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SUMMARY OF THE ZETA REGULARIZATION METHOD APPLIED TO THE CALCULATION OF DIVERGENT SERIES $\sum_{n=1}^{\infty} n^{s}$ AND DIVERGENT INTEGRALS $\int_{0}^{\infty} x^{s}dx$

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- ABSTRACT: We study a generalization of the zeta regularization method applied to the case of the regularization of divergent integrals $\int_{0}^{\infty} x^{s}dx$ for positive 's' , using the Euler Maclaurin summation formula, we manage to express a divergent integral in term of a linear combination of divergent series , these series can be regularized using the Riemann Zeta function $\zeta(s)$ s >0 , in the case of the pole at s=1 we use a property of the Functional determinant to obtain the regularization $\sum_{n=0}^{\infty}\frac{1}{(n+a)} = -\frac{\Gamma'}{\Gamma}(a)$, with the aid of the Laurent series in one and several variables we can extend zeta regularization to the cases of integrals $\int_{0}^{\infty} f(x)dx$, we believe this method can be of interest in the regularization of the divergent UV integrals in Quantum Field theory since our method would not have the problems of the Analytic regularization or dimensional regularization
- *Keywords: =* Riemann Zeta function, Functional determinant, Zeta regularization, divergent series .

ZETA REGULARIZATION FOR DIVERGENT INTEGRALS:

Sometimes in mathematics and physics , we must evaluate divergent series of the form $\sum_{n=1}^{\infty} n^{k}$, of course this series is divergent unles Re (k) >1 , however cases like k=1 or k=3 appear in several calculations of string theory and Casimir effect , for the case of

Casimir effect [3] the result $\sum_{n=1}^{\infty} n^3 = \frac{1}{120}$ appears to give the correct result for the Casimir force $\frac{F_c}{A} = -\frac{\hbar c \pi^2}{240 a^4}$ here A is the area and 'd' the separation between the 2 plates , c and $\hbar$ are the speed of ligth and the Planck's constant. The idea behind the Zeta regularization method is to take for granted that for every 's' the identity $\sum_{n=1}^{\infty} n^s = \zeta(s)$, follows although this formula is valid just for Re (s) > 1 , to extend the definition of the Riemann Zeta function to negative real numbers, one need to use the functional equation for the Riemann function

$$\zeta(1-s) = 2(2\pi)^{-s}\,\Gamma(s)\cos\left(\frac{\pi s}{2}\right)\zeta(s) \qquad \Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)} \qquad (1)$$

This gives the expressions $\sum_{n=i}^{\infty} n^0 = -\frac{1}{2}$, $\sum_{n=i}^{\infty} n = -\frac{1}{12}$ and $\sum_{n=i}^{\infty} n^2 = 0$ due to the pole at s=1 , the Harmonic series $\sum_{n=1}^{\infty} n^{-1}$ is NOT zeta regularizable, although it can be given a finite value $\sum_{n=1}^{\infty} n^{-1} = \gamma = 0.577215..$, this value can be justified by using the theory of Zeta-regularized infinite products (determinants) , as we shall see later in the paper

- *Zeta regularization for divergent integrals:*

Let be $f(x) = x^{m-s}$ with Re(m-s) < -1 , then the Euler-Maclaurin summation formula for this function reads

$$\begin{aligned} \int_a^{\infty} x^{m-s}dx &= \frac{m-s}{2}\int_a^{\infty} x^{m-1-s}dx + \zeta(s-m) - \sum_{i=1}^{a} i^{m-s} + a^{m-s} \\ &- \sum_{r=1}^{\infty} \frac{B_{2r}\Gamma(m-s+1)}{(2r)!\Gamma(m-2r+2-s)}(m-2r+1-s)\int_a^{\infty} x^{m-2r-s}dx \end{aligned} \qquad a>1 \qquad (2)$$

Here in formula (2) all the series and integrals are convergent, formula (2) is usually worthless , since it is trivial to prove that $\int_a^{\infty} x^{-k}dx = \frac{a^{1-k}}{k-1}$ for Re(k) >1 ,and the Riemann zeta function $\zeta(m-s) = \sum_{i=1}^{\infty} i^{m-s}$, so nothing new can be obtained from (2) , the idea is to use the Functional equation (1) for the Riemann and Zeta function to extend the definition of equation (2) to the whole complex plane except s=1 , in case (m-s) is positive there will be no pole at x=0 , so we can put a=0 and take the limit $s \to 0^+$

$$\int_0^\infty x^m dx = \frac{m}{2}\int_0^\infty x^{m-1}dx + \zeta(-m) - \sum_{r=1}^{\infty}\frac{B_{2r}m!(m-2r+1)}{(2r)!(m-2r+1)!}\int_0^\infty x^{m-2r}dx \qquad (3)$$

Formula (3) is the Analytic continuation of formula (2) with a=0 and can be used to obtain a finite definition for otherwise divergent integrals ,apparently this recurrence equation has an infinite number of terms but the Gamma function has a pole at x= 0 and at x being some negative integer , some examples of formula (3)

$$I_0 = \zeta(0) + 1 = \int_0^\infty dx \quad I_1 = \frac{I_0}{2} + \zeta(-1) = \int_0^\infty x dx$$

$$I_2 = \left(\frac{I_0}{2} + \zeta(-1)\right) - \frac{B_2}{2}a_{21}I_0 = \int_0^\infty x^2 dx \qquad (4)$$

$$I_3 = \frac{3}{2}\left(\frac{1}{2}\left(I_0 + \zeta(-1)\right) - \frac{B_2}{2}a_{21}I_0\right) + \zeta(-3) - B_2 a_{31} I_0 = \int_0^\infty x^3 dx$$

So our method can provide finite 'regularization' to divergent integrals , with the Aid of the zeta regularization algorithm. Also our formulae (2) (3) and (4) are consistent with the usual summation properties , in fact if $\int_0^\Lambda x^m dx$ is finite for finite Λ and we use the property of the Riemann and Hurwitz Zeta function [ ] to get the sum of the k-th powers of n on the interval $[0, \Lambda]$ $\sum_{i=0}^{\Lambda-1} i^m = \zeta(-m) - \zeta(-m,\Lambda)$, $\zeta(s,\Lambda) = \sum_{n=0}^{\infty}(n+\Lambda)^{-s}$ defined for Re(s) >1 (of course for positive 's' as $\Lambda \to \infty$ the second term goes to 0)

$$\int_0^\Lambda x^m dx = \frac{m}{2}\int_0^\Lambda x^{m-1}dx + \zeta(-m) - \zeta(-m,\Lambda) - \sum_{r=1}^{\infty}\frac{B_{2r}m!(m-2r+1)}{(2r)!(m-2r+1)!}\int_0^\Lambda x^{m-2r}dx \qquad (5)$$

For integer 'm' $\zeta_H(-m,x) = -\frac{B_{m+1}(x)}{m+1}$ we find the Bernoulli Polynomials , the powers of Λ would cancel the integral $\int_0^\Lambda x^m dx = \frac{\Lambda^{m+1}}{m+1}$, so in the end in formula (5) we would get the usual definition of Zeta regularization $\zeta_H(-m) = -\frac{B_{m+1}(0)}{m+1}$ for integer 'm'. Of course one could argue that a 'simpler' regularization of the divergent integrals should be $I(s) = \int_0^\infty dx(x+a)^s = -\frac{a^{s+1}}{s+1}$ and $I(-1) = \int_0^\infty dx(x+a)^{-1} = -\log a$, this is just dropping out the term proportional to $\log\infty$ or ∞^{s+1} inside the integral to make it finite, however if wi plugged this result into the Euler-Maclaurin summation formulae (2) (3) or (5) the terms involving 'a' would cancel and we would finally find that $\zeta_H(-m) = 0$ for every 'm' which clearly is against the definition of zeta regularization of a series, for the case of the logarihmic divergence , obtained from differentiation with respect to the external parameter ‘a’ this result of taking the finite part of the integral apparently works.

- *Zeta-regularized determinants and the Harmonic series:*

Given an operator A with an infinite set of nonzero Eigenvalues $\{\lambda_n\}_{n=0}^{\infty}$ we can define a Zeta function and a Zeta-regularized determinant , Voros [10]

$$Tr\left\{A^{-s}\right\} = \zeta_A(s) = \sum_{n=0}^{\infty} \lambda_n^{-s} \qquad \det(A) = \prod_{n=0}^{\infty} \lambda_n = \exp\left(-\frac{d\zeta_A(0)}{ds}\right) \qquad (6)$$

The proof of the second formula inside (6) is pretty easy, the derivative of the Generalized zeta function will be $\zeta_A{}'(s) = -\sum_{n=0}^{\infty} \frac{\log \lambda_n}{\lambda_n^{s}}$ now let s=0 , use the property of the logarithm $\log(a.b) = \log a + \log b$ and take the exponential on both sides.

For the case of the Eigenvalues of a simple Quantum Harmonic oscillator in one dimension [10] $\lambda_n = n + a$, the Zeta function is just the Hurwitz Zeta function, so we can define a zeta-regularized infinite product in the form

$$\prod_{n=0}^{\infty} (n+a) = \exp\left(-\frac{d\zeta_H(0,a)}{ds}\right) \qquad \frac{d\zeta_H(0,a)}{ds} = \log\Gamma(a) - \log\left(\sqrt{2\pi}\right) \qquad (7)$$

In case we put a=1 we find the zeta-regularized product of all the natural numbers $\prod_{n=0}^{\infty}(n+1) = \sqrt{2\pi}$,see [5] if we take the derivative with respect to ‘a’ , we would find the same regularized Value Ramanujan did [2] precisely $\sum_{n=0}^{\infty} \frac{1}{(n+a)} = -\frac{\Gamma'}{\Gamma}(a)$ a > 0

Harmonic series appear due to a logarithmic divergence of the integral $\int_0^{\infty} \frac{dx}{(n+a)}$, if we put m= -1 inside formula (2) , using a regulator ‘s’ , $s \to 0^+$ we have the Euler Maclaurin summation formula

$$\int_0^{\infty} \frac{dx}{(n+a)^{s+1}} = -\frac{1}{2a} + \sum_{n=0}^{\infty} \frac{1}{(n+a)^{1+s}} + \sum_{r=1}^{\infty} \frac{B_{2r}}{(2r)!} \frac{\partial^{2r-1}}{\partial u^{2r-1}} \left(\frac{1}{(x+a)^{s+1}}\right)_{x=0} \qquad (8)$$

Since s >0 the integral and the series inside (8) will be convergent, now we can integrate over ‘a’ inside (8) and use the definition of the logarithm $\lim_{s\to 0^+} \frac{x^s - 1}{s} = \log x$, to regularize the integral $\int_0^{\infty} \frac{dx}{(n+a)^{s+1}}$ as $s \to 0^+$ in terms of the function $-\frac{\Gamma'}{\Gamma}(a)$ plus some finite corrections due to the Euler-Maclaurin summation formula.

A faster method is just simple differentiate with respect to ‘a’ inside the integral $\int_0^\infty \frac{dx}{(n+a)^2} = -\frac{dI}{da}$, now this integral is convergent for every ‘a’ and equal to $\frac{1}{a}$, integration over ‘a’ again gives the value $-\log a + c$ plus a constant ‘c’ that will not depend on the value of a inside the integral in question , the proof that ‘c’ is unique no matter what a is comes from the fact that the difference $\int_0^\infty dx\left(\frac{1}{x+a}-\frac{1}{x+b}\right)=\log\left(\frac{b}{a}\right)$.

For the case a=0 , the derivative of the Hurwitz Zeta is $\frac{d\zeta_H(0,0)}{ds} = -\log\left(\sqrt{2\pi}\right)$ so if we approximate the divergent integral by a series, then we can get the regularized result $\int_0^\infty \frac{dx}{x} \approx \sum_{n=0}^\infty \frac{1}{n} = 0$. Apparently it seems that using two different regularizations we get some different results , the idea is that if we use the Stiriling asymptotic formula approximation for the logarithm of the Zeta function

$$\log\Gamma(z) = \left(z-\frac{1}{2}\right)\log z - z + \frac{1}{2}\log(2\pi) + \sum_{r=1}^\infty \frac{B_{2r}z^{1-2n}}{2r(2r-1)} \qquad (9)$$

If we take the derivative with respect to ‘z’ inside (9) , is now more apparent that for the logarithmic derivative $\int_0^\infty \frac{dx}{x+a} \approx \log\left(\frac{\mu}{a}\right)$ here $c = \log\mu$ is a constant obtained from differentiation with respect to ‘a’ to regularize the divergent integral , this constant ‘c’ must be related to some physical constant or in case the quantity ‘a’ has dimension of Energy then μ must have also dimensions of energy so the logarithm is dimensionless, this constant ‘c’ would be the only free adjustable parameter that would appear inside our calculations to regularize integrals.

- *Regularization of divergent integrals* $\int_0^\infty dx f(x)$ *:*

In general, the divergent integrals that appear in Quantum Field Theory [ ] are invariant under rotations, for example $\int \frac{d^4p}{\left(p^2+m^2\right)^2}$ or $\int \frac{d^4p}{\left((p-q)^2+m^2\right)}\frac{1}{p^2}$, if we use 4-dimesional polar coordinates we can reduce these integrals to the case $\frac{2\pi^{d/2}}{\Gamma(d/2)}\int_0^\infty dr f(r) r^{d-1}$ then the UV divergences appear when $r \to \infty$, here d=4 is the dimension of the spacetime, depending on the value of ‘d’ we can have several types of divergences $\int_0^\Lambda dr f(r) r^{d-1} \approx a\Lambda^{m+1} + b\log\Lambda$, if b =0 for m =2 the UV divergences are quadratic if m =0 the divergences are linear , in case a = 0 and b =1 the divergences are of logarithmic type , for example $\int \frac{d^4 p}{\left(p^2+m^2\right)^2}$ has only a logarithmic divergence in dimension 4 , for a lower value of the dimension (d=3) this integral exists.

To study the rate of divergence , we can expand the function into a Laurent series valid for $z \to \infty$, $f(x)=\sum_{n=-\infty}^{n=k} c_n (x+a)^n$ 'k' is a finite number and means that the function $f(x)$ has a power law divergence for big 'x' , then the idea to compute a divergent integral would be this, we add and substract a Polynomial plus a term proportional to $\frac{1}{x+a}$ to split the integral into a finite part and another divergent integrals

$$\int_0^\infty dx\left(f(x)-\sum_{n=0}^{k} b_n x^n - \frac{b_{-1}}{x+a}\right)+\sum_{n=0}^{k} b_n \int_0^\infty x^n dx + b_{-1}\int_0^\infty \frac{dx}{x+a}=\int_0^\infty f(x)dx \qquad (10)$$

The number of terms 'k' is chosen so the first integral is FINITE , this first integral can be computed by Numerical or exact methods and yields to a finite value , the rest of the integrals are just the logarithmic and power-law divergences, they can be regularized with the aid of formulae (2) (3) (4) (6) (8) to get a finite value involving a linear combination of $\zeta(-m)$ m=0,1,2,....,k and another value proportional to $\frac{\partial}{\partial s}\frac{\partial \zeta_H(a,0)}{\partial a}$ or $\int_0^\infty \frac{dx}{x+a} \approx \log\left(\frac{\mu}{a}\right)$ for example we can analyze this simple divergent integral a > 0

$$\begin{aligned}&\int_a^\infty \frac{x^2 dx}{x+1} = \int_a^\infty dx\left(\frac{x^2}{x+1}-1+x+\frac{1}{x}\right)+\frac{\zeta(0)}{2}-a+\frac{a^2}{2}+\frac{1}{2a}+ \\ &\frac{\Gamma'}{\Gamma}(a)-\sum_{r=1}^{\infty}\frac{B_{2r}}{(2r)!}\frac{\partial^{2r-1}}{\partial u^{2r-1}}\left(\frac{1}{x+a}\right)_{x=0}-\zeta(-1)+\frac{1}{2}\end{aligned} \qquad (11)$$

The first integral in (11) is convergent and have an exact value of $\log\left(\frac{a+1}{a}\right)$, in order to regularize the logarithmic integarl we have used the result $\sum_{n=0}^{\infty}\frac{1}{(n+a)}=-\frac{\Gamma'}{\Gamma}(a)$ plus the Euler-Maclaurin summation formula . The mathematical justification of this is the following, given a divergent integral $\int_a^\infty dx f(x)$ we introduce a regulator $F(s)=\int_a^\infty f(x)\frac{dx}{x^s}$ so the integral F(s) exists for some big 's' , if we add and substract powers of the form x^{k-s} for integer k and $\left(x+a\right)^{s+1}$, we can split F(s) into a convergent integral I (s) valid for $s \to 0^+$ and some divergent integrals of the form $\int_a^\infty x^{m-s}dx$ and

$\int_0^\infty \frac{dx}{(x+a)^{s+1}}$, using formulae (2) (3) (4) and (8) we can express these integrals in terms of the series $\sum_{n=0}^{\infty} \frac{1}{(n+a)^{s+1}}$ and $\sum_{n=0}^{\infty} \frac{1}{(n+a)^{s-m}}$, which will be convergent for $\mathrm{Re}(s-m) > 1$ and $\mathrm{Re}(s+1) > 1$, now using the Functional equation for the Hurwitz and Riemann Zeta function we can make the analytic continuation of both series to $s \to 0^+$ avoiding the pole at s=1 by the use of Riemann Zeta function at negative integers $\zeta(-n)$ plus some corrections involving $-\frac{\Gamma'}{\Gamma}(a)$ of course the rules for change of variable and still valid so $\int_0^\infty dx f(x+a) = \int_a^\infty du f(u)$ this can be used to avoid some IR divergences at x = 0 by splitting the integral into an IR divergent part and an UV divergent part $\int_0^\infty du = \int_0^a du + \int_a^\infty du$. For other types of divergent integrals like $\int_a^\infty dx \log^\beta(x) x^\alpha$ for positive α and β one could differentiate with respect to 'm' or 's' inside formula (2) in order to obtain a recurrence equation for the integrals $\int_a^\infty dx \log^\beta(x) x^\alpha$, this recurrence equation is finite (approximately) since for $\mathrm{Re}(p) > 1$ $\int_a^\infty dx \frac{\log^\beta(x)}{x^p}$ is finite and do not need to be regularized provided a > 0 . Other useful identities can be $(1+x)^{1/2} \approx 1 + \frac{x}{2} - \frac{x^2}{2.4}$ or the expansion of the logarithm valid for any x > 0 $\log x = 2\sum_{n=0}^{\infty} \frac{1}{2n+1}\left(\frac{x-1}{x+1}\right)^{2n+1}$ to make logarithms more tractable , also we could use Laurent expansions to handle complicate non-Polynomial expressions like $(x^n + \mu^n)^k$ by expanding it for big 'x' into asymptotic (inverse) power series.

- *Regularization of integrals in the form* $\int_0^\infty \frac{dx}{x^m}$ and $\int_0^\infty \frac{f(x)dx}{(x-a)^m (x-b)^m}$:

Until now, we have only considered the UV divergent integrals, the integrals whose integrand $f(x) \to \infty$ as $x \to \infty$, from the definition of an improper integral

$$\int_\varepsilon^\infty \frac{dx}{x^m} = \frac{\varepsilon^{m-1}}{m-1} = F_{reg}(m) \qquad \int_0^{1/\varepsilon} x^{m-2} dx = \frac{\varepsilon^{-(m-1)}}{m-1} = F_{reg}(2-m) \qquad \varepsilon = \frac{1}{N} \qquad (12)$$

As $N \to \infty$, this imply that in our regularization producedure $F_{reg}(s) = F_{reg}(2-s)$, for the case s > 0 we can use formulae (2) and (3) to regularize the divergent integral , and by the formula relating s and (2-s) one could also regularize IR (infrared) divergent integrals $\int_0^\infty \frac{dx}{x^m}$ in a similar way we did for $\int_0^\infty x^m dx$, except for the case m = 1 (logarithmic integral) ,for the case of this integral $\int_0^\infty \frac{dx}{x}$ one could split it into $\int_0^a \frac{dx}{x} + \int_a^\infty \frac{dx}{x}$ now we make the change of variables, $x \to x + a \quad x \to x + a^{-1}$ and $x \to \frac{1}{x}$ to rewrite this as $\int_0^\infty \frac{dx}{x+a} + \int_0^\infty \frac{dx}{x+1/a}$, these are again logarithmic divergent integrals and can be regularized with the aid of the zeta regularized product $\prod_{n=0}^\infty (n+b) = e^{-\zeta'_H(0,b)}$ plus the Euler-Maclaurin summation formula. For the case of a more general divergent integral like

$$\int_0^a \frac{f(x)dx}{(x-c)^m} \to \int_0^a \frac{f(x) - \sum_k f^{(k)}(c)(x-c)^k}{(x-c)^m} dx + \sum_k \int_0^a \frac{f^{(k)}(c)dx}{(x-c)^{m-k}} \qquad a > c > 0 \qquad (13)$$

First integral inside (13) is finite and after several manipulations the other divergent integrals can be written as $\int_0^\infty \frac{x^r dx}{(x^2 - (c+i\varepsilon)^2)^{m-k}}$ for some real and positive parameters 'r' 'c' 'm' and 'k', by multiplying both numerator and denominator by $(x+c)^{m-r}$

Another possibility is to avoid the pole at a certain point x = a by using the Analytic continuation of the integral involving several parameters

$$\int_0^\infty \frac{dx}{x^m} = \int_0^\infty \frac{dx(x+a)^m}{x^m (x+a)^m} = \sum_{i=0}^m \binom{m}{i} \int_0^\infty \frac{x^{m-i} a^i dx}{\left((x-\alpha_1)^2 + \alpha_2^2\right)^m} = F(\alpha_1, \alpha_2) \qquad (14)$$

The main idea is to calculate the integral (14) that will depend on two parameters α_i i =1,2 and finally set $\alpha_1 = -\frac{a}{2}$, $\alpha_2 = \pm\frac{ai}{2}$ if $F\left(-\frac{a}{2}, \frac{\pm ai}{2}\right)$ exists , this can be regarded as the regularized value of the integral, of course (14) may be divergent as $x \to \infty$ so we may need to add and substract terms to make it convergent in a similar way we did in (10) , another form to regularize (14) is defining $F\left(-\frac{a}{2}, \alpha_2\right)$ and then calculate this integral for a general value of α_2, in the end we would put $\alpha_2 = \pm\frac{ai}{2}$.
Another useful identity to regularize infrared divergences whenever x = a is (tables of integrals by Amabrowitz and Stegun [1])

$$\int_0^{\infty}\frac{x^m dx}{(x^2-(a+i\varepsilon)^2)^r}=\frac{(-1)^{r-1}\pi\left(-ia\right)^{m+1-2r}}{2\sin\left(\frac{m\pi+\pi}{2}\right)(r-1)!\,\Gamma\left(\frac{m+1}{3-r}\right)}\Gamma\left(\frac{m+1}{2}\right)=I(a,m,r) \qquad \varepsilon\to 0 \qquad (15)$$

Integral inside (14) will be convergent whenever 2r-m >1 if this is not the case we could use the Euclidean algorithm to split this integral into a convergent term defined as (14) and some divergent integrals $\int_0^{\infty} dx\left(x+a\right)^m$ m= -1 ,0,1,2,.... the main idea to justify why the infrared divergences are easier to regularize than ultraviolet ones , is that for the infrared, you could insert an small complex term $i\varepsilon$ to regularize it and make it convergent , so there are some complex values of 'a' that make (14) to be well-defined , however for the ultraviolet divergences this is not the case since there is no value of 'a' that makes $\int_0^{\infty}\frac{x^4 dx}{(x+a)^2}$ convergent , unless we use some kind of regularization

REGULARIZATION OF MULTIPLE INTEGRALS:

Until now, we have only considered integrals in one variable (after change to polar coordinates) , then it arises the question if one can apply our method of zeta regularization to more complicate integrals like

$$I(s)=\int d^4q_1\int d^4q_2..........\int d^4q_n\prod_{i=1}^{\infty}\frac{1}{\left(1+q_i^2\right)}F(q_1,q_2,.....,q_n)\left(R\left(q_1,q_2,.....,q_n\right)\right)^{-s} \qquad (16)$$

Here we have introduced a regulator depending on an external parameter 's' in order the integral (15) to converge for big 's' and then use the analytic regularization to take the limit $s\to 0^+$, this regulator must be chosen with care in order not to spoil any symmetries of the Physical system this regulator may be of the form

$$R\left(q_1,q_2,.....,q_n\right)=1+\sum_{i=1}^{n}q_i^2 \qquad R\left(q_1,q_2,.....,q_n\right)=\prod_{i=1}^{\infty}\left(1+q_i\right) \qquad (17)$$

Our first ansatz would be to define n-dimensional polar coordinates so we can rewrite (15) as a multiple integral depending on 'r' $\sqrt{\sum_{i=1}^{n}q_i^2}=r$ and several angles θ_i i= 1,2,3,4,..., n-1 , then (15) may be rewritten as

$$I(s)=\int_{\Omega}d\Omega\int_0^{\infty}drG(r,\theta_i)r^{n-1}\left(R(r,\theta_i)\right)^{-s} \qquad d\Omega=\prod_{i=1}^{n-1}d\theta_i\sin^{n-i-1}\left(\theta_i\right) \quad (18)$$

We may chase the first regulator inside (16) so it does not depend on the angular coordinates, the idea is that in case (16) has an ultraviolet divergence this divergence will appear whenever $r\to\infty$, so if we perform the integral over the angular variables

$d\Omega = \prod_{i=1}^{n-1} d\theta_i \sin^{n-i-1}(\theta_i)$ we are left with an integral $I(s) = \int_0^\infty drU(r)r^{n-1}(1+r)^{-s}$, in order to regularize this we define a convergent integral (by substraction) plus some divergent terms

$$I(s) = \int_0^\infty dr (1+r)^{-s} \left(U(r)r^{n-1} - \sum_{i=-1}^{k} a_i (1+r)^i \right) + \sum_{i=-1}^{k} a_i \int_0^\infty (1+r)^{i-s} \, dr \qquad (19)$$

U (r) is the function obtained after integration over the angles , and 'k' is a finite number to perform the minimal substraction of terms in order the first integral to be convergent even for s = 0 , if the integral over the angles is too complicate to have an exact form we could replace this integral over the angles by an approximate finite sum $d\Omega \to \sum_i$ in order to make the integral easier.

- *Substraction method:*

Once we have made the change of variable to spherical coordinates inside our integral $I(q_1, q_2, ..., q_n)$ one could substract some terms to render the integral finite

$$I(s) = \int_\Omega d\Omega \int_0^\infty dr \left(G(r, \theta_i) r^{n-1} - \sum_{j=-1}^{k} f_j(\theta_i)(1+r)^{j-s} \right) + \int_\Omega \sum_{j=-1}^{k} f_j(\theta_i) d\Omega \int_0^\infty dr (1+r)^{j-s} \quad (20)$$

We chose the number 'k' and the functions $f_j(\theta_i)$ so the first integral inside (20) is convergent , for the second integral we could perform integration over the angular variables and then use formulae (2) and (3) to regularize $\int_0^\infty (1+r)^m \, dr$.

Another method is to consider the multiple integral as an interate integral and then make the substraction for every variable for example

$$\int \partial q_n \left(F(q_1, q_2, ..., q_n) - \sum_{i=-1}^{k} a_i (q_1, q_2, ..., q_{n-1})(1+q_n)^i \right) + \int_0^\infty \partial q_n \sum_{i=-1}^{k} a_i (q_1, q_2, ..., q_{n-1})(1+q_n)^i$$

(21)

The symbol ∂q_n means that the integral is made over the variable q_n keeping the other variables constant , the number 'k' is chosen so the first integral is finite , this integral will depend on $I(q_1, q_2, ..., q_{n-1})$, the divergent integrals (even for the logarithmic case i=-1) can be regularized.

Now we have regularized the first integral, we have reduced in one variable the multiple integral, repeating the iterative process for the functions $a_i(q_1, q_2, ..., q_{n-1})$

$$\int \partial q_{n-1}\left(a_i(q_1,q_2,......,q_{n-1}) - \sum_{j=-1}^{k} b_j(q_1,q_2,.....,q_{n-2})(1+q_{n-1})^i \right) + \int_0^\infty \partial q_{n-1} \sum_{j=-1}^{k} b_i(q_1,q_2,.....,q_{n-2})(1+q_{n-1})^j$$

(22)

Using (21) and (22) for every step we can reduce the dimension of the integral until we reach to the one dimensional case, which is easier to handle. As an example

$$\int_0^\infty dx \int_0^\infty dy \frac{xy}{x+y+1} = \int_0^\infty dx \int_0^\infty dy \left(\frac{xy}{x+y+1} - x + \frac{x+x^2}{y+1} \right) + \int_0^\infty x dx \int_0^\infty dy - \int_0^\infty \left(x+x^2\right) dx \int_0^\infty \frac{dy}{y+1} \quad (23)$$

$$\int_0^\infty dx \left(f(x) - bx^2 + (a-b)x \right) \qquad f(x) = \int_0^\infty \frac{dy}{(y+1)} \frac{x^3+x^2}{\left(x+y+1\right)} \qquad (24)$$

With $a = \left(\int_0^\infty dx \right)_{reg}$ $b = \left(\int_0^\infty \frac{dy}{y+1} \right)_{reg}$ so for an initial given integral with an overlapping divergence as $x \to \infty$ $y \to \infty$ we have made a substraction to get a finite integral over 'y' (23) repeating the same process we can regularize the integral over 'x' , in order to integrate the finite part of the integral we can use several numerical methods.

If the integrand $F(q_1,q_2,.......,q_n)$ had no singularities for every $q_j > 0$, we may expand this integrand into a multiple Laurent series of several variables, and then perform the substraction $\sum_{m1,m2,.....,mn=-1}^{s1,s2,...,sn} C_{m1,m2,....,mn} \left(q_1+b_1\right)^{m1} \left(q_2+b_2\right)^{m2} ......\left(q_n+b_n\right)^{mn}$ in order to define a finite part of the integral

$$\int d^4q_1 \int d^4q_2 ....\int d^4q_n \left(F - \sum_{m1,m2,.....,mn=-1}^{s1,s2,...,sn} C_{m1,m2,....,mn} \left(q_1+b_1\right)^{m1} \left(q_2+b_2\right)^{m2} ......\left(q_n+b_n\right)^{mn} \right)$$

(25)

Plus some corrections due to divergent integrals $\int_0^\infty (q_i+b_i)^m dq_i$ m=-1,0,1,..... .In many cases although the integrals given in (21) and (22) are finite they will have no exact expression or the exact expression will be too complicate, in this case we can use the Gauss-Laguerre Quadrature formula (in case the interval is $[0,\infty)$) to approximate the integral by a sum over the zeros of Laguerre Polynomials $\sum_{i=0}^{n} w_i f(q_1,q_2,.....,q_{n-1},x_i)$ with the weigth expressed in terms of Laguerre Polynomials and their roots

$$w_i = \frac{x_i}{(n+1)^2 \left(L_{n+1}(x_i)\right)^2} \qquad L_n(x_i) = 0$$

CONCLUSIONS AND FINAL REMARKS:

We have extended the definition of the zeta regularization of a series to apply it to the Zeta regularization of a divergent integral $\int_0^\infty x^m dx$ m >0 by using the Zeta regularization technique combined with the Euler Maclaurin summation formula. For a good introduction to the Zeta regularization techniques , there is the book by Elizalde [4] or the Book by Brendt based on the mathematical discoveries of Ramanujan and its method of summation equivalent to the Zeta regularization algorithm [2] , another good reference (but a bit more advanced) is Zeidler [12] , for the case of Zeta-regularized determinants [7] is a good online reference describing also the process of Zeta regularization via analytic continuation and how it can be applied to prove the identity $\prod_{n=0}^{\infty}(n+1) = \log\sqrt{2\pi}$. Apparently there is a contradiction, since the Riemann Zeta funciton has a pole at s=1 so the Harmonic series could not be regularized, however using the definition of a functional determinant $\prod_{n=0}^{\infty}\frac{E_n}{\mu}$ $E_n = n + a$ one gets the finite result for the Harmonic (generalized) series $\sum_{n=0}^{\infty}\frac{1}{n+a} = -\frac{\Gamma'(a)}{\Gamma(a)}$, with the aid of the Euler-maclaurin summation formula this result for the Harmonic series can be used to give an approximate regularized value of the logarithmic integral $\int_0^\infty dx\frac{1}{x+a}$, for the case of other types of divergent integrals $\int_0^\infty dx\left(x+a\right)^m$ we can use again Euler-Maclaurin summatio formula to express this divergent integrals in terms of the negative values of the Hurwtiz or Riemann Zeta function $\zeta_H(s,1) = \zeta(s)$ $\zeta_H(-m,1)$ (UV) m= 0,1,2,3,4,..........and the value of the derivative of Hurwitz zeta function along s =0 $\partial_s\zeta_H(0,a)$ (logarithmic UV), these values encode the UV divergences [11] . For the case of the IR (infrared) divergences, $\int_0^\infty \frac{dx}{x^m}$ one could make a change of variable $x \to \frac{1}{x}$ to re-interpretate these integrals as $\int_0^\infty x^{m-2}dx$, We also believe that a similar procedure can be applied to extend our Zeta regularization algorithm to multiple (multi-loop) integrals $\int d^4q_1\int d^4q_2.........\int d^4q_n F(q_1,q_2,.......,q_n)$, one of the main advantages of this algorithm is that the dimension of the space does not appear explicitly so our method does not have the same problems as dimensional regularization, and can be used when the Dirac matrices $\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3$ appear

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