Cylindrical wave, wave equation, and method of separation of variables

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Abstract

It is shown that the wave equation cannot be solved for the general spreading of the cylindrical wave using the method of separation of variables. But an equation is presented in case of its solving the above act will have occurred. Also using this equation the above-mentioned general spreading of the cylindrical wave for large distances is obtained which contrary to what is believed consists of arbitrary functions.

1 Introduction

The wave equation $\partial^2\xi/\partial t^2 = v^2\nabla^2\xi$ is one of the most well-known equations in the classical physics. Particular solutions to this equation showing general spreading of plane and spherical waves are obtained easily using the method of guessing and trying (to these solutions is pointed in this paper). But by using this method finding the particular solution to this equation showing general spreading of a cylindrical wave has not been possible yet (we see this matter in the paper). Therefore, for finding this particular solution some of the physicists resort to another method named as the separation of variables (eg see Optics by Hecht and Zajak, Addison-Wesley, 1974, and Optics by Ajoy Ghatak, Tata McGraw Hill, 1977) and by using this method and the result obtained from it infer that the wave equation has been solved for the general spreading of the cylindrical wave excepting that contrary to the cases related to the plane and spherical waves there is no solution in terms of arbitrary functions in this case and eg for very far distances form of the wave function is restricted to only trigonometric functions (surprising that how a relation of wave motion can be restricted to only some particular functions).

In this article firstly it is shown that applying the method of separation of variables for obtaining the general spreading of the cylindrical wave from the wave equation is invalid, because with this act in fact only a particular state of spreading of cylindrical waves arising from interference of waves (producing nodes and bulges) can be obtained (of course if the boundary and initial conditions are satisfied) not general spreading of the cylindrical wave. Secondly, using the same method of guessing and trying we obtain the general spreading of the cylindrical wave for far distances from the wave equation such that includes arbitrary functions (and therefore there won't be necessary that in this case, contrary to other ones, to limit suddenly the arbitrary selection of functions for the wave function).

We implicitly get two results, one being that obtaining the general spreading of the cylindrical wave such that satisfies the wave equation appears to be an unsolved problem, and the other being that applying the method of separation of variables in the case of wave equations does not yield the general spreading of the waves but with satisfying the boundary and initial conditions only can show a particular state of the interference of waves providing that also in the process of solving the problem we don't encounter any contradictions, otherwise the problem can not be solved by this method at all. (This matter is important in general solving of the Schrodinger wave equation in which the method of separation of variables is used for obtaining the general spreading of the wave.)

2 Invalidity of the separation of variables for obtaining cylindrical wave function from the wave equation

Consider a stretched membrane fixed along its entire boundary in the xy-plane. The tension per unit length T and the mass per unit area m are constant. The deflection z of the membrane, supposing that is comparatively small, should be obtained from the following equation (see eg Differential Equations by Simmons, McGraw-Hill, 1972 or Advanced Engineering Mathematics by Kreyszig, John Wiley and Sons, 1979):

$$\frac{\partial^2 z}{\partial t^2} = a^2 (\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}) \ , \quad a = \sqrt{T/m} \tag{1}$$

If this membrane is a circular one of radius $\rho = \rho_0$ and the boundary condition is $z(\rho_0, \theta, t) = 0$ and the initial conditions are $z(\rho, \theta, 0) = f(\rho)$ and $\partial z/\partial t \mid_{t=0} = 0$ (ie it is fixed along its boundary and in $t = 0$ it is motionless and has the symmetric form of $f(\rho)$), then the solution of the equation (1) using the method of separation of variables and considering these conditions results in the following relation:

$$z = \sum_{n=1}^{\infty} a_n J_0(\frac{\lambda_n}{\rho_0}\rho) \cos(\frac{\lambda_n}{\rho_0} at) \ , \quad a_n = \frac{2}{\rho_0^2 J_1(\lambda_n)^2} \int_0^{\rho_0} \rho f(\rho) J_0(\frac{\lambda_n}{\rho_0}\rho) d\rho$$

Practically wherever the interference of waves and producing of standing waves are concerned, the method of separation of variables is efficacious for solving the wave equation. The reason of this matter can be seen cursorily in the result of the interference of the progressive wave $sink(\rho - vt)$ with the retrogressive wave $cosk(\rho + vt)$:

$$sink(\rho - vt) + cosk(\rho + vt) = (cosk\rho + sink\rho)(coskvt - sinkvt) \qquad (2)$$

As it is seen the variables are separated (the first parenthesis is a function of only ρ, and the second one is a function of only t, and we have obviously node situations). But while we are not faced by the phenomenon of interference of waves and the problem is only finding the relation of wave motion or in other words obtaining the general spreading of wave by proper particular solution of the wave equation, the method of separation of variables is wrong, because it is obvious that in this method we accept the existence of node situations implicitly, and anyway the relation of wave motion must have some arguments like $\rho \pm vt$ in order that it can demonstrate a wave motion and this is obviously contradictory to the separation of variables.

For clearing the above-mentioned material we try to solve the wave equation $\partial^2\xi/\partial t^2 = v^2\nabla^2\xi$ for the relation of cylindrical wave motion using the method of separation of variables and to see what the difficulty is. Suppose that source of the wave is the z-axis. Since the wave function ξ is independent of ϕ and z, the wave equation takes the form of

$$\frac{\partial^2\xi}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial\xi}{\partial\rho} = \frac{1}{v^2}\frac{\partial^2\xi}{\partial t^2}. \qquad (3)$$

Suppose that the general solution of this equation is $\xi(\rho, t) = \sum_n b_n u_n(\rho) w_n(t)$. Consider the general statement of this general solution, $\xi_n(\rho, t) = b_n u_n(\rho) w_n(t)$. Applying this ξ_n in the equation (3) yields

$$\frac{d^2u}{d\rho^2}\frac{1}{u} + \frac{1}{\rho}\frac{du}{d\rho}\frac{1}{u} = \frac{1}{v^2}\frac{d^2w}{dt^2}\frac{1}{w}$$

each side of which must be equal to a unique constant. We show this separation constant as $-\lambda_n^2$ (with $\lambda_n > 0$; it is easy to see why this constant cannot be non-negative). Therefore, the right side is solved as $w(t) = c_1 \cos\lambda_n vt + c_2 \sin\lambda_n vt$, and the left side results in Bessel's equation $\rho^2 u''(\rho) + \rho u'(\rho) + \lambda_n^2\rho^2 u(\rho) = 0$ which is solved as $u(\rho) = c_1' J_0(\lambda_n\rho) + c_2' Y_0(\lambda_n\rho)$.

Thus $\xi_n(\rho, t) = b_n(c_1' J_0(\lambda_n\rho) + c_2' Y_0(\lambda_n\rho))(c_1\cos(\lambda_n vt) + c_2\sin(\lambda_n vt))$ which for very large ρ's is reduced to

$$\xi_n(\rho, t) \approx b_n\sqrt{\frac{2}{\lambda_n\pi\rho}}(c_1'\cos(\lambda_n\rho - \frac{\pi}{4}) + c_2'\sin(\lambda_n\rho - \frac{\pi}{4}))(c_1\cos(\lambda_n vt) + c_2\sin(\lambda_n vt)).$$

The first parenthesis is a vibrating function of only ρ, and the second one is a vibrating function of only t, and obviously we have node situations, and the obtained form of ξ_n is rather similar to the cursory example (2) showing the result of interference of waves not spreading of a wave.

Maybe it is claimed hopelessly that although ξ_n does not show spreading of any wave (and can be result of some interference of waves), summation of all the ξ_n's can probably demonstrate spreading of a wave. But with some contemplation it can be understood that a theorem which is not true in case of the components, can not be true in case of the whole; in other words each ξ_n, as a particular solution, must demonstrate a physical independent wave. Furthermore, even if this matter is probable, for finding all the constant coefficients, the initial and boundary conditions must be applied and for application of the initial conditions we must have form of the wave in a definite time beforehand, while our problem is just finding the very form of the wave! This vicious circle in addition to all other material presented so far decisively shows that using the method of separation of variables for obtaining the general spreading of the cylindrical wave from the wave equation is invalid.

3 The way that the wave equation can be solved for the cylindrical wave

Equation $\partial^2\xi/\partial t^2 = v^2\nabla^2\xi$ appears in physics repeatedly wherever we know physically that the physical property ξ is being propagated with the speed v. Therefore, it is named as wave equation. So far, the general solution of this equation has not been obtained analytically such that generally it would have been proven that the obtained general solution is the same relation of wave motion causing propagation of the property ξ. (Of course in the one-dimensional case of this equation $\partial^2\xi/\partial t^2 = v^2\partial^2\xi/\partial x^2$, the general solution $\xi(x,t) = f_1(x - vt) + f_2(x + vt)$ with arbitrary functions f_1 and f_2 is obtained showing clearly a wave motion along the x-axis propagating the form of the arbitrary functions f_1 and f_2 with the speed v along this axis.) But since everywhere a wave motion is encountered this equation appears, we can be certain that the general solution to this equation is really a wave motion relation and each wave motion relation satisfies this equation. Therefore, eg we expect the motion relation of a plane wave propagating along the $\hat{u}$ with the speed v, ie $\xi(\mathbf{r},t) = f_1(\hat{u}\cdot\mathbf{r} - vt) + f_2(\hat{u}\cdot\mathbf{r} + vt)$, to satisfy the wave equation. (f_1 and f_2 are arbitrary functions. It is clear that in a plane wave the wave amplitude is constant.) Considering the direction cosines of the constant unit vector $\hat{u}$ and Cartesian components of $\mathbf{r}$, it can easily be seen that this relation satisfies the wave equation.

We also expect that a spherical or cylindrical wave (or other forms of wave eg an ellipsoid one) to satisfy the wave equation. A spherical wave has the form of

$$\xi_1(\mathbf{r},t) = g_1(\mathbf{r},t)(f_1(r - v_1t) + f_2(r + v_1t)), \tag{4}$$

and a cylindrical wave has the form of

$$\xi_2(\mathbf{r},t) = g_2(\rho,t)(f_3(\rho - v_2 t) + f_4(\rho + v_2 t)), \tag{5}$$

in which f_1, f_2, f_3 and f_4 are arbitrary functions, and g_1 and g_2 are amplitude coefficients (because it is clear that with wave spreading its amplitude decreases and probably is periodic in time in terms of the form of the wave).

We accept the general validity of the wave equation and apply it to the relations (4) and (5) in order that the amplitude coefficients will be obtained; then we shall justify the form of the waves obtained with these amplitudes physically.

If the independent variables in the spherical and cylindrical polar coordinates are (r,θ,ϕ) and (ρ,ϕ,z) respectively, because of the independence of ξ_1 from θ and ϕ and of ξ_2 from ϕ and z we shall have the following equations (using the Laplacian in its proper form in each coordinates):

$$\frac{\partial^2 \xi_1}{\partial t^2} = v_1^2 \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 \frac{\partial \xi_1}{\partial r}) \tag{6}$$

being the wave equation for the spherical wave ξ_1, and

$$\frac{\partial^2 \xi_2}{\partial t^2} = v_2^2 \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho \frac{\partial \xi_2}{\partial \rho}) \tag{7}$$

being the wave equation for the cylindrical wave ξ_2.

After some simple differentiations and algebraic operations, we shall have

$$\frac{\partial^2 g_1}{\partial t^2} + v_1 A_1 \frac{\partial g_1}{\partial t} = v_1^2(\frac{\partial^2 g_1}{\partial r^2} + (B_1 + \frac{2}{r})\frac{\partial g_1}{\partial r} + \frac{B_1}{r} g_1) \tag{8}$$

and

$$\frac{\partial^2 g_2}{\partial t^2} + v_2 A_2 \frac{\partial g_2}{\partial t} = v_2^2(\frac{\partial^2 g_2}{\partial \rho^2} + (2B_2 + \frac{1}{\rho})\frac{\partial g_2}{\partial \rho} + \frac{B_2}{\rho} g_2) \tag{9}$$

for the spherical and cylindrical waves respectively, in which

$$A_1 = A_1(r,t) = 2(-df_1/d(r - v_1 t) + df_2/d(r + v_1 t))/(f_1 + f_2),$$

$$B_1 = B_1(r,t) = 2(df_1/d(r - v_1 t) + df_2/d(r + v_1 t))/(f_1 + f_2),$$

$$A_2 = A_2(\rho,t) = 2(-df_3/d(\rho - v_2 t) + df_4/d(\rho + v_2 t))/(f_3 + f_4)$$

and $B_2 = B_2(\rho,t) = (df_3/d(\rho - v_2 t) + df_4/d(\rho + v_2 t))/(f_3 + f_4)$.

To obtain g_1 and g_2 the partial differential equations (8) and (9) must be solved. It is obvious that these equations can not be solved by the method of separation of variables. We shall now solve the equation (8) easily and also solve the equation (9) for when ρ approaches infinity, but its general solution should be found by interested physicists or mathematicians.

In order to solve (8) we try a solution that is independent of time (causing the left side of the equation to be zero) and its dependence on

r is such that the terms including B_1 cancel each other, ie some g_1 that satisfies the equation

$$B_1 \frac{\partial g_1}{\partial r} + \frac{B_1}{r} g_1 = 0. \tag{10}$$

Then for finding out that this solution is acceptable or not we must try it for other terms (excluding B_1) of the right side. If sum of them is zero, g_1 will be the acceptable solution of (8).

Thus, first of all we solve the equation (10). Its solution, considering being independent of time, is:

$$g_1 = \frac{1}{r} \tag{11}$$

Trying of this solution shows that sum of all the terms of the right side and also each term of the left side is zero. Then (11) is really the solution of (8). Therefore, the spherical wave has the form of $(1/r)(f_1(r-vt)+f_2(r+vt))$, and this is quite natural physically, because the conservation law of energy necessitates that since the sphere surface is proportional to r^2 causing the proportion of the surface density of energy to $1/r^2$, the wave amplitude is proportional to $\sqrt{1/r^2}$ or $1/r$.

Trying to solve the equation (9) we try the same method used for solving the equation (8). Then, we solve the equation $2B_2 \partial g_2/\partial\rho + (B_2/\rho)g_2 = 0$ and obtain the solution:

$$g_2 = \frac{1}{\sqrt{\rho}} \tag{12}$$

But trying of this solution yields the expression

$$v^2 \frac{1}{4} \rho^{-5/2} \tag{13}$$

for the right side of (9), while the left side will be zero. Then generally (12) is not an acceptable solution to (9), but when ρ approaches infinity, (13) approaches zero ie approaches being equal to the left side being zero. Thus, for infinite ρ's the solution of (9) is (12). In other words the cylindrical wave, when ρ approaches infinity, has the form of

$$\xi = \frac{1}{\sqrt{\rho}}(f_1(\rho - vt) + f_2(\rho + vt)), \tag{14}$$

and this is also natural physically, because the conservation law of energy necessitates that since the lateral area of the cylinder is proportional to ρ causing the proportion of the surface density of energy to $1/\rho$, the wave amplitude is proportional to $1/\sqrt{\rho}$. But, why is this physical justification true only for very large ρ's? Because generally all the energy produced from the cylinder axis is not propagated through the lateral surface, but some of it propagates through the two bases of the cylinder which this itself does not allow the coefficient to be $1/\sqrt{\rho}$ exactly. (Visualize the wavelets produced from each point of the axis which necessarily spread through the bases.) Furthermore, it is comprehensible that the part of the energy that passes through the bases depends also on the form (or shape) of the wave which this itself justifies the dependence of the amplitude on the time which probably we shall observe after obtaining the general solution of (9).

But when ρ increases, the bases area increases proportional to ρ^2, while the lateral area increases proportional to ρ. If we suppose that the corner wavelets (at the circumferences of the bases) transmit the same energy through the bases as through the lateral surface, then we conclude that when ρ increases the surface density of the energy of the waves passing through the bases decreases proportional to $1/\rho^2$, while the surface density of the energy of the same waves passing through the lateral surface decreases proportional to $1/\rho$. It is obvious that when ρ increases very much the importance of the wave energy passing through the bases decreases very much in comparison with the one passing through the lateral surface.

The other similar manner to obtain (14) as the cylindrical wave for infinite ρ's is trying (14) in each side of the wave equation $\partial^2\xi/\partial t^2 = v^2\nabla^2\xi$. For the left and right sides the following expressions are obtained respectively:

$$\frac{\partial^2\xi}{\partial t^2} = v^2\rho^{-1/2}(\frac{d^2 f_1}{d(\rho - vt)^2} + \frac{d^2 f_2}{d(\rho + vt)^2}) \tag{15}$$

$$v^2\nabla^2\xi = v^2\rho^{-1/2}(\frac{d^2 f_1}{d(\rho - vt)^2} + \frac{d^2 f_2}{d(\rho + vt)^2}) + \frac{v^2}{4}\rho^{-5/2}(f_1 + f_2) \tag{16}$$

It is clear that when ρ increases very much the importance of the second term of the right side of the relation (16) decreases in comparison with the first term, and when ρ approaches infinity we can relinquish it, and then deduce from (15) and (16) that (14) satisfies the wave equation for infinite ρ's.