# Mathematical derivation of the fine structure constant from fundamental properties of natural numbers

Otto G. Piringer

FABES Forschungs-GmbH

e-mail: otto.piringer@fabes-online.de

## Abstract

Recent publications discussed a possible change with time of Sommerfeld's fine structure constant $\alpha$, in which several of the fundamental constants of Nature are combined. The problem of a changing nature of $\alpha$ raises the question whether its value is ultimately a result of chance or reveals an objective law of nature. If the value of the fine structure constant is independent of human reason, a derivation of it may be possible from basic numbers, like $e$ and $\pi$, which appear in the logical development of mathematics [1].

In the following investigation a pure mathematical derivation of the fine structure constant is described, starting from a fundamental property of natural numbers. The constant $\alpha$ results as a limit value in an algorithm with exponential structures where the value $w = \exp(2\pi/\mathrm{e})$ plays a decisive role.

## 1. Introduction

The dimensionless electromagnetic coupling constant $\alpha = q_e^2 / 2\varepsilon_0 hc$ with the approximate value 1/137 combines the electric charge $q_e$ and electric-permittivity constant $\varepsilon_0$ in vacuum, with Planck's constant $h$ and the velocity of light $c$. If such a fundamental dimensionless constant of Nature can be obtained by numerical calculation it may be possible from the fundamental relations between natural numbers. A characteristic feature of stable elementary particles of matter is their uniformity. This uniformity can be regarded as a property of Nature resulted by following mathematical relations between natural numbers into directions of extreme values. The guiding principle in this investigation was the search for a simple, fundamental property of natural numbers which supports this interpretation and allows further correlations with material systems.

The basic property used is the relation $g \le m$ between the arithmetical mean $m = (x_1+x_2+...+x_n)/n$ and the geometrical mean $g = (x_1 \cdot x_2 \cdot ... x_n)^{1/n}$ between any number $n$ of positive quantities, denoted by $x_1, x_2,...x_n$. The general theorem states that the extreme $g=m$ is valid only if all the $x_i$ are equal, $x_1 = x_2 = ... = x_n$. This property is used as starting point.

But from the extreme value $g = m$ further relations with new extremes follow immediately. If a natural number $n$ can be written in different ways as a sum of $i$ identical prime numbers $p$, then the corresponding product of the primes, $p^i$, has a maximum for the number $p = 3$ as seen in the following example with $n = 30$:

| $p$ | $i$ | $n = i \times p$ | $p^i$ |
|---|---|---|---|
| 2 | 15 | $30 = 15 \times 2$ | $2^{15} = 32768 = 2^{n/2}$ |
| 3 | 10 | $30 = 10 \times 3$ | $3^{10} = 59049 = 3^{n/3}$ |
| 5 | 6 | $30 = 6 \times 5$ | $5^6 = 15625 = 5^{n/5}$ |

This property can be generalized for any number and a function

$$y = x^{q/x} \tag{1}$$

with the positive real numbers $x, q \in R^+$ and a maximum value $y_{max} = e^{q/e}$ for $x = e$ results.

In Figure 1 the graph of the function (1) with $q = 2\pi$ is shown. The number $2\pi$ has been selected because it appears as a proportional constant in all relations concerning periodical phenomena and objects with spherical symmetry. The maximum value, $e^{2\pi/e} = w = 10.089\ldots$, plays a fundamental role in the following investigation.

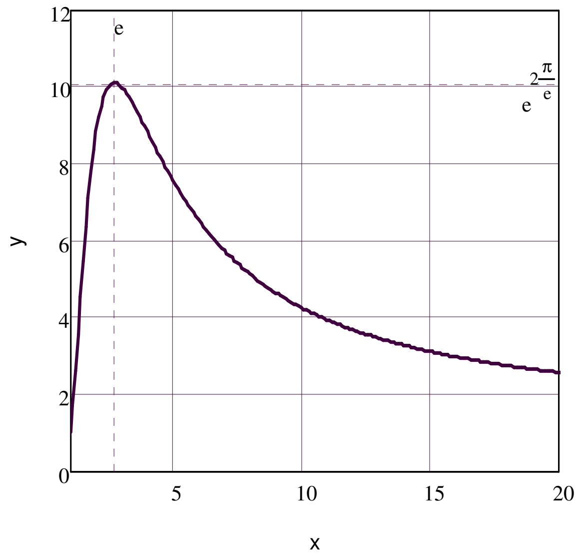


Figure 1. Graph of the function $y = x^{2\pi/x}$

The maximum value, $e^{2\pi/e} = w$, appears also as the limit value of the following power sequence:

$$w_{i,n} = \left(1 + \frac{2\pi}{i}\right)^{\frac{i}{(1+1/n)^n}} \tag{2}$$

with lim $w_{i,n} = w_{i,e}$ if $n \to \infty$ and lim $w_{i,e} = w$ if $i \to \infty$.

The sequence $w_{i,n}$ and its particular form $w_{i,e}$ are denoted as interaction functions. These functions can be used for modeling of emergent properties in multi-particle systems.

## 2. Correlation of the limit value $w$ with the fine structure constant

The number $e^{1/e}$ obtained from (1) for $x = e$ and $q = 1$ can be presented as resulting formally in two steps: $y_1 = e^q = e^1$ and $y_2 = exp(q/e^q) = exp(1/e^1) = exp(1/y_1) = e^{1/e}$. By proceeding in this way an algorithm is produced which provides an oscillating sequence, $y_n$:

$y_n$

$y_1 = exp(q)$

$y_2 = exp(q/y_1)$

$y_3 = exp(q/y_2)$ (3)

……………..

$y_i = \exp(q/y_{i-1})$.

Starting with $q = 1$, one monotone decreasing sequence $y^I_{2n-1}$ (Figure 2, trace 1) and one monotone increasing sequence $y^{II}_{2n}$ (Figure 2, trace 2) result, which reach asymptotically the same limit value (Figure 2 trace 3), $y^I_\infty = y^{II}_\infty = y_\infty(1) = 1.763222834351897... >$ $e^{1/e}$ = 1.44466…

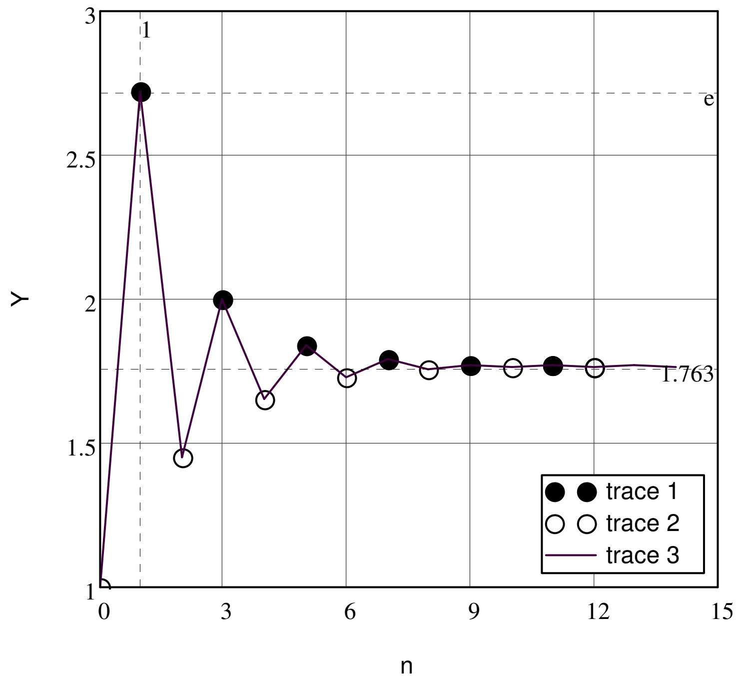


Figure 2. Graph (trace 3) of the oscillating sequence $y_n$ in (3) for $q$ =1: $y^I_{2n-1}$ (trace 1) and $y^{II}_{2n}$ (trace 2) with the same limit value $y^I_\infty = y^{II}_\infty = y_\infty(1) = 1.763...$

The algorithm (3) based on the exponential function is used as backbone in this investigation with $y_\infty(1) = 1.763222834351897...$ as the first corner point.

The limit values $y_\infty(q)$ calculated with the algorithm (3) as functions of $q \geq 0$ are shown in Figure 3.

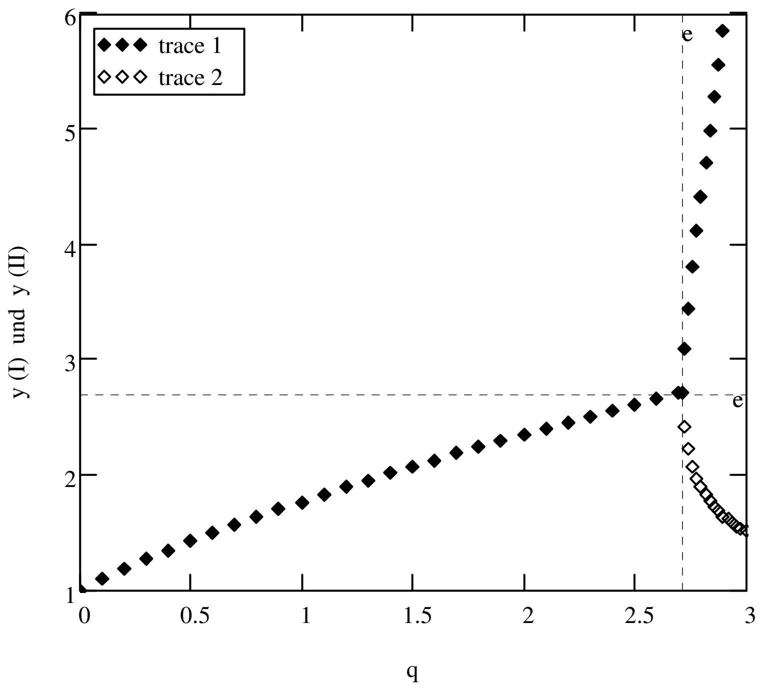


Figure 3. Graph of the limit values *y(I)* = $y_\infty^I(q)$ (trace 1) and *y(II)* = $y_\infty^{II}(q)$ (trace 2).

In this figure two domains with remarkable different properties appear. In a first domain with $0 \leq q \leq e$ a maximal limit value $y_\infty(e) = e$ for $q = e$ results asymptotically. In a second domain with $q > e$ two limit values, $\lim(y_{2n-1}) = y_\infty^I(q) > \lim(y_{2n}) = y_\infty^{II}(q)$, with $Y_\infty(q) = y_\infty^I(q) / y_\infty^{II}(q)$ are obtained with the algorithm (3).

A special role plays the result obtained with the algorithm (3) if $q = w$. The limit values $y_\infty^I(w) =$ 23976.90006232883…> $y_\infty^{II}(w)$ = 1.000420872319414….are obtained with their ratio $Y_\infty(w) = y_\infty^I(w) / y_\infty^{II}(w)$ = 23966.81309411294… The value $Y_\infty(w)$ is the second corner point in this investigation.

Of special interest for the following relations is the limit value

$$Y_\infty(1, w) = \exp\left(\frac{y_\infty(1)}{Y_\infty(w)}\right) = 1.000073572055. \tag{4}$$

Figure 3 illustrates a model of collective self-organization, starting with natural numbers, with the result of limit values with exponential structures. Remarkable is the limit $y_\infty(e) = e$ obtained with $q = e$ as starting value.

From three consecutive steps which form one cycle in the algorithm (3), $y_i$, $y_{i+1}$ and $y_{i+2}$, and with $q = e$ the following relation is obtained for $i >> 1$:

$$y_{i+2} = \exp\left(\frac{q}{y_{i+1}}\right) = \exp\left(\frac{q}{\exp\left(\frac{q}{y_i}\right)}\right) = \exp\left(\frac{e}{\exp\left(\frac{e}{y_\infty(e)}\right)}\right) = e \tag{5}$$

with $y_\infty(e) = e$.

If the structure of equation (5) with the limit $e$ is maintained as an invariant postulate, but including also the cases $Y_\infty(q) > y_\infty(e) = e$ from the second domain ($q > e$), an enlargement of equation (5) can be used by introducing the term $A \cdot B$, with $B = Y_\infty(q) > e$, instead of $y_\infty(e) = e$:

$$\exp\left(\frac{e \cdot (A \cdot B)}{\exp\left(\frac{e \cdot (A \cdot B)}{Y_\infty(q)}\right)}\right) = \exp\left(\frac{e \cdot (A \cdot B)}{\exp(e \cdot A)}\right) = e \ . \tag{6}$$

The structure of equation (5) resulting with the algorithm (3) with $q = e$ and the enlargement (6) with the invariant limit value $e$ form the third corner point of this investigation.

Whereas $A = 1/e$ results with $B = y_\infty(e) = e$ and equation (5) is obtained again, Figure (4) shows the graph of $B = e^{e \cdot A} / e \cdot A$ in the environment of $B = w^2$ in order to fulfill equation (6). From this figure a relation between the number $w^2$ and $\alpha/2$ is suggested. Due to this connection, $A = A^*/2$ and $B = w^2$ are introduced into equation (6):

$$\exp\left(\frac{e\cdot\left(\frac{A^*}{2}\cdot w^2\right)}{\exp\left(e\cdot\frac{A^*}{2}\right)}\right)=e\,. \tag{7}$$

Equation (7) is fulfilled for $A^*$ = 1/136.98… This number exceeds the value $\alpha = 7.2973525698\cdot 10^{-3} = 1/137.035999074$ by $4.1\cdot 10^{-4}$.

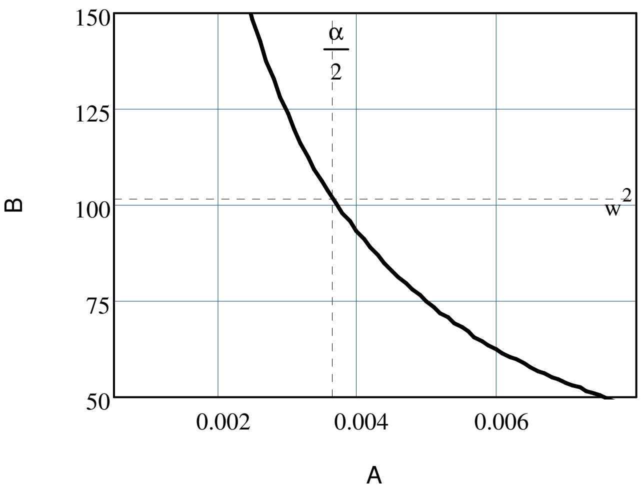


Figure 4. Graph of $B = e^{e\cdot A}/e\cdot A$ in the environment of $B = w^2$.

In the next step a connection to the limit value $Y_\infty(w)$ is realized. Enlargement of $Y_\infty(w)$ with the exponent $2\pi/e$ from $w$ gives:

$$Y_\infty(w)=\frac{2\cdot\pi}{e}\cdot\left(\frac{Y_\infty(w)}{\frac{2\cdot\pi}{e}}\right). \tag{8}$$

Instead of $B = w^2$ = 101.789…and $A^*$ the numbers resulting from the limit value $Y_\infty(w)$ and $2\pi/e$:

$$B_1=\left(\frac{Y_\infty(w)}{2\cdot\pi/e}\right)^{\frac{1}{2}}=101.826... \tag{9}$$

and

$$A_1 = \frac{2\cdot\pi\cdot a_0}{e}\cdot\left(\frac{Y_\infty(w)}{2\cdot\pi/e}\right)^{\frac{1}{2}} \tag{10}$$

are introduced into equation (7). In this way the new equation (11):

$$\exp\left(\frac{\frac{e\cdot A_1}{2}\cdot B_1}{\exp\left(\frac{e\cdot A_1}{2}\right)}\right) = e \tag{11}$$

is fulfilled for $x = 3.100507352...\cdot 10^{-5}$. The corresponding value $A_1$ = 1/137.03118… exceeds $\alpha$ by $3.5\cdot 10^{-5}$.

The result obtained at this stage of the investigation already demonstrates the existence of a deeper connection between the fine structure constant $\alpha$ and the number *w* in combination with the invariant structure (6) where the first corner point $Y_\infty(w)$ has been used.

The number $a_0$ in (10) is related to $Y_\infty(w)$. The term $2\pi a_0/e$ can be regarded as analogous to the exponent $2\pi/e$ in $w$ = $\exp(2\cdot\pi/e)$ and gives the value $\exp(2\cdot\pi\cdot a_0/e)$ = 1.000071669….The corresponding limit value for this number is $Y_\infty(1,w)$ defined in (4). This limit value is taken into account with the following enlargement, where instead of $A_1$ and $B_1$ in (11), the numbers

$$a = \frac{2\cdot\pi\cdot a_0}{e}\cdot\left(\frac{Y_\infty(w)\cdot Y_\infty(1,w)}{(2\cdot\pi/e)}\right)^{\frac{1}{2}} = \frac{2\cdot\pi\cdot a_0}{e}\cdot b^* \tag{12}$$

and

$$b = \left(\frac{Y_\infty(w)\cdot Y_\infty(1,w)}{(2\cdot\pi/e)}\right)^{\frac{1}{2}}\cdot\frac{\exp\left(\frac{2\cdot\pi\cdot a_0}{e}\right)}{Y_\infty(1,w)} = b^*\cdot\frac{\exp\left(\frac{2\cdot\pi\cdot a_0}{e}\right)}{Y_\infty(1,w)} \tag{13}$$

are used. With this enlargement the number $\exp(2\cdot\pi\cdot a_0/e)$ resulting from $2\pi a_0/e$ is related to the limit value (4). With *a* and *b* instead of $A_1$ and $B_1$ the new equation (14) results:

$$\exp\left(\frac{\frac{e\cdot a}{2}\cdot b}{\exp\left(\frac{e\cdot a}{2}\right)}\right)=e\,. \tag{14}$$

This equation is fulfilled for $a_0$ = 3.1002840896...$\cdot 10^{-5}$ and gives $a$ = 7.29735215...$\cdot 10^{-3}$ and $1/a$ = 137.0360069…, respectively. This $a$-value is by $-5.2\times 10^{-8}$ smaller than $\alpha$. This difference is within the order of magnitude in which possible variations of the fine-structure constant with time are discussed [2]. It is believed that $\alpha$ has been smaller in the past.

The relation $a_0 = e\cdot a/2\cdot\pi\cdot b^*$ from (12) in combination with equation (14) allows the determination of the number $a$ as the limit value of the following sequence:

$$a_{n+1}=\frac{2}{e}\cdot\left(\frac{Y_\infty(w)\cdot Y_\infty(1,w)}{2\cdot\pi/e}\right)^{-1/2}\cdot\frac{Y_\infty(1,w)\cdot\exp(a_n\cdot e/2)}{\exp\left(a_n\cdot\left(\frac{Y_\infty(w)\cdot Y_\infty(1,w)}{2\cdot\pi/e}\right)^{-1/2}\right)}\,. \tag{15}$$

With the starting value $a_1=(2/e)\cdot(Y_\infty(w)\cdot Y_\infty(1,w)\cdot e/2\pi)^{-1/2}=1/138.4...$ the final value of $a$ results by iteration $a$ = 1/137.0360069….
With the above procedure the fine structure constant is obtained from the numbers 2, $e$, $\pi$ and the limit values $Y_\infty(w)$ and $Y_\infty(1,w)$ which also result from these basic numbers.

### 3. Conclusions

The backbone in this investigation is the algorithm (3) in combination with the two numbers $e^{1/e}$ and $e^{2\pi/e}$ = $w$, which result as maximum values of the function (1) for $q$ = 1 and $q$ = $2\pi$.
The decisive step is the introduction of the limit values $y_\infty(1)$, $Y_\infty(w)$ and $Y_\infty(1,w)$ into equation (5). From this invariant mathematical structure the value of $a$ for the fine structure constant is obtained by iteration from equation (15) as a limit value.
Now the question is raised if the constant $b$ can also be connected with fundamental constants of Nature. A positive answer may be regarded as a confirmation for the correctness of the way followed above for the calculation of the fine structure constant.

The constant $b$ contains in its definition in (13) the square roots of $Y_\infty(w)$ and consequently both values $\pm b$ may be considered. The square root $Y_\infty(w)^{1/2}$ used for $a$ is positive because the constant $a$ is also positive. But depending on the sign of $b$ in (13) equation (14) can be written as:

$$\left(e^{\pm b}\right)^{\frac{e\cdot a}{2}\cdot e^{\frac{-e\cdot a}{2}}} = e^{\pm 1} \tag{16}$$

with $e^{+b} = 1.67647037\cdot 10^{44}$ and $e^{-b} = 5.96491303\cdot 10^{-45}$.

The negative value of $b$ suggests a connection of it to a further dimensionless constant of nature, the coupling constant for the gravity interaction:

$$\alpha_G = G\cdot m_e^2\cdot\frac{2\cdot\pi}{h\cdot c} = 1.7516885\cdot 10^{-45}\,. \tag{17}$$

With $G$ the constant of gravity and with $m_e$ the mass of the electron are denoted, respectively. The ratio of gravity to electromagnetic coupling gives the following dimensionless number:

$$\frac{\alpha_G}{\alpha} = 2.40044385\cdot 10^{-43} = \frac{1}{4.165896\cdot 10^{42}}\,. \tag{18}$$

These numbers express the interactions between the lightest elementary particles with no internal structure but with known masses [3].

A further combination of fundamental constants of nature connects the Fermi-constant with the mass of W-Bosons to the dimensionless number:

$$M_W^2\cdot c^4 = \frac{4\cdot\pi\cdot\alpha}{8\cdot\sin^2\Theta_W}\cdot\frac{\sqrt{2}\left(\frac{h\cdot c}{2\pi}\right)^3}{G_F}\,. \tag{19}$$

The number $\Theta_W \cong 28.7°$ represents the Weinberg-angle with the $\sin^2(\Theta_W) \cong 0.2306$ which relates the electric charge with the weak charge. The weak interaction is a function of the energy and depending of the method used for its measurement different values of $\sin^2(\Theta_W)$ between 0.2223 and 0.2314 are obtained [4].

The constant $b$ can now be correlated with a combination of $\alpha_G$, $\alpha$, and (19) into a dimensionless number:

$$\beta = \frac{1}{2} \cdot \frac{4 \cdot \pi \cdot \alpha}{8 \cdot \sin^2 \Theta_W} \cdot \frac{G \cdot m_e^2}{\alpha} \cdot \frac{2 \cdot \pi}{h \cdot c} = \frac{1}{2} \cdot \frac{\pi \cdot G \cdot m_e^2}{2 \cdot \sin^2 \Theta_W} \cdot \frac{2\pi}{h \cdot c} = 5.964913... \cdot 10^{-45} \tag{20}$$

with the following values of the natural constants [5]:
Planck's constant $h =$ $6.62606957 \cdot 10^{-34}$ Js,
velocity of light $c =$ $2.99792458 \cdot 10^{8}$ $\mathrm{ms^{-1}}$,
mass of the electron $m_e =$ $9.10938291 \cdot 10^{-31}$ kg,
constant of gravitation $G =$ $6.67384 \cdot 10^{-11}\,\mathrm{m^3kg^{-1}s^{-2}}$,
fine structure constant $\alpha = 7.2973525698 \cdot 10^{-3} = 1/137.035999074$ by $4.1 \cdot 10^{-4}$.
The Weinberg angle is used as a free parameter with the value $\sin^2(\Theta_W) = 0.2306442$, within the above mentioned limits. With $\alpha$ and $\beta$, as well as with $a$ and $b$ from (16), the following expression is obtained finally:

$$\ln(\beta) \cdot \frac{e \cdot \alpha}{2} \cdot e^{\frac{-e \cdot \alpha}{2}} = -1 = -b \cdot \frac{e \cdot a}{2} \cdot e^{\frac{-e \cdot a}{2}} = e^{\pi \cdot i} . \tag{21}$$

The left side of this equation contains all fundamental constants of Nature necessary for characterization of the four fundamental interactions occurring in stable matter. The limit value $e$ is considered here as the dimensionless coupling constant $\alpha_S$ for the strong interaction in stable nucleons. The right side represents the fundamental relation between the two transcendental numbers $\pi$ and e. Equation (21) can be regarded as the bridge between fundamental constants of Nature and a fundamental mathematical relation. This result supports the assumption of general validity of dimensionless constants of Nature.

It is well known that the coupling constant of strong interaction and the coupling constant for weak interaction are in fact not constants, due to their "running" character [4]. Whereas at high energies $\alpha_S < 1$, an increase above 1 occurs at low energy values, as the case in stable matter. For this stable state one single value for interaction is needed.

If only the first domain ($q \leq e$) is considered, than equation (14) reduces to equation (5) with only one limit value, the number $e$. This situation can be regarded as the limit where the four interactions are asymptotically unified in the strong interaction.

## References


[1]. Rosenthal-Schneider I., *Reality and Scientific Truth: Discussions with Einstein, von Laue and Planck.* Wayne State University Press, Detroit (1980).

[2]. Barrow J. D., *The Constants of Nature. From Alpha to Omega.* Jonathan Cape, London 2002.

[3]. Feynman R., *The Feynman Lectures on Physics.* Vol.1. Addison-Wesley Publishing Company, 1963.

[4]. Griffith, D., Introduction to elementary particles, Second, Revised Edition. WILEY-VCH, 2008.

[5]. CODATA internationally recommended values of the fundamental physical constants, (2010).


München, 10.12.2012.