Can Differentiable Description of Physical Reality Be Considered Complete?

Wang Xiong *

Abstract

How to relate the physical *real* reality with the logical *true* abstract mathematics concepts is nothing but pure postulate. The most basic postulates of physics are by using what kind of mathematics to describe the most fundamental concepts of physics. Main point of relativity theories is to remove incorrect and simplify the assumptions about the nature of space-time. There are plentiful bonus of doing so, for example gravity emerges as natural consequence of curvature of spacetime. We argue that the Einstein version of general relativity is not complete, since it can't explain quantum phenomenon. If we want to reconcile quantum, we should give up one implicit assumption we tend to forget: the differentiability. What would be the benefits of these changes? It has many surprising consequences. We show that the weird uncertainty principle and non-commutativity become straightforward in the circumstances of non-differentiable functions. It's just the result of the divergence of usual definition of *velocity*. All weirdness of quantum mechanics are due to we are trying to making sense of nonsense. Finally, we proposed a complete relativity theory in which the spacetime are non-differentiable manifold, and physical law takes the same mathematical form in all coordinate systems, under arbitrary differentiable or non-differentiable coordinate transformations. Quantum phenomenon emerges as natural consequence of non-differentiability of spacetime.

*Email:wangxiong8686@gmail.com

Contents

1 What are the most basic postulates of physics?

Physics is the study of matter and its motion through space and time. So the fundamental concepts of physics are *matter*, *space-time*, and *motion*.

The fundamental questions of physics are: What is the physical world made of? What is the structure of space-time? How do things change through space-time?

Mathematics is considered as the language of science. It's believed that god created the universe according to precise, perfect and elegant math. It's gradually become a firm belief that natural law is written by elegant mathematics, and we gain many benefits from this belief. Some perfect mathematical theory can bring up new understanding or even new research fields for physics. Mathematics becomes the key of understanding the physics.

But how to relate the *real* physical reality with the logical *true* abstract mathematics concepts is nothing but pure postulate. Although sometimes the implicit assumptions are so obvious and with no doubt that we tend to forget we have postulated them.

In fact, the assumptions are the very basic foundations of our physics theory. Any theory needs to start from these foundations that we believe are not further questioned.

The most basic postulates of physics are by using what kind of mathematics to describe the most fundamental concepts of physics:

1. What is the physical world made of? Field or particle? What kind of field, tensor or spinor or even operator?

2. What is the structure of space-time? Flat or curved? 3+1D or more? Manifold? What kind of manifold, and why?

3. How to describe change through space-time? ODE or PDE? Linear or nonlinear? Is the classical calculus good enough?

2 Main point of relativity theories is about assumptions of space-time

Einstein stated that the theory of relativity belongs to the class of "principle-theories". The main point of relativity theories is to remove incorrect and simplify the assumptions about the nature of space-time.

2.1 Special relativity

Special theory of relativity applies to all inertial physical phenomena without gravity. Special relativity is based on two postulates which are contradictory in classical mechanics:

- The laws of physics are the same for all observers in uniform motion relative to one another (principle of relativity).
- The speed of light in a vacuum is the same for all observers, regardless of their relative motion or of the motion of the source of the light.

Before Einstein, space and time are only considered as the stage on which physical events play out, the various dynamical entities being the actors. Special relativity changes this picture only marginally, and spacetime is a 4D manifold M with a flat Lorentzian metric $\eta_{\mu\nu} = diag[1, -1, -1, -1]$. This tensor, $\eta_{\mu\nu}$, enters all physical equations, representing the determinant influence of the stage and of its metrical properties on the motion of anything. Absolute acceleration is deviation of the world line of a particle from the straight lines defined by $\eta_{\mu\nu}$.

What would be the benefits of these changes? It has many surprising consequences, see the table 1.

2.2 General relativity

The profound change comes with general relativity (GR). The central novelty of GR is that spacetime become a quite dynamical object, and the dynamic of spacetime is exactly the cause of some physical phenomenon, such as the gravity. The spacetime in GR is curved pseudo-Riemannian manifold with a metric tensor $g_{\mu\nu}$ which itself is dynamical and determined by Einstein field equation

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu}.$$

The development of general relativity began with the equivalence principle, under which the states of accelerated motion and being at rest in a gravitational field (for example when standing on the surface of the Earth) are physically identical.

A number of lines of reasoning led Einstein to make the following assumptions. These assumptions are sufficient to define a theory of gravity (though as we will see they do not define a *unique* theory).

- **The Principle of Equivalence**
  In any gravitational field we can choose a local freely falling inertial frame in which the laws of nature take the same form as in unaccelerated Cartesian coordinate systems in the absence of gravity.

- **The laws of Physics are covariant** This means the following:

  ○ We can find a transformation from a general coordinate system x^μ to a locally inertial system ξ^μ
  $$\frac{\partial \xi^\alpha}{\partial x^\mu}.$$
  (These transformations will be *non-linear* in general).

  ○ The equations of physics *must* agree with the laws of Special Relativity in the absence of gravitation.

  ○ Physical laws must have a *tensorial* form, so that they can be stated in a form that is invariant under a general coordinate transformation $x^\mu \to x'^\mu$.

- **Riemannian geometry** We need a mathematical description of non-Cartesian geometry for spaces described by a metric
  $$ds^2 = g_{\mu\nu} dx^\mu dx^\nu.$$

2.3 General relativity is not complete

The essential idea of relativity principle is that coordinates do not exist a priori in nature, but are only artifices used in describing nature, and hence should play no role in the formulation of fundamental physical laws.

Einstein's believe of the general relativity principle is that all physical law should takes the same mathematical form in all coordinate systems, for all observers in any motion relative to one another.

However in general relativity, general covariance is the invariance of the form of physical laws under arbitrary differentiable coordinate transformations, and is usually expressed in terms of tensor fields. But in fact, both the transformation and its inverse are unreasonablely assumed to be smooth, in the sense of being differentiable an arbitrary number of times. Only coordinate systems related through sufficiently differentiable transformations are considered. This has no physical reason! Only because mathematicians can't deal with general arbitrary coordinate transformations.

Since arbitrary differentiable coordinate transformations is really just a measure zero subset of arbitrary coordinate transformations, so the relativity principle in version of Einstein's general relativity is not really fulfilled.

So, general relativity is not completed.

3 Quantum anomalous

There are two cornerstones of quantum theory: the *Particle-wave duality* and the *Principle of uncertainty.*

3.1 Why uncertainty?

The Heisenberg uncertainty principle states precise inequalities that constrain certain pairs of physical properties, such as measuring the present position while determining future momentum; both cannot be simultaneously done to arbitrarily high precision. That is, the more precisely one property is measured, the less precisely the other can be controlled or determined. The principle states that a minimum exists for the product of the uncertainties in these properties that is equal to or greater than one half of the reduced Planck constant $\hbar$:

$$\Delta p \Delta x \sim h.$$

3.2 Why non-commutative operator?

In quantum mechanics, position and momentum are *conjugate variables.* The Heisenberg uncertainty principle defines limits on how accurately the momentum and position of a single observable system can be known at once. In quantum mechanics, momentum is defined as an operator on the wave function.

For a single particle described in the position basis the momentum operator can be written as

$$\mathbf{p} = \frac{\hbar}{i}\nabla = -i\hbar\nabla.$$

But, here the physical meaning of *value* p is not clear from this definition.

In quantum mechanics each dynamical variable (e.g. position, translational momentum, orbital angular momentum, spin, total angular momentum, energy, etc.) is associated with a Hermitian operator that acts on the state of the quantum system and whose eigenvalues correspond to the possible values of the dynamical variable.

In classical physics, all observables commute and the commutator would be zero. In quantum physics, the canonical commutation relation is the relation between canonical conjugate quantities (quantities which are related by definition such that one is the Fourier transform of another), for example:

$$[x, p_x] = i\hbar.$$

All in all, quantum world is queer. How to *really understand* all these bizarreness instead of just accepting them as facts? What assumptions of our fundamental picture of nature would we be forced to give up if we took all these seriously?

4 Differentiability: the implicit assumptions we tend to forget

4.1 Why we forget?

For research convenient, mathematicians often made excessive hypothesis and the approximations. Mathematicians spent too much time and interest on the *differentiable* or even *smooth* kind of geometry object. But in fact, the real geometry of physical world is most often non-smooth, non-differentiable. If the mathematicians try to put their math book down for a while and go around and have a look at the real world, the trees, the cloud, the mountains, the rivers, they must find that all around in nature are fractal [1], not smooth or differentiable at all.

For centuries we can only do calculus to a relatively rare and special kind of sufficiently well-behaved functions, i.e. the differentiable function or even smooth function. But later we come to realize that the typical and prevalent continuous functions are nowhere-differentiable functions like the Weierstrass function.

Image that if let the price changes continuously through time, the price curve must fluctuate all the time and have a self-similar property which is in fact a fractal. The daily price curve fluctuates heavily while the moving average curves run much more smoothly. Why financial workers study moving average line instead of the "real" fractal curve of . The reason is simply that we don't know how to deal with fractal and thus make a kind of approximation. This is also the case for mathematicians and physicists. Why the space-time manifold is smooth or differentiable? **No other physical reasons but just because the limitation of classical calculus.** The classical calculus with integer order can just deal with smooth function and powerless on nowhere-differentiable fractal functions.

In physics, the Heisenberg uncertainty principle is really erratic, while the fractal function is also erratic in the mathematical world. I will show the similarity between these two concepts.

4.2 What's velocity?

The velocity vector $\mathbf{v}$ of an object that has positions $\mathbf{x}(t)$ at time t and $\mathbf{x}(t+\Delta t)$ at time $t+\Delta t$, can be computed as the derivative of position:

$$\lim_{\Delta t \to 0} \frac{X(t+\Delta t) - X(t)}{\Delta t}.$$

Noticed that this definition is only appropriate when the trajectory is smooth, i.e. the $X(t+\Delta t) - X(t)$ and Δt are of the same order. So, here is a big problem:

what if the trajectory is not smooth or differentiable, i.e. the $X(t+\Delta t) - X(t)$ and Δt are not of the same order, and this limit doesn't exist?

The fact is that for almost all continuous function $X(t)$, this limit does not exist,i.e. **almost all continuous functions are non-differentiable** according to the above definition.

The term $\dfrac{X(t+\Delta t) - X(t))}{\Delta t}$ will fluctuate with no limitation when $\Delta t \to 0$. This is exactly the analogy *if you knowing the position with higher precision, the fluctuation of velocity will become bigger*; both cannot be simultaneously done to arbitrarily high precision. So the weird uncertainty principle becomes straightforward in the circumstances of non-differentiable functions. It's just the result of the divergence of usual definition of *velocity*.

4.3 Differentiability of function

The most basic functions $x^n, e^x, \sin x$ are smooth function. For smooth function we can use power series to approximate and express any smooth function.

In mathematical analysis, a differentiability class is a classification of functions according to the properties of their derivatives. Higher order differentiability classes correspond to the existence of more derivatives. Functions that have derivatives of all orders are called smooth.

Definition 1. *The function f is said to be of class C^k if the derivatives $f', f'', ..., f^{(k)}$ exist and are continuous (except for a finite number of points or a measure zero set of points). The function f is said to be of class C^∞, or smooth, if it has derivatives of all orders.*

Under the above definition, the class C^0 consists of all continuous functions. The class C^1 consists of all differentiable functions whose derivative is continuous; such functions are called continuously differentiable. Thus, a C^1 function is exactly a function whose derivative exists and is of class C^0.

The relation hold: $C^\infty \subset ... \subset C^n \subset ... \subset C^2 \subset C^1 \subset C^0 \subset G$, where G denote the generalized function. In particular, C^k is contained in C^{k-1} for every k, and there are examples to show that this containment is strict.

One interesting property is if $f \in C^m$ and $g \in C^n$, then $f + g \in C^{-(m,n)}$.

4.4 Non-differentiable function

It's interesting that, in the above relation C^{k+1} is always a proper (in fact, a measure zero) subset of C^k. So could there be some non-trivial examples which are $\in C^0$ but not $\in C^1$ almost everywhere? The answer is yes, Weierstrass type of functions are typical such examples, see [4].

A simple example of such a parametrization is given by

$$W_\lambda^s(t) = \sum_{k=1}^{\infty} \lambda^{(D-2)k} \sin \lambda^k t$$

where $\lambda > 1$ and $1 < D < 2$. The graph of $W_\lambda^s(t)$ is known to be a fractal curve with box-dimension D.

What's more, the Weierstrass function is far from being an isolated or special example. In fact, it is the dominant and typical continuous functions, in the math sense.

Here I want to give a remark about Nottale's work on *The theory of scale relativity* [5]. It's a great idea but some of the fundamental concepts remain unclear. In Nottale's approach, the non-differentiability comes from the inequality of left and right derivative. In fact, it's not the case for general non-differentiable functions which the truly non-differentiability comes from the $X(t + \Delta t) - X(t)$ and Δt are not of the same order. So both left and right derivative will diverge.

The non-differentiability of function like $X(t) = |t|$ comes from the inequality of left and right derivative at the single point $t = 0$. This kind of non-differentiability at most happens at a set of countable infinite points which is in fact of measure zero. But the Weierstrass-like function is nowhere differentiable.

5 Give up differentiability to reconcile quantum

Quantum mechanics is on-differentiable what assumptions would we be forced to give up if we took them seriously The definition of velocity is not well-defined in quantum mechanics. This is the root of all evil.

5.1 Trajectories in Quantum mechanics is non-differentiable

Feynman and Hibbs have already noted in ([3],p.176-177) that typical path of quantum-mechanical particle is continuous and non-differentiable. More precisely, there exist a quadratic velocity, i.e. if $X(t)$ denotes the particle trajectory, then

$$\lim_{t\to t'} \frac{(X(t)-X(t'))^2}{t-t'} \text{ exists.}$$

The interesting question is: *If we take this new definition of velocity, will the uncertainty principle still hold?*

5.2 Insight from path integral

The path integral formulation of quantum mechanics is a description of quantum theory which generalizes the action principle of classical mechanics. It replaces the classical notion of a single, unique trajectory for a system with a sum, or functional integral, over an infinity of possible trajectories to compute a quantum amplitude.

Interestingly, the basic idea of the path integral formulation can be traced back to Norbert Wiener, who introduced the Wiener integral for solving problems in diffusion and Brownian motion.

The path integral representation gives the quantum amplitude to go from point x to point y as an integral over all paths. For a free particle action ($m=\hbar=1$):

$$S=\int \frac{\dot{x}^2}{2}dt$$

the integral can be evaluated explicitly.

To do this, it is convenient to start without the factor i in the exponential, so that large deviations are suppressed by small numbers, not by cancelling oscillatory contributions.

$$K=\int_{x(0)=x}^{x(T)=y} \exp\left\{-\int_0^T \frac{\dot{x}^2}{2}dt\right\} Dx$$

Splitting the integral into time slices:

$$K=\int_{x(0)=x}^{x(T)=y} \Pi_t \exp\left\{-\frac{1}{2}\left(\frac{x(t+\epsilon)-x(t)}{\epsilon}\right)^2 \epsilon\right\} Dx$$

where the Dx is interpreted as a finite collection of integrations at each integer multiple of ϵ.

We can see that the main contribution of quantum fluctuation comes from the path which satisfy

$$\left(\frac{x(t+\epsilon)-x(t)}{\epsilon}\right)^2 \epsilon \sim 1.$$

This implies that the average displacement $x(t+\epsilon)-x(t)$ within a time step ϵ is proportional not to ϵ but to $\sqrt{\epsilon}$, just the same as for a random walk.

5.3 Uncertainty relation comes from non-differentiability

Suppose the quantum path is a non-differentiable fractal curve. We can understand uncertainty relations as *if you knowing the position with higher precision, the fluctuation of velocity will become bigger; both cannot be simultaneously done to arbitrarily high precision.*

Let Δx, Δt and Δp be the precision of the measurement of the position x, time t and *momentum* p of a given particle. The Heisenberg uncertainty relation on momentum and position is

$$\Delta p \Delta x \sim h.$$

So

$$m\Delta v \Delta x = m\frac{(\Delta x)^2}{\Delta t} \sim h$$

$$\Delta x \sim (\Delta t)^{\frac{1}{2}}$$

Thus, we get the same result as from the path integral approach, and this result means the quantum path is fractal of Hausdorff dimension 2.

There is another way to understand the uncertainty relation between x and $\frac{dx}{dt}$.

In classical mechanics, the dynamic equation is of order two, so the zero order derivative x and the first derivative $\frac{dx}{dt}$ are totally independent. We have to input x and $\frac{dx}{dt}$ independently as initial condition to solve the dynamic equation. But now things become interesting for quantum mechanics, the dynamic equation is not of order two, and the dynamic variable is not x anymore, but the abstract wave function. The schrodinger equation is of order one as for the wave function.

But by analogy, we can say that quantum equation must of order 1.5 as for x. So there is less than two, but more than one independent variables. This is why the x and $\frac{dx}{dt}$ are somehow not independent to each other, but are *conjugate variables* with uncertainty relation between them.

5.4 Non-commutativity comes from non-differentiability

Since the velocity in quantum mechanics is ill-defined, so talking about *velocity* $\frac{dx}{dt}$ is actually ***making sense of nonsense***. So we need a strange way to make this sensible.

Notice that the rigorous *velocity* $\frac{dx}{dt}$ is actually diverged, so what make sense is just a kind of approximation $\frac{\Delta x}{\Delta t}$. Thinking about the quality $xv \equiv x\frac{\Delta x}{\Delta t}$, since x and $\frac{\Delta x}{\Delta t}$ cannot be simultaneously fixed to arbitrarily high precision, now the order become a problem:

1. The first way is to determine $x(t)$ very near some point t first, so we have a higher precision of $x(t)$ (but not infinite high precision, otherwise we can't talk about v). Then in the region near t we can find $t + \epsilon$, and then calculate the $\frac{(x(t+\epsilon)-x(t))}{\epsilon}$ as v. The result will be:
$$vx \equiv \frac{\Delta x}{\Delta t}x = \frac{(x(t+\epsilon) - x(t))}{\epsilon}x(t)$$
2. The other way is to determine $\frac{(x(t+\epsilon)-x(t))}{\epsilon}$ as $v(t)$ first, so we have to choose point in a wider region near t we can find two points in this wider region, and then calculate the $\frac{(x(t+\epsilon)-x(t))}{\epsilon}$ as v. But now we can't talk about exactly the $x(t)$ very near t, but only some point in the wider region far from t, we represent it as $x(t + \epsilon)$. So the result will be:
$$xv \equiv x\frac{\Delta x}{\Delta t} = x(t+\epsilon)\frac{(x(t+\epsilon) - x(t))}{\epsilon}$$

In elementary calculus for differentiable function, this won't cause any problem. The two are only different by an amount which goes to zero as ϵ goes to zero. But in non-differentiable case, the difference between the two is not zero:

$$[2] - [1] = \frac{(x(t+\epsilon) - x(t))^2}{\epsilon} \approx \frac{\epsilon}{\epsilon}$$

So the order to determine the two conjugate variable x and $\frac{dx}{dt}$ is importat:

$$[x, \dot{x}] = x\frac{dx}{dt} - \frac{dx}{dt}x = 1$$

We can see that the weird non-commutativity between conjugate variables becomes straightforward in the circumstances of non-differentiable functions. In quantum physics, the canonical commutation relation between x and p_x:

$$[x, p_x] = i\hbar.$$

6 Toward a complete theory of relativity

We argue that the Einstein version of general relativity is not complete. The reasons are twofold. In physic, GR can't explain quantum phenomenon. In math, the now general covariance in GR is the invariance of the form of physical laws under arbitrary differentiable coordinate transformations, which is really just a measure zero subset of arbitrary coordinate transformations.

So we need to fulfill the relativity principle in a complete theory of relativity, in which the spacetime are non-differentiable manifold, and physical law takes the same mathematical form in all coordinate systems, under arbitrary differentiable or non-differentiable coordinate transformations. Quantum phenomenon emerges as natural consequence of non-differentiability of spacetime. This may finally fulfill the dream of Einstein to reveal an elegant unified theory of everything, of every scale.

Let me end this article by the following new assumptions for a new physics:

What is the physical world made of? Non-differentiable field

What is the structure of space-time? Non-differentiable manifold

How to describe change through space-time? Fractal calculus

References

[1] Mandelbrot, B. (1983). *The fractal geometry of nature*. Wh Freeman.

[2] Abbott L, Wise M, Dimension of a quantum-mechanical path, Am. J. Phys. 49(1), Jan.1981.

[3] Feynman R, Hibbs A, *Quantum mechanics and path integrals*, MacGraw-Hill,1965.

[4] Berry, M. and Lewis, Z. (1980). On the weierstrass-mandelbrot fractal function. *Proceedings of the Royal Society of London. A. Mathematical and Physical Sciences*, 370(1743), 459–484.

[5] Nottale L, The theory of scale relativity, Int. J. Modern Phys. A, Vol. 7,

Theory of relativity	Assumption or principle	Spacetime geometry	Bonus of new assumption
Galilean relativity	*The laws of mechanics* are the same for all observers in uniform motion relative to one another	Galilean Spacetime: Space is the 3-D Euclidean space S. Time is an 1-D metric space T Spacetime is the direct product space called a fibre bundle	
Special relativity	*All laws of physics* are the same for all observers in uniform motion relative to one another The speed of light in a vacuum is the same for all observers	Minkowski manifold: Spacetime is a four-dimensional, smooth, flat Lorentzian manifold with metric $\eta_{\mu\nu}$.	Relativity of simultaneity, Time dilation, Length contraction, Mass-energy equivalence, Maximum speed is finite
General relativity	*All laws of physics* are the same for all observers in uniform accelerating motion relative to one another The Principle of Equivalence	Riemannian manifold: Spacetime is a four-dimensional, smooth, curved pseudo-Riemannian manifold with a metric tensor $g_{\mu\nu}$ which itself is dynamical and determined by Einstein field equation $R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi G}{c^4}T_{\mu\nu}$.	Gravity emerges as natural consequence of curvature of spacetime, Gravitational time dilation and frequency shift, Gravitational waves, Orbital effects, Black holes, Big Bang, and modern Cosmology
Complete relativity	All Physical laws are the same for all observers in any motion relative to one another.	Fractal manifold: A pure geometric description of matter and spacetime A unified field equation take Einstein's field equation as classical limit case	Quantum phenomenon emerges as natural consequence of non-differentiability of spacetime, Unification of general relativity and quantum theory or Perhaps everything...

Table 1: The road toward a complete theory of relativiy

Bio of the author

WANG Xiong has obtained his B.Sc. majoring in math from the Shanghai Jiao Tong University, China. Currently, he is a research student at City University Hong Kong at the Centre for Chaos and Complex Networks. From undergraduate time, he continue an independent solitary quest for a unified foundation for mathematics and physics.