Fine Structure Constant $\alpha \sim 1/137.036$ and Blackbody Radiation Constant $\alpha_{\mathbf{R}} \sim 1/157.555$

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Abstract

The fine structure constant $\alpha = \mathbf{e}^2/\hbar c \approx 1/137.036$ and the blackbody radiation constant $\alpha_R = \mathbf{e}^2(a_R/k_B^4)^{1/3} \approx 1/157.555$ are linked by prime numbers. The blackbody radiation constant is a new method to measure the fine structure constant. It also links the fine structure constant to the Boltzmann constant.

Planck and Einstein noted respectively in 1905 and 1909 that $\mathbf{e}^2/c \sim h$ have the same order and dimension.[1, 2] This was before the introduction of the fine structure constant $\alpha = \mathbf{e}^2/\hbar c$ by Sommerfeld in 1916.[3] Therefore, the search for a mathematical relationship between $\mathbf{e}^2/c \sim h$ was started from the blackbody radiation. The Stefan-Boltzmann law states that the radiative flux density or irradiance is $J = \sigma T^4$ $[\mathrm{erg} \cdot \mathrm{cm}^{-2} \cdot \mathrm{s}^{-1}]$ in CGS units or $[\mathrm{W} \cdot \mathrm{m}^{-2}]$ in SI units. From the Planck law, the Stefan-Boltzmann constant $\sigma = 5.670400(40) \times 10^{-5}$ $[\mathrm{erg} \cdot \mathrm{cm}^{-2}\mathrm{K}^{-4}\mathrm{s}^{-1}]$ is

$$\begin{aligned} \sigma &= 2\pi \int_0^\infty \frac{x^3 dx}{e^x - 1} \frac{ck_B^4}{(hc)^3} = \frac{2\pi^5}{15} \frac{ck_B^4}{(hc)^3} \\ &= 2\pi\Gamma(4)\zeta(4) \frac{ck_B^4}{(hc)^3} = \frac{4^2\pi^5}{5!} \frac{ck_B^4}{(hc)^3} \end{aligned} \tag{1}$$

The Stefan-Boltzmann law can be expressed as the volume energy density of the blackbody $\varepsilon_T = \boldsymbol{a}_R T^4$ $[\mathrm{erg} \cdot \mathrm{cm}^{-3}]$, where the radiation density constant $\boldsymbol{a}_R$ is linked to the Stefan-Boltzmann constant

$$\boldsymbol{a}_R = \frac{4\sigma}{c} = \frac{4^3\pi^5}{5!} \frac{k_B^4}{(hc)^3} \tag{2}$$

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In 1914, Lewis and Adams noticed that the dimension of the radiation density constant divided by the 4^{th} power Boltzmann constant $\boldsymbol{a}_R/k_B^4$ is (energy $\times$ length)$^{-3}$, while $\mathbf{e}^2$ is (energy $\times$ length). However, they obtained an incorrect result equivalent to $\alpha^{-1} = \hbar c/\mathbf{e}^2 = 32\pi\left(\pi^5/5!\right)^{1/3} = 137.348$.[4] In 1915, Allen rewrote it as $\alpha = \mathbf{e}^2/\hbar c = (15/\pi^2)^{1/3}/(4\pi)^2$.[5]

In CGS units, $\mathbf{e}^2 = \left(4.80320427(12) \times 10^{-10}\right)^2$ [erg·cm], $\boldsymbol{a}_R = 7.56576738 \times 10^{-15}$ [erg·cm^{-3}K^{-4}], and $k_B^4 = \left(1.3806504(24) \times 10^{-16}\right)^4$ [erg^{4}K^{-4}]. We get the experimental dimensionless constant[6]

$$\begin{aligned} \alpha_R &= \mathbf{e}^2\left(\frac{\boldsymbol{a}_R}{k_B^4}\right)^{1/3} = \frac{1}{\mathbf{157.5548787}} \\ &= \mathbf{0.00634699482} \end{aligned} \tag{3}$$

This is the dimensionless blackbody radiation constant α_R,[1] which is on the same order of the fine structure constant $\alpha = \mathbf{e}^2/\hbar c$

$$\begin{aligned} \alpha_R &= \frac{2}{\pi}\left(\frac{\pi^5}{5!}\right)^{1/3}\alpha = \left(\frac{\Gamma(4)\zeta(4)}{\pi^2}\right)^{1/3}\alpha \\ &= \left(\frac{\pi^2}{15}\right)^{1/3}\alpha = 0.8697668 \cdot \alpha \end{aligned} \tag{4}$$

Therefore, $\alpha_R \neq \alpha$, both α and α_R are experimental results incapable of producing the α math formula. Physically, the fine structure constant α is obtained from the atomic *discrete* spectra, while the blackbody radiation constant α_R is obtained from the thermal radiation of a 3D cavity in the *continuous* spectra. However, their relationship can be given by the Riemann zeta-function or by the modification of Euler's product formula (1737)

$$\begin{aligned} \frac{\alpha_R^3}{\alpha^3} &= \frac{\pi^2}{15} = \frac{\zeta(4)}{\zeta(2)} = \frac{\sum_{n=1}^{\infty}\frac{1}{n^4}}{\sum_{n=1}^{\infty}\frac{1}{n^2}} = \frac{\prod_{\mathrm{p}}\left(1-\frac{1}{\mathrm{p}^2}\right)}{\prod_{\mathrm{p}}\left(1-\frac{1}{\mathrm{p}^4}\right)} = \prod_{\mathrm{p}}\left(\frac{\mathrm{p}^2}{\mathrm{p}^2+1}\right) \\ &= \frac{2^2}{(2^2+1)}\frac{3^2}{(3^2+1)}\frac{5^2}{(5^2+1)}\frac{7^2}{(7^2+1)}\cdots = \frac{4}{5}\cdot\frac{9}{10}\cdot\frac{25}{26}\cdot\frac{49}{50}\cdot\frac{121}{122}\cdots \end{aligned} \tag{5}$$

where the Euler product extends over all the *prime* numbers. In other words, the fine structure constant and the blackbody radiation constant can be linked by the prime numbers. From (5), the Stefan-Boltzmann law as the volume energy density of the blackbody ε_T is related to the fine structure constant α and the oscillating charge $\mathbf{e}^2$ with the different resonating frequencies in a cavity

[1] Do not confuse with Stefan-Boltzmann constant σ or hc/k (blackbody radiation constant)

$$\begin{aligned}\varepsilon_T &= \boldsymbol{a}_R T^4 = \frac{4\sigma}{c} T^4 = \frac{4^3 \pi^5}{5!} \frac{k_B^4}{(hc)^3} T^4 \\ &= \frac{\zeta(4)}{\zeta(2)} \left(\frac{\alpha}{\mathbf{e}^2}\right)^3 k_B^4 T^4 = \left(\frac{\alpha_R}{\mathbf{e}^2}\right)^3 k_B^4 T^4\end{aligned} \tag{6}$$

and the radiative flux density is

$$J = \sigma T^4 = \frac{\zeta(4)}{\zeta(2)} \left(\frac{\alpha}{\mathbf{e}^2}\right)^3 \frac{c}{4} k_B^4 T^4 = \frac{c}{4} \left(\frac{\alpha_R}{\mathbf{e}^2}\right)^3 k_B^4 T^4 \tag{7}$$

and the total brightness of a blackbody is

$$B = \frac{J}{\pi} = \frac{\zeta(4)}{\zeta(2)} \left(\frac{\alpha}{\mathbf{e}^2}\right)^3 \frac{c}{4\pi} k_B^4 T^4 = \frac{c}{4\pi} \left(\frac{\alpha_R}{\mathbf{e}^2}\right)^3 k_B^4 T^4 \tag{8}$$

and the inner wall pressure of the blackbody cavity is

$$P = \frac{4\sigma}{3c} T^4 = \frac{\zeta(4)}{\zeta(2)} \left(\frac{\alpha}{\mathbf{e}^2}\right)^3 \frac{1}{3} k_B^4 T^4 = \frac{1}{3} \left(\frac{\alpha_R}{\mathbf{e}^2}\right)^3 k_B^4 T^4 \tag{9}$$

According to the Bose-Einstein model of photon-gas,[7] the free energy of the thermodynamics is

$$F = -PV = -\frac{4\sigma}{3c} V T^4 = -\frac{\zeta(4)}{\zeta(2)} \left(\frac{\alpha}{\mathbf{e}^2}\right)^3 \frac{V}{3} k_B^4 T^4 = -\frac{V}{3} \left(\frac{\alpha_R}{\mathbf{e}^2}\right)^3 k_B^4 T^4 \tag{10}$$

and the total radiation energy is

$$E = -3F = 3PV = \frac{4\sigma}{c} V T^4 = \frac{\zeta(4)}{\zeta(2)} \left(\frac{\alpha}{\mathbf{e}^2}\right)^3 V k_B^4 T^4 = V \left(\frac{\alpha_R}{\mathbf{e}^2}\right)^3 k_B^4 T^4 \tag{11}$$

where the photon gas $E = 3PV$ is the same as the extreme relativistic electron gas, and the entropy is

$$S = -\frac{\partial F}{\partial T} = \frac{16\sigma}{3c} V T^3 = \frac{\zeta(4)}{\zeta(2)} \left(\frac{\alpha}{\mathbf{e}^2}\right)^3 \frac{4V}{3} k_B^4 T^3 = \frac{4V}{3} \left(\frac{\alpha_R}{\mathbf{e}^2}\right)^3 k_B^4 T^3 \tag{12}$$

and the specific heat of the radiation is

$$C_V = \left(\frac{\partial E}{\partial T}\right)_V = \frac{16\sigma}{c} V T^3 = \frac{\zeta(4)}{\zeta(2)} \left(\frac{\alpha}{\mathbf{e}^2}\right)^3 4V k_B^4 T^3 = 4V \left(\frac{\alpha_R}{\mathbf{e}^2}\right)^3 k_B^4 T^3 \tag{13}$$

Landau assumed that the volume V in (10)$\sim$(13) must be sufficiently large in order to change from *discrete* to *continuous* spectra. Experimentally, solids or dense-gas have the continuous spectra, and hot low-density gas emits the discrete atomic spectra. The pattern of Planck spectra is given by $f(x) = x^3/(e^x - 1)$

where photon $h\nu$ is hidden in $x = h\nu/k_BT$. The *photon* integral in (1) is equal to a dimensionless constant (Fig. 1)

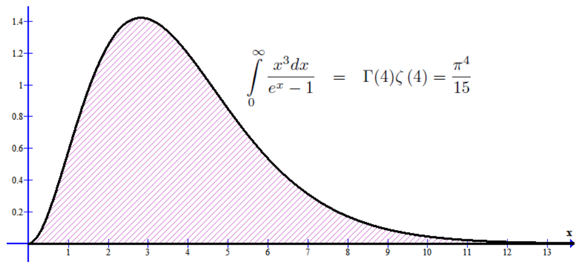

Figure 1: Photon integral is a dimensionless number $\Gamma(4)\zeta(4) = \frac{\pi^4}{15} = 6.4939394$

$$\begin{aligned}
\int_0^\infty \frac{x^3dx}{e^x-1} &= \Gamma(4)\zeta(4) = \frac{\pi^4}{15} = 2\cdot 3\cdot \prod_{\mathrm{p}}\left(\frac{\mathrm{p}^4}{\mathrm{p}^4-1}\right) \qquad (14)\\
&= 2\cdot 3\cdot \frac{2^4}{(2^4-1)}\frac{3^4}{(3^4-1)}\frac{5^4}{(5^4-1)}\frac{7^4}{(7^4-1)}\cdots\\
&= \frac{3^35^2}{2^3\cdot 13}\cdot\frac{7^4}{(7^4-1)}\frac{11^4}{(11^4-1)}\cdots = \frac{3^35^2}{2^3\cdot 13}\prod_{\mathrm{p}>5}\left(\frac{\mathrm{p}^4}{\mathrm{p}^4-1}\right)
\end{aligned}$$

where the Euler product extends over all the *prime* numbers. The photon distribution integral (14) yields a zeta-function that is linked to the Euler prime products, therefore, there is no $h\nu$ in (6)∼(13). (14) shows clearly how the fine structure constant α for the discrete spectra in (4) is converted to the blackbody radiation constant α_R for the continuous spectra by multiplying a dimensionless constant. (5) and (14) indicate that this dimensionless constant can be expressed as the Euler infinite prime number product.

In (6)∼(13), the oscillators of the thermal electrons $\alpha/\mathbf{e}^2 = 1/\hbar c$ or $\alpha_R/\mathbf{e}^2$ play a critical role in the electromagnetic coupling on a 3D surface (Fig. 2).

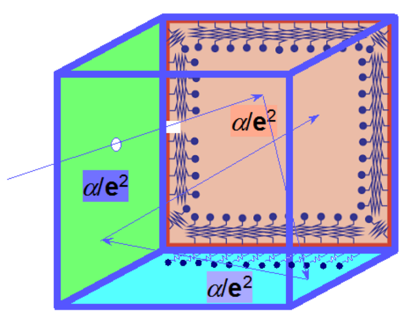

Figure 2: Blackbody radiation is related to α and $\mathbf{e}^2$ in a 3D cavity.

This 3D box (or sphere) model does not necessarily have solid walls, the plasma gas layer of a star can have the same effect. We are not reinventing the blackbody radiation law, but instead pointing out that charge is the oscillator, and it is related to the fine structure constant. This links the quantum theory to the classical theory of blackbody radiation with or without using the Planck constant

$$\begin{aligned} \boldsymbol{a}_R &= \left(\frac{\alpha_R}{\mathbf{e}^2}\right)^3 k_B^4 = \frac{\zeta(4)}{\zeta(2)}\left(\frac{\alpha}{\mathbf{e}^2}\right)^3 k_B^4 = \frac{\zeta(4)}{\zeta(2)}\left(\frac{2\pi}{hc}\right)^3 k_B^4 \\ &= \left(\frac{\alpha}{\mathbf{e}^2}\right)^3 k_B^4 \cdot \frac{4}{5}\cdot\frac{9}{10}\cdot\frac{25}{26}\cdot\frac{49}{50}\cdot\frac{121}{122}\cdot\frac{169}{170}\cdot\frac{289}{290}\cdot\frac{361}{362}\cdots \end{aligned} \tag{15}$$

The Planck constant h with the revolutionary concept of energy quanta is a bridge between classical physics and quantum physics. Einstein's proposal of the light quanta $h\nu$ in 1905 was based on the Planck constant, however, Planck always had reservations due to the continuous spectra of blackbody radiation and the wave-particle duality. In 1951, Einstein said that, "All these fifty years of conscious brooding have brought me no nearer to the answer to the question, 'What are light quanta?'"[7] In QED, the photon is treated as a gauge boson, and the perturbation theory involves the finite power series in α. The discrete-continuous spectra is bridged by the Bose-Einstein distribution, and the *prime* sequences link the fine structure constant α to the blackbody radiation constant α_R. The blackbody radiation constant is a new method to measure the fine structure constant. It also links the fine structure constant to the Boltzmann constant.

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