Finding Shortest Hamiltonian Path is in P

Dhananjay P. Mehendale
Sir Parashurambhau College, Tilak Road, Pune 411030,
India

Abstract

The problem of finding shortest Hamiltonian path in a weighted complete graph belongs to the class of NP-Complete problems [1]. In this paper we will show that we can obtain shortest Hamiltonian path in a given weighted complete graph in polynomial time! We will be discussing a very simple but useful idea of applying certain chosen sequence of permutations (actually transpositions) on given weighted adjacency matrix corresponding to the complete graph, on *p* points say, under consideration. This simple and novel algorithm essentially consists of applying certain transpositions that will transform the weighted adjacency matrix in such a way that its vertices are now relabeled and in this relabeled weighted complete graph the algorithm terminates decisively in producing the shortest Hamiltonian path, and this shortest Hamiltonian path will be

$$1 \rightarrow 2 \rightarrow 3 \rightarrow \cdots \rightarrow j \rightarrow (j+1) \rightarrow \cdots \rightarrow (p-1) \rightarrow p$$

Introduction: Let *G* be a weighted complete graph with the vertex set $V(G)$ and edge set $E(G)$ respectively:

$$V(G) = \{v_1, v_2, \cdots, v_p\} \text{ and}$$
$$E(G) = \{e_1, e_2, \cdots, e_q\}$$

Let $A_G = [w_{ij}]_{p \times p}$ denotes the weighted adjacency matrix of *G*.

Note: Applying transposition (m, n) on A_G is essentially equivalent to interchanging rows as well as columns, m and n. That is replace m-th row in A_G by n-th row and vice versa and then in thus transformed matrix replace m-th column by n-th column and vice versa (order of these operations, i.e.

whether you interchange rows first and then interchange columns or you interchange columns first and then interchange rows, is immaterial as it produce same end result). Note that this transformation essentially produces a new weighted adjacency matrix that will result due to interchanging labels of vertices v_m, v_n in the original weighted complete graph.

Algorithm:

(1) If entry at position (1, 2) in the matrix, i.e. weight w_{12} is already smallest in the first row then proceed to step 2. Else, among the weights $w_{1j}, j = 2,3,..., p$, find minimum weight, say w_{1j_1}. Apply transposition $(2, j_1)$ on A_G, producing new weighted adjacency matrix, say A_{G_1}.

(2) If entry at position (2, 3) in the matrix, i.e. weight w_{23} is already smallest in the second row then proceed to step 3. Else, among the weights $w_{2j}, j = 3,4,..., p$, find minimum weight, say w_{2j_2}. Now apply transposition $(3, j_2)$ on A_{G_1}, producing new weighted adjacency matrix, say A_{G_2}.

(3) If entry at position (3, 4) in the matrix, i.e. weight w_{34} is already smallest in the third row then proceed to step 4. Else, among the weights $w_{3j}, j = 4,5,..., p$, find minimum weight, say w_{3j_3}. Now apply transposition $(4, j_3)$ on A_{G_2}, producing new weighted adjacency matrix, say A_{G_3}.

(4) Continue this procedure applying appropriate transpositions till we finally reach (*p*-2)-th row and among the weights $w_{(p-2)j}, j = (p-1), p$, find minimum weight, say $w_{(p-2)j_{(p-2)}}$. Now apply transposition $((p-1), j_{(p-2)})$ on $A_{G_{(p-3)}}$, producing new weighted adjacency matrix, say $A_{G_{(p-2)}}$.

(5) Find the sum of weights of edges in the Hamiltonian path

$$1 \to 2 \to 3 \to \cdots \to j \to (j+1) \to \cdots \to (p-1) \to p$$

Theorem: After carrying out algorithm 2.5.1 on given weighted complete graph the Hamiltonian path

$$1 \to 2 \to 3 \to \cdots \to j \to (j+1) \to \cdots \to (p-1) \to p$$

represents the shortest Hamiltonian path in the given (and conveniently relabeled) weighted complete graph.

Proof: The algorithm begins with application of permutation (transposition) which brings smallest weight entry in the first row at position (1, 2) in the weighted adjacency matrix. This is achieved by transposition of type $(2, j_1)$, where $j_1 > 2$. The algorithm then applies transposition which brings smallest weight entry in the second row at position (2, 3), in the transformed weighted adjacency matrix that results after applying transposition mentioned above. This is achieved by transposition of type $(3, j_2)$, where $j_2 > 3$. Note that because of its special form this second transposition doesn't affect the smallest entry achieved at position (1, 2) while bringing smallest entry (weight) in the second row at position (2, 3) by this second transposition! This story continues, i.e. the later applied transpositions doesn't affect the results of earlier transpositions because of the special choice of the transpositions and at end achieves smallest possible weights in the rows at positions on the **diagonal neighboring the principle diagonal**, i.e. at positions (1, 2), (2, 3), …., (p-1, p), of the evolved weighted adjacency matrix, evolved through the successive transpositions of specially chosen type. Note that this neighboring diagonal represents the weights on the Hamiltonian path

$$1 \to 2 \to 3 \to \cdots \to j \to (j+1) \to \cdots \to (p-1) \to p$$

□

Example: We consider following weighted adjacency matrix representing a weighted complete graph and find the shortest Hamiltonian path in its relabeled copy will be in the form

$$1 \to 2 \to 3 \to \cdots \to j \to (j+1) \to \cdots \to (p-1) \to p$$

Consider following weighted adjacency matrix in which entries are actually the weights of the corresponding edges:

$$\begin{bmatrix} 0 & 1 & 6 & 8 & 4 \\ 1 & 0 & 8 & 5 & 6 \\ 6 & 8 & 0 & 9 & 7 \\ 8 & 5 & 9 & 0 & 8 \\ 4 & 6 & 7 & 8 & 0 \end{bmatrix}$$

(1) Since entry at position (1, 2) is already smallest in the first row we proceed to next step.

(2) Since entry in position (2, 4) = 5 is smallest in second row we apply transposition (3, 4) on the above matrix that results into matrix

$$\begin{bmatrix} 0 & 1 & 8 & 6 & 4 \\ 1 & 0 & 5 & 8 & 6 \\ 8 & 5 & 0 & 9 & 8 \\ 6 & 8 & 9 & 0 & 7 \\ 4 & 6 & 8 & 7 & 0 \end{bmatrix}$$

(3) Since entry in position (3, 5) = 8 is smallest in third row we apply transposition (4, 5) on the above matrix that results into matrix

$$\begin{bmatrix} 0 & 1 & 8 & 4 & 6 \\ 1 & 0 & 5 & 6 & 8 \\ 8 & 5 & 0 & 8 & 9 \\ 4 & 6 & 8 & 0 & 7 \\ 6 & 8 & 9 & 7 & 0 \end{bmatrix}$$

Clearly, in this transformed weighted adjacency matrix the Hamiltonian path

$$1 \to 2 \to 3 \to 4 \to 5$$

will be the shortest and has total weight $\sum_{i=1}^{4} w_{i,(i+1)} = 21$

Conclusion: It is clear to see that this algorithm requires checking at most (p-2) rows of gradually decreasing lengths p, (p-1), (p-2), ….,etc for finding the minimum entry in these rows. The algorithm further needs at most (p-2) transposition operations to be carried out on the weighted adjacency matrix under consideration. The algorithm is clearly in P, i.e. of polynomial time complexity!

References

1. Christos H. Papadimitriou, Computational Complexity, Page 190, Addison-Wesley Publishing Company, Inc.,1994.