both the classical wave equation and the Schrödinger Wave Equation. Therefore, the hypothesis presented in Part 1 is not rejected. Necessary conditions to satisfy both wave equations are Equations 13.1 and 13.2.

Equation 13.1:

$$\alpha = \frac{2mc}{\left(\frac{h}{2\pi}\right)} = 5.686 \cdot 10^{42}\, m \left(\frac{1}{meter \cdot kg}\right)$$

Equation 13.2:

$$\beta = \frac{\theta}{x} = \frac{\pi mc}{\left(\frac{h}{2\pi}\right)\ln \pi} \cong 7.802 \cdot 10^{42}\, m \left(\frac{radian}{meter \cdot kg}\right)$$

These are thought to be fundamental properties of the wave medium. Equation 13.1 results from application of the separation of variables method to the Schrödinger Wave Equation. The value α is the separation of variables constant for the system. It has units of ($length^{-1}$).

There are at least two ways to express the differential equations associated with this problem. One method is as Equation 13. Another method is as sin(αx) for separation of variables. Equation 13.2 results from requiring that the values within the parentheses be equal for these two methods (i.e., αx = (2/π)θlnπ). The value β is simply a proportionality factor between the two methods. It has units of

(length^{-1}). In some ways, it expresses the wave nature of matter and the nature of the wave medium itself.

A possible wave function for the electron is therefore:

Equation 13.3:

$$(\boldsymbol{\Psi}_E - \psi_0) = \pm\sin\left(\frac{2}{\pi}\theta\ln(\pi)\right)e^{-\mathbf{i}\alpha ct} = \pm\sin(\alpha x)\,e^{-\mathbf{i}\alpha ct}$$

Equation 13.2 may be used to estimate the size of the proton by setting θ = 2π. The resulting value is 4.815 x 10^{-16} meters. This is the distance from the center of the proton to the center of a face along one of the 3 axes. The diameter - actually the distance between opposite corners - is then determined by multiplying by 2 x sqrt(3). The resulting value is 1.668 x 10^{-15} meters. The accepted value[5] is 1.755(102) x 10^{-15} meters for diameter. This is perhaps the best evidence favoring an aether.

Equation 13.4:

$$d = 4\sqrt{3}\left(\frac{1}{mc}\right)\left(\frac{h}{2\pi}\right)\ln(\pi)$$

Rigorous Analysis:

In this section, the author will develop generic quaternion based solutions to the wave equations. The author will then specify values for the elements of the quaternion to match the reasoning presented in **General Ideas** above. The equations that must be satisfied are the classical wave equation and the Schrödinger Wave Equation. These are expressed as follows:

Classical[6]:

$$\nabla^2\psi = \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}$$

Schrödinger[7]:

$$-\frac{\left(\frac{h}{2\pi}\right)^2}{2m}\nabla^2\psi = \mathbf{i}\left(\frac{h}{2\pi}\right)\frac{\partial\psi}{\partial t}$$

Appendix B presents a simple introduction to quaternions and vector rotation. Quaternions were developed by Sir William Rowan Hamilton[8] in the mid 1800's as a method to determine the ratio between non-collinear vectors. Quaternions are used to rotate and scale vectors.

Euler's Equation[9] is typically written as follows:

$$e^{\mathbf{i}x} = \cos x + \mathbf{i}\sin x$$

or as

$$e^{-\mathbf{i}x} = \cos x - \mathbf{i}\sin x$$

Comparison of Euler's Equation with the results from Appendix B leads to the conclusion that Euler's Equation is actually a quaternion that performs rotation. As such, the value x is actually an angle rather than a linear distance. So, when Euler's Equation is used as a basis to solve a differential equation such as the wave equation, it is necessary to provide a means to convert between angular rotation and linear distance. Therefore, a constant of proportionality is required to make this conversion. When separation of variables is used, the separation constant α serves this purpose.

In the **General Ideas** section above, several equations are presented that have an exponential term of the form (1 + **i**)x. This is an exponential of a quaternion. Suppose that a quaternion **Q** is defined as follows:

$$\mathbf{Q} = q_0 + q_i\mathbf{i} + q_j\mathbf{j} + q_k\mathbf{k} = q_0 + \mathbf{q}$$

Capitalized bold text is used to designate a quaternion.

Next define a function **X** as follows:

Equation 14:

$$\boldsymbol{X} = e^{q_0+\mathbf{q}x} = e^{q_0}[\cos(q_i x) + \mathbf{i}\sin(q_i x)]\big[\cos\big(q_j x\big) + \mathbf{j}\sin\big(q_j x\big)\big][\cos(q_k x) + \mathbf{k}\sin(q_k x)]$$

In this section, the author has switched from ψ to **X** (or **R**) in anticipation of using the separation of variables method. In the above **X**, the vector portion **q** is multiplied by x but the scalar portion q_0 is not. The wave function ψ will then be written **X**(x)T(t) or **R**(r)T(t).

The 1'st and 2'nd derivatives of Equation 14 are:

1'st Derivative:

$$\frac{d\boldsymbol{X}}{dx} = \mathbf{q}e^{q_0+\mathbf{q}x} = \mathbf{q}\boldsymbol{X}$$

2'nd Derivative:

$$\frac{d^2\boldsymbol{X}}{dx^2} = \mathbf{q}^2 e^{q_0+\mathbf{q}x} = \mathbf{q}^2\boldsymbol{X} = -\left(q_i^2 + q_j^2 + q_k^2\right)\boldsymbol{X}$$

This illustrates the power and convenience of quaternions. Would the reader prefer to take the derivatives of the various sines and cosines of Equation 14, or of the complex exponential instead? The complex exponential is certainly easier and more compact.

Separation of variables is a standard method of solving partial differential equations. Essentially, the solution is assumed to be the result of multiplying several functions together. Each of these functions is a function of one of the independent variables. So, for a problem such as this one, the wave function would be assumed to have a form as follows:

$$\boldsymbol{\psi} = \mathbf{X}(x)T(t) \ or \ \boldsymbol{\psi} = \mathbf{R}(r)T(t)$$

The various derivatives are determined for this and then substituted into the equation to be solved. The most important feature is that the cross derivatives are zero because each function is a function of only one independent variable.

When separation of variables is used to solve a partial differential equation, a separation constant is produced. For a 2'nd order problem, this is typically designated as $-\alpha^2$. The negative sign is typically used to invoke complex solutions but is not essential. A requirement of satisfying the classical wave equation is the following:

$$\frac{1}{\boldsymbol{X}}\frac{d^2\boldsymbol{X}}{dx^2} = -\alpha^2 = \frac{1}{c^2}\frac{1}{T}\frac{d^2T}{dt^2}$$

$$\mathbf{q}^2 = -\alpha^2$$

There are an unlimited number of ways to satisfy this. A simple solution is to set $q_i = \pm\alpha$ and $q_j = q_k = 0$. The value of q_0 is not a factor here. It is easier to produce a quaternion solution than non-quaternion one since the user has some degree of freedom.

Next consider the conjugate of the above quaternion **Q**. Define **Q*** as follows:

$$\mathbf{Q}^* = q_0 - q_i\mathbf{i} - q_j\mathbf{j} - q_k\mathbf{k} = q_0 - \mathbf{q} = \mathbf{Q} - 2(q_i\mathbf{i} + q_j\mathbf{j} + q_k\mathbf{k}) = \mathbf{Q} - 2\mathbf{q}$$

It is easy to show the following quaternion identities:

$$\mathbf{Q} + \mathbf{Q}^* = 2q_0$$

$$\mathbf{Q} - \mathbf{Q}^* = 2\mathbf{q}$$

$$\mathbf{Q}^2 = (\mathbf{Q}^*)^2 = q_0^2 - q_i^2 - q_j^2 - q_k^2$$

$$\mathbf{Q}\mathbf{Q}^* = q_0^2 + q_i^2 + q_j^2 + q_k^2$$

and

$$\frac{1}{2}(\mathbf{Q} + \mathbf{Q}^*) = q_0 = \mathbf{Q} - \mathbf{q}$$

Next, define a function similar to Equation 14 except use both **Q** and its conjugate **Q***.

Equation 15:

$$\mathbf{F} = \left(e^{\mathbf{Q}x} - e^{\mathbf{Q}^*x}\right) = \left(e^{q_0+\mathbf{q}x} - e^{q_0-\mathbf{q}x}\right)$$

Equation 15 will also satisfy the classical wave equation. In addition, it will satisfy the spherical version of the classical wave equation when multiplied by 1/x.

It would be valid to pre-multiply the exponential terms of Equation 15 by -1. At first glimpse this appears to produce two solutions. What it actually does is swap the positions of **Q** and **Q***. Whether or not those are unique or redundant is another question. The author believes they are redundant.

Now consider the form of T. It must be a complex exponential in order to satisfy Schrödinger. By solving both the Schrödinger Equation and the classical wave equation for the Laplacian, and then setting the two equations equal to each other, the following is obtained:

Equation 16:

$$\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = -\mathbf{i}\frac{2m}{\left(\frac{h}{2\pi}\right)}\frac{\partial\psi}{\partial t}$$

Define T(t) as follows:

Equation 17:

$$T(t) = e^{-\mathbf{i}\alpha ct}$$

1'st Derivative:

$$\frac{\partial T}{\partial t} = -\mathbf{i}\alpha ce^{-\mathbf{i}\alpha ct} = -\mathbf{i}\alpha cT$$

2'nd Derivative:

$$\frac{\partial^2 T}{\partial t^2} = i^2\alpha^2c^2e^{-i\alpha ct} = -\alpha^2c^2T$$

Substitution of the 1'st and 2'nd derivatives into Equation 17 produces the following:

$$-\frac{1}{c^2}\alpha^2c^2T = \mathbf{i}^2\frac{2m}{\left(\frac{h}{2\pi}\right)}\alpha cT$$

This simplifies to:

Equation 18:

$$\alpha = +\frac{2mc}{\left(\frac{h}{2\pi}\right)}$$

If the exponential uses +**i**αct instead of -**i**αct, then the following result is obtained:

Equation 19:

$$T(t) = e^{+\mathbf{i}\alpha ct}$$

Equation 20:

$$\alpha = -\frac{2mc}{\left(\frac{h}{2\pi}\right)}$$

It is noteworthy that the time exponentials of +**i**αct and -**i**αct produce valid solutions when taken individually but not when taken together as a sum or difference. When taken together as a sum or difference, they will satisfy the classical wave equation but not the Schrödinger Wave Equation. In fact, only the -**i**αct form will satisfy Schrödinger for positive values of α and m because of the -**i** in Schrödinger.

It would be equally valid to define Equation 17 and Equation 19 as the negative of the exponential. Therefore, there are four forms of T that will satisfy the wave equations. These may be compactly written as follows:

Equation 21:

$$\mathbf{T}(t) = \pm\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}\mathbf{T}_1\\\mathbf{T}_2\end{bmatrix}$$

Where

$$\mathbf{T}_1 = e^{+\mathbf{i}\alpha ct}$$

and

$$\mathbf{T}_2 = e^{-\mathbf{i}\alpha ct}$$

In principle, the form of **T** results in 4 solutions when multiplied by **R**. In practice, the +**i**αct form is not a solution. Therefore, there are only two valid solutions.

As was mentioned previously, it is possible to satisfy the wave equations using a quaternion exponential in an unlimited number of ways. Appendix C confirms that the above generic quaternion wave function satisfies the wave equations. Appendix C also presents 3 general solutions. These solutions are 1-D, 2-D, and 3-D. The value of q_0 can be any real number. Below, the author briefly discusses each. The objective is to determine which solutions best match the electron and proton with the requirement that the proton must be $6\pi^5$ more massive at rest than the electron. The task for each problem will be to determine the values for the coefficients of the quaternion and the resulting wave function. All of the coefficients have ± values. The author's preference is to use positive values for **Q** and negative values for **Q***. The values for α and β are based upon Equations 13.1 and 13.2 from the **General Ideas** section. The values for α and β are the same for all of the following wave functions. In these formulations, **Q** and **Q*** have been replaced by (q_0 + **q**α) and (q_0 - **q**α) respectively.

$$\alpha = \frac{2mc}{\left(\frac{h}{2\pi}\right)} ; \beta = \frac{\pi mc}{\left(\frac{h}{2\pi}\right)\ln\pi} ; \frac{\alpha}{\beta} = \frac{2\ln(\pi)}{\pi}$$

Case	q_i	q_j	q_k
1	±1	0	0
2	0	$\pm\sqrt{\frac{1}{2}}$	$\pm\sqrt{\frac{1}{2}}$
3	$\pm\sqrt{\frac{1}{3}}$	$\pm\sqrt{\frac{1}{3}}$	$\pm\sqrt{\frac{1}{3}}$

Table 1 - Example Solutions

Case 1:

$$(\boldsymbol{\Psi} - \psi_0) = \pm\frac{1}{r}\left(e^{q_0+\mathbf{i}\alpha r} - e^{q_0-\mathbf{i}\alpha r}\right)e^{-\mathbf{i}\alpha ct}$$

Case 2:

$$(\boldsymbol{\Psi} - \psi_0) = \pm\frac{1}{r}\left(e^{q_0+(\pm\mathbf{j}\pm\mathbf{k})\frac{\alpha}{\sqrt{2}}r} - e^{q_0-(\pm\mathbf{j}\pm\mathbf{k})\frac{\alpha}{\sqrt{2}}r}\right)e^{-\mathbf{i}\alpha ct}$$

Case 3:

$$(\boldsymbol{\Psi} - \psi_0) = \pm\frac{1}{r}\left(e^{q_0+(\pm\mathbf{i}\pm\mathbf{j}\pm\mathbf{k})\frac{\alpha}{\sqrt{3}}r} - e^{q_0-(\pm\mathbf{i}\pm\mathbf{j}\pm\mathbf{k})\frac{\alpha}{\sqrt{3}}r}\right)e^{-\mathbf{i}\alpha ct}$$

1-D Solution:

This is a typical solution presented for this type of differential equation. There are two wave functions represented here. These are designated by the ± sign in front of the 1/r. Wolff[4] presents a real solution very similar to this and states his belief that it represents the electron / positron. Wolff's solution satisfies the classical wave equation but not the Schrödinger Wave Equation.

2-D Solution:

The next wave function presented is a 2-D vibration. There is a trap here. Take another look at Equation 14. For the purpose of the definition of **Q**, the order of addition for **i**, **j**, and **k** makes no difference. But in Equation 14, the order of addition equates with the order of multiplication of the various Euler's Equations. Basically, the trap is as follows:

$$e^{+(\mathbf{j}+\mathbf{k})x} \neq e^{+(\mathbf{k}+\mathbf{j})x} ; e^{-(\mathbf{j}+\mathbf{k})x} \neq e^{-(\mathbf{k}+\mathbf{j})x}$$

See Appendix D for further details. What is most noteworthy from Appendix D is that it is possible to combine the above four relations so as to selectively eliminate groups of terms. It also allows for the use of the double angle or half angle identities from trigonometry. This may be of use later.

There are four wave functions presented here. One of the ± symbols in the exponential is redundant if the ± in front of the 1/r is kept.

3-D Solution:

The same type of transposition trap exists here. For this solution there are 8 wave functions. One of the ± symbols in the exponential is redundant if the ± in front of the 1/r is kept.

The Electron:

From the **General Ideas** section, the electron is a 1-D solution with no scalar term (i.e., case 1 with q_0 = 0). Therefore, it could be of the form:

$$(\mathbf{\Psi} - \psi_0) = \pm\frac{1}{r}\left(e^{+\mathbf{i}\alpha r} - e^{-\mathbf{i}\alpha r}\right)e^{-\mathbf{i}\alpha ct}$$

or it could simply be:

$$(\mathbf{\Psi} - \psi_0) = \pm\frac{1}{r}\sin(\alpha r)\, e^{-\mathbf{i}\alpha ct}$$

Both of the above were shown to be valid solutions in Appendix A.

The difference between the complex exponentials can be simplified as follows:

$$(\mathbf{\Psi} - \psi_0) = \pm\frac{2\mathbf{i}}{r}\sin(\alpha r)\, e^{-\mathbf{i}\alpha ct}$$

So, the only question regarding the electron is whether or not to include the 2**i** term. To meet the $6\pi^5$ requirement, it is easier not to use the 2**i**.

Equation 22:

$$(\mathbf{\Psi_E} - \psi_0) = \pm\frac{1}{r}\sin(\alpha r)\, e^{-\mathbf{i}\alpha ct}$$

This definition for the electron wave function produces a scalar value. This is consistent with the arguments of Part 1. Now consider the limit of Equation 22 as r tends to zero.

$$\lim_{r\to 0}(\mathbf{\Psi_E} - \psi_0) = \lim_{r\to 0}\left[\pm\frac{1}{r}\sin(\alpha r)\, e^{-\mathbf{i}\alpha ct}\right] = \pm\alpha e^{-\mathbf{i}\alpha ct} = \pm\frac{2m_E c}{\left(\frac{h}{2\pi}\right)} e^{-\mathbf{i}\alpha ct}$$

This can be rearranged as follows:

$$\pm\left(\frac{1}{2}\right)\left(\frac{h}{2\pi}\right)\left(\frac{1}{c}\right)(\mathbf{\Psi_E} - \psi_0)e^{+\mathbf{i}\alpha ct} = m_E$$

Now compare the above with Equation 1. It appears to be reasonable to infer the following:

Equation 22.1:

$$\gamma = \frac{1}{2}\frac{h}{2\pi}$$

This is very similar to the vacuum energy and links the wave function of the electron to vacuum energy. Viewed in this way, it seems obvious that the vacuum is a wave medium that is waiting to vibrate as an electron. This is consistent with the argument presented in Part 1. This interpretation is a direct assault upon both the Copenhagen Interpretation and the Uncertainty Principal and it adds a new twist to the Casimir Effect. The author's best guess is that the amplitude of ψ is determined by mass and the wave length of ψ is determined by momentum.

The Proton:

From the **General Ideas** section, the proton must have a scalar term in the exponential. Also, three different functions are to be added together to make the proton. That excludes case 3 since there is only one way to express it. That leaves case 1 and case 2 as possibilities.

The author will guess that the electron and proton should be synchronized. Therefore, the vector coefficients of the quaternions should be equal. That rejects case 2 and makes case 1 the best candidate. Making them synchronized probably gives the simplest solution.

The next question then is what is the value of q_0? It must scale the solution by a factor of π^4. Since it is in the exponential, it must be 4ln(π). If Equation 13.1 is combined with Equation 13.2, it is simple to show that:

$$\frac{\alpha}{\beta} = \frac{2\ln(\pi)}{\pi}$$

Therefore,

$$q_0 = 4\ln(\pi) = 2\pi\frac{\alpha}{\beta}$$

The three wave functions are then:

$$\psi_i = \pm\frac{1}{r}\left(e^{q_0+\mathbf{i}\alpha r} - e^{q_0-\mathbf{i}\alpha r}\right)e^{-\mathbf{i}\alpha ct} = \pm\frac{2\mathbf{i}}{r}e^{q_0}\sin(\alpha r)\,e^{-\mathbf{i}\alpha ct}$$

$$\psi_j = \pm\frac{1}{r}\left(e^{q_0+\mathbf{j}\alpha r} - e^{q_0-\mathbf{j}\alpha r}\right)e^{-\mathbf{i}\alpha ct} = \pm\frac{2\mathbf{j}}{r}e^{q_0}\sin(\alpha r)\,e^{-\mathbf{i}\alpha ct}$$

$$\psi_k = \pm\frac{1}{r}\left(e^{q_0+\mathbf{k}\alpha r} - e^{q_0-\mathbf{k}\alpha r}\right)e^{-\mathbf{i}\alpha ct} = \pm\frac{2\mathbf{k}}{r}e^{q_0}\sin(\alpha r)\,e^{-\mathbf{i}\alpha ct}$$

The proton is then:

Equation 23:

$$(\mathbf{\Psi_P} - \psi_0) = \pi(\mathbf{i} + \mathbf{j} + \mathbf{k})\left(\psi_i + \psi_j + \psi_k\right)$$

There is at least one possible problem with this definition of the proton. That is the de Broglie wavelength. Equation 23 uses the same value for α as the electron. Therefore, it would be expected to have the same de Broglie wavelength as the electron. But that is a contradiction of the conventional understanding from Schrödinger. It would be interesting to conduct the 2-slit experiment using protons rather than electrons - assuming that it is possible using protons.

Conclusion

Two wave functions are presented to represent the electron and proton at rest. The ratio between these functions is $6\pi^5$ as required by Part 1. Therefore, the hypothesis from Part 1 concerning movement through an aether is not rejected. An unforeseen result of this work is the ability to calculate the size of the proton by using Equation 13.4. The predicted value is within the accepted measured range for the size of the proton. Another unforeseen result is the ability to link vacuum energy with the wave function of the electron. The author believes that taken together, $m_p/m_e = 6\pi^5$ combined with the other two results constitute good circumstantial evidence in support of an aether.

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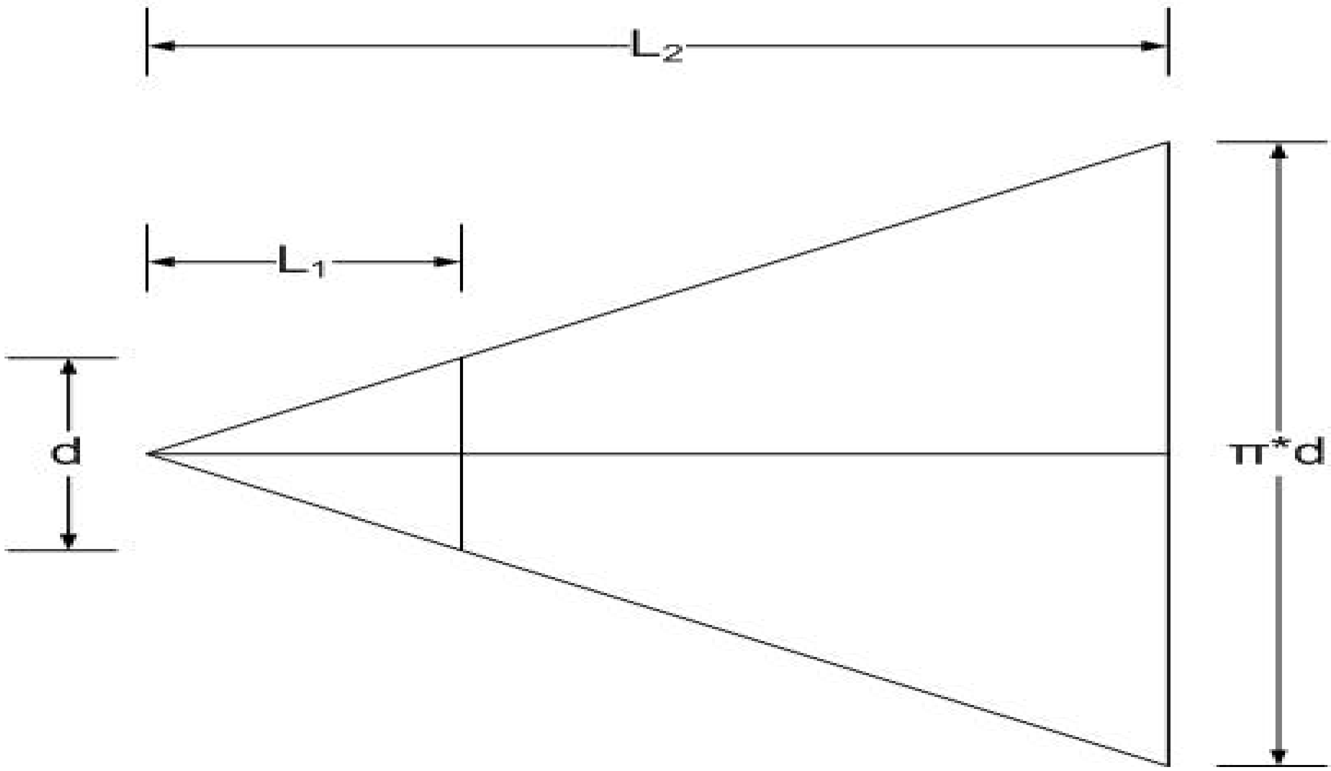
Figure 1



Appendix A

The objective of this Appendix is to confirm that Equation 13 of **General Ideas** in the main text satisfies the classical wave equation and the Schrödinger Wave Equation. Begin by restating Equation 13:

Equation A.1:

$$(\mathbf{\Psi}_E - \psi_0) = \sin\left(\frac{2}{\pi}\theta \ln(\pi)\right)$$

The wave equations to be satisfied are the classical wave equation and the Schrödinger Wave Equation. These are as follows:

Classical[5]:

$$\nabla^2\psi = \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}$$

Schrödinger[6]:

$$-\frac{\left(\frac{h}{2\pi}\right)^2}{2m}\nabla^2\psi = \mathbf{i}\left(\frac{h}{2\pi}\right)\frac{\partial\psi}{\partial t}$$

$$\nabla^2\psi = -\mathbf{i}\frac{2m}{\left(\frac{h}{2\pi}\right)}\frac{\partial\psi}{\partial t}$$

In the main text up to the point of Equation 13, there had been no mention or consideration of time. All that is considered is angle of rotation (θ). This is equated with space (i.e., x or r). The standard method of solving partial differential equations is separation of variables. Therefore, Equation A.1 is actually the X or R function used by that method. Since Equation A.1 uses θ, it will be re-written as follows for use by the separation of variables method:

Equation A.2:

$$X = \sin(\alpha x)$$

Separation of variables is then written as follows:

Equation A.3:

$$\psi = X(x)T(t)$$

The 1'st and 2'nd derivatives with respect to x are as follows:

1'st derivative:

$$\frac{d\psi}{dx} = \frac{dX}{dx}T$$

2'nd derivative:

$$\frac{d^2\psi}{dx^2} = \frac{d^2X}{dx^2}T$$

The 1'st and 2'nd derivatives with respect to t are as follows:

1'st derivative:

$$\frac{d\psi}{dt} = X\frac{dT}{dt}$$

2'nd derivative:

$$\frac{d^2\psi}{dt^2} = X\frac{d^2T}{dt^2}$$

Classical Wave Equation:

Substitution of the two 2'nd derivatives into the classical wave equation produces the following:

$$T\frac{d^2X}{dx^2} = \frac{1}{c^2}X\frac{d^2T}{dt^2}$$

Equation A.4

$$\frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{c^2}\frac{1}{T}\frac{d^2T}{dt^2} = -\alpha^2$$

Now it is simply a question of finding X and T to satisfy the above. The usual solutions here are sin(αx), cos(αx), and $e^{\pm i\alpha x}$ for space and sin(αct), cos(αct), and $e^{\pm i\alpha ct}$ for time. For both time and space, it is also possible to use the sum or the difference of the complex exponentials. Since Equation 13 from the main text uses the sine, it will also be used for X here.

$$X = \sin(\alpha x)$$

$$\frac{dX}{dx} = \alpha\cos(\alpha x)$$

$$\frac{d^2X}{dx^2} = -\alpha^2\sin(\alpha x) = -\alpha^2 X$$

Application of Equation A.4 produces the following:

$$\frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{\sin(\alpha x)}[-\alpha^2 \sin(\alpha x)] = -\alpha^2$$

It is also possible to use the difference between the complex exponentials as follows:

$$X = e^{+\mathbf{i}\alpha x} - e^{-\mathbf{i}\alpha x} = 2\mathbf{i}\sin(\alpha x)$$

$$\frac{dX}{dx} = \mathbf{i}\alpha e^{+\mathbf{i}\alpha x} + \mathbf{i}\alpha e^{-\mathbf{i}\alpha x}$$

$$\frac{d^2X}{dx^2} = \mathbf{i}^2\alpha^2 e^{+\mathbf{i}\alpha x} - \mathbf{i}^2\alpha^2 e^{-\mathbf{i}\alpha x} = \mathbf{i}^2\alpha^2\left(e^{+\mathbf{i}\alpha x} - e^{-\mathbf{i}\alpha x}\right) = -\alpha^2 X$$

Application of Equation A.4 produces the following:

$$\frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{X}(-\alpha^2 X) = -\alpha^2$$

This form can be used in conjunction with a complex form of T such that the **i** in the 2**i**sin(αx) combines with the **i** in complex T to produce a real solution. This form can also be used with the spherical form of the wave equation. Both forms for X will satisfy the Laplacian part of the classical wave equation.

The time function T is considered next. It must be a complex exponential in order to satisfy Schrödinger. Use both +**i**αct and -**i**αct.

Use +**i**αct.

$$T = e^{+\mathbf{i}\alpha ct}$$

$$\frac{dT}{dt} = \mathbf{i}\alpha c e^{+\mathbf{i}\alpha ct} = \mathbf{i}\alpha c T$$

$$\frac{d^2T}{dt^2} = \mathbf{i}^2\alpha^2 c^2 e^{+\mathbf{i}\alpha ct} = -\alpha^2 c^2 T$$

Application of Equation A.4 produces the following:

$$\frac{1}{c^2}\frac{1}{T}\frac{d^2T}{dt^2} = \frac{1}{c^2}\frac{1}{T}(-\alpha^2 c^2 T) = -\alpha^2$$

Use -**i**αct.

$$T = e^{-\mathbf{i}\alpha ct}$$

$$\frac{dT}{dt} = -\mathbf{i}\alpha c e^{-\mathbf{i}\alpha ct} = -\mathbf{i}\alpha c T$$

$$\frac{d^2T}{dt^2} = \mathbf{i}^2\alpha^2c^2e^{-\mathbf{i}\alpha ct} = -\alpha^2c^2T$$

Application of Equation A.4 produces the following:

$$\frac{1}{c^2}\frac{1}{T}\frac{d^2T}{dt^2} = \frac{1}{c^2}\frac{1}{T}(-\alpha^2c^2T) = -\alpha^2$$

Therefore, both ±**i**αct forms will satisfy the classical wave equation. It is also possible to use the difference between the complex exponentials since it simplifies to 2**i**sin(αct).

$$T = e^{+\mathbf{i}\alpha ct} - e^{-\mathbf{i}\alpha ct} = 2\mathbf{i}\sin(\alpha ct)$$

$$\frac{dT}{dt} = +\mathbf{i}\alpha ce^{+\mathbf{i}\alpha ct} + \mathbf{i}\alpha ce^{-\mathbf{i}\alpha ct} = 2\mathbf{i}\alpha c\cos(\alpha ct)$$

$$\frac{d^2T}{dt^2} = +\mathbf{i}^2\alpha^2c^2e^{+\mathbf{i}\alpha ct} - \mathbf{i}^2\alpha^2c^2e^{-\mathbf{i}\alpha ct} = -2\mathbf{i}\alpha^2c^2\sin(\alpha ct) = -\alpha^2c^2T$$

Application of Equation A.4 produces the following:

$$\frac{1}{c^2}\frac{1}{T}\frac{d^2T}{dt^2} = \frac{1}{c^2}\frac{1}{T}(-\alpha^2c^2T) = -\alpha^2$$

Therefore, the difference between the complex exponentials also will satisfy the classical wave equation.

Therefore, a wave function ψ that satisfies the classical wave equation can be produced from any combination of the following:

$$\psi(x,t) = X(x)T(t)$$

$$X(x) = \pm\sin(\alpha x)\,; X(x) = \pm\left(e^{+\mathbf{i}\alpha x} - e^{-\mathbf{i}\alpha x}\right) = \pm 2\mathbf{i}\sin(\alpha x)$$

$$T(t) = \pm e^{+\mathbf{i}\alpha ct}; T(t) = \pm e^{-\mathbf{i}\alpha ct}; T(t) = \pm\left(e^{+\mathbf{i}\alpha ct} - e^{-\mathbf{i}\alpha ct}\right) = \pm 2\mathbf{i}\sin(\alpha ct)$$

The next task is to apply separation of variables to the Schrödinger Wave Equation.

Schrödinger Wave Equation:

$$\nabla^2\psi = -\mathbf{i}\frac{2m}{\left(\frac{h}{2\pi}\right)}\frac{\partial\psi}{\partial t}$$

Application of separation of variables to the Schrödinger Wave Equation produces the following:

$$\frac{d^2X}{dx^2}T = -\mathbf{i}\frac{2m}{\left(\frac{h}{2\pi}\right)}X\frac{dT}{dt}$$

Equation A.5:

$$\frac{1}{X}\frac{d^2X}{dx^2} = -\mathbf{i}\frac{2m}{\left(\frac{h}{2\pi}\right)}\frac{1}{T}\frac{dT}{dt} = -\alpha^2$$

Define T as follows:

$$T = e^{-\mathbf{i}\alpha ct}$$

$$\frac{dT}{dt} = -\mathbf{i}\alpha ce^{-\mathbf{i}\alpha ct} = -\mathbf{i}\alpha cT$$

Application of Equation A.5 produces the following:

$$-\mathbf{i}\frac{2m}{\left(\frac{h}{2\pi}\right)}\frac{1}{T}\frac{dT}{dt} = -\mathbf{i}\frac{2m}{\left(\frac{h}{2\pi}\right)}\frac{1}{T}(-\mathbf{i}\alpha cT) = -\alpha^2$$

$$-\frac{2m}{\left(\frac{h}{2\pi}\right)}c = -\alpha$$

Equation A.6:

$$\frac{2mc}{\left(\frac{h}{2\pi}\right)} = \alpha$$

Since the units for Planck's Constant h are ($length^2$ x mass / time), the resulting units for α are ($length^{-1}$). This is consistent with α being in the exponential of both X and T as follows:

$$X = e^{\pm\mathbf{i}\alpha x}$$

and

$$T = e^{\pm\mathbf{i}\alpha ct}$$

Now compare Equations A.1, A.2, and A.6. For all of these relations to be true, the following must be true:

$$\alpha x = \frac{2mc}{\left(\frac{h}{2\pi}\right)}x = \frac{2}{\pi}\theta\ln\pi$$

Therefore, the following must be true:

Equation A.7:

$$\beta = \frac{\theta}{x} = \frac{\pi mc}{\left(\frac{h}{2\pi}\right)\ln\pi}$$

The author believes that Equation A.7 describes a fundamental property of space that is necessary for the classical wave equation and the Schrödinger Wave Equation to both be true at the same time.

Next define T using the positive complex exponential.

$$T = e^{+\mathbf{i}\alpha ct}$$

$$\frac{dT}{dt} = +\mathbf{i}\alpha c e^{+\mathbf{i}\alpha ct} = +\mathbf{i}\alpha cT$$

Application of Equation A.5 produces the following:

$$-\mathbf{i}\frac{2m}{\left(\frac{h}{2\pi}\right)}\frac{1}{T}\frac{dT}{dt} = -\mathbf{i}\frac{2m}{\left(\frac{h}{2\pi}\right)}\frac{1}{T}(+\mathbf{i}\alpha cT) = -\alpha^2$$

$$+\frac{2m}{\left(\frac{h}{2\pi}\right)}c = -\alpha$$

Therefore, for the +**i**αct form of T to be valid, α must be negative. This is redundant to simply using the negative **i**αct form of T. Therefore, only the -**i**αct form of T will be used.

Lastly, a wave function ψ that satisfies both the classical wave equation and Schrödinger can be produced from any combination of the following:

$$\psi(x,t) = X(x)T(t)$$

$$X(x) = \pm\sin(\alpha x)\,; X(x) = \pm\left(e^{+\mathbf{i}\alpha x} - e^{-\mathbf{i}\alpha x}\right) = \pm 2\mathbf{i}\sin(\alpha x)$$

$$T(t) = \pm e^{-\mathbf{i}\alpha ct}$$

Appendix B

Quaternions:

A quaternion **Q** is defined by Hamilton[5] as follows:

Equation B.1

$$\mathbf{Q} = q_0 + q_i\mathbf{i} + q_j\mathbf{j} + q_k\mathbf{k}$$

Hamilton[5] further made the following definitions:

Equation B.2

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$$

It is essential to understand that within Hamilton's system, the sequence of multiplication is important. Therefore, **ij** ≠ **ji** but rather **ij** = -**ji**. The identities within equation B.2 allow the following statements:

Equation B.3

$$\mathbf{i}(\mathbf{ijk}) = -\mathbf{i}\ therefore\ \mathbf{jk} = \mathbf{i}$$

Equation B.4

$$(\mathbf{ijk})\mathbf{k} = -\mathbf{k}\ therefore\ \mathbf{ij} = \mathbf{k}$$

Equation B.5

$$\mathbf{i}(\mathbf{ijk})\mathbf{k} = -\mathbf{ik}\ therefore\ \mathbf{j} = -\mathbf{ik}$$

Now consider a vector $r_0\mathbf{i}$ in the **i**-**j** plane. It is desired to rotate this vector counter-clockwise about the **k** axis to form a new vector of length r_0. The problem is constructed as follows:

$$r_0\mathbf{iQ} = r_0\mathbf{i}\cos\theta_k + r_0\mathbf{j}\sin\theta_k$$

$$r_0\mathbf{i}(q_0 + q_k\mathbf{k}) = r_0\mathbf{i}\cos(\theta_k) + r_0\mathbf{j}\sin(\theta_k)$$

In the quaternion above, q_1 and q_2 are zero because the rotation is only about the **k** axis. The r_0 term can be eliminated by dividing both sides of the equation by r. This gives the following:

$$q_0\mathbf{i} + q_k\mathbf{ik} = \mathbf{i}\cos(\theta_k) + \mathbf{j}\sin(\theta_k)$$

Since **ik** = -**j**, it follows that:

$$q_0\mathbf{i} - q_k\mathbf{j} = \mathbf{i}\cos(\theta_k) + \mathbf{j}\sin(\theta_k)$$

By inspection, $q_0 = \cos(\Theta_k)$ and $q_k = -\sin(\Theta_k)$. Substituting these values back into the rotation yields:

Equation B.6

$$\mathbf{i}(\cos(\theta_k) - \mathbf{k}\sin(\theta_k)) = \mathbf{i}\cos(\theta_k) + \mathbf{j}\sin(\theta_k)$$

If these steps are repeated for a clock-wise rotation about **k**, the result is as follows:

Equation B.7

$$\mathbf{i}(\cos(\theta_k) + \mathbf{k}\sin(\theta_k)) = \mathbf{i}\cos(\theta_k) - \mathbf{j}\sin(\theta_k)$$

Next consider clockwise rotation of the vector $r_0\mathbf{i}$ in the **i**-**k** plane about the **j** axis. The problem is stated as follows:

$$r_0\mathbf{i}\mathbf{Q} = r_0\mathbf{i}\cos(\theta_j) + r_0\mathbf{k}\sin(\theta_j)$$

$$r_0\mathbf{i}(q_0 + q_j\mathbf{j}) = r_0\mathbf{i}\cos(\theta_j) + r_0\mathbf{k}\sin(\theta_j)$$

Repeating the process described above produces:

$$q_0\mathbf{i} + q_j\mathbf{ij} = \mathbf{i}\cos(\theta_j) + \mathbf{k}\sin(\theta_j)$$

By inspection, $q_0 = \cos(\Theta_j)$ and since **ij** = **k**, $q_j = \sin(\Theta_j)$. Substitution into the rotation yields:

Equation B.8

$$\mathbf{i}(\cos(\theta_j) + \mathbf{j}\sin(\theta_j)) = \mathbf{i}\cos(\theta_j) + \mathbf{k}\sin(\theta_j)$$

Repeating this exercise for a counter-clockwise rotation produces:

Equation B.9

$$\mathbf{i}(\cos(\theta_j) - \mathbf{j}\sin(\theta_j)) = \mathbf{i}\cos(\theta_j) - \mathbf{k}\sin(\theta_j)$$

The four rotations presented above change the orientation of the starting vector $r_0\mathbf{i}$ but not its length. The quaternion problem for scaling is stated as follows:

$$r_0\mathbf{i}(q_0 + q_k\mathbf{k}) = r\mathbf{i}\cos(\theta_k) + r\mathbf{j}\sin(\theta_k)$$

The length r_0 on the right hand side has simply been replaced with a more general length r. Dividing both sides by r_0 produces:

$$\mathbf{i}(q_0 + q_k\mathbf{k}) = \frac{r}{r_0}[\mathbf{i}\cos(\theta_k) + \mathbf{j}\sin(\theta_k)]$$

So, the resulting coefficients in the quaternion are simply multiplied by the ratio r/r_0. This is simple enough provided that r_0 does not equal zero. Unfortunately, for the method described in the main text, r_0 does equal zero. Beware of division by zero. For the scaling function proposed in the main text, it appears that the r/r_0 ratio has a finite limit (possibly equal to one) as r_0 goes to zero. What is needed then is something akin to sin(x)/x since it has a limit at x = 0.

The scaling factor can be incorporated into the exponential of the rotation as follows: Let $f = r/r_0$. Write the ratio as the exponential of the natural log (the two operations cancel each other).

$$f = \frac{r}{r_0}$$

$$f = e^{\ln f}$$

Then multiply the exponential of Euler's Equation by the exponential form of f.

$$e^{\ln f} e^{\mathbf{i}\theta_i} = e^{\ln f + \mathbf{i}\theta_i}$$

It should be noted that the scaling portion of the exponential is not multiplied by the unit vector. This is an exponential of a quaternion!

Appendix C

This Appendix will work through the math to demonstrate that the equation provided for the generic quaternion based wave function satisfies both the classical wave equation and the Schrödinger Wave Equation.

Begin with the (+) version of the -**i**αct time form of the wave function:

Equation C.1

$$(\mathbf{\Psi} - \psi_0) = +\frac{1}{r}(e^{q_0+\mathbf{q}\alpha r} - e^{q_0-\mathbf{q}\alpha r})e^{-\mathbf{i}\alpha ct}$$

The wave equations to be satisfied are as follows:

Classical Wave Equation:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) = \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}$$

Schrödinger Wave Equation:

$$-\frac{\left(\frac{h}{2\pi}\right)^2}{2m}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) = \mathbf{i}\left(\frac{h}{2\pi}\right)\frac{\partial\psi}{\partial t}$$

The spherical form[10] of the Laplacian is used in both wave equations.

The required differentials are the 1'st and 2'nd time derivatives and the 1'st and 2'nd derivatives with respect to r.

Begin with r. Take the 1'st derivative of Equation C.1

$$\frac{\partial\psi}{\partial r} = \frac{1}{r}(\mathbf{q}\alpha e^{q_0+\mathbf{q}\alpha r} + \mathbf{q}\alpha e^{q_0-\mathbf{q}\alpha r})e^{-\mathbf{i}\alpha ct} - \frac{1}{r^2}(e^{q_0+\mathbf{q}\alpha r} - e^{q_0-\mathbf{q}\alpha r})e^{-\mathbf{i}\alpha ct}$$

$$\frac{\partial\psi}{\partial r} = \frac{1}{r}(\mathbf{q}\alpha e^{q_0+\mathbf{q}\alpha r} + \mathbf{q}\alpha e^{q_0-\mathbf{q}\alpha r})e^{-\mathbf{i}\alpha ct} - \frac{1}{r}\psi$$

Equation C.1.r.1

$$\frac{\partial\psi}{\partial r} = \frac{1}{r}\left[(\mathbf{q}\alpha e^{q_0+\mathbf{q}\alpha r} + \mathbf{q}\alpha e^{q_0-\mathbf{q}\alpha r})e^{-\mathbf{i}\alpha ct} - \psi\right]$$

Now take the 2'nd derivative.

$$\frac{\partial^2\psi}{\partial r^2} = \frac{1}{r}\left[\frac{\partial}{\partial r}(\mathbf{q}\alpha e^{q_0+\mathbf{q}\alpha r} + \mathbf{q}\alpha e^{q_0-\mathbf{q}\alpha r})e^{-\mathbf{i}\alpha ct} - \frac{\partial\psi}{\partial r}\right] - \frac{1}{r^2}\left[(\mathbf{q}\alpha e^{q_0+\mathbf{q}\alpha r} + \mathbf{q}\alpha e^{q_0-\mathbf{q}\alpha r})e^{-\mathbf{i}\alpha ct} - \psi\right]$$

$$\frac{\partial^2\psi}{\partial r^2} = \frac{1}{r}\left[\left(\mathbf{q^2}\alpha^2 e^{q_0+\mathbf{q}\alpha r} - \mathbf{q^2}\alpha^2 e^{q_0-\mathbf{q}\alpha r}\right)e^{-\mathbf{i}\alpha ct} - \frac{\partial\psi}{\partial r}\right] - \frac{1}{r}\frac{\partial\psi}{\partial r}$$

$$\frac{\partial^2\psi}{\partial r^2} = \frac{1}{r}\left[\mathbf{q^2}\alpha^2(e^{q_0+\mathbf{q}\alpha r} - e^{q_0-\mathbf{q}\alpha r})e^{-\mathbf{i}\alpha ct} - \frac{\partial\psi}{\partial r}\right] - \frac{1}{r}\frac{\partial\psi}{\partial r}$$

Equation C.1.r.2

$$\frac{\partial^2\psi}{\partial r^2} = \mathbf{q^2}\alpha^2\psi - \frac{2}{r}\frac{\partial\psi}{\partial r}$$

First, demonstrate that the classical wave equation is satisfied.

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) = \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}$$

$$\frac{1}{r^2}\left(2r\frac{\partial\psi}{\partial r} + r^2\frac{\partial^2\psi}{\partial r^2}\right) = \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}$$

$$\left(\frac{2}{r}\frac{\partial\psi}{\partial r} + \frac{\partial^2\psi}{\partial r^2}\right) = \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}$$

$$\left(\frac{2}{r}\frac{\partial\psi}{\partial r} + \mathbf{q^2}\alpha^2\psi - \frac{2}{r}\frac{\partial\psi}{\partial r}\right) = \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}$$

$$\left(\mathbf{q^2}\alpha^2\psi\right) = \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}$$

$$\mathbf{q^2}\alpha^2\psi = \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}$$

$$\mathbf{q^2}\alpha^2\psi = \frac{1}{c^2}(-\alpha^2c^2\psi)\,;(\textit{see time derivatives below})$$

$$\mathbf{q^2}\alpha^2\psi = -\alpha^2\psi$$

$$\mathbf{q^2} = -1$$

Therefore, the classical wave equation is satisfied provided the terms within the vector portion **q** satisfy the following:

$$-\left(q_i^2 + q_j^2 + q_k^2\right) = -1$$

q_i	q_j	q_k
±1	0	0
0	$\pm\sqrt{\frac{1}{2}}$	$\pm\sqrt{\frac{1}{2}}$
$\pm\sqrt{\frac{1}{3}}$	$\pm\sqrt{\frac{1}{3}}$	$\pm\sqrt{\frac{1}{3}}$

Table C-1

Table C-1 presents a few simple ways to specify **q** that satisfy the classical wave equation. The solutions are 1-D, 2-D, and 3-D.

The time derivatives are more simple. Take the 1'st time derivative.

$$\frac{\partial\psi}{\partial t} = \frac{1}{r}\left(e^{\mathbf{Q}\alpha r} - e^{\mathbf{Q}^*\alpha r}\right)\left(-\mathbf{i}\alpha c e^{-\mathbf{i}\alpha c t}\right)$$

Equation C.1.t.1

$$\frac{\partial\psi}{\partial t} = -\mathbf{i}\alpha c\psi$$

Take the 2'nd derivative.

$$\frac{\partial^2\psi}{\partial t^2} = -\mathbf{i}\alpha c\frac{\partial\psi}{\partial t} = +\mathbf{i}^2\alpha^2 c^2\psi$$

Equation C.1.t.2

$$\frac{\partial^2\psi}{\partial t^2} = -\alpha^2 c^2\psi$$

Repeat this exercise for the Schrödinger Wave Equation. The Laplacian portion is not changed from the classical version.

$$-\frac{\left(\frac{h}{2\pi}\right)^2}{2m}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) = \mathbf{i}\left(\frac{h}{2\pi}\right)\frac{\partial\psi}{\partial t}$$

$$-\frac{\left(\frac{h}{2\pi}\right)^2}{2m}\left(\mathbf{q}^2\alpha^2\psi\right) = \mathbf{i}\left(\frac{h}{2\pi}\right)\left(-\mathbf{i}\alpha c\psi\right)$$

$$\frac{\left(\frac{h}{2\pi}\right)}{2m}\left(\mathbf{q}^{2}\alpha^{2}\right) = \mathbf{i}^{2}(\alpha c)$$

$$\frac{\left(\mathbf{q}^{2}\alpha^{2}\right)}{\alpha} = -\frac{2mc}{\left(\frac{h}{2\pi}\right)}$$

To satisfy the classical wave equation, $\mathbf{q}^2 = -1$.

$$\frac{(-\alpha^{2})}{\alpha} = -\frac{2mc}{\left(\frac{h}{2\pi}\right)}$$

Equation C.1.1

$$\alpha = \frac{2mc}{\left(\frac{h}{2\pi}\right)}$$

Therefore, the Schrödinger Wave Equation and the classical wave equation will be satisfied for Equation C.1 provided α satisfies the above equation.

Now repeat the above for the +**i**αct time form. The r derivatives will be the same. Only the t derivatives will change.

Equation C.2

$$(\mathbf{\Psi} - \psi_0) = \frac{1}{r}\left(e^{\mathbf{Q}\alpha r} - e^{\mathbf{Q}^{*}\alpha r}\right)e^{+\mathbf{i}\alpha ct}$$

Take the 1'st time derivative.

$$\frac{\partial\psi}{\partial t} = \frac{1}{r}\left(e^{\mathbf{Q}\alpha r} - e^{\mathbf{Q}^{*}\alpha r}\right)\left(+\mathbf{i}\alpha c e^{+\mathbf{i}\alpha ct}\right)$$

Equation C.2.t.1

$$\frac{\partial\psi}{\partial t} = +\mathbf{i}\alpha c\psi$$

Take the 2'nd derivative.

$$\frac{\partial^{2}\psi}{\partial t^{2}} = +\mathbf{i}\alpha c\frac{\partial\psi}{\partial t} = +\mathbf{i}^{2}\alpha^{2}c^{2}\psi$$

Equation C.2.t.2

$$\frac{\partial^2 \psi}{\partial t^2} = -\alpha^2 c^2 \psi$$

Regarding Schrödinger, repeating the above exercise produces the following:

Equation C.2.1

$$\alpha = -\frac{2mc}{\left(\frac{h}{2\pi}\right)}$$

The thing to recognize regarding the time derivatives of the functions $T_1 = +\mathbf{i}\alpha ct$ and $T_2 = -\mathbf{i}\alpha ct$ is that the 2'nd derivatives are the same. Therefore, the classical wave equation is satisfied by either one. However, the 1'st derivatives are opposite to each other. Since Schrödinger uses the 1'st time derivative, only one of these time functions can be true for a given value of α unless m is allowed to have both positive and negative values. Since Schrödinger uses -**i**, only the -**i**αct exponential produces a valid solution.

It is also possible to satisfy both wave equations using the following:

Equation C.3:

$$(\mathbf{\Psi} - \psi_0) = +\frac{1}{r}\left(e^{\mathbf{Q}\alpha r} - e^{\mathbf{Q}^*\alpha r}\right)e^{-\mathbf{i}\alpha ct}$$

The difference between this generic solution and what results from Equation C.1 is that it is necessary for $\mathbf{Q}^2 = -1$ instead of $\mathbf{q}^2 = -1$. Therefore, q_0 must be considered as follows:

$$q_0^2 - q_i^2 - q_j^2 - q_k^2 = -1$$

The author believes that Equation C.1 is the better solution because it maintains q_0 as a constant scalar field.

Appendix D

Consider the following:

Part 1:

$$e^{+(\mathbf{j}+\mathbf{k})x} = \left[\cos(q_j x) + \mathbf{j}\sin(q_j x)\right]\left[\cos(q_k x) + \mathbf{k}\sin(q_k x)\right]$$

$$e^{+(\mathbf{j}+\mathbf{k})x} = \cos(q_j x)\cos(q_k x) + \mathbf{j}\sin(q_j x)\cos(q_k x) + \mathbf{k}\cos(q_j x)\sin(q_k x) + \mathbf{jk}\sin(q_j x)\sin(q_k x)$$

$$e^{+(\mathbf{j}+\mathbf{k})x} = \cos(q_j x)\cos(q_k x) + \mathbf{j}\sin(q_j x)\cos(q_k x) + \mathbf{k}\cos(q_j x)\sin(q_k x) + \mathbf{i}\sin(q_j x)\sin(q_k x)$$

Now transpose **j** and **k** in the exponential.

Part 2:

$$e^{+(\mathbf{k}+\mathbf{j})x} = \left[\cos(q_k x) + \mathbf{k}\sin(q_k x)\right]\left[\cos(q_j x) + \mathbf{j}\sin(q_j x)\right]$$

$$e^{+(\mathbf{k}+\mathbf{j})x} = \cos(q_k x)\cos(q_j x) + \mathbf{k}\sin(q_k x)\cos(q_j x) + \mathbf{j}\cos(q_k x)\sin(q_j x) + \mathbf{kj}\sin(q_k x)\sin(q_j x)$$

$$e^{+(\mathbf{k}+\mathbf{j})x} = \cos(q_k x)\cos(q_j x) + \mathbf{k}\sin(q_k x)\cos(q_j x) + \mathbf{j}\cos(q_k x)\sin(q_j x) - \mathbf{i}\sin(q_k x)\sin(q_j x)$$

The only difference between the two exponentials due to transposition is that +**i**$\sin(q_j x)\sin(q_k x)$ in one becomes -**i**$\sin(q_j x)\sin(q_k x)$ in the other.

The same holds true for the conjugates as follows:

Part 3:

$$e^{-(\mathbf{j}+\mathbf{k})x} = \left[\cos(q_j x) - \mathbf{j}\sin(q_j x)\right]\left[\cos(q_k x) - \mathbf{k}\sin(q_k x)\right]$$

$$e^{-(\mathbf{j}+\mathbf{k})x} = \cos(q_j x)\cos(q_k x) - \mathbf{j}\sin(q_j x)\cos(q_k x) - \mathbf{k}\cos(q_j x)\sin(q_k x) + \mathbf{jk}\sin(q_j x)\sin(q_k x)$$

$$e^{-(\mathbf{j}+\mathbf{k})x} = \cos(q_j x)\cos(q_k x) - \mathbf{j}\sin(q_j x)\cos(q_k x) - \mathbf{k}\cos(q_j x)\sin(q_k x) + \mathbf{i}\sin(q_j x)\sin(q_k x)$$

Now transpose **j** and **k** in the exponential.

Part 4:

$$e^{-(\mathbf{k}+\mathbf{j})x} = \left[\cos(q_k x) - \mathbf{k}\sin(q_k x)\right]\left[\cos(q_j x) - \mathbf{j}\sin(q_j x)\right]$$

$$e^{-(\mathbf{k}+\mathbf{j})x} = \cos(q_k x)\cos(q_j x) - \mathbf{k}\sin(q_k x)\cos(q_j x) - \mathbf{j}\cos(q_k x)\sin(q_j x) + \mathbf{kj}\sin(q_k x)\sin(q_j x)$$

$$e^{-(\mathbf{k}+\mathbf{j})x} = \cos(q_k x)\cos(q_j x) - \mathbf{k}\sin(q_k x)\cos(q_j x) - \mathbf{j}\cos(q_k x)\sin(q_j x) - \mathbf{i}\sin(q_k x)\sin(q_j x)$$

The difference between the conjugates pairs is that the sign of the cross terms changes. So, +**j**sin(q_jx)cos(q_kx) becomes -**j**sin(q_jx)cos(q_kx) and +**k**cos(q_jx)sin(q_kx) becomes -**k**cos(q_jx)sin(q_kx).

Next, combine the various forms so as to eliminate groups of terms.

Taking the sum of the four forms produces the following:

Equation D.1:

$$\left[e^{+(\mathbf{j}+\mathbf{k})x} + e^{-(\mathbf{j}+\mathbf{k})x}\right] + \left[e^{+(\mathbf{k}+\mathbf{j})x} + e^{-(\mathbf{k}+\mathbf{j})x}\right] = 4\cos(q_j x)\cos(q_k x)$$

Taking the sum of conjugate pairs produces the following:

Equation D.2:

$$e^{+(\mathbf{j}+\mathbf{k})x} + e^{-(\mathbf{j}+\mathbf{k})x} = 2\cos(q_j x)\cos(q_k x) + 2\mathbf{i}\sin(q_j x)\sin(q_k x)$$

Equation D.3:

$$e^{+(\mathbf{k}+\mathbf{j})x} + e^{-(\mathbf{k}+\mathbf{j})x} = 2\cos(q_k x)\cos(q_j x) - 2\mathbf{i}\sin(q_k x)\sin(q_j x)$$

Taking the difference between conjugate pairs produces the following:

Equation D.4:

$$e^{+(\mathbf{j}+\mathbf{k})x} - e^{-(\mathbf{j}+\mathbf{k})x} = 2\mathbf{j}\sin(q_j x)\cos(q_k x) + 2\mathbf{k}\cos(q_j x)\sin(q_k x)$$

Equation D.5:

$$e^{+(\mathbf{k}+\mathbf{j})x} - e^{-(\mathbf{k}+\mathbf{j})x} = 2\mathbf{k}\sin(q_k x)\cos(q_j x) + 2\mathbf{j}\cos(q_k x)\sin(q_j x)$$

Equation D.6:

$$e^{+(\mathbf{j}+\mathbf{k})x} - e^{-(\mathbf{j}+\mathbf{k})x} = e^{+(\mathbf{k}+\mathbf{j})x} - e^{-(\mathbf{k}+\mathbf{j})x}$$

A form composed of sines only can be produced from Equation D.2 and Equation D.3 as follows:

Equation D.7:

$$\left[e^{+(\mathbf{j}+\mathbf{k})x} + e^{-(\mathbf{j}+\mathbf{k})x}\right] - \left[e^{+(\mathbf{k}+\mathbf{j})x} + e^{-(\mathbf{k}+\mathbf{j})x}\right] = 4\mathbf{i}\sin\left(q_j x\right)\sin(q_k x)$$

There are many trigonometric identities[11] that might be useful here. The two that seem most promising relate to Equation D.1 and Equation D.7. These are as follows:

$$sin^2(u) = \frac{1 - \cos(2u)}{2}$$

$$cos^2(u) = \frac{1 + \cos(2u)}{2}$$

There is also a double angle identity as follows:

$$\sin(2u) = 2\sin(u)\cos(u)$$

The citation for these identities is http://sosmath.com/trig/Trig5/trig5/trig5.html.