TGD AS A GENERALIZED NUMBER THEORY

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0.1 Background

T(opological) G(eometro)D(ynamics) is one of the many attempts to find a unified description of basic interactions. The development of the basic ideas of TGD to a relatively stable form took time of about half decade [16]. The great challenge is to construct a mathematical theory around these physically very attractive ideas and I have devoted the last twenty-three years for the realization of this dream and this has resulted in seven online books [TGDview, TGDgeom, TGDquant, TGDnumber, TGDclass, TGDpad, TGDfree] about TGD and eight online books about TGD inspired theory of consciousness and of quantum biology [TGDconsc, TGDselforg, TGDware, TGDholo, TGDgeme, TGDeeg, TGDmagn, 15].

Quantum T(opological)D(ynamics) as a classical spinor geometry for infinite-dimensional configuration space, p-adic numbers and quantum TGD, and TGD inspired theory of consciousness have been for last decade of the second millenium the basic three strongly interacting threads in the tapestry of quantum TGD.

For few yeas ago the discussions with Tony Smith generated a fourth thread which deserves the name 'TGD as a generalized number theory'. The work with Riemann hypothesis made time ripe for realization that the notion of infinite primes could provide, not only a reformulation, but a deep generalization of quantum TGD. This led to a thorough and extremely fruitful revision of the basic views about what the final form and physical content of quantum TGD might be.

The fifth thread came with the realization that by quantum classical correspondence TGD predicts an infinite hierarchy of macroscopic quantum systems with increasing sizes, that it is not at all clear whether standard quantum mechanics can accommodate this hierarchy, and that a dynamical quantized Planck constant might be necessary and certainly possible in TGD framework. The identification of hierarchy of Planck constants whose values TGD "predicts" in terms of dark matter hierarchy would be natural. This also led to a solution of a long standing puzzle: what is the proper interpretation of the predicted fractal hierarchy of long ranged classical electro-weak and color gauge fields. Quantum classical correspondences allows only single answer: there is infinite hierarchy of p-adically scaled up variants of standard model physics and for each of them also dark hierarchy. Thus TGD Universe would be fractal in very abstract and deep sense.

TGD forces the generalization of physics to a quantum theory of consciousness, and represent TGD as a generalized number theory vision leads naturally to the emergence of p-adic physics as physics of cognitive representations. The seven online books [TGDview, TGDgeom, TGDquant, TGDnumber, TGDclass, TGDpad, TGDfree] about TGD and eight online books about TGD inspired theory of consciousness and of quantum biology [TGDconsc, TGDselforg, TGDware, TGDholo, TGDgeme, TGDeeg, TGDmagn, 15] are warmly recommended to the interested reader.

0.2 Basic Ideas of TGD

The basic physical picture behind TGD was formed as a fusion of two rather disparate approaches: namely TGD is as a Poincare invariant theory of gravitation and TGD as a generalization of the old-fashioned string model.

0.2.1 TGD as a Poincare invariant theory of gravitation

The first approach was born as an attempt to construct a Poincare invariant theory of gravitation. Space-time, rather than being an abstract manifold endowed with a pseudo-Riemannian structure, is regarded as a surface in the 8-dimensional space $H = M^4_+ \times CP_2$, where M^4_+ denotes the interior of the future light cone of the Minkowski space (to be referred as light cone in the sequel) and $CP_2 = SU(3)/U(2)$ is the complex projective space of two complex dimensions [2, 18, 19, 5]. The identification of the space-time as a submanifold [21, 22] of $M^4 \times CP_2$ leads to an exact Poincare invariance and solves the conceptual difficulties related to the definition of the energy-momentum in General Relativity [Misner-Thorne-Wheeler, Logunov *et al*]. The actual choice $H = M^4_+ \times CP_2$ implies the breaking of the Poincare invariance in the cosmological scales but only at the quantum level. It soon however turned out that submanifold geometry, being considerably richer in structure than the abstract manifold geometry, leads to a geometrization of all basic interactions. First, the geometrization of the elementary particle quantum numbers is achieved. The geometry of CP_2 explains electro-weak and color quantum numbers. The different H-chiralities of H-spinors correspond to the conserved baryon and lepton numbers. Secondly, the geometrization of the field concept results. The projections of the CP_2 spinor connection, Killing vector fields of CP_2 and of H-metric to four-surface define classical electro-weak, color gauge fields and metric in X^4.

0.2.2 TGD as a generalization of the hadronic string model

The second approach was based on the generalization of the mesonic string model describing mesons as strings with quarks attached to the ends of the string. In the 3-dimensional generalization 3-surfaces correspond to free particles and the boundaries of the 3- surface correspond to partons in the sense that the quantum numbers of the elementary particles reside on the boundaries. Various boundary topologies (number of handles) correspond to various fermion families so that one obtains an explanation for the known elementary particle quantum numbers. This approach leads also to a natural topological description of the particle reactions as topology changes: for instance, two-particle decay corresponds to a decay of a 3-surface to two disjoint 3-surfaces.

0.2.3 Fusion of the two approaches via a generalization of the space-time concept

The problem is that the two approaches seem to be mutually exclusive since the orbit of a particle like 3-surface defines 4-dimensional surface, which differs drastically from the topologically trivial macroscopic space-time of General Relativity. The unification of these approaches forces a considerable generalization of the conventional space-time concept. First, the topologically trivial 3-space of General Relativity is replaced with a "topological condensate" containing matter as particle like 3-surfaces "glued" to the topologically trivial background 3-space by connected sum operation. Secondly, the assumption about connectedness of the 3-space is given up. Besides the "topological condensate" there is "vapor phase" that is a "gas" of particle like 3-surfaces (counterpart of the "baby universies" of GRT) and the nonconservation of energy in GRT corresponds to the transfer of energy between the topological condensate and vapor phase.

0.3 The five threads in the development of quantum TGD

The development of TGD has involved four strongly interacting threads: physics as infinite-dimensional geometry; p-adic physics; TGD inspired theory of consciousness and TGD as a generalized number theory. In the following these five threads are briefly described.

0.3.1 Quantum TGD as configuration space spinor geometry

A turning point in the attempts to formulate a mathematical theory was reached after seven years from the birth of TGD. The great insight was "Do not quantize". The basic ingredients to the new approach have served as the basic philosophy for the attempt to construct Quantum TGD since then and are the following ones:

a) Quantum theory for extended particles is free(!), classical(!) field theory for a generalized Schrödinger amplitude in the configuration space CH consisting of all possible 3-surfaces in H. "All possible" means that surfaces with arbitrary many disjoint components and with arbitrary internal topology and also singular surfaces topologically intermediate between two different manifold topologies are included. Particle reactions are identified as topology changes [23, 24, 25]. For instance, the decay of a 3-surface to two 3-surfaces corresponds to the decay $A \rightarrow B + C$. Classically this corresponds to a path of configuration space leading from 1-particle sector to 2-particle sector. At quantum level this corresponds to the dispersion of the generalized Schrödinger amplitude localized to 1-particle sector to two-particle sector. All coupling constants should result as predictions of the theory since no nonlinearities are introduced.

b) Configuration space is endowed with the metric and spinor structure so that one can define various metric related differential operators, say Dirac operator, appearing in the field equations of the theory.

0.3.2 p-Adic TGD

The p-adic thread emerged for roughly ten years ago as a dim hunch that p-adic numbers might be important for TGD. Experimentation with p-adic numbers led to the notion of canonical identification mapping reals to p-adics and vice versa. The breakthrough came with the successful p-adic mass calculations using p-adic thermodynamics for Super-Virasoro representations with the super-Kac-Moody algebra associated with a Lie-group containing standard model gauge group. Although the details of the calculations have varied from year to year, it was clear that p-adic physics reduces not only the ratio of proton and Planck mass, the great mystery number of physics, but all elementary particle mass scales, to number theory if one assumes that primes near prime powers of two are in a physically favored position. Why this is the case, became one of the key puzzless and led to a number of arguments with a common gist: evolution is present already at the elementary particle level and the primes allowed by the p-adic length scale hypothesis are the fittest ones.

It became very soon clear that p-adic topology is not something emerging in Planck length scale as often believed, but that there is an infinite hierarchy of p-adic physics characterized by p-adic length scales varying to even cosmological length scales. The idea about the connection of p-adics with cognition motivated already the first attempts to understand the role of the p-adics and inspired 'Universe as Computer' vision but time was not ripe to develop this idea to anything concrete (p-adic numbers are however in a central role in TGD inspired theory of consciousness). It became however obvious that the p-adic length scale hierarchy somehow corresponds to a hierarchy of intelligences and that p-adic prime serves as a kind of intelligence quotient. Ironically, the almost obvious idea about p-adic regions as cognitive regions of space-time providing cognitive representations for real regions had to wait for almost a decade for the access into my consciousness.

There were many interpretational and technical questions crying for a definite answer. What is the relationship of p-adic non-determinism to the classical non-determinism of the basic field equations of TGD? Are the p-adic space-time region genuinely p-adic or does p-adic topology only serve as an effective topology? If p-adic physics is direct image of real physics, how the mapping relating them is constructed so that it respects various symmetries? Is the basic physics p-adic or real (also real TGD seems to be free of divergences) or both? If it is both, how should one glue the physics in different number field together to get *The Physics*? Should one perform p-adicization also at the level of the configuration space of 3-surfaces? Certainly the p-adicization at the level of super-conformal representation is necessary for the p-adic mass calculations. Perhaps the most basic and most irritating technical problem was how to precisely define p-adic definite integral which is a crucial element of any variational principle based formulation of the field equations. Here the frustration was not due to the lack of solution but due to the too large number of solutions to the problem, a clear symptom for the sad fact that clever inventions rather than real discoveries might be in question.

Despite these frustrating uncertainties, the number of the applications of the poorly defined p-adic physics growed steadily and the applications turned out to be relatively stable so that it was clear that the solution to these problems must exist. It became only gradually clear that the solution of the problems might require going down to a deeper level than that represented by reals and p-adics.

0.3.3 TGD as a generalization of physics to a theory consciousness

General coordinate invariance forces the identification of quantum jump as quantum jump between entire deterministic quantum histories rather than time=constant snapshots of single history. The new view about quantum jump forces a generalization of quantum measurement theory such that observer becomes part of the physical system. Thus a general theory of consciousness is unavoidable outcome. This theory is developed in detail in the books [TGDconsc, TGDselforg, TGDware, TGDholo, TGDgeme, TGDeeg, TGDmagn, 15].

Quantum jump as a moment of consciousness

The identification of quantum jump between deterministic quantum histories (configuration space spinor fields) as a moment of consciousness defines microscopic theory of consciousness. Quantum jump involves the steps

$$\Psi_i \rightarrow U\Psi_i \rightarrow \Psi_f \quad ,$$

where U is informational "time development" operator, which is unitary like the S-matrix characterizing the unitary time evolution of quantum mechanics. U is however only formally analogous to Schrödinger time evolution of infinite duration although there is *no* real time evolution involved. It is not however clear whether one should regard U-matrix and S-matrix as two different things or not: U-matrix is a completely universal object characterizing the dynamics of evolution by self-organization whereas S-matrix is a highly context dependent concept in wave mechanics and in quantum field theories where it at least formally represents unitary time translation operator at the limit of an infinitely long interaction time. The S-matrix understood in the spirit of superstring models is however something very different and could correspond to U-matrix.

The requirement that quantum jump corresponds to a measurement in the sense of quantum field theories implies that each quantum jump involves localization in zero modes which parameterize also the possible choices of the quantization axes. Thus the selection of the quantization axes performed by the Cartesian outsider becomes now a part of quantum theory. Together these requirements imply that the final states of quantum jump correspond to quantum superpositions of space-time surfaces which are macroscopically equivalent. Hence the world of conscious experience looks classical. At least formally quantum jump can be interpreted also as a quantum computation in which matrix U represents unitary quantum computation which is however not identifiable as unitary translation in time direction and cannot be 'engineered'.

The notion of self

The concept of self is absolutely essential for the understanding of the macroscopic and macro-temporal aspects of consciousness. Self corresponds to a subsystem able to remain un-entangled under the sequential informational 'time evolutions' U. Exactly vanishing entanglement is practically impossible in ordinary quantum mechanics and it might be that 'vanishing entanglement' in the condition for self-property should be replaced with 'subcritical entanglement'. On the other hand, if space-time decomposes into p-adic and real regions, and if entanglement between regions representing physics in different number fields vanishes, space-time indeed decomposes into selves in a natural manner.

It is assumed that the experiences of the self after the last 'wake-up' sum up to single average experience. This means that subjective memory is identifiable as conscious, immediate short term memory. Selves form an infinite hierarchy with the entire Universe at the top. Self can be also interpreted as mental images: our mental images are selves having mental images and also we represent mental images of a higher level self. A natural hypothesis is that self S experiences the experiences of its subselves as kind of abstracted experience: the experiences of subselves S_i are not experienced as such but represent kind of averages $\langle S_{ij} \rangle$ of sub-subselves S_{ij}. Entanglement between selves, most naturally realized by the formation of join along boundaries bonds between cognitive or material space-time sheets, provides a possible a mechanism for the fusion of selves to larger selves (for instance, the fusion of the mental images representing separate right and left visual fields to single visual field) and forms wholes from parts at the level of mental images.

Relationship to quantum measurement theory

The third basic element relates TGD inspired theory of consciousness to quantum measurement theory. The assumption that localization occurs in zero modes in each quantum jump implies that the world of conscious experience looks classical. It also implies the state function reduction of the standard quantum measurement theory as the following arguments demonstrate (it took incredibly long time to realize this almost obvious fact!).

a) The standard quantum measurement theory a la von Neumann involves the interaction of brain with the measurement apparatus. If this interaction corresponds to entanglement between microscopic degrees of freedom m with the macroscopic effectively classical degrees of freedom M characterizing the reading of the measurement apparatus coded to brain state, then the reduction of this entanglement in quantum jump reproduces standard quantum measurement theory provide the unitary time evolution operator U acts as flow in zero mode degrees of freedom and correlates completely some orthonormal basis of configuration space spinor fields in non-zero modes with the values of the zero modes. The flow property guarantees that the localization is consistent with unitarity: it also means 1-1 mapping of quantum state basis to classical variables (say, spin direction of the electron to its orbit in the external magnetic field).

b) Since zero modes represent classical information about the geometry of space-time surface (shape, size, classical Kähler field,...), they have interpretation as effectively classical degrees of freedom and are the TGD counterpart of the degrees of freedom M representing the reading of the measurement apparatus. The entanglement between quantum fluctuating non-zero modes and zero modes is the TGD counterpart for the $m-M$ entanglement. Therefore the localization in zero modes is equivalent with a quantum jump leading to a final state where the measurement apparatus gives a definite reading.

This simple prediction is of utmost theoretical importance since the black box of the quantum measurement theory is reduced to a fundamental quantum theory. This reduction is implied by the replacement of the notion of a point like particle with particle as a 3-surface. Also the infinite-dimensionality of the zero mode sector of the configuration space of 3-surfaces is absolutely essential. Therefore the reduction is a triumph for quantum TGD and favors TGD against string models.

Standard quantum measurement theory involves also the notion of state preparation which reduces to the notion of self measurement. Each localization in zero modes is followed by a cascade of self measurements leading to a product state. This process is obviously equivalent with the state preparation process. Self measurement is governed by the so called Negentropy Maximization Principle (NMP) stating that the information content of conscious experience is maximized. In the self measurement the density matrix of some subsystem of a given self localized in zero modes (after ordinary quantum measurement) is measured. The self measurement takes place for that subsystem of self for which the reduction of the entanglement entropy is maximal in the measurement. In p-adic context NMP can be regarded as the variational principle defining the dynamics of cognition. In real context self measurement could be seen as a repair mechanism allowing the system to fight against quantum thermalization by reducing the entanglement for the subsystem for which it is largest (fill the largest hole first in a leaking boat).

Selves self-organize

The fourth basic element is quantum theory of self-organization based on the identification of quantum jump as the basic step of self-organization [I1]. Quantum entanglement gives rise to the generation of long range order and the emergence of longer p-adic length scales corresponds to the emergence of larger and larger coherent dynamical units and generation of a slaving hierarchy. Energy (and quantum entanglement) feed implying entropy feed is a necessary prerequisite for quantum self-organization. Zero modes represent fundamental order parameters and localization in zero modes implies that the sequence of quantum jumps can be regarded as hopping in the zero modes so that Haken's classical theory of self organization applies almost as such. Spin glass analogy is a further important element: self-organization of self leads to some characteristic pattern selected by dissipation as some valley of the "energy" landscape.

Dissipation can be regarded as the ultimate Darwinian selector of both memes and genes. The mathematically ugly irreversible dissipative dynamics obtained by adding phenomenological dissipation terms to the reversible fundamental dynamical equations derivable from an action principle can be understood as a phenomenological description replacing in a well defined sense the series of reversible quantum histories with its envelope.

Classical non-determinism of Kähler action

The fifth basic element are the concepts of association sequence and cognitive space-time sheet. The huge vacuum degeneracy of the Kähler action suggests strongly that the absolute minimum space-time is not always unique. For instance, a sequence of bifurcations can occur so that a given space-time branch can be fixed only by selecting a finite number of 3-surfaces with time like(!) separations on the orbit of 3-surface. Quantum classical correspondence suggest an alternative formulation. Space-time surface decomposes into maximal deterministic regions and their temporal sequences have interpretation a space-time correlate for a sequence of quantum states defined by the initial (or final) states of quantum jumps. This is consistent with the fact that the variational principle selects preferred extremals of Kähler action as generalized Bohr orbits.

In the case that non-determinism is located to a finite time interval and is microscopic, this sequence of 3-surfaces has interpretation as a simulation of a classical history, a geometric correlate for contents of consciousness. When non-determinism has long lasting and macroscopic effect one can identify it as volitional non-determinism associated with our choices. Association sequences relate closely with the cognitive space-time sheets defined as space-time sheets having finite time duration and psychological time can be identified as a temporal center of mass coordinate of the cognitive space-time sheet. The gradual drift of the cognitive space-time sheets to the direction of future force by the geometry of the future light cone explains the arrow of psychological time.

p-Adic physics as physics of cognition and intentionality

The sixth basic element adds a physical theory of cognition to this vision. TGD space-time decomposes into regions obeying real and p-adic topologies labelled by primes $p = 2, 3, 5, ...$ p-Adic regions obey the same field equations as the real regions but are characterized by p-adic non-determinism since the functions having vanishing p-adic derivative are pseudo constants which are piecewise constant functions. Pseudo constants depend on a finite number of positive pinary digits of arguments just like numerical predictions of any theory always involve decimal cutoff. This means that p-adic space-time regions are obtained by gluing together regions for which integration constants are genuine constants. The natural interpretation of the p-adic regions is as cognitive representations of real physics. The freedom of imagination is due to the p-adic non-determinism. p-Adic regions perform mimicry and make possible for the Universe to form cognitive representations about itself. p-Adic physics space-time sheets serve also as correlates for intentional action.

A more more precise formulation of this vision requires a generalization of the number concept obtained by fusing reals and p-adic number fields along common rationals (in the case of algebraic extensions among common algebraic numbers). This picture is discussed in [E1]. The application this notion at the level of the imbedding space implies that imbedding space has a book like structure with various variants of the imbedding space glued together along common rationals (algebraics). The implication is that genuinely p-adic numbers (non-rationals) are strictly infinite as real numbers so that most points of p-adic space-time sheets are at real infinity, outside the cosmos, and that the projection to the real imbedding space is discrete set of rationals (algebraics). Hence cognition and intentionality are almost completely outside the real cosmos and touch it at a discrete set of points only.

This view implies also that purely local p-adic physics codes for the p-adic fractality characterizing long range real physics and provides an explanation for p-adic length scale hypothesis stating that the primes $p \simeq 2^k$, k integer are especially interesting. It also explains the long range correlations and short term chaos characterizing intentional behavior and explains why the physical realizations of cognition are always discrete (say in the case of numerical computations). Furthermore, a concrete quantum model for how intentions are transformed to actions emerges.

The discrete real projections of p-adic space-time sheets serve also space-time correlate for a logical thought. It is very natural to assign to p-adic pinary digits a p-valued logic but as such this kind of logic does not have any reasonable identification. p-Adic length scale hypothesis suggest that the $p = 2^k - n$ pinary digits represent a Boolean logic B^k with k elementary statements (the points of the k-element set in the set theoretic realization) with n taboos which are constrained to be identically true.

0.3.4 TGD as a generalized number theory

Quantum T(opological)D(ynamics) as a classical spinor geometry for infinite-dimensional configuration space, p-adic numbers and quantum TGD, and TGD inspired theory of consciousness, have been for last ten years the basic three strongly interacting threads in the tapestry of quantum TGD. For few yeas ago the discussions with Tony Smith generated a fourth thread which deserves the name 'TGD as a generalized number theory'. It involves three separate threads: the fusion of real and various p-adic physics to a single coherent whole by requiring number theoretic universality discussed already, the formulation of quantum TGD in terms of hyper-counterparts of classical number fields identified as sub-spaces of complexified classical number fields with Minkowskian signature of the metric defined by the complexified inner product, and the notion of infinite prime.

The role of classical number fields

The vision about the physical role of the classical number fields relies on the notion of number theoretic compactifiction stating that space-time surfaces can be regarded as surfaces of either M^8 or $M^4 \times CP_2$.

As surfaces of M^8 identifiable as space of hyper-octonions they are hyper-quaternionic or co-hyper-quaternionic- and thus maximally associative or co-associative. This means that their tangent space is either hyper-quaternionic plane of M^8 or an orthogonal complement of such a plane. These surface can be mapped in natural manner to surfaces in $M^4 \times CP_2$ [E2] provided one can assign to each point of tangent space a hyper-complex plane $M^2(x) \subset M^4$. One can also speak about $M^8 - H$ duality.

This vision has very strong predictive power. It predicts that the extremals of Kähler action correspond to either hyper-quaternionic or co-hyper-quaternionic surfaces such that one can assign to tangent space at each point of space-time surface a hyper-complex plane $M^2(x) \subset M^4$. As a consequence, the M^4 projection of space-time surface at each point contains $M^2(x)$ and its orthogonal complement. These distributions are integrable implying that space-time surface allows dual slicings defined by string world sheets Y^2 and partonic 2-surfaces X^2. The existence of this kind of slicing was earlier deduced from the study of extremals of Kähler action and christened as Hamilton-Jacobi structure. The physical interpretation of $M^2(x)$ is as the space of non-physical polarizations and the plane of local 4-momentum.

One can fairly say, that number theoretical compactification is responsible for most of the understanding of quantum TGD that has emerged during last years. This includes the realization of Equivalence Principle at space-time level, dual formulations of TGD as Minkowskian and Euclidian string model type theories, the precise identification of preferred extremals of Kähler action as extremals for which second variation vanishes (at least for deformations representing dynamical symmetries) and thus providing space-time correlate for quantum criticality, the notion of number theoretic braid implied by the basic dynamics of Kähler action and crucial for precise construction of quantum TGD as almost-topological QFT, the construction of configuration space metric and spinor structure in terms of second quantized induced spinor fields with modified Dirac action defined by Kähler action realizing automatically the notion of finite measurement resolution and a connection with inclusions of hyper-finite factors of type II_1 about which Clifford algebra of configuration space represents an example.

Infinite primes

The discovery of the hierarchy of infinite primes and their correspondence with a hierarchy defined by a repeatedly second quantized arithmetic quantum field theory gave a further boost for the speculations about TGD as a generalized number theory. The work with Riemann hypothesis led to further ideas.

After the realization that infinite primes can be mapped to polynomials representable as surfaces geometrically, it was clear how TGD might be formulated as a generalized number theory with infinite primes forming the bridge between classical and quantum such that real numbers, p-adic numbers, and various generalizations of p-adics emerge dynamically from algebraic physics as various completions of the algebraic extensions of rational (hyper-)quaternions and (hyper-)octonions. Complete algebraic, topological and dimensional democracy would characterize the theory.

What is especially satisfying is that p-adic and real regions of the space-time surface could emerge automatically as solutions of the field equations. In the space-time regions where the solutions of field equations give rise to in-admissible complex values of the imbedding space coordinates, p-adic solution can exist for some values of the p-adic prime. The characteristic non-determinism of the p-adic differential equations suggests strongly that p-adic regions correspond to 'mind stuff', the regions of space-time where cognitive representations reside. This interpretation implies that p-adic physics is physics of cognition. Since Nature is probably extremely brilliant simulator of Nature, the natural idea is to study the p-adic physics of the cognitive representations to derive information about the real physics. This view encouraged by TGD inspired theory of consciousness clarifies difficult interpretational issues and provides a clear interpretation for the predictions of p-adic physics.

0.3.5 Dynamical quantized Planck constant and dark matter hierarchy

By quantum classical correspondence space-time sheets can be identified as quantum coherence regions. Hence the fact that they have all possible size scales more or less unavoidably implies that Planck constant must be quantized and have arbitrarily large values. If one accepts this then also the idea about dark matter as a macroscopic quantum phase characterized by an arbitrarily large value of Planck constant emerges naturally as does also the interpretation for the long ranged classical electro-weak and color fields predicted by TGD. Rather seldom the evolution of ideas follows simple linear logic, and this was the case also now. In any case, this vision represents the fifth, relatively new thread in the evolution of TGD and the ideas involved are still evolving.

Dark matter as large $\hbar$ phase

D. Da Rocha and Laurent Nottale [53] have proposed that Schrödinger equation with Planck constant $\hbar$ replaced with what might be called gravitational Planck constant $\hbar_{gr} = \frac{GmM}{v_0}$ ($\hbar = c = 1$). v_0 is a velocity parameter having the value $v_0 = 144.7 \pm .7$ km/s giving $v_0/c = 4.6 \times 10^{-4}$. This is rather near to the peak orbital velocity of stars in galactic halos. Also subharmonics and harmonics of v_0 seem to appear. The support for the hypothesis coming from empirical data is impressive.

Nottale and Da Rocha believe that their Schrödinger equation results from a fractal hydrodynamics. Many-sheeted space-time however suggests astrophysical systems are not only quantum systems at larger space-time sheets but correspond to a gigantic value of gravitational Planck constant. The gravitational (ordinary) Schrödinger equation would provide a solution of the black hole collapse (IR catastrophe) problem encountered at the classical level. The resolution of the problem inspired by TGD inspired theory of living matter is that it is the dark matter at larger space-time sheets which is quantum coherent in the required time scale [D6].

Already before learning about Nottale's paper I had proposed the possibility that Planck constant is quantized [E9] and the spectrum is given in terms of logarithms of Beraha numbers: the lowest Beraha number B_3 is completely exceptional in that it predicts infinite value of Planck constant. The inverse of the gravitational Planck constant could correspond a gravitational perturbation of this as $1/\hbar_{gr} = v_0/GMm$. The general philosophy would be that when the quantum system would become non-perturbative, a phase transition increasing the value of $\hbar$ occurs to preserve the perturbative character and at the transition $n = 4 \rightarrow 3$ only the small perturbative correction to $1/\hbar(3) = 0$ remains. This would apply to QCD and to atoms with $Z > 137$ as well.

TGD predicts correctly the value of the parameter v_0 assuming that cosmic strings and their decay remnants are responsible for the dark matter. The harmonics of v_0 can be understood as corresponding to perturbations replacing cosmic strings with their n-branched coverings so that tension becomes n^2-fold: much like the replacement of a closed orbit with an orbit closing only after n turns. $1/n$-sub-harmonic would result when a magnetic flux tube split into n disjoint magnetic flux tubes. Also a model for the formation of planetary system as a condensation of ordinary matter around quantum coherent dark matter emerges [D6].

Dark matter as a source of long ranged weak and color fields

Long ranged classical electro-weak and color gauge fields are unavoidable in TGD framework. The smallness of the parity breaking effects in hadronic, nuclear, and atomic length scales does not however seem to allow long ranged electro-weak gauge fields. The problem disappears if long range classical electro-weak gauge fields are identified as space-time correlates for massless gauge fields created by dark matter. Also scaled up variants of ordinary electro-weak particle spectra are possible. The identification explains chiral selection in living matter and unbroken $U(2)_{ew}$ invariance and free color in bio length scales become characteristics of living matter and of bio-chemistry and bio-nuclear physics. An attractive solution of the matter antimatter asymmetry is based on the identification of also antimatter as dark matter.

p-Adic and dark matter hierarchies and hierarchy of moments of consciousness

Dark matter hierarchy assigned to a spectrum of Planck constant having arbitrarily large values brings additional elements to the TGD inspired theory of consciousness.

a) Macroscopic quantum coherence can be understood since a particle with a given mass can in principle appear as arbitrarily large scaled up copies (Compton length scales as $\hbar$). The phase transition to this kind of phase implies that space-time sheets of particles overlap and this makes possible macroscopic quantum coherence.

b) The space-time sheets with large Planck constant can be in thermal equilibrium with ordinary ones without the loss of quantum coherence. For instance, the cyclotron energy scale associated with EEG turns out to be above thermal energy at room temperature for the level of dark matter hierarchy corresponding to magnetic flux quanta of the Earth's magnetic field with the size scale of Earth and a successful quantitative model for EEG results [M3].

Dark matter hierarchy leads to detailed quantitative view about quantum biology with several testable predictions [M3]. The applications to living matter suggests that the basic hierarchy corresponds to a hierarchy of Planck constants coming as $\hbar(k) = \lambda^k(p)\hbar_0$, $\lambda \simeq 2^{11}$ for $p = 2^{127-1}$, $k = 0, 1, 2, ...$ [M3]. Also integer valued sub-harmonics and integer valued sub-harmonics of λ might be possible. Each p-adic length scale corresponds to this kind of hierarchy and number theoretical arguments suggest a general formula for the allowed values of Planck constant λ depending logarithmically on p-adic prime [C6]. Also the value of $\hbar_0$ has spectrum characterized by Beraha numbers $B_n = 4cos^2(\pi/n)$, $n \geq 3$, varying by a factor in the range $n > 3$ [C6]. It must be however emphasized that the relation of this picture to the model of quantized gravitational Planck constant h_{gr} appearing in Nottale's model is not yet completely understood.

The general prediction is that Universe is a kind of inverted Mandelbrot fractal for which each bird's eye of view reveals new structures in long length and time scales representing scaled down copies of standard physics and their dark variants. These structures would correspond to higher levels in self hierarchy. This prediction is consistent with the belief that 75 per cent of matter in the universe is dark.

1. Living matter and dark matter

Living matter as ordinary matter quantum controlled by the dark matter hierarchy has turned out to be a particularly successful idea. The hypothesis has led to models for EEG predicting correctly the band structure and even individual resonance bands and also generalizing the notion of EEG [M3]. Also a generalization of the notion of genetic code emerges resolving the paradoxes related to the standard dogma [L2, M3]. A particularly fascinating implication is the possibility to identify great leaps in evolution as phase transitions in which new higher level of dark matter emerges [M3].

It seems safe to conclude that the dark matter hierarchy with levels labelled by the values of Planck constants explains the macroscopic and macro-temporal quantum coherence naturally. That this explanation is consistent with the explanation based on spin glass degeneracy is suggested by following observations. First, the argument supporting spin glass degeneracy as an explanation of the macro-temporal quantum coherence does not involve the value of $\hbar$ at all. Secondly, the failure of the perturbation theory assumed to lead to the increase of Planck constant and formation of macroscopic quantum phases could be precisely due to the emergence of a large number of new degrees of freedom due to spin glass degeneracy. Thirdly, the phase transition increasing Planck constant has concrete topological interpretation in terms of many-sheeted space-time consistent with the spin glass degeneracy.

2. Dark matter hierarchy and the notion of self

The vision about dark matter hierarchy leads to a more refined view about self hierarchy and hierarchy of moments of consciousness [J6, M3]. The larger the value of Planck constant, the longer the subjectively experienced duration and the average geometric duration $T(k) \propto \lambda^k$ of the quantum jump.

Quantum jumps form also a hierarchy with respect to p-adic and dark hierarchies and the geometric durations of quantum jumps scale like $\hbar$. Dark matter hierarchy suggests also a slight modification of the notion of self. Each self involves a hierarchy of dark matter levels, and one is led to ask whether the highest level in this hierarchy corresponds to single quantum jump rather than a sequence of quantum jumps. The averaging of conscious experience over quantum jumps would occur only for sub-selves at lower levels of dark matter hierarchy and these mental images would be ordered, and single moment of consciousness would be experienced as a history of events. The quantum parallel dissipation at the lower levels would give rise to the experience of flow of time. For instance, hadron as a macro-temporal quantum system in the characteristic time scale of hadron is a dissipating system at quark and gluon level corresponding to shorter p-adic time scales. One can ask whether even entire life cycle could be regarded as a single quantum jump at the highest level so that consciousness would not be completely lost even during deep sleep. This would allow to understand why we seem to know directly that this biological body of mine existed yesterday.

The fact that we can remember phone numbers with 5 to 9 digits supports the view that self corresponds at the highest dark matter level to single moment of consciousness. Self would experience the average over the sequence of moments of consciousness associated with each sub-self but there would be no averaging over the separate mental images of this kind, be their parallel or serial. These mental images correspond to sub-selves having shorter wake-up periods than self and would be experienced as being time ordered. Hence the digits in the phone number are experienced as separate mental images and ordered with respect to experienced time.

3. The time span of long term memories as signature for the level of dark matter hierarchy

The simplest dimensional estimate gives for the average increment τ of geometric time in quantum jump $\tau \sim 10^4$ CP_2 times so that $2^{127}-1 \sim 10^{38}$ quantum jumps are experienced during secondary p-adic time scale $T_2(k=127) \simeq 0.1$ seconds which is the duration of physiological moment and predicted to be fundamental time scale of human consciousness [L1]. A more refined guess is that $\tau_p = \sqrt{p}\tau$ gives the dependence of the duration of quantum jump on p-adic prime p. By multi-p-fractality predicted by TGD and explaining p-adic length scale hypothesis, one expects that at least $p=2$-adic level is also always present. For the higher levels of dark matter hierarchy τ_p is scaled up by $\hbar/\hbar_0$. One can understand evolutionary leaps as the emergence of higher levels at the level of individual organism making possible intentionality and memory in the time scale defined τ [L2].

Higher levels of dark matter hierarchy provide a neat quantitative view about self hierarchy and its evolution. For instance, EEG time scales corresponds to $k=4$ level of hierarchy and a time scale of .1 seconds [J6], and EEG frequencies correspond at this level dark photon energies above the thermal threshold so that thermal noise is not a problem anymore. Various levels of dark matter hierarchy would naturally correspond to higher levels in the hierarchy of consciousness and the typical duration of life cycle would give an idea about the level in question.

The level would determine also the time span of long term memories as discussed in [M3]. $k=7$ would correspond to a duration of moment of conscious of order human lifetime which suggests that $k=7$ corresponds to the highest dark matter level relevant to our consciousness whereas higher levels would in general correspond to transpersonal consciousness. $k=5$ would correspond to time scale of short term memories measured in minutes and $k=6$ to a time scale of memories measured in days.

The emergence of these levels must have meant evolutionary leap since long term memory is also accompanied by ability to anticipate future in the same time scale. This picture would suggest that the basic difference between us and our cousins is not at the level of genome as it is usually understood but at the level of the hierarchy of magnetic bodies [L2, M3]. In fact, higher levels of dark matter hierarchy motivate the introduction of the notions of super-genome and hyper-genome. The genomes of entire organ can join to form super-genome expressing genes coherently. Hyper-genomes would result from the fusion of genomes of different organisms and collective levels of consciousness would express themselves via hyper-genome and make possible social rules and moral.

0.4 Bird's eye of view about the topics of the book

The focus of this book is the number theoretical vision about physics. This vision involves three loosely related parts.

1. The fusion of real physic and various p-adic physics to a single larger whole by generalizing the number concept by fusing real numbers and various p-adic number fields along common rationals. Extensions of p-adic number fields can be introduced by gluing them along common algebraic numbers to reals. Algebraic continuation of the physics from rationals and their their extensions to various number fields (completion of rational physics to physics in various number fields) is the key idea and the challenge is to understand whether how one could achieve this dream. A very profound implication is that purely local p-adic physics codes for the p-adic fractality of long length length scale real physics and vice versa. As a consequence one can understand the origins of p-adic length scale hypothesis and ends up with a very concrete view about space-time correlates of cognition and intentionality.

2. Second part of the vision involves what I call hyper counterparts of the classical number fields defined as subspaces of their complexifications with Minkowskian signature of the metric. The hypothesis is that allowed space-time surfaces correspond to what might be called hyper-qunternionic sub-manifolds of a hyper-octonionic space. Second hypothesis is that space-time surfaces can be also regarded hyper-quaternionic sub-manifolds of the hyper-octonionic imbedding space. This means that one can assign to each point of space-time surface a hyper-quanternionic 4-plane which is the plane defined by the modified gamma matrices and co-incides

with tangent plane only for action defined by the metric determinant. Hence the basic variational principle of TGD would have deep number theoretic content. Reduction to a closed form would also mean that classical TGD would define a generalized topological field theory with Noether charges defining topological invariants.

3. The third part of the vision involves infinite primes, which can be identified in terms of an infinite hierarchy of second quantized arithmetic quantum fields theories on one hand, and as having representations as space-time surfaces analogous to zero surfaces of polynomials on the other hand. In this framework space-time surface would represent an infinite number. This vision leads also the conclusion that single point of space-time has an infinitely complex structure since real unity can be represented as a ratio of infinite numbers in infinitely many manners each having its own number theoretic anatomy. Thus single space-time point is in principle able to represent in its structure the quantum state of the entire universe. This number theoretic variant of Brahman=Atman identity also means that Universe is an algebraic hologram.

   Besides this holy trinity I will discuss also loosely related topics. Included are the possible applications of the category theory in TGD and in TGD inspired theory of consciousness; various TGD inspired considerations related to Riemann hypothesis - in particular, a strategy for proving Riemann hypothesis using a modification of Hilbert-Polya conjecture replacing quantum states with coherent states of a unique conformally invariant physical system; topological quantum computation in TGD Universe; and TGD inspired approach to Langlands program.

The seven online books about TGD [TGDview, TGDgeom, TGDquant, TGDnumber, TGDclass, TGDpad, TGDfree] and eight online books about TGD inspired theory of consciousness and quantum biology [TGDconsc, TGDselforg, TGDware, TGDholo, TGDgeme, TGDeeg, TGDmagn, 15] are warmly recommended for the reader willing to get overall view about what is involved.

0.5 The contents of the book

0.5.1 PART I: Number theoretical vision

TGD as a Generalized Number Theory I: p-Adicization Program

The vision about a number theoretic formulation of quantum TGD is based on the gradual accumulation of wisdom coming from different sources. The attempts to find a formulation allowing to understand real and p-adic physics as aspects of some more general scenario have been an important stimulus and generated a lot of, not necessarily mutually consistent ideas, some of which might serve as building blocks of the final formulation.

The first part of the 3-part chapter is devoted to the p-adicization program attempting to construct physics in various number fields as an algebraic continuation of physics in the field of rationals (or appropriate extension of rationals). The program involves in essential manner the generalization of number concept obtained by fusing reals and p-adic number fields to a larger structure by gluing them together along common rationals. Highly non-trivial number theoretic conjectures are an i outcome of the program.

1. Real and p-adic regions of the space-time as geometric correlates of matter and mind

The solutions of the equations determining space-time surfaces are restricted by the requirement that the imbedding space coordinates are real. When this is not the case, one might apply instead of a real completion with some rational-adic or p-adic completion: this is how rational-adic p-adic physics could emerge from the basic equations of the theory. One could interpret the resulting rational-adic or p-adic regions as geometrical correlates for 'mind stuff'.

p-Adic non-determinism implies extreme flexibility and therefore makes the identification of the p-adic regions as seats of cognitive representations very natural. Unlike real completion, p-adic completions preserve the information about the algebraic extension of rationals and algebraic coding of quantum numbers must be associated with 'mind like' regions of space-time. p-Adics and reals are in the same relationship as map and territory.

The implications are far-reaching and consistent with TGD inspired theory of consciousness: p-adic regions are present even at elementary particle level and provide some kind of model of 'self' and external world. In fact, p-adic physics must model the p-adic cognitive regions representing real elementary particle regions rather than elementary particles themselves!

2. The generalization of the notion of number

The unification of real physics of material work and p-adic physics of cognition and intentionality leads to the generalization of the notion of number field. Reals and various p-adic number fields are glued along their common rationals (and common algebraic numbers too) to form a fractal book like structure. Allowing all possible finite-dimensional extensions of p-adic numbers brings additional pages to this "Big Book".

At space-time level the book like structure corresponds to the decomposition of space-time surface to real and p-adic space-time sheets. This has deep implications for the view about cognition. For instance, two points infinitesimally near p-adically are infinitely distant in real sense so that cognition becomes a cosmic phenomenon.

3. Number theoretical Universality and number theoretical criticality

Number theoretic universality has been one of the basic guide lines in the construction of quantum TGD. There are two forms of the principle.

1. The strong form of number theoretical universality states that physics for any system should effectively reduce to a physics in algebraic extension of rational numbers at the level of M-matrix (generalization of S-matrix) so that an interpretation in both real and p-adic sense (allowing a suitable algebraic extension of p-adics) is possible. One can however worry whether this principle only means that physics is algebraic so that there would be no need to talk about real and p-adic physics at the level of M-matrix elements. It is not possible to get rid of real and p-adic numbers at the level of classical physics since calculus is a prerequisite for the basic variational principles used to formulate the theory. For this option the possibility of completion is what poses conditions on M-matrix.

2. The weak form of principle requires only that both real and p-adic variants of physics make sense and that the intersection of these physics consist of physics associated with various algebraic extensions of rational numbers. In this rational physics would be like rational numbers allowing infinite number of algebraic extensions and real numbers and p-adic number fields as its completions. Real and p-adic physics would be completions of rational physics. In this framework criticality with respect to phase transitions changing number field - number theoretical criticality - becomes a viable concept. This form of principle allows also purely p-adic phenomena such as p-adic pseudo non-determinism assigned to imagination and cognition. Genuinely p-adic physics does not however allow definition of notions like conserved quantities since the notion of definite integral is lacking and only the purely local form of real physics allows p-adic counterpart.

Experience has taught that it is better to avoid too strong statements and perhaps the weak form of the principle is enough.

4. p-Adicization by algebraic continuation

One general idea which results as an outcome of the generalized notion of number is the idea of a universal function continuable from a function mapping rationals to rationals or to a finite extension of rationals to a function in any number field. It must be however emphasized that for weaker form of number theoretical universality this restriction applies only at number theoretical quantum criticality. This algebraic continuation is analogous to the analytical continuation of a real analytic function to the complex plane. Rational functions with rational coefficients are obviously functions satisfying this constraint. Algebraic functions with rational coefficients satisfy this requirement if appropriate finite-dimensional algebraic extensions of p-adic numbers are allowed. Exponent function is such a function.

For instance, residue calculus might be generalized so that the value of an integral along the real axis could be calculated by continuing it instead of the complex plane to any number field via its values in the subset of rational numbers forming the rim of the book like structure having number fields as its pages. If the poles of the continued function in the finitely extended number field allow interpretation as real numbers it might be possible to generalize the residue formula. One can also imagine of extending residue calculus to any algebraic extension. An interesting situation arises when the poles correspond to extended p-adic rationals common to different pages of the "great book". Could this mean that the integral could be calculated at any page having the pole common. In particular, could a p-adic residue integral be calculated in the ordinary complex plane by utilizing the fact that in this case numerical approach makes sense.

Algebraic continuation is the basic tool of p-adicization program. Entire physics of the TGD Universe should be algebraically continuable to various number fields. Real number based physics would define the physics of matter and p-adic physics would describe correlates of cognition and intentionality. The basic stumbling block of this program is integration and algebraic continuation should allow to circumvent this difficulty. Needless to say, the requirement that the continuation exists must pose immensely tight constraints on the physics.

Due to the fact that real and p-adic topologies are fundamentally different, ultraviolet and infrared cutoffs in the set of rationals are unavoidable notions and correspond to a hierarchy of different physical phases on one hand and different levels of cognition on the other hand. Two types of cutoffs are predicted: p-adic length scale cutoff and a cutoff due to phase resolution. Zero energy ontology provides a natural realization for the p-adic length scale cutoff. The latter cutoff seems to correspond naturally to the hierarchy of algebraic extensions of p-adic numbers and quantum phases $exp(i2\pi/n)$, $n \geq 3$, coming as roots of unity and defining extensions of rationals and p-adics allowing to define p-adically sensible trigonometric functions These phases relate closely to the hierarchy of quantum groups, braid groups, and II_1 factors of von Neumann algebra..

5. Number theoretic democracy

The interpretation allows all finite-dimensional extensions of p-adic number fields and perhaps even infinite-P p-adics. The notion arithmetic quantum theory generalizes to include Gaussian and Eisenstein variants of infinite primes and corresponding arithmetic quantum field theories. Also the notion of p-adicity generalizes: it seems that one can indeed assign to Gaussian and Eisenstein primes what might be called G-adic and E-adic numbers.

p-Adicization by algebraic continuation gives hopes of continuing quantum TGD from reals to various p-adic number fields. The existence of this continuation poses extremely strong constraints on theory.

TGD as a Generalized Number Theory II: Quaternions, Octonions and their Hyper Counterparts

This chapter is the second part of the multi-chapter devoted to the vision about TGD as a generalized number theory.

1. Hyper-quaternions and octonions

The original idea was that space-time surfaces could be regarded as four-surfaces in 8-D imbedding space with the property that the tangent spaces of these spaces can be locally regarded as 4- *resp.* 8-dimensional number fields of quaternions and octonions.

The difficulties caused by the Euclidian metric signature of the number theoretical norm have however forced to give up the original idea as such, and to introduce complexified octonions and quaternions resulting by extending quaternionic and octonionic algebra by adding imaginary units multiplied with $\sqrt{-1}$. This spoils the number field property but the notion of prime is not lost. The sub-space of hyper-quaternions *resp.* -octonions is obtained from the algebra of ordinary quaternions and octonions by multiplying the imaginary part with $\sqrt{-1}$. The transition is the number theoretical counterpart of the transition from Riemannian to pseudo-Riemannin geometry performed already in Special Relativity.

The problem is that $H = M^4 \times CP_2$ cannot be endowed with a hyper-octonionic manifold structure. Indeed, space-time surfaces are assumed to be hyper-quaternionic or co-hyper-quaternionic 4-surfaces of 8-dimensional Minkowski space M^8 identifiable as the hyper-octonionic space HO. Since the hyper-quaternionic sub-spaces of HO with fixed complex structure are labelled by CP_2, each (co)-hyper-quaternionic four-surface of HO defines a 4-surface of $M^4 \times CP_2$. One can say that the number-theoretic analog of spontaneous compactification occurs.

2. Space-time-surface as a HQ or CHQ sub-manifold of hyper-octonionic imbedding space?

Space-time identified as a hyper-quaternionic (HQ) or co-hyper-quaternionic (coHQ) sub-manifold of the hyper-octonionic space in the sense that the tangent space or normal space of the space-time surface defines a hyper-quaternionic sub-algebra of the hyper-octonionic tangent space of H at each space-time point, looks an attractive idea. Second possibility is that the tangent space-algebra of the space-time surface is either associative or co-associative at each point.

One can also consider possibility that the dynamics of the space-time surface is determined from the requirement that space-time surface is algebraically closed in the sense that tangent space at each point has this property. Also the possibility that the property in question is associated with the normal space at each point of X^4 can be considered. Some delicacies are caused by the question whether the induced algebra at X^4 is just the hyper-octonionic product or whether the algebra product is projected to the space-time surface. If normal part of the product is projected out the space-time algebra closes automatically.

The first guess would be that space-time surfaces are hyper-quaternionic (HQ) or co-hyper-quaternionic (coHQ) sub-manifolds of hyper-octonionic space $HO = M^8$ with the property that complex structure is fixed and same at all points of space-time surface. This corresponds to a global selection of a preferred octonionic imaginary unit. The automorphisms leaving this selection invariant form group $SU(3)$ identifiable as color group. The selections of hyper-quaternionic sub-space under this condition are parameterized by CP_2. This means that each 4-surface in HO defines a 4-surface in $M^4 \times CP_2$ and one can speak about number-theoretic analog of spontaneous compactification having of course nothing to do with dynamics. It would be possible to make physics in two radically different geometric pictures: HO picture and $H = M^4 \times CP_2$ picture.

For a theoretical physicists of my generation it is easy to guess that the next step is to realize that it is possible to fix the preferred octonionic imaginary at each point of HO separately so that local $S^6 = G_2/SU(3)$, or equivalently the local group G_2 subject to $SU(3)$ gauge invariance, characterizes the possible choices of hyper-quaternionic structure with a preferred imaginary unit. $G_2 \subset SO(7)$ is the automorphism group of octonions, and appears also in M-theory. This local choice has interpretation as a fixing of the plane of non-physical polarizations and rise to degeneracy which is a good candidate for the ground state degeneracy caused by the vacuum extremals.

$OH - -M^4 \times CP_2$ duality allows to construct a foliation of HO by hyper-quaternionic space-time surfaces in terms of maps $HO \rightarrow SU(3)$ satisfying certain integrability conditions guaranteing that the distribution of hyper-quaternionic planes integrates to a foliation by 4-surfaces. In fact, the freedom to fix the preferred imaginary unit locally extends the maps to $HO \rightarrow G_2$ reducing to maps $HO \rightarrow SU(3) \times S^6$ in the local trivialization of G_2. This foliation defines a four-parameter family of 4-surfaces in $M^4 \times CP_2$ for each local choice of the preferred imaginary unit. The dual of this foliation defines a 4-parameter family co-hyper-quaternionic space-time surfaces and it turns out that also these surfaces are needed.

Hyper-octonion analytic functions $HO \rightarrow HO$ with real Taylor coefficients provide a physically motivated ansatz satisfying the integrability conditions. The basic reason is that hyper-octonion analyticity is not plagued by the complications due to non-commutativity and non-associativity. Indeed, this notion results also if the product is Abelianized by assuming that different octonionic imaginary units multiply to zero. A good candidate for the HO dynamics is free massless Dirac action with Weyl condition for an octonion valued spinor field using octonionic representation of gamma matrices and coupled to the G_2 gauge potential defined by the tensor 7×7 tensor product of the imaginary parts of spinor fields.

The basic conjecture is that HQ and $coHQ$ surfaces correspond to preferred extremals of Kähler action. This conjecture has several variants. It could be that only asymptotic behavior corresponds to HQ analytic function but that HQ and $coHQ$ is a generic property. It could also be that maxima of Kähler function correspond to this kind of 4-surfaces. The encouraging hint is the fact that Hamilton-Jacobi coordinates appear naturally also in the construction of general solutions of field equations.

3. The notion of Kähler calibration

Calibration is a closed p-form, whose value for a given p-plane is not larger than its volume in the induced metric. What is important that if it is maximum for tangent planes of p-sub-manifold, minimal surface with smallest volume in its homology equivalence class results.

The idea of Kähler calibration is based on a simple observation. A hyper-octonionic spinor field defines a map $M^8 \rightarrow H = M^4 \times CP_2$ allowing to induce metric and Kähler form of H to M^8. Also Kähler action is well defined for the local hyper-quaternion plane.

The idea is that the non-closed 4-form associated the wedge product of unit tangent vectors of HQ plane in M^8 and saturating to volume for it becomes closed by multiplication with Kähler action density L_K. If L_K is minimal for hyper-quaternion plane, hyper-quaternionic manifolds define extremals of Kähler action for which the magnitudes of positive and negative contributions to the action are separately minimized.

In $coHQ$ case dual of the Kähler calibration results. In this case L_K would be most naturally maximal for HQ normal plane. There is also an alternative option but it is not favored by physical considerations.

This variational principle is not equivalent with the absolute minimization of Kähler action. Rather, in HQ case Universe would do its best to save energy, being as near as possible to vacuum. Also vacuum extremals would become physically relevant (they carry non-vanishing density gravitational energy). The non-determinism of the vacuum extremals would have an interpretation in terms of the ability of Universe to engineer itself. The attractiveness of the number theoretical variational principle from the point of calculability of TGD would be that the initial values for the time derivatives of the imbedding space coordinates at X^3 at light-like 7-D causal determinant could be computed by requiring that the energy of the solution is minimized. This could mean a computerizable construction of Kähler function.

In $coHQ$ phase Universe would obviously maximize fluctuations and contrasts in accordance with quantum criticality. One might say that these two phases give Universe kind of hawk-dove polarity.

One can assign to a given 3-surface both HQ and cHQ 4-surface in the generic case and the equivalence of descriptions requires that corresponding Kähler functions differ by the real part of a holomorphic function of CH coordinates.

4. Generalizing the notion of $HO-H$ duality to quantum level

The obvious question is how the $HO-H$ duality could generalize to quantum level. Number theoretical considerations combined with the general vision about generalized Feynman diagrams as a generalization of braid diagrams lead to general formulas for vertices in HO picture.

Simple arguments lead to the conclusion that strict duality can make sense only if the hyper-octonionic spinor field is second quantized in some sense. One can imagine two, not necessarily mutually exclusive, manners to quantize.

1. The construction of the spinor structure for the configuration space of 3-surfaces in HO forces to conclude that HO spinor fields induced to $X^4 \subset HO$ are second quantized as usual and define configuration space gamma matrices as super generators. The classical real-analytic HO spinor fields would represent analogs of zero modes of H spinor fields. The second quantized part of hyper-octonionic spinor fields induced to $X^4 \subset HO$ would have $1+1+3+\overline{3}$ decomposition having interpretation in terms of quarks and leptons and second quantized oscillator operators would commute with hyper-octonionic units. The detailed realization of $HO-H$ duality suggests that the induced spinor fields at $X^4 \subset H$ *resp.* $X^4 \subset HO$ are restrictions of H *resp.* HO spinor fields. This would hold for zero modes and could hold for second quantized part too.

2. The original idea was that the real Laurent coefficients correspond to a complete set of mutually commuting Hermitian operators having interpretation as observables. This is not enough for configuration space geometry but is favored by quantum classical correspondence. Space-time concept would be well defined only for the eigen states of these operators and physical states are mapped to space-time surfaces. The Hermitian operators would naturally correspond to the state space spanned by super Kac-Moody and super-canonical algebras, and quantum states would have precise space-time counterparts in accordance with quantum-classical correspondence.

The regions inside which the power series representing HO analytic function and matrix elements of G_2 rotation converge are identified as counterparts of maximal deterministic regions of the space-time surface. The Hermitian operators replacing Laurent coefficients are assumed to commute inside these regions identifiable also as coherence regions for the generalized Schrödinger amplitude represented by the HO spinor field.

By quantum classical correspondence these regions would be correlates for the final states of quantum jumps. The space-like 3-D causal determinants X^3 bounding adjacent regions of this kind represent quantum jumps. The hyper-octonionic part of the inner of the hyper-octonionic spinor fields at the two sides of the discontinuity defined as an integral over X^3 would give a number identifiable as complex number when imaginary unit is identified appropriately. The inner product would be identified as a representation of S-matrix element for an internal transition of particle represented by 3-surface, that is 2-vertex.

For the generalized Feynman diagrams n-vertex corresponds to a fusion of n 4-surfaces along their ends at X^3. 3-vertex can be represented number theoretically as a triality of three hyper-octonion spinors integrated over the 3-surface in question. Higher vertices can be defined as composite functions of triality with a map $(h_1, h_2) \rightarrow \overline{h}_3$ defined by octonionic triality and by duality given by the inner product. More concretely, $m+n$ vertex corresponds in HO picture to the inner product for the local hyper-octonionic products of m outgoing and n incoming hyper-octonionic spinor fields integrated over the 3-surface defining the vertex. Both 2-vertices representing internal transitions and $n > 2$ vertices are completely fixed. This should give some idea about the power of the number theoretical vision.

One can raise objections against the need for non-conventional quantization. The number theoretic prescription does not apply to the second quantized parts of HO spinor fields and S-matrix elements can be constructed using them so that two equivalent prescriptions of S-matrix would emerge. On the other hand, TGD inspired quantum measurement theory suggests dual codings S-matrix elements based on either quantum states or classical observables (zero modes) in 1-1 correspondence with them.

5. Does TGD reduce to 8-D WZW string model in HO picture?

Conservation laws suggests that in the case of non-vacuum extremals the dynamics of the local G_2 automorphism is dictated by field equations of some kind. The experience with WZW model suggests that in the case of non-vacuum extremals G_2 element could be written as a product $g = g_L(h)g_R^{-1}(\overline{h})$ of hyper-octonion analytic and anti-analytic complexified G_2 elements. g would be determined by the data at hyper-complex 2-surface for which the tangent space at a given point is spanned by real unit and preferred hyper-octonionic unit. Also Dirac action would be naturally restricted to this surface. The theory would reduce in HO picture to 8-D WZW string model both classically and quantally since vertices would reduce to integrals over 1-D curves. The interpretation of generalized Feynman diagrams in terms of generalized braid/ribbon diagrams and the unique properties of G_2 provide further support for this picture. In particular, G_2 is the lowest-dimensional Lie group allowing to realize full-powered topological quantum computation based on generalized braid diagrams and using the lowest level k=1 Kac Moody representation. Even if this reduction would occur only in special cases, such as asymptotic solutions for which Lorentz Kähler force vanishes or maxima of Kähler function, it would mean enormous simplification of the theory.

6. Why hyper-quaternionicity corresponds to the minimization of Kähler action?

The resulting over all picture leads also to a considerable understanding concerning the basic questions why (co)-hyper-quaternionic 4-surfaces define extrema of Kähler action and why WZW strings would provide a dual for the description using Kähler action. The answer boils down to the realization that the extrema of Kähler action minimize complexity, also algebraic complexity, in particular non-commutativity. A measure for non-commutativity with a fixed preferred hyper-octonionic imaginary unit is provided by the commutator of 3 and $\overline{3}$ parts of the hyper-octonion spinor field defining an antisymmetric tensor in color octet representation: very much like color gauge field.

Color action is a natural measure for the non-commutativity minimized when the tangent space algebra closes to complexified quaternionic, instead of complexified octonionic, algebra. On the other hand, Kähler action is nothing but color action for classical color gauge field defined by projections of color Killing vector fields. That WZW + Dirac action for hyper-octonionic strings would correspond to Kähler action would in turn be the TGD counterpart for the proposed string-YM dualities.

7. Various dualities and their physical counterparts

$HO - H$ duality is only one representative in a family of dualities characterizing TGD. It is not equivalent with $HQ - coHQ$ duality, which seems however to be equivalent with the electric-magnetic duality known for long. This duality relates descriptions based on space-like partonic 2-surfaces and time-like string orbits. $HO - H$ and $HQ - coHQ$ dualities seem to be closely correlated in the sense that HO picture is natural in HQ phase and H picture in $coHQ$ phase.

At configuration space level $HO - H$ duality means roughly following. In H picture spin and ew spin are spin-like quantum numbers whereas color is orbital quantum number and cannot be seen at space-time level directly. In HO picture the roles of these quantum numbers are changed. One could say that $HO - H$ duality acts as a super-symmetry permuting spin and orbital degrees of freedom of configuration space spinor fields. This duality allows a surprisingly detailed understanding of almost paradoxical dualities of hadron physics, and also explains proton spin crisis from first principles.

It seems possible to interpret $HO - H$ and $HQ - coHQ$ dualities as analogs of wave-particle duality in the infinite-dimensional context. For $HO - H$ duality the cotangent bundle of configuration space CH would be the unifying notion. Position q in CH would be represented by 3-surface whereas canonical momentum p would correspond to the same 3-surface but as a surface in CHO with induced metric and Kähler structure inherited from HO defining the tangent space of H. The notion of stringy configuration space might allow to understand also M-theory dualities in this manner.

TGD as a Generalized Number Theory III: Infinite Primes

Infinite primes are besides p-adicization and the representation of space-time surface as a hyper-quaternionic sub-manifold of hyper-octonionic space, basic pillars of the vision about TGD as a generalized number theory and will be discussed in the third part of the multi-chapter devoted to the attempt to articulate this vision as clearly as possible.

1. Why infinite primes are unavoidable

Suppose that 3-surfaces could be characterized by p-adic primes characterizing their effective p-adic topology. p-Adic unitarity implies that each quantum jump involves unitarity evolution U followed by a quantum jump. Simple arguments show that the p-adic prime characterizing the 3-surface representing the entire universe increases in a statistical sense. This leads to a peculiar paradox: if the number of quantum jumps already occurred is infinite, this prime is most naturally infinite. On the other hand, if one assumes that only finite number of quantum jumps have occurred, one encounters the problem of understanding why the initial quantum history was what it was. Furthermore, since the size of the 3-surface representing the entire Universe is infinite, p-adic length scale hypothesis suggest also that the p-adic prime associated with the entire universe is infinite.

These arguments motivate the attempt to construct a theory of infinite primes and to extend quantum TGD so that also infinite primes are possible. Rather surprisingly, one can construct what might be called generating infinite primes by repeating a procedure analogous to a quantization of a super symmetric quantum field theory. At given level of hierarchy one can identify the decomposition of space-time surface to p-adic regions with the corresponding decomposition of the infinite prime to primes at a lower level of infinity: at the basic level are finite primes for which one cannot find any formula.

2. Two views about the role of infinite primes and physics in TGD Universe

Two different views about how infinite primes, integers, and rationals might be relevant in TGD Universe have emerged.

1. The first view is based on the idea that infinite primes characterize quantum states of the entire Universe. 8-D hyper-octonions make this correspondence very concrete since 8-D hyper-octonions have interpretation as 8-momenta. By quantum-classical correspondence also the decomposition of space-time surfaces to p-adic space-time sheets should be coded by infinite hyper-octonionic primes. Infinite primes could even have a representation as hyper-quaternionic 4-surfaces of 8-D hyper-octonionic imbedding space.

2. The second view is based on the idea that infinitely structured space-time points define space-time correlates of mathematical cognition. The mathematical analog of Brahman=Atman identity would however suggest that both views deserve to be taken seriously.

3. Infinite primes and infinite hierarchy of second quantizations

The discovery of infinite primes suggested strongly the possibility to reduce physics to number theory. The construction of infinite primes can be regarded as a repeated second quantization of a super-symmetric arithmetic quantum field theory. Later it became clear that the process generalizes so that it applies in the case of quaternionic and octonionic primes and their hyper counterparts. This hierarchy of second quantizations means an enormous generalization of physics to what might be regarded a physical counterpart for a hierarchy of abstractions about abstractions about.... The ordinary second quantized quantum physics corresponds only to the lowest level infinite primes. This hierarchy can be identified with the corresponding hierarchy of space-time sheets of the many-sheeted space-time.

One can even try to understand the quantum numbers of physical particles in terms of infinite primes. In particular, the hyper-quaternionic primes correspond four-momenta and mass squared is prime valued for them. The properties of 8-D hyper-octonionic primes motivate the attempt to identify the quantum numbers associated with CP_2 degrees of freedom in terms of these primes. The representations of color group $SU(3)$ are indeed labelled by two integers and the states inside given representation by color hyper-charge and iso-spin.

It turns out that associativity constraint allows only rational infinite primes. One can however decompose rational infinite primes to hyper-octonionic infinite primes at lower level of the hierarchy. Physically this would mean that the number theoretic 8-momenta have only time-component. This decomposition is completely analogous to the decomposition of hadrons to its colored constituents and might be even interpreted in terms of color confinement. The interpretation of the decomposition of rational primes to primes in the algebraic extensions of rationals, hyper-quaternions, and hyper-octonions would have an interpretation as an increase of number theoretical resolution and the principle of number theoretic confinement could be seen as a fundamental physical principle implied by associativity condition.

4. Infinite primes as a bridge between quantum and classical

An important stimulus came from the observation stimulated by algebraic number theory. Infinite primes can be mapped to polynomial primes and this observation allows to identify completely generally the spectrum of infinite primes whereas hitherto it was possible to construct explicitly only what might be called generating infinite primes.

This in turn led to the idea that it might be possible represent infinite primes (integers) geometrically as surfaces defined by the polynomials associated with infinite primes (integers).

Obviously, infinite primes would serve as a bridge between Fock-space descriptions and geometric descriptions of physics: quantum and classical. Geometric objects could be seen as concrete representations of infinite numbers providing amplification of infinitesimals to macroscopic deformations of space-time surface. We see the infinitesimals as concrete geometric shapes!

5. Conjecture about various equivalent characterizations of space-times as surfaces

One can imagine several number-theoretic characterizations of the space-time surface.

1. The approach based on octonions and quaternions suggests that space-time surfaces correspond to associative, or equivalently, hyper-quaternionic surfaces of hyper-octonionic imbedding space HO. Also co-associative, or equivalently, co-hyper-quaternionic surfaces are possible. These foliations can be mapped in a natural manner to the foliations of $H = M^4 \times CP_2$ by space-time surfaces which are identified as preferred extremals of the Kähler action (absolute minima or maxima for regions of space-time surface in which action density has definite sign). These views are consistent if hyper-quaternionic space-time surfaces correspond to so called Kähler calibrations [E2].

2. Hyper-octonion real-analytic surfaces define foliations of the imbedding space to hyper-quaternionic 4-surfaces and their duals to co-hyper-quaternionic 4-surfaces representing space-time surfaces.

3. Rational infinite primes can be mapped to rational functions of n arguments. For hyper-octonionic arguments non-associativity makes these functions poorly defined unless one assumes that arguments are related by hyper-octonion real-analytic maps so that only single independent variable remains. These hyper-octonion real-analytic functions define foliations of HO to space-time surfaces if b) holds true.

The challenge of optimist is to prove that these characterizations are equivalent.

6. The representation of infinite hyper-octonionic primes as 4-surfaces

The difficulties caused by the Euclidian metric signature of the number theoretical norm forced to give up the idea that space-time surfaces could be regarded as quaternionic sub-manifolds of octonionic space, and to introduce complexified octonions and quaternions resulting by extending quaternionic and octonionic algebra by adding imaginary units multiplied with $\sqrt{-1}$. This spoils the number field property but the notion of prime is not lost. The sub-space of hyper-quaternions *resp.* -octonions is obtained from the algebra of ordinary quaternions and octonions by multiplying the imaginary part with $\sqrt{-1}$. The transition is the number theoretical counterpart for the transition from Riemannian to pseudo-Riemannin geometry performed already in Special Relativity.

The commutative $\sqrt{-1}$ relates naturally to the algebraic extension of rationals generalized to an algebraic extension of rational quaternions and octonions and conforms with the vision about how quantum TGD could emerge from infinite dimensional Clifford algebra identifiable as a hyper-finite factor of type II_1 [C6, A9].

The notions of hyper-quaternionic and octonionic manifold make sense but it is implausible that $H = M^4 \times CP_2$ could be endowed with a hyper-octonionic manifold structure. Indeed, space-time surfaces can be assumed to be hyper-quaternionic or co-hyper-quaternionic 4-surfaces of 8-dimensional Minkowski space M^8 identifiable as the hyper-octonionic space HO. Since the hyper-quaternionic sub-spaces of HO with a locally fixed complex structure (preferred imaginary unit contained by tangent space at each point of HO) are labelled by CP_2, each (co)-hyper-quaternionic four-surface of HO defines a 4-surface of $M^4 \times CP_2$. One can say that the number-theoretic analog of spontaneous compactification occurs.

Any hyper-octonion analytic function $HO \rightarrow HO$ defines a function $g : OH \rightarrow SU(3)$ acting as the group of octonion automorphisms leaving a preferred imaginary unit invariant, and g in turn defines a foliation of HO and $H = M^4 \times CP_2$ by space-time surfaces. The selection can be local which means that G_2 appears as a local gauge group.

Since the notion of prime makes sense for the complexified octonions, it makes sense also for the hyper-octonions. It is possible to assign to infinite prime of this kind a hyper-octonion analytic polynomial $P : HO \rightarrow HO$ and hence also a foliation of HO and $H = M^4 \times CP_2$ by 4-surfaces. Therefore space-time surface can be seen as a geometric counterpart of a Fock state. The assignment is not unique but determined only up to an element of the local octonionic automorphism group G_2 acting in HO and fixing the local choices of the preferred imaginary unit of the hyper-octonionic tangent plane. In fact, a map $HO \rightarrow S^6$ characterizes the choice since $SO(6)$ acts effectively as a local gauge group.

The construction generalizes to all levels of the hierarchy of infinite primes if one poses the associativity requirement implying that hyper-octonionic variables are related by hyper-octonion real-analytic maps, and produces also representations for integers and rationals associated with hyper-octonionic numbers as space-time surfaces. A close relationship with algebraic geometry results and the polynomials define a natural hierarchical structure in the space of 3-surfaces. By the effective 2-dimensionality naturally associated with infinite primes represented by real polynomials 4-surfaces are determined by data given at partonic 2-surfaces defined by the intersections of 3-D and 7-D light-like causal determinants. In particular, the notions of genus and degree serve as classifiers of the algebraic geometry of the 4-surfaces. The great dream is to prove that this construction yields the solutions to the absolute minimization of Kähler action.

7. Generalization of ordinary number fields: infinite primes and cognition

Both fermions and p-adic space-time sheets are identified as correlates of cognition in TGD Universe. The attempt to relate these two identifications leads to a rather concrete model for how bosonic generators of super-algebras correspond to either real or p-adic space-time sheets (actions and intentions) and fermionic generators to pairs of real space-time sheets and their p-adic variants obtained by algebraic continuation (note the analogy with fermion hole pairs).

The introduction of infinite primes, integers, and rationals leads also to a generalization of real numbers since an infinite algebra of real units defined by finite ratios of infinite rationals multiplied by ordinary rationals which are their inverses becomes possible. These units are not units in the p-adic sense and have a finite p-adic norm which can be differ from one. This construction generalizes also to the case of hyper- quaternions and -octonions although non-commutativity and in case of octonions also non-associativity pose technical problems to which the reduction to ordinary rational is simplest cure which would however allow interpretation as decomposition of infinite prime to hyper-octonionic lower level constituents. Obviously this approach differs from the standard introduction of infinitesimals in the sense that sum is replaced by multiplication meaning that the set of real units becomes infinitely degenerate.

Infinite primes form an infinite hierarchy so that the points of space-time and imbedding space can be seen as infinitely structured and able to represent all imaginable algebraic structures. Certainly counter-intuitively, single space-time point is even capable of representing the quantum state of the entire physical Universe in its structure. For instance, in the real sense surfaces in the space of units correspond to the same real number 1, and single point, which is structure-less in the real sense could represent arbitrarily high-dimensional spaces as unions of real units.

One might argue that for the real physics this structure is completely invisible and is relevant only for the physics of cognition. On the other hand, one can consider the possibility of mapping the configuration space and configuration space spinor fields to the number theoretical anatomies of a single point of imbedding space so that the structure of this point would code for the world of classical worlds and for the quantum states of the Universe. Quantum jumps would induce changes of configuration space spinor fields interpreted as wave functions in the set of number theoretical anatomies of single point of imbedding space in the ordinary sense of the word, and evolution would reduce to the evolution of the structure of a typical space-time point in the system. Physics would reduce to space-time level but in a generalized sense. Universe would be an algebraic hologram, and there is an obvious connection both with Brahman=Atman identity of Eastern philosophies and Leibniz's notion of monad.

0.5.2 PART II: TGD and p-Adic Numbers

p-Adic Numbers and Generalization of Number Concept

In this chapter the general TGD inspired mathematical ideas related to p-adic numbers are discussed. The extensions of the p-adic numbers including extensions containing transcendentals, the correspondences between p-adic and real numbers, p-adic differential and integral calculus, and p-adic symmetries and Fourier analysis belong the topics of the chapter.

The basic hypothesis is that p-adic space-time regions correspond to cognitive representations for the real physics appearing already at the elementary particle level. The interpretation of the p-adic physics as a physics of cognition is justified by the inherent p-adic non-determinism of the p-adic differential equations making possible the extreme flexibility of imagination.

p-Adic canonical identification and the identification of reals and p-adics by common rationals are the two basic identification maps between p-adics and reals and can be interpreted as two basic types of cognitive maps. The concept of p-adic fractality is defined and p-adic fractality is the basic property of the cognitive maps mapping real world to the p-adic internal world. Canonical identification is not general coordinate invariant and at the fundamental level it is applied only to map p-adic probabilities and predictions of p-adic thermodynamics to real numbers. The correspondence via common rationals is general coordinate invariant correspondence when general coordinate transformations are restricted to rational or extended rational maps: this has interpretation in terms of fundamental length scale unit provided by CP_2 length.

A natural outcome is the generalization of the notion of number. Different number fields form a book like structure with number fields and their extensions representing the pages of the book glued together along common rationals representing the rim of the book. This generalization forces also the generalization of the manifold concept: both imbedding space and configuration space are obtained as union of copies corresponding to various number fields glued together along common points, in particular rational ones. Space-time surfaces decompose naturally to real and p-adic space-time sheets. In this framework the fusion of real and various p-adic physics reduces more or less to to an algebraic continuation of rational number based physics to various number fields and their extensions.

p-Adic differential calculus obeys the same rules as real one and an interesting outcome are p-adic fractals involving canonical identification. Perhaps the most crucial ingredient concerning the practical formulation of the p-adic physics is the concept of the p-adic valued definite integral. Quite generally, all general coordinate invariant definitions are based on algebraic continuation by common rationals. Integral functions can be defined using just the rules of ordinary calculus and the ordering of the integration limits is provided by the correspondence via common rationals. Residy calculus generalizes to p-adic context and also free Gaussian functional integral generalizes to p-adic context and is expected to play key role in quantum TGD at configuration space level.

The special features of p-adic Lie-groups are briefly discussed: the most important of them being an infinite fractal hierarchy of nested groups. Various versions of the p-adic Fourier analysis are proposed: ordinary Fourier analysis generalizes naturally only if finite-dimensional extensions of p-adic numbers are allowed and this has interpretation in terms of p-adic length scale cutoff. Also p-adic Fourier analysis provides a possible definition of the definite integral in the p-adic context by using algebraic continuation.

Fusion of p-Adic and Real Variants of Quantum TGD to a More General Theory

The mathematical aspects of p-adicization of quantum TGD are discussed. In a well-defined sense Nature itself performs the p-adicization and p-adic physics can be regarded as physics of cognitive regions of space-time which in turn provide representations of real space-time regions. Cognitive representations presumably involve the p-adicization of the geometry at the level of the space-time and imbedding space by a mapping of a real space time region to a p-adic one. One can differentiate between two kinds of maps: the identification induced by the common rationals of real and p-adic space time region and the representations of the external real world to internal p-adic world induced by a canonical identification type maps.

Only the identification by common rationals respects general coordinate invariance, and it leads to a generalization of the number concept. Different number fields form a book like structure with number fields and their extensions representing the pages of the book glued together along common rationals representing the rim of the book. This generalization forces also the generalization of the manifold concept: both imbedding space and configuration space are obtained as union of copies corresponding to various number fields glued together along common points, in particular rational ones. Space-time surfaces decompose naturally to real and p-adic space-time sheets. In this framework the fusion of real and various p-adic physics reduces more or less to to an algebraic continuation of rational number based physics to various number fields and their extensions.

The program makes sense only if also extensions containing transcendentals are allowed: the p-dimensional extension containing powers of e is perhaps the most important transcendental extension involved. Entire cognitive hierarchy of extension emerges and the dimension of extension can be regarded as a measure for the cognitive resolution and the higher the dimension the shorter the length scale of resolution. Cognitive resolution provides also number theoretical counterpart for the notion of length scale cutoff unavoidable in quantum field theories: now the length scale cutoffs are part of the physics of cognition rather than reflecting the practical limitations of theory building.

There is a lot of p-adicizing to do.

1. The p-adic variant of classical TGD must be constructed. Field equations make indeed sense also in the p-adic context. The strongest assumption is that real space time sheets have the same functional form as real space-time sheet so that there is non-uniqueness only due to the hierarchy of dimensions of extensions.

2. Probability theory must be generalized. Canonical identification playing central role in p-adic mass calculations using p-adic thermodynamics maps genuinely p-adic probabilities to their real counterparts. p-Adic entropy can be defined and one can distinguish between three kinds of entropies: real entropy, p-adic entropy mapped to its real counterpart by canonical identification, and number theoretical entropies applying when probabilities are in finite-dimensional extension of rationals. Number theoretic entropies can be negative and provide genuine information measures, and it turns that bound states should correspond in TGD framework to entanglement coefficients which belong to a finite-dimensional extension of rationals and have negative number theoretic entanglement entropy. These information measures generalize by quantum-classical correspondence to space-time level.

3. p-Adic quantum mechanics must be constructed. p-Adic unitarity differs in some respects from its real counterpart: in particular, p-adic cohomology allows unitary S-matrices $S = 1 + T$ such that T is hermitian and nilpotent matrix. p-Adic quantum measurement theory based on Negentropy Maximization Principle (NMP) leads to the notion of monitoring, which might have relevance for the physics of cognition.

4. Generalized quantum mechanics results as fusion of quantum mechanics in various number fields using algebraic continuation from the field of rational as a basic guiding principle. It seems possible to generalize the notion of unitary process in such a manner that unitary matrix

leads from rational Hilbert space H_Q to a formal superposition of states in all Hilbert spaces H_F, where F runs over number fields. If this is accepted, state function reduction is a pure number theoretical necessity and involves a reduction to a particular number field followed by state function reduction and state preparation leading ultimately to a state containing only entanglement which is rational or finitely-extended rational and because of its negative number theoretic entanglement entropy identifiable as bound state entanglement stable against NMP.

5. Generalization of the configuration space and related concepts is also necessary and again gluing along common rationals and algebraic continuation is the basic guide line also now. Configuration space is a union of symmetric spaces and this allows an algebraic construction of the configuration space Kähler metric and spinor structure, whose definition reduces to the super canonical algebra defined by the function basis at the light cone boundary. Hence the algebraic continuation is relatively straightforward. Even configuration space functional integral could allow algebraic continuation. The reason is that symmetric space structure together with Duistermaat Hecke theorem suggests strongly that configuration space integration with the constraints posed by infinite-dimensional symmetries on physical states is effectively equivalent to Gaussian functional integration in free field theory around the unique maximum of Kähler function using contravariant configuration space metric as a propagator. Algebraic continuation is possible for a subset of rational valued zero modes if Kähler action and Kähler function are rational functions of configuration space coordinates for rational values of zero modes.

0.5.3 PART III: Related topics

Category theory, quantum TGD and TGD inspired theory of consciousness

Category theory has been proposed as a new approach to the deep problems of modern physics, in particular quantization of General Relativity. Category theory might provide the desired systematic approach to fuse together the bundles of general ideas related to the construction of quantum TGD proper. Category theory might also have natural applications in the general theory of consciousness and the theory of cognitive representations.

1. The ontology of quantum TGD and TGD inspired theory of consciousness based on the trinity of geometric, objective and subjective existences could be expressed elegantly using the language of the category theory. Quantum classical correspondence might allow a mathematical formulation in terms of structure respecting functors mapping the categories associated with the three kinds of existences to each other.

2. Cognition is categorizing and category theory suggests itself as a tool for understanding cognition and self hierarchies and the abstraction processes involved with conscious experience.

3. Categories possess inherent generalized logic based on set theoretic inclusion which in TGD framework is naturally replaced with topological condensation: the outcome is quantum variants for the notions of sieve, topos, and logic. This suggests the possibility of geometrizing the logic of both geometric, objective and subjective existences and perhaps understand why ordinary consciousness experiences the world through Boolean logic and Zen consciousness experiences universe through three-valued logic. Also the right-wrong logic of moral rules and beautiful-ugly logic of aesthetics seem to be too naive and might be replaced with a more general quantum logic.

Riemann hypothesis and physics

Riemann hypothesis states that the nontrivial zeros of Riemann Zeta function lie on the axis $x = 1/2$. Since Riemann zeta function allows interpretation as a thermodynamical partition function for a quantum field theoretical system consisting of bosons labelled by primes, it is interesting to look Riemann hypothesis from the perspective of physics. Quantum TGD and also TGD inspired theory of consciousness provide additional view points to the hypothesis and suggests sharpening of Riemann hypothesis, detailed strategies of proof of the sharpened hypothesis, and heuristic arguments for why the hypothesis is true.

The idea that the evolution of cognition involves the increase of the dimensions of finite-dimensional extensions of p-adic numbers associated with p-adic space-time sheets emerges naturally in TGD inspired theory of consciousness. A further input that led to a connection with Riemann Zeta was the work of Hardmuth Mueller [48] suggesting strongly that e and its $p-1$ powers at least should belong to the extensions of p-adics. The basic objects in Mueller's approach are so called logarithmic waves $exp(iklog(u))$ which should exist for $u = n$ for a suitable choice of the scaling momenta k.

Logarithmic waves appear also as the basic building blocks (the terms $n^s = exp(log(n)(Re[s] + iIm[s]))$ in Riemann Zeta. This inspires naturally the hypothesis that also Riemann Zeta function is universal in the sense that it is defined at is zeros $s = 1/2 + iy$ not only for complex numbers but also for all p-adic number fields provided that an appropriate finite-dimensional extensions involving also transcendentals are allowed. This allows in turn to algebraically continue Zeta to any number field. The zeros of Riemann zeta are determined by number theoretical quantization and are thus universal and should appear in the physics of critical systems. The hypothesis $log(p) = \frac{q_1(p)exp[q_2(p)]}{\pi}$ explains the length scale hierarchies based on powers of e, primes p and Golden Mean.

Mueller's logarithmic waves lead also to an elegant concretization of the Hilbert Polya conjecture and to a sharpened form of Riemann hypothesis: the phases q^{-iy} for the zeros of Riemann Zeta belong to a finite-dimensional extension of R_p for any value of primes q and p and any zero $1/2 + iy$ of ζ. The question whether the imaginary parts of the Riemann Zeta are linearly independent (as assumed in the previous work) or not is of crucial physical significance. Linear independence implies that the spectrum of the super-canonical weights is essentially an infinite-dimensional lattice. Otherwise a more complex structure results. The numerical evidence supporting the translational invariance of the correlations for the spectrum of zeros together with p-adic considerations leads to the working hypothesis that for any prime p one can express the spectrum of zeros as the product of a subset of Pythagorean prime phases and of a fixed subset U of roots of unity. The spectrum of zeros could be expressed as a union over the translates of the same basic spectrum defined by the roots of unity translated by the phase angles associated with a subset of Pythagorean phases: this is consistent with what the spectral correlations strongly suggest. That decompositions defined by different primes p yield the same spectrum would mean a powerful number theoretical symmetry realizing p-adicities at the level of the spectrum of Zeta.

A second strategy is based on, what I call, Universality Principle. The function, that I refer to as $\hat{\zeta}$, is defined by the product formula for ζ and exists in the infinite-dimensional algebraic extension Q_∞ of rationals containing all roots of primes. $\hat{\zeta}$ is defined for all values of s for which the partition functions $1/(1 - p^{-z})$ appearing in the product formula have value in Q_∞. Universality Principle states that $|\hat{\zeta}|^2$, defined as the product of the p-adic norms of $|\hat{\zeta}|^2$ by reversing the order of producting in the adelic formula, equals to $|\zeta|^2$ and, being an infinite dimensional vector in Q_∞, vanishes only if it contains a rational factor which vanishes. This factor is present only provided an infinite number of partition functions appearing in the product formula of $\hat{\zeta}$ have rational valued norm squared: this locates the plausible candidates for the zeros on the lines $Re[s] = n/2$.

Universality Principle implies the following stronger variant about sharpened form of the Riemann hypothesis: the real part of the phase p^{-iy} is rational for an infinite number of primes for zeros of ζ. Universality Principle, even if proven, does not however yield a proof of the Riemann hypothesis. The failure of the Riemann hypothesis becomes however extremely implausible. An important outcome of this approach is the realization that super-conformal invariance is a natural symmetry associated with ζ (not surprisingly, since the symmetry group of complex analysis is in question!).

Super-conformal invariance inspires a strategy for proving the Riemann hypothesis. The vanishing of the Riemann Zeta reduces to an orthogonality condition for the eigenfunctions of a non-Hermitian operator D^+ having the zeros of Riemann Zeta as its eigenvalues. The construction of D^+ is inspired by the conviction that Riemann Zeta is associated with a physical system allowing super-conformal transformations as its symmetries and second quantization in terms of the representations of the super-conformal algebra. The eigenfunctions of D^+ are analogous to coherent states of a harmonic oscillator and in general they are not orthogonal to each other. The states orthogonal to a vacuum state (having a negative norm squared) correspond to the zeros of Riemann Zeta. The physical states having a positive norm squared correspond to the zeros of Riemann Zeta at the critical line. Riemann hypothesis follows both from the hermiticity and positive definiteness of the metric in the space of states corresponding to the zeros of ζ. Also conformal symmetry in appropriate sense implies Riemann hypothesis and after one year from the discovery of the basic idea it became clear that one can actually construct a rigorous twenty line long analytic proof for the Riemann hypothesis using a standard argument from Lie group theory.

These approaches concretize the vision about TGD based physics as a generalized number theory. Two new realizations of the super-conformal algebra result and the second realization has direct application to the modelling of $1/f$ noise. The zeros of ζ code for the states of an arithmetic quantum field theory coded also by infinite primes: also the hierarchical structure of the many-sheeted space-time is coded. Even some basic quantum numbers of particles of TGD Universe might have number theoretical representation.

Topological Quantum Computation in TGD Universe

Topological quantum computation (TQC) is one of the most promising approaches to quantum computation. The coding of logical qubits to the entanglement of topological quantum numbers promises to solve the de-coherence problem whereas the S-matrices of topological field theories (modular functors) providing unitary representations for braids provide a realization of quantum computer programs with gates represented as simple braiding operations. Because of their effective 2-dimensionality anyon systems are the best candidates for realizing the representations of braid groups.

TGD allows several new insights related to quantum computation. TGD predicts new information measures as number theoretical negative valued entanglement entropies defined for systems having extended rational entanglement and characterizes bound state entanglement as bound state entanglement. Negentropy Maximization Principle and p-adic length scale hierarchy of space-time sheets encourage to believe that Universe itself might do its best to resolve the de-coherence problem. The new view about quantum jump suggests strongly the notion of quantum parallel dissipation so that thermalization in shorter length scales would guarantee coherence in longer length scales. The possibility of negative energies and communications to geometric future in turn might trivialize the problems caused by long computation times: computation could be iterated again and again by turning the computer on in the geometric past and TGD inspired theory of consciousness predicts that something like this occurs routinely in living matter.

The absolute minimization of Kähler action is the basic variational principle of classical TGD and predicts extremely complex but non-chaotic magnetic flux tube structures, which can get knotted and linked. The dimension of CP_2 projection for these structures is $D = 3$. These structures are the corner stone of TGD inspired theory of living matter and provide the braid structures needed by TQC.

Anyons are the key actors of TQC and TGD leads to detailed model of anyons as systems consisting of track of a periodically moving charged particle realized as a flux tube containing the particle inside it. This track would be a space-time correlate for the outcome of dissipative processes producing the asymptotic self-organization pattern. These tracks in general carry vacuum Kähler charge which is topologized when the CP_2 projection of space-time sheet is $D = 3$. This explains charge fractionization predicted to occur also for other charged particles. When a system approaches chaos periodic orbits become slightly aperiodic and the correlate is flux tube which rotates N times before closing. This gives rise to Z_N valued topological quantum number crucial for TQC using anyons ($N = 4$ holds true in this case). Non-Abelian anyons are needed by TQC, and the existence of long range classical electro-weak fields predicted by TGD is an essential prerequisite of non-Abelianity.

Negative energies and zero energy states are of crucial importance of TQC in TGD. The possibility of phase conjugation for fermions would resolve the puzzle of matter-antimatter asymmetry in an elegant manner. Anti-fermions would be present but have negative energies. Quite generally, it is possible to interpret scattering as a creation of pair of positive and negative energy states, the latter representing the final state. One can characterize precisely the deviations of this Eastern world view with respect to the Western world view assuming an objective reality with a positive definite energy and understand why the Western illusion apparently works. In the case of TQC the initial *resp.* final state of braided anyon system would correspond to positive *resp.* negative energy state.

The light-like boundaries of magnetic flux tubes are ideal for TQC. The point is that 3-dimensional light-like quantum states can be interpreted as representations for the time evolution of a two-dimensional system and thus represented self-reflective states being "about something". The light-likeness (no geometric time flow) is a space-time correlate for the ceasing of subjective time flow during macro-temporal quantum coherence. The S-matrices of TQC can be coded to these light-like states such that each elementary braid operation corresponds to positive energy anyons near the boundary of the magnetic flux tube A and negative energy anyons with opposite topological charges residing near the boundary of flux tube B and connected by braided threads representing the quantum gate. Light-like boundaries also force Chern-Simons action as the only possible general coordinate invariant action since the vanishing of the metric determinant does not allow any other candidate. Chern-Simons action indeed defines the modular functor for braid coding for a TQC program.

Langlands Program and TGD

Number theoretic Langlands program can be seen as an attempt to unify number theory on one hand and theory of representations of reductive Lie groups on the other hand. So called automorphic functions to which various zeta functions are closely related define the common denominator. Geometric Langlands program tries to achieve a similar conceptual unification in the case of function fields. This program has caught the interest of physicists during last years.

TGD can be seen as an attempt to reduce physics to infinite-dimensional Kähler geometry and spinor structure of the "world of classical worlds" (WCW). Since TGD ce be regarded also as a generalized number theory, it is difficult to escape the idea that the interaction of Langlands program with TGD could be fruitful.

More concretely, TGD leads to a generalization of number concept based on the fusion of reals and various p-adic number fields and their extensions implying also generalization of manifold concept, which inspires the notion of number theoretic braid crucial for the formulation of quantum TGD. TGD leads also naturally to the notion of infinite primes and rationals. The identification of Clifford algebra of WCW as a hyper-finite factors of type II_1 in turn inspires further generalization of the notion of imbedding space and the idea that quantum TGD as a whole emerges from number theory. The ensuing generalization of the notion of imbedding space predicts a hierarchy of macroscopic quantum phases characterized by finite subgroups of $SU(2)$ and by quantized Planck constant. All these new elements serve as potential sources of fresh insights.

1. The Galois group for the algebraic closure of rationals as infinite symmetric group?

The naive identification of the Galois groups for the algebraic closure of rationals would be as infinite symmetric group S_∞ consisting of finite permutations of the roots of a polynomial of infinite degree having infinite number of roots. What puts bells ringing is that the corresponding group algebra is nothing but the hyper-finite factor of type II_1 (HFF). One of the many avatars of this algebra is infinite-dimensional Clifford algebra playing key role in Quantum TGD. The projective representations of this algebra can be interpreted as representations of braid algebra B_∞ meaning a connection with the notion of number theoretical braid.

2. Representations of finite subgroups of S_∞ as outer automorphisms of HFFs

Finite-dimensional representations of $Gal(\overline{Q}/Q)$ are crucial for Langlands program. Apart from one-dimensional representations complex finite-dimensional representations are not possible if S_∞ identification is accepted (there might exist finite-dimensional l-adic representations). This suggests that the finite-dimensional representations correspond to those for finite Galois groups and result through some kind of spontaneous breaking of S_∞ symmetry.

1. Sub-factors determined by finite groups G can be interpreted as representations of Galois groups or, rather infinite diagonal imbeddings of Galois groups to an infinite Cartesian power of S_n acting as outer automorphisms in HFF. These transformations are counterparts of global gauge transformations and determine the measured quantum numbers of gauge multiplets and thus measurement resolution. All the finite approximations of the representations are inner automorphisms but the limit does not belong to S_∞ and is therefore outer. An analogous picture applies in the case of infinite-dimensional Clifford algebra.

2. The physical interpretation is as a spontaneous breaking of S_∞ to a finite Galois group. One decomposes infinite braid to a series of n-braids such that finite Galois group acts in each n-braid in identical manner. Finite value of n corresponds to IR cutoff in physics in the sense that longer wave length quantum fluctuations are cut off. Finite measurement resolution is crucial. Now it applies to braid and corresponds in the language of new quantum measurement theory to a sub-factor $\mathcal{N} \subset \mathcal{M}$ determined by the finite Galois group G implying non-commutative physics with complex rays replaced by $\mathcal{N}$ rays. Braids give a connection to topological quantum field

theories, conformal field theories (TGD is almost topological quantum field theory at parton level), knots, etc..

3. TGD based space-time correlate for the action of finite Galois groups on braids and for the cutoff is in terms of the number theoretic braids obtained as the intersection of real partonic 2-surface and its p-adic counterpart. The value of the p-adic prime p associated with the parton is fixed by the scaling of the eigenvalue spectrum of the modified Dirac operator (note that renormalization group evolution of coupling constants is characterized at the level free theory since p-adic prime characterizes the p-adic length scale). The roots of the polynomial would determine the positions of braid strands so that Galois group emerges naturally. As a matter fact, partonic 2-surface decomposes into regions, one for each braid transforming independently under its own Galois group. Entire quantum state is modular invariant, which brings in additional constraints.

   Braiding brings in homotopy group aspect crucial for geometric Langlands program. Another global aspect is related to the modular degrees of freedom of the partonic 2-surface, or more precisely to the regions of partonic 2-surface associated with braids. $Sp(2g, R)$ (g is handle number) can act as transformations in modular degrees of freedom whereas its Langlands dual would act in spinorial degrees of freedom. The outcome would be a coupling between purely local and and global aspects which is necessary since otherwise all information about partonic 2-surfaces as basic objects would be lost. Interesting ramifications of the basic picture about why only three lowest genera correspond to the observed fermion families emerge.

3. Correspondence between finite groups and Lie groups

The correspondence between finite and Lie group is a basic aspect of Langlands.

1. Any amenable group gives rise to a unique sub-factor (in particular, compact Lie groups are amenable). These groups act as genuine outer automorphisms of the group algebra of S_∞ rather than being induced from S_∞ outer automorphism. If one gives up uniqueness, it seems that practically any group G can define a sub-factor: G would define measurement resolution by fixing the quantum numbers which are measured. Finite Galois group G and Lie group containing it and related to it by Langlands correspondence would act in the same representation space: the group algebra of S_∞, or equivalently configuration space spinors. The concrete realization for the correspondence might transform a large number of speculations to theorems.

2. There is a natural connection with McKay correspondence which also relates finite and Lie groups. The simplest variant of McKay correspondence relates discrete groups $G \subset SU(2)$ to ADE type groups. Similar correspondence is found for Jones inclusions with index $\mathcal{M} : \mathcal{N} \leq 4$. The challenge is to understand this correspondence.

   i) The basic observation is that ADE type compact Lie algebras with n-dimensional Cartan algebra can be seen as deformations for a direct sum of n SU(2) Lie algebras since SU(2) Lie algebras appear as a minimal set of generators for general ADE type Lie algebra. The algebra results by a modification of Cartan matrix. It is also natural to extend the representations of finite groups $G \subset SU(2)$ to those of $SU(2)$.

   ii) The idea would that is that n-fold Connes tensor power transforms the direct sum of n SU(2) Lie algebras by a kind of deformation to a ADE type Lie algebra with n-dimensional Cartan Lie algebra. The deformation would be induced by non-commutativity. Same would occur also for the Kac-Moody variants of these algebras for which the set of generators contains only scaling operator L_0 as an additional generator. Quantum deformation would result from the replacement of complex rays with $\mathcal{N}$ rays, where $\mathcal{N}$ is the sub-factor.

   iii) The concrete interpretation for the Connes tensor power would be in terms of the fiber bundle structure $H = M^4_\pm \times CP_2 \to H/G_a \times G_b$, $G_a \times G_b \subset SU(2) \times SU(2) \subset SL(2,C) \times SU(3)$, which provides the proper formulation for the hierarchy of macroscopic quantum phases with a quantized value of Planck constant. Each sheet of the singular covering would represent single factor in Connes tensor power and single direct $SU(2)$ summand. This picture has an analogy with brane constructions of M-theory.

4. Could there exist a universal rational function giving rise to the algebraic closure of rationals?

One could wonder whether there exists a universal generalized rational function having all units of the algebraic closure of rationals as roots so that S_∞ would permute these roots. Most naturally it would be a ratio of infinite-degree polynomials.

With motivations coming from physics I have proposed that zeros of zeta and also the factors of zeta in product expansion of zeta are algebraic numbers. Complete story might be that non-trivial zeros of Zeta define the closure of rationals. A good candidate for this function is given by $(\xi(s)/\xi(1-s)) \times (s-1)/s)$, where $\xi(s) = \xi(1-s)$ is the symmetrized variant of ζ function having same zeros. It has zeros of zeta as its zeros and poles and product expansion in terms of ratios $(s-s_n)/(1-s+s_n)$ converges everywhere. Of course, this might be too simplistic and might give only the algebraic extension involving the roots of unity given by $exp(i\pi/n)$. Also products of these functions with shifts in real argument might be considered and one could consider some limiting procedure containing very many factors in the product of shifted ζ functions yielding the universal rational function giving the closure.

5. What does one mean with S_∞?

There is also the question about the meaning of S_∞. The hierarchy of infinite primes suggests that there is entire infinity of infinities in number theoretical sense. Any group can be formally regarded as a permutation group. A possible interpretation would be in terms of algebraic closure of rationals and algebraic closures for an infinite hierarchy of polynomials to which infinite primes can be mapped. The question concerns the interpretation of these higher Galois groups and HFFs. Could one regard these as local variants of S_∞ and does this hierarchy give all algebraic groups, in particular algebraic subgroups of Lie groups, as Galois groups so that almost all of group theory would reduce to number theory even at this level?

Be it as it may, the expressive power of HFF:s seem to be absolutely marvellous. Together with the notion of infinite rational and generalization of number concept they might unify both mathematics and physics!

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Part I

NUMBER THEORETICAL VISION

Chapter 1

TGD as a Generalized Number Theory I: p-Adicization Program

1.1 Introduction

The vision about a number theoretic formulation of quantum TGD is based on the gradual accumulation of wisdom coming from different sources. The attempts to find a formulation allowing to understand real and p-adic physics as aspects of some more general scenario have been an important stimulus and generated a lot of, not necessarily mutually consistent ideas, some of which might serve as building blocks of the final formulation. The original chapter representing the number theoretic vision as a consistent narrative grew so massive that I decided to divide it to three parts.

The first part is devoted to the p-adicization program attempting to construct physics in various number fields as an algebraic continuation of physics in the field of rationals (or appropriate extension of rationals). The program involves in essential manner the generalization of number concept obtained by fusing reals and p-adic number fields to a larger structure by gluing them together along common rationals. Highly non-trivial number theoretic conjectures are an i outcome of the program.

Second part focuses on the idea that the tangent spaces of space-time and imbedding space can be regarded as 4- *resp.* 8-dimensional algebras such that space-time tangent space defines sub-algebra of imbedding space. The basic candidates for the pair of algebras are hyper-quaternions and hyper-octonions.

The great idea is that space-time surfaces X^4 correspond to hyper-quaternionic or co-hyper-quaternionic sub-manifolds of $HO = M^8$. The possibility to assign to X^4 a surface in $M^4 \times CP_2$ means a number theoretic analog for spontaneous compactification. Of course, nothing dynamical is involved: a dual relation between totally different descriptions of the physical world are in question.

The third part is devoted to infinite primes. Infinite primes are in one-one correspondence with the states of super-symmetric arithmetic quantum field theories. The infinite-primes associated with hyper-quaternionic and hyper-octonionic numbers are the most natural ones physically because of the underlying Lorentz invariance, and the possibility to interpret them as momenta with mass squared equal to prime. Most importantly, the polynomials associated with hyper-octonionic infinite primes have automatically space-time surfaces as representatives so that space-time geometry becomes a representative for the quantum states.

1.1.1 The painting is the landscape

The work with TGD inspired theory of consciousness has led to a vision about the relationship of mathematics and physics. Physics is not in this view a model of reality but objective reality itself: painting is the landscape. One can also equate mathematics and physics in a well defined sense and the often implicitly assumed Cartesian theory-world division disappears. Physical realities are mathematical ideas represented by configuration space spinor fields (quantum histories) and quantum jumps between quantum histories give rise to consciousness and to the subjective existence of mathematician.

The concrete realization for the notion algebraic hologram based on the notion of infinite prime is a second new element. The notion of infinite rationals leads to the generalization of also the notion of finite number since infinite-dimensional space of real units obtained from finite rational valued ratios q of infinite integers divided by q. These units are not units in p-adic sense. The generalization to the quaternionic and octonionic context means that ordinary space-time points become infinitely structured and space-time point is able to represent even the quantum physical state of the Universe in its algebraic structure. Single space-time point becomes the Platonia not visible at the level of real physics but essential for mathematical cognition.

In this view evolution becomes also evolution of mathematical structures, which become more and more self-conscious quantum jump by quantum jump. The notion of p-adic evolution is indeed a basic prediction of quantum TGD but even this vision might be generalized by allowing rational-adic topologies for which topology is defined by a ring with unit rather than number field.

1.1.2 Real and p-adic regions of the space-time as geometric correlates of matter and mind

The solutions of the equations determining space-time surfaces are restricted by the requirement that the components of quaternions are real. When this is not the case, one might apply instead of a real completion with some rational-adic or p-adic completion: this is how rational-adic p-adic physics emerges from basic equations of the theory. One can interpret the resulting rational-adic or p-adic regions as geometrical correlates for 'mind stuff'.

p-Adic non-determinism implies extreme flexibility and therefore makes the identification of the p-adic regions as seats of cognitive representations very natural. Unlike real completion, p-adic completions preserve the information about the algebraic extension of rationals and algebraic coding of quantum numbers must be associated with 'mind like' regions of space-time. p-Adics and reals are in the same relationship as map and territory.

The implications are far-reaching and consistent with TGD inspired theory of consciousness: p-adic regions are present even at elementary particle level and provide some kind of model of 'self' and external world. In fact, p-adic physics must model the p-adic cognitive regions representing real elementary particle regions rather than elementary particles themselves!

1.1.3 The generalization of the notion of number

The unification of real physics of material work and p-adic physics of cognition and intentionality leads to the generalization of the notion of number field. Reals and various p-adic number fields are glued along their common rationals (and common algebraic numbers too) to form a fractal book like structure. Allowing all possible finite-dimensional extensions of p-adic numbers brings additional pages to this "Big Book".

At space-time level the book like structure corresponds to the decomposition of space-time surface to real and p-adic space-time sheets. This has deep implications for the view about cognition. For instance, two points infinitesimally near p-adically are infinitely distant in real sense so that cognition becomes a cosmic phenomenon.

1.1.4 Zero energy ontology, cognition, and intentionality

One could argue that conservation laws forbid p-adic-real phase transitions in practice so that cognitions (intentions) realized as real-to-padic (p-adic-to-real) transitions would not be possible. The situation changes if one accepts what might be called zero energy ontology [C1, C2].

Zero energy ontology classically

In TGD inspired cosmology [D5] the imbeddings of Robertson-Walker cosmologies are vacuum extremals. Same applies to the imbeddings of Reissner-Nordström solution [D3] and in practice to all solutions of Einstein's equations imbeddable as extremals of Kähler action. Since four-momentum currents define a collection of vector fields rather than a tensor in TGD, both positive and negative signs for energy corresponding to two possible assignments of the arrow of the geometric time to a given space-time surface are possible. This leads to the view that all physical states have vanishing net energy classically and that physically acceptable universes are creatable from vacuum.

The result is highly desirable since one can avoid unpleasant questions such as "What are the net values of conserved quantities like rest mass, baryon number, lepton number, and electric charge for the entire universe?", "What were the initial conditions in the big bang?", "If only single solution of field equations is selected, isn't the notion of physical theory meaningless since in principle it is not possible to compare solutions of the theory?". This picture fits also nicely with the view that entire universe understood as quantum counterpart 4-D space-time is recreated in each quantum jump and allows to understand evolution as a process of continual re-creation.

Zero energy ontology at quantum level

Also the construction of S-matrix [C2] leads to the conclusion that all physical states possess vanishing conserved quantum numbers. Furthermore, the entanglement coefficients between positive and negative energy components of the state define a unitary S-matrix. S-matrix thus becomes a property of the zero energy state and physical states code by their structure what is usually identified as quantum dynamics.

Also the transitions between zero energy states are possible but general arguments lead to the conclusion that the corresponding S-matrix is almost trivial. This finding, which actually forced the new view about S-matrix, is highly desirable since it explains why positive energy ontology works so well if one forgets effects related to intentional action.

At space-time level this would mean that positive energy component and negative energy component are at a temporal distance characterized by an appropriate p-adic time scale and the integer characterizing the value of Planck constant for the state in question. The scale in question would also characterize the geometric duration of quantum jump and the size scale of space-time region contributing to the contents of conscious experience. The interpretation in terms of a mini bang followed by a mini crunch suggests itself also.

Hyper-finite factors of type II_1 and new view about S-matrix

The representation of S-matrix as unitary entanglement coefficients would not make sense in ordinary quantum theory but in TGD the von Neumann algebra in question is not a type I factor as for quantum mechanics or a type III factor as for quantum field theories, but what is called hyper-finite factor of type II_1 [C6]. This algebra is an infinite-dimensional algebra with the almost defining, and at the first look very strange, property that the infinite-dimensional unit matrix has unit trace. The infinite dimensional Clifford algebra spanned by the configuration space gamma matrices (configuration space understood as the space of 3-surfaces, the "world of classical worlds") is indeed very naturally algebra of this kind since infinite-dimensional Clifford algebras provide a canonical representations for hyper-finite factors of type II_1.

The new view about quantum measurement theory

This mathematical framework leads to a new kind of quantum measurement theory. The basic assumption is that only a finite number of degrees of freedom can be quantum measured in a given measurement and the rest remain untouched. What is known as Jones inclusions $\mathcal{N} \subset \mathcal{M}$ of von Neumann algebras allow to realize mathematically this idea [C6]. $\mathcal{N}$ characterizes measurement resolution and quantum measurement reduces the entanglement in the non-commutative quantum space $\mathcal{M}/\mathcal{N}$. The outcome of the quantum measurement is still represented by a unitary S-matrix but in the space characterized by $\mathcal{N}$. It is not possible to end up with a pure state with a finite sequence of quantum measurements.

The obvious objection is that the replacement of a universal S-matrix coding entire physics with a state dependent unitary entanglement matrix is too heavy a price to be paid for the resolution of the above mentioned paradoxes. Situation could be saved if the S-matrices have fractal structure. The quantum criticality of TGD Universe indeed implies fractality. The possibility of an infinite sequence of Jones inclusions for hyperfinite type II_1 factors isomorphic as von Neumann algebras expresses this fractal character algebraically. Thus one can hope that the S-matrix appearing as entanglement coefficients is more or less universal in the same manner as Mandelbrot fractal looks more or less the same in all length scales and for all resolutions. Whether this kind of universality must be posed as an additional condition on entanglement coefficients or is an automatic consequence of unitarity in type II_1 sense is an open question.

The S-matrix for p-adic-real transitions makes sense

In zero energy ontology conservation laws do not forbid p-adic-real transitions and one can develop a relatively concrete vision about what happens in these kind of transitions. The starting point is the generalization of the number concept obtained by gluing p-adic number fields and real numbers along common rationals (expressing it very roughly). At the level of the imbedding space this means that p-adic and real space-time sheets intersect only along common rational points of the imbedding space and transcendental p-adic space-time points are infinite as real numbers so that they can be said to be infinite distant points so that intentionality and cognition become cosmic phenomena.

In this framework the long range correlations characterizing p-adic fractality can be interpreted as being due to a large number of common rational points of imbedding space for real space-time sheet and p-adic space-time sheet from which it resulted in the realization of intention in quantum jump. Thus real physics would carry direct signatures about the presence of intentionality. Intentional behavior is indeed characterized by short range randomness and long range correlations.

One can even develop a general vision about how to construct the S-matrix elements characterizing the process [C2]. The basic guideline is the vision that real and various p-adic physics as well as their hybrids are continuable from the rational physics. This means that these S-matrix elements must be characterizable using data at rational points of the imbedding space shared by p-adic and real space-time sheets so that more or less same formulas describe all these S-matrix elements. Note that also $p_1 \rightarrow p_2$ p-adic transitions are possible.

1.1.5 What number theoretical universality might mean?

Number theoretic universality has been one of the basic guide lines in the construction of quantum TGD. There are two forms of the principle.

1. The strong form of number theoretical universality states that physics for any system should effectively reduce to a physics in algebraic extension of rational numbers at the level of M-matrix so that an interpretation in both real and p-adic sense (allowing a suitable algebraic extension of p-adics) is possible. One can however worry whether this principle only means that physics is algebraic so that there would be no need to talk about real and p-adic physics at the level of M-matrix elements. It is not possible to get rid of real and p-adic numbers at the level of classical physics since calculus is a prerequisite for the basic variational principles used to formulate the theory. For this option the possibility of completion is what poses conditions on M-matrix.

2. The weak form of principle requires only that both real and p-adic variants of physics make sense and that the intersection of these physics consist of physics associated with various algebraic extensions of rational numbers. In this rational physics would be like rational numbers allowing infinite number of algebraic extensions and real numbers and p-adic number fields as its completions. Real and p-adic physics would be completions of rational physics. In this framework criticality with respect to phase transitions changing number field becomes a viable concept. This form of principle allows also purely p-adic phenomena such as p-adic pseudo non-determinism assigned to imagination and cognition. Genuinely p-adic physics does not however allow definition of notions like conserved quantities since the notion of definite integral is lacking and only the purely local form of real physics allows p-adic counterpart.

Experience has taught that it is better to avoid too strong statements and perhaps the weak form of the principle is enough. It is however clear that number theoretical criticality could provide important insights to quantum TGD: p-adic thermodynamics is excellent example of this. In consciousness theory the transitions transforming intentions to actions and actions to cognitions would be key applications. Needless to say, zero energy ontology is absolutely essential: otherwise this kind of transitions would not make sense. In the original version of this chapter number theoretical universality was identified as number theoretical criticality and this leads to so strong conditions that they might not be possible to satisfy.

1.1.6 p-Adicization by algebraic continuation

The basic challenges of the p-adicization program are following.

1. The first problem -the conceptual one- is the identification of preferred coordinates in which functions are algebraic and for which algebraic values of coordinates are in preferred position. This problem is encountered both at the level of space-time, imbedding space, and configuration space. Here the group theoretical considerations play decisive role and the selection of preferred coordinates relates closely to the selection of quantization axes. This selection has direct physical correlates at the level of imbedding space and the hierarchy of Planck constants has interpretation as a correlate for the selection of quantization axes [A9].

   Algebraization does not necessarily mean discretization at space-time level: for instance, the coordinates characterizing partonic 2-surface can be algebraic so that algebraic point of the configuration space results and surface is not discretized. If this kind of function spaces are finite-dimensional, it is possible to fix X^2 completely data for a finite number of points only.

2. Local physics generalizes as such to p-adic context (field equations, etc...). The basic stumbling block of this program is integration already at space-time (Kähler action etc..). The problem becomes really horrible looking at configuration space level (functional integral). Algebraic continuation could allow to circumvent this difficulty. Needless to say, the requirement that the continuation exists must pose immensely tight constraints on the physics. For instance, at configuration space level radiative corrections to the functional integral should vanish and the resulting perturbation theory using propagators and vertices could make sense p-adically.

One general idea which results as an outcome of the generalized notion of number is the idea of a universal function continuable from a function mapping rationals to rationals or to a finite extension of rationals to a function in any number field. This algebraic continuation is analogous to the analytical continuation of a real analytic function to the complex plane.

1. Rational functions with rational coefficients are obviously functions satisfying this constraint. Algebraic functions with rational coefficients satisfy this requirement if appropriate finite-dimensional algebraic extensions of p-adic numbers are allowed. Exponent function is such a function.

2. For instance, residue calculus essential in the construction of N-point functions of conformal field theory might be generalized so that the value of an integral along the real axis could be calculated by continuing it instead of the complex plane to any number field via its values in the subset of rational numbers forming the rim of the book like structure having number fields as its pages. If the poles of the continued function in the finitely extended number field allow interpretation as real numbers it might be possible to generalize the residue formula. One can also imagine of extending residue calculus to any algebraic extension. An interesting situation arises when the poles correspond to extended p-adic rationals common to different pages of the "great book". Could this mean that the integral could be calculated at any page having the pole common. In particular, could a p-adic residue integral be calculated in the ordinary complex plane by utilizing the fact that in this case numerical approach makes sense.

3. Algebraic continuation is the basic tool of p-adicization program. Entire physics of the TGD Universe should be algebraically continuable to various number fields. Real number based physics would define the physics of matter and p-adic physics would describe correlates of cognition and intentionality.

4. For instance, the idea that number theoretically critical partonic 2-surfaces are expressible in terms of rational functions with rational or algebraic coefficients so that also p-adic variants of these surfaces make sense, is very attractive.

5. Finite sums and products respect algebraic number property and the condition of finiteness is coded naturally by the notion of finite measurement resolution in terms of the notion of (number theoretic) braid. This simplifies dramatically the algebraic continuation since configuration space reduces to a finite-dimensional space and the space of configuration space spinor fields reduces to finite-dimensional function space.

The real configuration space can well contain sectors for which p-adicization does not make sense. For instance, if the exponent of Kähler function and Kähler are not expressible in terms of algebraic functions with rational or at most algebraic functions or more general functions making sense p-adically, the continuation is not possible. p-Adic non-determinism in p-adic sectors makes also impossible the continuation to real sector. All this is consistent with vision about rational and algebraic physics as as analog of rational and algebraic numbers allowing completion to various continuous number fields.

Due to the fact that real and p-adic topologies are fundamentally different, ultraviolet and infrared cutoffs in the set of rationals are unavoidable notions and correspond to a hierarchy of different physical phases on one hand and different levels of cognition on the other hand. For instance, most points p-adic space-time sheets reside at infinity in real sense and p-adically infinitesimal is infinite in real sense. Two types of cutoffs are predicted: p-adic length scale cutoff and a cutoff due to phase resolution related to the hierarchy of Planck constants. Zero energy ontology provides natural realization for the p-adic length scale cutoff. The latter cutoff seems to correspond naturally to the hierarchy of algebraic extensions of p-adic numbers and quantum phases $exp(i2\pi/n)$, $n \geq 3$, coming as roots of unity and defining extensions of rationals and p-adics allowing to define p-adically sensible trigonometric functions These phases relate closely to the hierarchy of quantum groups, braid groups, and II_1 factors of von Neumann algebra.

1.1.7 For the reader

Most of this chapter has been written for about decade before the above discussion of number theoretical universality and criticality. Therefore the chapter in its original form reflects the first violent burst of ideas of an innocent novice rather than the recent more balanced vision about the role of number theory in quantum TGD. For instance, in the original view about number theoretic universality is the strong one and is un-necessarily restricting. Although I have done my best to update the sections, the details of the representation may still reflect in many aspects quantum TGD as I understood it for a decade ago and the recent vision differs dramatically from this view.

The plan of the chapter is following. In the first one half I describe general ideas as they emerged years ago in a rather free flowing "Alice in the Wonderland" mood. I also describe phenomenological applications, such as conjectures about number theoretic anatomy of coupling constants which are now at rather firm basis. The chapter titled "The recent view about Quantum TGD" represents kind of turning point and introduces quantum TGD in its recent formulation in the real context. The remaining chapters are devoted to the challenge of understanding p-adic counterpart of this general theory.

1.2 How p-adic numbers emerge from algebraic physics?

The new algebraic vision leads to several generalization of the p-adic philosophy. Besides p-adic topologies more general rational-adic topologies are possible. Topology is purely dynamically determined and -adic topologies are quite 'real'. There is a physics oriented review article by Brekke and Freund [29]. The books of Gouvêa and Khrennikov give more mathematics-oriented views about p-adics [30, 28].

This section is written before the discovery that it is possible to generalize the notion of the number field by the fusion reals and various p-adic numbers fields and their extensions together along common rationals (and also common algebraic numbers) to form a book like structure. The interpretation of p-adic physics as physics of intention and cognition removes interpretational problems. This vision provides immediately an answer to many questions raised in the text. In particular, it leads naturally to a complete algebraic democracy. The introduction of infinite primes, which are discussed in next chapter, extends the algebraic democracy even further and gives hopes of describing mathematically also mathematical cognition.

1.2.1 Basic ideas and questions

It is good to list the basic ideas and pose the basic question before more detailed considerations.

Topology is dynamical

The dynamical emergence of p-adicity is strongly supported both by the applications of p-adic and algebraic physics. The solutions of polynomial equations involving more than one variable involve roots of polynomials. Only roots in the real algebraic extensions of rationals are allowed since the components of quaternions must be real numbers. When the root is complex in real topology, one can however introduce p-adic topology such that the root exists as a number in a real extension of p-adics. In p-adic context only a finite-dimensional algebraic extension of rational numbers is needed. The solutions of the derivative conditions guaranteing Lagrange manifold property involve p-adic pseudo constants so that the p-adic solutions are non-deterministic. The interpretation is that real roots of polynomials correspond to geometric correlates of matter whereas p-adic regions are geometric correlates of mind in consistency with the p-adic non-determinism.

Does this picture imply the physically attractive working hypothesis stating that the decomposition of infinite prime into primes of lower level corresponds to a decomposition of the space-time surface to various p-adic regions appearing in the definition of the infinite prime? Generating infinite primes correspond to quaternionic rationals and these rationals contain powers of quaternionic primes defining the infinite prime. The convergence of the power series solution of the polynomial equations defining space-time surface might depend crucially on the norms of these rationals in the p-adic topology used. This could actually force in a given space-time region p-adic topology associated with some prime involved in the expansion. This is in complete accordance with the idea that p-adic topologies are topologies of sensory experience and real topology is the topology of reality.

Various generalizations of p-adic topologies

p-Adicized quaternions is not a number field anymore. One could allow also rational-adic extensions [28] for which pinary expansions are replaced by expansions in powers of rational. These extensions give rise to rings with unit but not to number fields. In this approach p-adic, or more generally rational-adic, topology determined by the algebraic number field on a given space-time sheet would be absolutely 'real' rather than mere effective topology. Space-time surface decomposes into regions which look like fractal dust when seen by an observer characterized by different number field unless the observer uses some resolution.

This approach suggests even further generalizations. The original observation stimulated by the work with Riemann hypothesis was that the primes associated with the algebraic extensions of rationals, in particular Gaussian primes and Eisenstein primes, have very attractive physical interpretation. Quaternionic primes and rationals might in turn define what might be regarded as noncommutative generalization of the p-adic and rational-adic topology.

...-Adic topology measures the complexity of the quantum state

The higher the degree of the polynomial, and thus the number of particles in the physical state and its complexity, the higher the algebraic dimension of the rational quaternions. A complete algebraic and quaternion and octonion-dimensional democracy would prevail. Accordingly, space-time topology would be completely dynamical in the sense that space-time contains both rational-adic, p-adic regions, and real regions. Physical evolution could be seen as evolution of mathematical structures in this framework: p-adic topologies would be obviously winners over rational-adic topologies and p-adic length scale hypothesis would select the surviving p-adic topologies. For instance, Gaussian-adic and Eisenstein-adic topologies would in turn be higher level survivers possibly associated with biological systems.

Dimensional democracy would be realized in the sense that one can regard the space-time sheets defining n-sheeted topological condensate also as a $4n$-dimensional surface in H^n. This hypothesis fixes the interactions associated with the topological condensation, and the hierarchical structure of the topological condensate conforms with the hierarchical ordering of the quaternionic arguments of the polynomials to which infinite primes are mapped. Polynomials (infinite integers) at a given level of hierarchy in turn can be interpreted in terms of formation of bound states by the formation of join along boundaries bonds.

Is adelic principle consistent with the dynamical topology?

There is competing, and as it seems, almost diametrically opposite view. Just like adelic formula allows to express the norm of a rational number as product of its p-adic norms, various algebraic number fields and even more general structures such as quaternions allowing the notion of prime, provide a collection of incomplete but hopefully calculable views about physics. The net description gives rise to quantum TGD formulated using real numbers. These descriptions would be like summary over all experiences about world of conscious experiencers characterized by p-adic completions of various four-dimensional algebraic number rationals. What is important is that the descriptions using algebraic number fields or their generalization might be calculable. This view need not be conflict with the dynamical view and one could indeed claim that the p-adic physics associated with various algebraic extensions of rational quaternions provide a model about physics constructed by various conscious observers. For a given quantum state there would be however minimal algebraic extension containing all points of the space-time surface in it.

1.2.2 Are more general adics indeed needed?

The considerations related to Riemann hypothesis inspired the notion of G- and E-adic numbers in which rational prime p is replaced with Gaussian or Eisenstein prime. The notion of Eisenstein prime is so attractive because it makes possible to circumvent the complexification of p-adic numbers for $p\ mod\ 4 = 1$ for which $\sqrt{-1}$ exists as a p-adic number. What forces to take the notion of G-adics very seriously is that Gaussian Mersennes correspond to the p-adic length scale of atomic nucleus and to important biological length scales in the range between 10 nanometers and few micrometers. Also the key role of Golden Mean τ in biology and self-organizing systems could be understood if $Q(\tau, i)$ defines D-adic topology. Thus there is great temptation to believe that the notion of p-adic number generalizes in these sense that any irreducible associated with real or complex algebraic extension defines generalization of p-adic numbers and that these extensions appear in the algebraic extensions of quaternions.

Thus one must consider seriously also generalized p-adic numbers, D-adics as they were called in [E8]. D-adics would correspond to powers series of a prime belonging to a complex algebraic extension of rationals. Quaternions decompose naturally in longitudinal and transversal part and transversal part can be interpreted as a complex algebraic extension of rationals in the case of both M^4 and CP_2. Thus some irreducibles of this complex extension could define a generalization of p-adic numbers used to define the algebraic extension of rational quaternions reduced to a pair of complex coordinates.

Perhaps one could go even further: quaternion-adics defined as power series of quaternionic primes of norm p suggest themselves. What would be nice that this prime could perhaps be interpreted as a representation for the momentum of corresponding space-time sheets. The components of the prime belong to algebraic extension of rationals and would even code information about external world if the proposed interpretations are correct. One can also ask whether quaternionic primes could define what might be called quaternion-adic algebras and whether these algebras might be a basic element of algebraic physics.

This would mean that space-time topology would code information about the quantum numbers of a physical state. Rings with unit rather than number fields are in question since the p-adic counterparts of quaternionic integers in general fail to have inverse. It must be emphasized that the field property might not be absolutely essential. For instance 'rational-adics' [28], for which prime p is replaced with a rational q such that norm comes as a power of q, exists as rings with unit and define topology. Rational-adic topologies could have also quaternionic counterparts.

The idea of q-rational topologies is supported by the physical picture about the correspondence between Fock states and space-time sheets. Single 3-surface can in principle carry arbitrarily high fermion and boson numbers but is unstable to a topological decay to 3-surfaces carrying single fermion and boson states. The translation of this statement to ...-adic context would be that the Fock states associated with infinite primes which correspond to rational-adic quaternionic topologies are unstable against decay to states described by polynomial primes in which each factor corresponds to prime (bosons) or its inverse (fermions) in algebraic extension of quaternions. This tendency to evolve to prime-adic topologies could be seen also as a manifestation of p-adic evolution and self-organization. Rational-adic topologies would be simply losers in the fight for survival against topologies defining number fields. Since also quaternion-adic topologies fail to define number fields they are expected to be losers in the fight for survival. Winners would be ...-adic topologies defining number fields. At the level of Fock states this would mean the instability of states which contain more than one prime: that this is indeed the case, is one of the basic assumptions of quantum TGD forced by the experimental fact that elementary particles correspond to simplest Fock states associated with configuration space spinors.

1.2.3 Why completion to p-adics necessarily occurs?

There is rather convincing argument in favor of ...-adic physics. Typically one must find zeros of rational functions of several variables. Simplifying somewhat, at the first level one must find zeros of polynomials $P(x_1, x_2)$. Newton's theorem states that the monic polynomial $P_n(y, x) = y^n + a_{n-1}x^{n-1} + ..$ allows a factorization in an algebraically closed number field

$$P(y, x^m) = \prod_k (y - f_k(x)) \ . \tag{1.2.1}$$

Here f_k are polynomials and m is integer which divides n and equals to n for an irreducible polynomial P. Since the multiplication of x by m:th root of unity (ζ_m) leaves left hand side invariant it must permute the factors on right hand side. Thus one can express the formula also as

$$P(y, x) = \prod_{k=1,..m} (y - f_k(\zeta_m^k x^{1/m})) \ . \tag{1.2.2}$$

When number field is not algebraically closed this means that one must introduce an algebraic extension by m:th roots of all rationals.

The problem is that these roots are not real in general and one cannot solve the problem by using a completion to complex numbers since only real extensions for the components of quaternion are possible. Only in the region where some of the roots of the polynomial are real, this is possible. The only manner to achieve consistency with the reality requirement is to allow p-adic topology or possibly rational-adic topology: in this case also the algebraic extension allowing m:th roots is always finite-dimensional. For instance, for $m = 2$ p-adic extension of rationals would be 4-dimensional for $p > 2$. The situation is similar for rational-adic topology.

If this argument is correct, one can conclude that real topology is possible only in the regions where real roots of the polynomial equation are possible: in the regions where all roots are complex, p-adicization gives rise to roots in the algebraic extension of p-adics and p-adic topology emerges naturally. This picture provides a precise view about how the space-time surface defined by the polynomial of quaternions decomposes to real and p-adic regions. Also a connection with catastrophe theory [31] emerges: the boundaries of the catastrophe regions where some roots coincide, serve also as boundaries between ...-adic and real regions.

1.2.4 Decomposition of space-time to ...-adic regions

Number-theoretic constraints are important in determining which ...-adic topologies are possible in a given space-time region. There is no hope of building any unique vision unless one poses some general principles. Complete algebraic and topological democracy and the generalization of the notion of p-adic evolution to what might be called rational-adic evolution allow to build plausible and sufficiently general working hypothesis not requiring too much ad hoc assumptions and allowing at least mathematical testing. A further natural principle states that the topology for a given region is such that complex extension of rationals is not needed and that the series defining the normal quaternionic coordinate as function of the space-time quaternionic coordinate converges and gives rise to a smooth surface.

The power series defining solutions of polynomial equations must converge in some topology

The roots of polynomials of several variables can be expressed as Taylor series. When the root is complex, real topology is not possible and some p-adic topology must be considered. This suggests a very attractive dynamical mechanism of p-adicization. In the regions where the root belongs to a complex extension of rationals in the real topology, one could find those values of p for which the series converges p-adically. The rational numbers characterizing the polynomials associated with the generating infinite primes certainly determine the convergence and the primes for which p-adic convergence occurs are certainly functions of these rationals. Hence it could occur that the p-adic topologies for which convergence occurs correspond to the primes appearing as factors in these rationals.

In this approach topology is a result of dynamics. Note that also the notion of symmetry depends on the region of space-time. Contrary to the basic working hypothesis, ...-adic topology of a given space-time sheet is its 'real' topology rather than being only an effective topology and the topology of space-time is completely dynamical being dictated by algebraic physics and smoothness requirement.

It is also possible that convergence does not occur with respect to any ...-adic topology and in this case the topology would be discrete. This situation would correspond to primordial chaos but still the algebraic formulation and Fock space description of the theory would make sense.

Space-time surfaces must be smooth in the completion

The completion must give rise to a smooth or at least continuous ....-adic or real surface satisfying absolute minimization of Kähler action. This requirement might allow only finite number of...-adic topologies for a given space-time region. If the completion involves functions expandable in powers of a (possibly quaternionic) rational $q = m/n$, then the prime factors of m define natural p-adic number fields for which completion is possible. Also q itself could define rational-adic topology. Since the space-time surface decomposes into regions labeled by rationals in an algebraic extension of rationals q_1, there is interesting possibility that q_1 as such defines the rational-adic topology so that there would be no need to understand why the space-time region labeled by q decomposes into space-time sheets labeled by the prime factors of q.

Whatever the details of the coding are, the coding would mean that the quantum numbers associated with the space-time sheet would determine the generalized ...-adic topology associated with it. The information about quantum systems would be mapped to space-time physics and the coding of quantum numbers to ...-adic topology would solve at a general level the problem how the information about quantum state is coded into the structure of space-time.

1.2.5 Universe as an algebraic hologram?

Quaternionic primes have a natural indentification as four-momenta. If the Minkowski norm for the quaternion is defined using the algebraic norm of the real extension of rationals involved with the state, mass squared is integer-valued as in super-conformal theories. The use of the algebraic norm means a loss of information carried by the units of the real algebraic extension $K(\theta)$ (see the appendix of this chapter). Hence one can say that besides ordinary elementary particle quantum numbers there are algebraic quantum numbers which presumably carry algebraic information. Very effective coding of information about quantum numbers becomes possible and these quantum numbers commute with ordinary quantum numbers. This information does not become manifest for matter-like regions where a real completion of rationals are used. In p-adic regions representing geometric correlates of mind the situation is different since p-adic number field in question is a finite algebraic extension of rationals.

Almost every calculation is approximation and completion to reals or p-adics makes possible to measure how good the approximation is. Real numbers are extremely practical in this respect but the failure of the real number based physics is that it reduces number to a mere quantity having a definite size but no number-theoretical properties. This is practical from the point of view of numerics but means huge loss of capacity for information storage and representation. In algebraic number theory number contains representation for its construction recipe. It seems that the correct manner to see numbers is as elements of the state space provided by the algebraic extension. p-Adic physics using p-adic versions of the algebraic extensions does not lead to a loss of this information unlike real physics. Thus the basic topology of the space-time sheet could code the quantum numbers associated with it.

Since the algebraic extension of rationals, and hence also of p-adics, depends on the number of particles present in the Fock state coded by the infinite prime, the only possible interpretation is that the additional quantum numbers code information about the many-particle state. Hence the idea about 'cognitive representation' of the fractal quantum numbers of particles of the external world suggests itself naturally. In particular, the degree of the minimal polynomial for the real extension $Q(\theta)$ is n, where n is the number of particles in the Fock state in the casethe resulting state represents infinite prime. This means that there are $n-1$ quantum numbers represented by fractal scalings (see Appendix for Dirichlet's unit theorem). The interpretation as a representation for the fractal quantum numbers representing information about states of other particles in the system suggests itself. One cannot exclude the possibility that the fractal quantum numbers represent momenta or some other quantum numbers of other particles.

If this rather un-orthodox interpretation is correct, then cognitive representations are present already at the elementary particle level in p-adic regions associated with particles and are realized as algebraic holograms. Universe as a Computer consisting of sub-computers mimicking each other would be realized already at the elementary particle level. This view is consistent with the TGD inspired theory of consciousness. Algebraic physics would also make possible kind of a Gödelian loop by providing a representation for how the information about the structure of a physical system is coded into its properties.

This view has also immediate implications for complexity theory. The dimension of the minimal algebraic extension containing the algebraic number is a unique measure for its complexity. More concretely: the degree of the minimal polynomial measures the complexity. Everyone can solve second order polynomial but very few of us remembers formulas for the roots of fourth order polynomials. For higher orders quadratures do not even exist. Of course, numbers represent typically coordinates and this is consistent with the general coordinate invariance only if some preferred coordinates exist. In TGD based physics these coordinates exist: imbedding space allows (apart from isometries) unique coordinates in which the components of the metric tensor are rational functions of the coordinates.

Similar realization is fundamental in the second almost-proof of Riemann hypothesis described in [E8]. In this case ζ is interpreted as an element in an infinite-dimensional algebraic extension of rationals allowing all roots of rationals. The vanishing of ζ requires that all components of this infinite-dimensional vector contain a common rational factor which vanishes. This is possible only if an infinite number of partition functions in the product representation of the modulus squared of ζ are rational and their product vanishes. This implies Riemann hypothesis. The assumption that only square roots of rationals are needed is very probably wrong and must be replaced with the assumption that p^{iy} is algebraic numbers when $z = 1/2 + iy$ is zero of ζ for any prime p. It is quite possible that the almost-proof survives this generalization.

The notion of Platonia discussed already in the introduction adds cognition to this picture and allows to understand where all those mathematical structures continually invented by mathematicians but not realized physically in the conventional sense of the word reside. This notion takes also the notion of algebraic hologram to its extreme by making space-time points infinitely structured.

1.2.6 How to assign a p-adic prime to a given real space-time sheet?

p-Adic mass calculations force to assign p-adic prime also to the real space-time sheets and the longstanding problem is how this p-adic prime, or possibly many of them, are determined. Number theoretic view about information concept provides a possible solution of this long-standing problem.

Number theoretic information concept

The notion of information in TGD framework differs in some respects from the standard notion.

1. The definition of the entropy in p-adic context is based on the notion p-adic logarithm depending on the p-adic norm of the argument only ($Log_p(x) = Log_p(|x|_p) = n$) [H2]. For rational- and even algebraic number valued probabilities this entropy can be regarded as a real number. The entanglement entropy defined in this manner can be negative so that the entanglement can carry genuine positive information. Rationally/algebraically entangled p-adic system has a positive information content only if the number of the entangled state pairs is proportional to a positive power of the p-adic prime p.

2. This kind of definition of entropy works also in the real-rational/algebraic case and makes always sense for finite ensembles. This would have deep implications. For ordinary definition of the entropy NMP [H2] states that entanglement is minimized in the state preparation process. For the number theoretic definition of entropy entanglement could be generated during state preparation for both p-adic and real sub-systems, and NMP forces the emergence of p-adicity (say the number of entangled state is power of prime). The fragility of quantum coherence is the basic problem of quantum computations and the good news would be that Nature itself (according to TGD) tends to stabilize quantum coherence both in the real and p-adic contexts.

3. Quantum-classical correspondence suggests that the notion of information is well defined also at the space-time level. In the presence of the classical non-determinism of Kähler action and p-adic

non-determinism one can indeed define ensembles, and therefore also probability distributions and entropies. For a given space-time sheet the natural ensemble consists of the deterministic pieces of the space-time sheet regarded as different states of the same system.

Life as islands of rational/algebraic numbers in the seas of real and p-adic continua?

The possibility to define entropy differently for rational/algebraic entangl3ment raises deep questions.

1. Is physics rational/algebraic at Hilbert space level or does the rational/algebraic entanglement represent only a special kind of entanglement for which the number theoretic definition of entropy makes sense? If rational/algebraic entanglement corresponds to a bound state entanglement then the second option seems more sensible and has quite dramatic implications. For instance, bound-unbound and living-dead dichotomies would correspond to rational/irrational or algebraic/transcendental dichotomy. Life would correspond to islands of rationality/algebraicity in the seas of real and p-adic continua.

2. Life would metaphorically reside at the rational/algebraic intersection of reals and p-adics/algebraic extensions of p-adics. Does this plus quantum-classical correspondence mean that life is a boundary phenomenon at the space-time level: real and p-adic space-time sheets, action and intention, meet along common rational/algebraic points at the boundaries of the real space-time sheets?

3. Does life corresponds to rational or algebraic entanglement? Algebraic option would maximize the size of the living sector of the state space. Rational numbers are common for reals and all p-adics: in algebraic case this holds true only if one introduces algebraic extensions of p-adics. This might make rationals preferred.

Does space-time sheet represent integer and its prime factorization?

A long-standing problem of quantum TGD is how to associate to a given real space-time sheet a (not necessarily) unique p-adic prime as required by the p-adic length scale hypothesis. One could achieve this by requiring that for this prime the negentropy associated with the ensemble is maximal. The simplest hypothesis is that a real space-time sheet consisting of N deterministic pieces corresponds to p-adic prime defining the largest factor of N. One could also consider a more general possibility. If N contains p^n as a factor, then the real fractality above n-ary p-adic length scale $L_p(n) = p^{(n-1)/2}L_p$ corresponds to smoothness in the p-adic topology. This option is more attractive since it predicts that the fundamental p-adic length scale L_p for a given p can be effectively replaced by any integer multiple NL_p, such that N is not divisible by p. There is indeed a considerable evidence for small p p-adicity in long length scales. For instance, genetic code and the appearance of binary pairs like cell membrane consisting of liquid layers suggests 2-adicity in nano length scales. This view means that the fractal structure of a given real space-time sheet represents both an integer N and its decomposition to prime factors physically. This obviously conforms with the physics as a generalized number theory vision.

Quantum-classical correspondence suggests that quantum computation processes might have counterparts at the level of space-time. An especially interesting process of this kind is the factorization of integers to prime factors. The classical cryptography relies on the fact that the factorization of large integers to prime factors is a very slow process using classical computation: the time needed to factor 100 digit number using modern computer would take more than the recent age of the universe. For quantum computers the factorization is achieved very rapidly using the famous Shor's algorithm. Does the factorization process indeed have a space-time counterpart?

Suppose that one can map the integer N to be factored to a real space-time sheet with N deterministic pieces. If one can measure the powers $p_i^{n_i}$ of primes p_i for which the fractality above the appropriate p-adic length scale looks smoothness in the p-adic topology, it is possible to deduce the factorization of N by direct physical measurements of the p-adic length scales characterizing the representative space-time sheet (say from the resonance frequencies of the radiation associated with the space-time sheet). If only the p-adic topology corresponding to the largest prime p_1 is realized in this manner, one can deduce first it, and repeat the process for N/p_1^n, and so on, until the full factorization is achieved. A possible test is to generate resonant radiation in a wave guide of having length which is an integer multiple of the fundamental p-adic length scale and to see whether frequencies which correspond to the factors of N appear spontaneously.

1.2.7 Gaussian and Eistenstein primes and physics

Gaussian and Eisenstein primes could give rise to what might be called G- and E-adicities and also these -adicities might be of physical interest.

Gaussian and Eisenstein primes and elementary particle quantum numbers

The properties of Gaussian and Eisenstein primes have intriguing parallels with quantum TGD at the level of elementary particle quantum numbers.

1. The lengths of the complex vectors defined by the non-degenerate Gaussian and Eisenstein primes are square roots of primes as are also the preferred p-adic length scales L_p: this suggests a direct connection with quantum TGD.

2. Each non-degenerate (purely real or imaginary) Gaussian prime of given norm p corresponds to 8 different complex numbers $G = \pm r \pm is$ and $G = \pm s \pm ir$. This is the number of different spin states for the imbedding space spinors and also for the color states of massless gluons (note that in TGD quark color is not spin like quantum number but is analogous to orbital angular momentum). Complex conjugation might be interpreted as a representation of charge conjugation and multiplication by $\pm 1, \pm i$ could give rise to different spin states. The 4-fold degeneracy associated with the $p\ mod\ 4 = 3$ Gaussian primes could correspond to the quartet of massless electro-weak gauge bosons with a given helicity $[(\gamma, Z^0) \leftrightarrow \pm p)$ and $(W^+, W^-) \leftrightarrow \pm ip]$.

3. For Eisenstein prime E_{p_1} the multiplication by $\pm i$ does not respect the rationality of the real part of $|Z_{p_1}|^2$ and the number of states is reduced to four. Eisenstein primes $r + isw$ and $s + irw$ have however the same norm squared so that also now the 8-fold degeneracy is present. When p_1^{iy} is of the general form $r + i\sqrt{k}s$ this degeneracy is not present.

4. The basic character of the quark color is triality realized as phases w which are third roots of unity. The fact that the phases are associated with the Eisenstein primes suggests that they might provide a representation of quark color. One can indeed multiply any Eisenstein prime in the product decomposition by factor 1, w or $\overline{w}$ and the interpretation is that the three primes represent three color states of quark. The obvious interpretation is that each factor Z_{p_1} with $p_1\ mod\ 4 = 1$ could represent 8 possible leptonic states. Each factor Z_{p_1} satisfying $p_1\ mod\ 4 = 3$ and $p_1\ mod\ 3 = 1$ conditions simultaneously would correspond to a product of Eisenstein prime with Eisenstein phase and each prime p_i associated with Eisenstein phase would correspond to one color state of quark. Even a number theoretical counterpart of color confinement could be imagined.

   There is also a further interesting analogy supporting the idea about number theoretical counterpart of the quark color. ζ decomposes into a product $\zeta_1 \times \zeta_3$, such that ζ_1 is the product of $p\ mod\ 4 = 1$ partition functions and ζ_3 the product of $p\ mod\ 4 = 3$ partition functions. This decomposition reminds of the leptonic color singlets and color triplet of quarks. Rather interestingly, leptons and quarks correspond to Ramond and Neveu-Schwartz type super Virasoro representations and the fields of N-S representation indeed contain square roots of complex variable existing p-adically for $p\ mod\ 4 = 3$.

5. What about the most general factors $r + is\sqrt{k}$? Can one assign some kind of color degeneracy also with these factors? It seems that this is the case. One can always find phase factors of type $U_\pm = (r \pm is\sqrt{k})/n$ with minimal values of n $(r^2 + s^2k = n^2)$. The factors $1, U_\pm$ clearly give rise to a 3-fold degeneracy analogous to color degeneracy.

6. What about interpretation of the components of the complex integers? For Super Virasoro representations basic quantum numbers of this kind correspond to energy and longitudinal momentum. This suggests the interpretation of $r^2 + s^2k$ as energy, $r^2 - s^2k$ as mass, and $2rs\sqrt{k}$ as momentum. For the squares $r^2 - s^2 + (2rs - s^2)w$ of Eisenstein primes $r^2 - s^2/2 - rs$ corresponds to mass, $r^2 + s^2 - rs$ to energy, and $(2rs - s^2)\sqrt{3}/2$ to momentum. Note that the sign of mass changes for Gaussian primes in the interchange $r \leftrightarrow s$. The fact that the hexagonal lattice defined by Eisenstein integers correspond to the root lattice of $SU(3)$ group means that energy, momentum and mass corresponds to the sides of the triangles in the root lattice of color group.

G-adic, E-adic and even more general fractals?

Still one line of thoughts relates to the possibility to generalize the notion of p-adicity so that could speak about G-adic and E-adic number fields. The properties of the Gaussian and Einsenstein primes indeed strongly suggest a generalization for the notion of p-adic numbers to include what might be called G-adic or E-adic numbers. In fact, the argument generalizes to the case of all nine $\sqrt{-d}$ type extensions of rationals allowing a unique prime decomposition so that one might perhaps speak about D-adics.

1. Consider for definiteness Gaussian primes. The basic point is that the decomposition into a product of prime factors is unique. For a given Gaussian prime one could consider the representation of the algebraic extension involved (complex integers in the case of Gaussian primes) as a ring formed by the formal power series

$$G = \sum_n z_n G_p^n \quad . \tag{1.2.3}$$

Here z_n is Gaussian integer with norm smaller than $|G_p|$, which equals to p for $p\ mod\ 4 = 3$ and $\sqrt{p}$ for $p\ mod\ 4 = 1$.

2. If any Gaussian integer z has a unique expansion in powers of G_p such that coefficients have norm squared smaller than p, modulo G arithmetics makes sense and one can construct the inverse of G and number field results. This is the case if Gaussian integers behave with respect to modulo G_p arithmetics like finite field $G(p, 2)$. For $p\ mod\ 4 = 1$ the extension of the p-adic numbers by introducing $\sqrt{-1}$ as a unit is not possible since $\sqrt{-1}$ exists as a p-adic number: the proposed structure might perhaps provide the counterpart of the p-adic complex numbers in the case$p\ mod\ 4 = 1$. Thus the question is whether one could regard Gaussian p-adic numbers as a natural complexification of p-adics for $p\ mod\ 4 = 1$, perhaps some kind of square root of R_p, and if they indeed form a number field, do they reduce to some known algebraic extension of R_p?

3. In the case of Eisenstein numbers one can identify the coefficients z_n in the formal power series $E = \sum z_n E_p^n$ as Eisenstein numbers having modulus square smaller than p associated with E_p and similar argument works also in this case.

4. As already noticed, in the case of complex extensions of form $r + \sqrt{-d}s$ a unique prime factorization is obtained only in nine cases corresponding to $d = 1, 2, 3, 7, 11, 19, 46, 67, 163$ [26]. The poor man's argument above does not distinguish between G- and E-adics ($d = 1, 3$) and these extensions.One might perhaps call this extensions generally D-adics. This suggests that generalized p-adics could exist also in this case. In fact, the generalization p-adics could make sense also for higher-dimensional algebraic extensions allowing unique prime decomposition. For $d = 2$ complex algebraic primes are of form $r + s\sqrt{-2}$ satisfying the condition $r^2 + 2s^2 = p$. For $d > 2$ complex algebraic primes are of form $(r + s\sqrt{-d})/2$ such that both r and s are even or odd. Quite generally, the condition $p\ mod\ d = k^2$ must be satisfied. $\sqrt{-d}$ corresponds to a root of unity only for $d = 1$ and $d = 3$ so that the powers of a complex primes in this case have a finite number of possible phase angles: this might make G- and E-adics physically special.

TGD suggests rather interesting physical applications of D-adics.

1. What is interesting from the physics point of view is that for $p\ mod\ 4 = 1$ the points D_p^n are on the logarithmic spiral $z_n = p^{n/2} exp(in\phi_0/2)$, where ϕ is the phase associated with D_p^2. The logarithmic spiral can be written also as $\rho = exp(nlog(p)\phi/\phi_0)$. This reminds strongly of the logarithmic spirals, which are fractal structures frequently encountered in self-organizing systems: D-adics might provide the mathematics for the modelling of these structures.

2. p-Adic length scale hypothesis should hold true also for Gaussian primes, in particular, Gaussian Mersennes of form $(1 \pm i)^k - 1$ should be especially interesting from TGD point of view.

i) The integers k associated with the lowest Gaussian Mersennes are following: $2, 3, 5, 7, 11, 19, 29, 47, 73, 79, 11$
$k = 113$ corresponds to the p-adic length scale associated with the atomic nucleus and muon. Thus all known charged leptons, rather than only e and τ, as well as nuclear physics length scale, correspond to Mersenne primes in the generalized sense.

ii) The primes $k = 151, 157, 163, 167$ define perhaps the most fundamental biological length scales: $k = 151$ corresponds to the thickness of the cell membrane of about ten nanometers and $k = 167$ to cell size about 2.56 μm. This strongly suggests that cellular organisms have evolved to their present form through four basic stages.

iii) $k = 239, 241, 283, 353, 367, 379, 457$ associated with the next Gaussian Mersennes define astronomical length scales. $k = 239$ and $k = 241$ correspond to the p-adic time scales .55 ms and 1.1 ms: basic time scales associated with nerve pulse transmission are in question. $k = 283$ corresponds to the time scale of 38.6 min. An interesting question is whether this period could define a fundamental biological rhythm. The length scale $L(353)$ corresponds to about 2.6×10^6 light years, roughly the size scale of galaxies. The length scale $L(367) \simeq \times 3.3 \times 10^8$ light years is of same order of magnitude as the size scale of the large voids containing galaxies on their boundaries (note the analogy with cells). $T(379) \simeq 2.1 \times 10^{10}$ years corresponds to the lower bound for the order of the age of the Universe. $T(457) \sim 10^{22}$ years defines a completely super-astronomical time and length scale.

3. Eisenstein integers form a hexagonal lattice equivalent with the root lattice of the color group $SU(3)$. Microtubular surface defines a hexagonal lattice on the surface of a cylinder which suggests an interpretation in terms of E-adicity. Also the patterns of neural activity form often hexagonal lattices.

Gaussian and Eisenstein versions of infinite primes

The vision about quantum TGD as a generalized number theory generates a further line of thoughts.

1. As has been found, the zeros of ζ code for the physical states of a super-symmetric arithmetic quantum field theory. As a matter fact, the arithmetic quantum field theory in question can be identified as arithmetic quantum field theory in which single particle states are labeled by Gaussian primes. The properties of the Gaussian primes imply that the single particle states of this theory have 8-fold degeneracy plus the four-fold degeneracy related to the $\pm i$ or ± 1-factor which could be interpreted as a phase factor associated with any $p \ mod \ 4 = 3$ type Gaussian prime. Also Eisenstein primes could allow the construction of a similar arithmetic quantum field theory.

2. The construction of the infinite primes reduces to a repeated second quantization of an arithmetic quantum field theory. A straightforward generalization of the procedure of the previous chapter allows to define also the notion of infinite Gaussian and Eisenstein primes. Since each infinite prime is in a well-defined sense a composite of finite primes playing the role of elementary particles, this would mean that each composite prime in the expansion of an infinite prime has either four-fold degeneracy or eight-fold degeneracy. The interpretation of infinite primes could thus literally be as many-particle states of quantum TGD.

1.2.8 p-Adic length scale hypothesis and quaternionic primality

p-Adic length scale hypothesis states that fundamental length scales correspond to the so called p-adic length scales proportional to $\sqrt{p}$, p prime. Even more: the p-adic primes $p \simeq 2^k$, k prime or possibly power of prime, are especially interesting physically. The so called elementary particle-blackhole analogy gives strong support for this hypothesis. Elementary particles correspond to the so called CP_2 type extremals in TGD. Elementary particle horizon can be defined as a surface at which the Euclidian signature of the metric of the space-time surface containing topologically condensed CP_2 type extremal, changes to Minkowskian signature. The generalization of the Hawking-Bekenstein formula relates the real counterpart of the p-adic entropy associated with the elementary particle to the area of the elementary particle horizon. If one requires that the radius of the elementary particle horizon corresponds to a p-adic length scale: $R = L(k)$ or $k^{n/2}L(k)$ where k is prime, then p is automatically near to 2^{k^n} and p-adic length scale hypothesis is reproduced! The proportionality of length scale to $\sqrt{p}$, rather than p, follows from p-adic thermodynamics for mass squared (!) operator and from Uncertainty Principle.

What Tony Smith [20] suggested, was a beautiful connection with number theory based on the generalization of the concept of a prime number. In the so called D^4 lattice regarded as consisting of integer quaternions, one can identify prime quaternions as the generators of the multiplicative algebra of the integer quaternions. From the basic properties of the quaternion norm it follows directly that prime quaternions correspond to the 3-dimensional spheres $R^2 = p$, p prime. The crucial point from the TGD:eish point of view is the appearance of the *square* of the norm instead of the norm. One can even define the product of spheres $R^2 = n_1$ and $R^2 = n_2$ by defining the product sphere with norm squared $R^2 = n_1 n_2$ to consist of the quaternions, which are products of quaternions with norms squared n_1 and n_2 respectively. Prime spheres correspond to $n = p$. The powers of sphere p correspond to a multiplicatively closed structure consisting of powers p^n of the sphere p. It is also possible to speak about the multiplication of balls and prime balls in the case of integer quaternions.

p-Adic length scale hypothesis follows if one assumes that the Euclidian piece of the space-time surrounding the topologically condensed CP_2 type extremal can be approximated with a quaternion integer lattice with radius squared equal to $r^2 = k^n$, k prime. One manner to understand the finiteness in the time direction is that topological sum contacts of CP_2 type extremal are not static 3-dimensional topological sum contacts but genuinely four-dimensional: 3-dimensional contact is created, expands to a maximum size and is gradually reduced to point. The Euclidian space-time volume containing the contact would correspond to an Euclidian region $R^2 = k^n$ of space-time. The distances of the lattice points would be measured using the induced metric. These contacts could have arbitrarily long duration from the point of view of external observer since classical gravitational fields give rise to strong time dilation effects (strongest on the boundary of the Euclidian region where the metric becomes degenerate with the emergence of a light like direction).

Lattice structure is essential for the argument. Lattice structures of type D^4 indeed emerge naturally in the p-adic QFT limit of TGD as also in the construction of the p-adic counterparts of the space-time surfaces as p-adically analytic surfaces. The essential idea is to construct the p-adic surface by first discretizing space-time surface using a p-adic cutoff in k:th pinary digit and mapping this surface to its p-adic counterpart and complete this to a unique smooth p-adically analytic surface. This leads to a fractal construction in which a given interval is decomposed to p smaller intervals, when the resolution is increased. In the 4-dimensional case one naturally obtains a fractal hierarchy of nested D^4 lattices. The interior of the elementary particle horizon with Euclidian signature corresponds to some subset of the quaternionic integer lattice D^4: an attractive possibility is that the absolute minimization of the Kähler action and the maximization of the Kähler function force this set to be a ball $R^2 \leq k^n$, k prime.

1.3 Scaling hierarchies and physics as a generalized number theory

The scaling hierarchies defined by powers of Φ and primes p probably reflect something very profound. Mueller has proposed also a scaling law in powers of e [48]. This scaling law can be however questioned since $\Phi^2 = 2.6180..$ is rather near to $e = 2.7183...$ Note that powers of e define p-dimensional extension of R_p since e^p exists as a p-adic number in this case.

The interpretation of the p-adic as physics of cognition and the vision about reduction of physics to rational physics continuable algebraically to various extensions of rationals and p-adic number fields is an attractive general framework allowing to understand how p-adic fractality could emerge in real physics. In this section it will be found that this vision provides a concrete tool in principle allowing to construct global solutions of field equations by reducing long length scale real physics to short length scale p-adic physics. Also p-adic length scale hypothesis can be understood and the notion of multi-p p-fractality can be formulated in precise sense in this framework. This vision leads also to a concrete quantum model for how intentions are transformed to actions and the S-matrix for the process has the same general form as the ordinary S-matrix.

The fractal hierarchy associated with Golden mean cannot be understood in a manner analogous to p-adic fractal hierarchies. Rather, the understanding of Golden Mean and Fibonacci series could reduce to the hypothesis that space-time surfaces, and thus the geometry of physical systems, provide a representations for the hierarchy of Fibonacci numbers characterizing the Jones inclusions of infinite-dimensional Clifford sub-algebras of configuration space spinors identifiable as infinite-dimensional von Neumann algebras known as hyper-finite factors of type II_1 (not that configuration space corresponds here to the "world of classical worlds"). The emergence of powers of e has been discussed in [E8] and will not be discussed here.

1.3.1 p-Adic physics and the construction of solutions of field equations

The number theoretic vision about physics relies on the idea that physics or, rather what we can know about it, is basically rational number based. One interpretation would be that space-time surfaces, the induced spinors at space-time surfaces, configuration space spinor fields, S-matrix, etc..., can be obtained by algebraically continuing their values in a discrete subset of rational variant of the geometric structure considered to appropriate completion of rationals (real or p-adic). The existence of the algebraic continuation poses very strong additional constraints on physics but has not provided any practical means to solve quantum TGD.

In the following it is however demonstrated that this view leads to a very powerful iterative method of constructing global solutions of classical field equations from local data and at the same time gives justification for the notion of p-adic fractality, which has provided very successful approach not only to elementary particle physics but also physics at longer scales. The basic idea is that mere p-adic continuity and smoothness imply fractal long range correlations between rational points which are very close p-adically but far from each other in the real sense and vice versa.

The emergence of a rational cutoff

For a given p-adic continuation only a subset of rational points is acceptable since the simultaneous requirements of real and p-adic continuity can be satisfied only if one introduces ultraviolet cutoff length scale. This means that the distances between subset of rational points fixing the dynamics of the quantities involved are above some cutoff length scale, which is expected to depend on the p-adic number field R_p as well as a particular solution of field equations. The continued quantities coincide only in this subset of rationals but not in shorter length scales.

The presence of the rational cutoff implies that the dynamics at short scales becomes effectively discrete. Reality is however not discrete: discreteness and rationality only characterize the inherent limitations of our knowledge about reality. This conforms with the fact that our numerical calculations are always discrete and involve finite set of points.

The intersection points of various p-adic continuations with real space-time surface should code for all actual information that a particular p-adic physics can give about real physics in classical sense. There are reasons to believe that real space-time sheets are in the general case characterized by integers n decomposing into products of powers of primes p_i. One can expect that for p_i-adic continuations the sets of intersection points are especially large and that these p-adic space-time surfaces can be said to provide a good discrete cognitive mimicry of the real space-time surface.

Adelic formula represents real number as product of inverse of its p-adic norms. This raises the hope that taken together these intersections could allow to determine the real surface and thus classical physics to a high degree. This idea generalizes to quantum context too.

The actual construction of the algebraic continuation from a subset of rational points is of course something which cannot be done in practice and this is not even necessary since much more elegant approach is possible.

Hierarchy of algebraic physics

One of the basic hypothesis of quantum TGD is that it is possible to define exponent of Kähler action in terms of fermionic determinants associated with the modified Dirac operator derivable from a Dirac action related super-symmetrically to the Kähler action.

If this is true, a very elegant manner to define hierarchy of physics in various algebraic extensions of rational numbers and p-adic numbers becomes possible. The observation is that the continuation to various p-adic numbers fields and their extensions for the fermionic determinant can be simply done by allowing only the eigenvalues which belong to the extension of rationals involved and solve field equations for the resulting Kähler function. Hence a hierarchy of fermionic determinants results.

The value of the dynamical Planck constant characterizes in this approach the scale factor of the M^4 metric in various number theoretical variants of the imbedding space $H = M^4 \times CP_2$ glued together along subsets of rational points of H. The values of $\hbar$ are determined from the requirement of quantum criticality [C6] meaning that Kähler coupling strength is analogous to critical temperature.

In this approach there is no need to restrict the imbedding space points to the algebraic extension of rationals and to try to formulate the counterparts of field equations in these discrete imbedding spaces.

p-Adic short range physics codes for long range real physics and vice versa

One should be able to construct global solutions of field equations numerically or by engineering them from the large repertoire of known exact solutions [D1]. This challenge looks formidable since the field equations are extremely non-linear and the failure of the strict non-determinism seems to make even in principle the construction of global solutions impossible as a boundary value problem or initial value problem.

The hope is that short distance physics might somehow code for long distance physics. If this kind of coding is possible at all, p-adicity should be crucial for achieving it. This suggests that one must articulate the question more precisely by characterizing what we mean with the phrases "short distance" and "long distance". The notion of short distance in p-adic physics is completely different from that in real physics, where rationals very close to each other can be arbitrary far away in the real sense, and vice versa. Could it be that in the statement "Short length scale physics codes for long length scale physics" the attribute "short"/"long" could refer to p-adic/real norm, real/p-adic norm, or both depending on the situation?

The point is that rational imbedding space points very near to each other in the real sense are in general at arbitrarily large distances in p-adic sense and vice versa. This observation leads to an elegant method of constructing solutions of field equations.

1. Select a rational point of the imbedding space and solve field equations in the real sense in an arbitrary small neighborhood U of this point. This can be done with an arbitrary accuracy by choosing U to be sufficiently small. It is possible to solve the linearized field equations or use a piece of an exact solution going through the point in question.

2. Select a subset of rational points in U and interpret them as points of p-adic imbedding space and space-time surface. In the p-adic sense these points are in general at arbitrary large distances from each and real continuity and smoothness alone imply p-adic long range correlations. Solve now p-adic field equations in p-adically small neighborhoods of these points. Again the accuracy can be arbitrarily high if the neighborhoods are choose small enough. The use of exact solutions of course allows to overcome the numerical restrictions.

3. Restrict the solutions in these small p-adic neighborhoods to rational points and interpret these points as real points having arbitrarily large distances. p-Adic smoothness and continuity alone imply fractal long range correlations between rational points which are arbitrary distant in the real sense. Return to 1) and continue the loop indefinitely.

In this manner one obtains even in numerical approach more and more small neighborhoods representing almost exact p-adic and real solutions and the process can be continued indefinitely.

Some comments about the construction are in order.

1. Essentially two different field equations are in question: real field equations fix the local behavior of the real solutions and p-adic field equations fix the long range behavior of real solutions. Real/p-adic global behavior is transformed to local p-adic/real behavior. This might be the deepest reason why for the hierarchy of p-adic physics.

2. The failure of the strict determinism for the dynamics dictated by Kähler action and p-adic non-determinism due to the existence of p-adic pseudo constants give good hopes that the construction indeed makes it possible to glue together the (not necessarily) small pieces of space-time surfaces inside which solutions are very precise or exact.

3. Although the full solution might be impossible to achieve, the predicted long range correlations implied by the p-adic fractality at the real space-time surface are a testable prediction for which p-adic mass calculations and applications of TGD to biology provide support.

4. It is also possible to generalize the procedure by changing the value of p at some rational points and in this manner construct real space-time sheets characterized by different p-adic primes.

5. One can consider also the possibility that several p-adic solutions are constructed at given rational point and the rational points associated with p-adic space-time sheets labeled by $p_1, ..., p_n$ belong to the real surface. This would mean that real surface would be multi-p p-adic fractal.

I have earlier suggested that even elementary particles are indeed characterized by integers and that only particles for which the integers have common prime factors interact by exchanging particles characterized by common prime factors. In particular, the primes $p = 2, 3, ..., 23$ would be common to the known elementary particles and appear in the expression of the gravitational constant. Multi-p p-fractality leads also to an explanation for the weakness of the gravitational constant. The construction recipe for the solutions would give a concrete meaning for these heuristic proposals.

This approach is not restricted to space-time dynamics but is expected to apply also at the level of say S-matrix and all mathematical object having physical relevance. For instance, p-adic four-momenta appear as parameters of S-matrix elements. p-Adic four-momenta very near to each other p-adically restricted to rational momenta define real momenta which are not close to each other and the mere p-adic continuity and smoothness imply fractal long range correlations in the real momentum space and vice versa.

p-Adic length scale hypothesis

Approximate p_1-adicity implies also approximate p_2-adicity of the space-time surface for primes $p \simeq p_1^k$. p-Adic length scale hypothesis indeed states that primes $p \simeq 2^k$ are favored and this might be due to simultaneous $p \simeq 2^k$- and 2-adicity. The long range fractal correlations in real space-time implied by 2-adicity would indeed resemble those implied by $p \simeq 2^k$ and both $p \simeq 2^k$-adic and 2-adic space-time sheets have larger number of common points with the real space-time sheet.

If the scaling factor λ of $\hbar$ appearing in the dark matter hierarchy is in good approximation $\lambda = 2^{11}$ also dark matter hierarchy comes into play in a resonant manner and dark space-time sheets at various levels of the hierarchy tend to have many intersection points with each other.

There is however a problem involved with the understanding of the origin of the p-adic length scale hypothesis if the correspondence via common rationals is assumed.

1. The mass calculations based on p-adic thermodynamics for Virasoro generator L_0 predict that mass squared is proportional to $1/p$ and Uncertainty Principle implies that L_p is proportional to $\sqrt{p}$ rather than p, which looks more natural if common rationals define the correspondence between real and p-adic physics.

2. It would seem that length $d_p \simeq pR$, R or order CP_2 length, in the induced space-time metric must correspond to a length $L_p \simeq \sqrt{p}R$ in M^4. This could be understood if space-like geodesic lines at real space-time sheet obeying effective p-adic topology are like orbits of a particle performing Brownian motion so that the space-like geodesic connecting points with M^4 distance r_{M^4} has a length $r_{X^4} \propto r_{M^4}^2$. Geodesic random walk with randomness associated with the motion in CP_2 degrees of freedom could be in question. The effective p-adic topology indeed induces a strong local wiggling in CP_2 degrees of freedom so that r_{X^4} increases and can depend non-linearly on r_{M^4}.

3. If the size of the space-time sheet associated with the particle has size $d_p \sim pR$ in the induced metric, the corresponding M^4 size would be about $L_p \propto \sqrt{p}R$ and p-adic length scale hypothesis results.

4. The strongly non-perturbative and chaotic behavior $r_{X^4} \propto r_{M^4}^2$ is assumed to continue only up to L_p. At longer length scales the space-time distance d_p associated with L_p becomes the unit of space-time distance and geodesic distance r_{X^4} is in a good approximation given by

$$r_{X^4} = \frac{r_{M^4}}{L_p} d_p \propto \sqrt{p} \times r_{M^4} \ , \tag{1.3.1}$$

and is thus linear in M^4 distance r_{M^4}.

Does cognition automatically solve real field equations in long length scales?

In TGD inspired theory of consciousness p-adic space-time sheets are identified as space-time correlates of cognition. Therefore our thoughts would have literally infinite size in the real topology if p-adics and reals correspond to each other via common rationals (also other correspondence based on the separate canonical identification of integers m and n in $q = m/n$ with p-adic numbers).

The cognitive solution of field equations in very small p-adic region would solve field equations in real sense in a discrete point set in very long real length scales. This would allow to understand why the notions of Universe and infinity are a natural part of our conscious experience although our sensory input is about an infinitesimally small region in the scale of universe.

The idea about Universe performing mimicry at all possible levels is one of the basic ideas of TGD inspired theory of consciousness. Universe could indeed understand and represent the long length scale real dynamics using local p-adic physics. The challenge would be to make quantum jumps generating p-adic surfaces having large number of common points with the real space-time surface. We are used to call this activity theorizing and the progress of science towards smaller real length scales means progress towards longer length scales in p-adic sense. Also real physics can represent p-adic physics: written language and computer represent examples of this mimicry.

1.3.2 A more detailed view about how local p-adic physics codes for p-adic fractal long range correlations of the real physics

The vision just described gives only a rough heuristic view about how the local p-adic physics could code for the p-adic fractality of long range real physics. There are highly non-trivial details related to the treatment of M^4 and CP_2 coordinates and to the mapping of p-adic H-coordinates to their real counterparts and vice versa.

How real and p-adic space-time regions are glued together?

The first task is to visualize how real and p-adic space-time regions relate to each other. It is convenient to start with the extension of real axis to contain also p-adic points. For finite rationals $q = m/n$, m and n have finite power expansions in powers of p and one can always write $q = p^k \times r/s$ such that r and s are not divisible by p and thus have pinary expansion of in powers of p as $x = x_0 + \sum_1^N x_n p^n$, $x_i \in \{0, p\}$, $x_0 \neq 0$.

One can always express p-adic number as $x = p^n y$ where y has p-adic norm 1 and has expansion in non-negative powers of p. When x is rational but not integer the expansion contains infinite number of terms but is periodic. If the expansion is infinite and non-periodic, one can speak about *strictly p-adic* number having infinite value as a real number.

In the same manner real number x can be written as $x = p^n y$, where y is either rational or has infinite non-periodic expansion $y = r_0 + \sum_{n>0} r_n p^{-n}$ in negative powers of p. As a p-adic number y is infinite. In this case one can speak about strictly real numbers.

This gives a visual idea about what the solution of field equations locally in various number fields could mean and how these solutions are glued together along common rationals. In the following I shall be somewhat sloppy and treat the rational points of the imbedding space as if they were points of real axis in order to avoid clumsy formulas.

1. The p-adic variants of field equations can be solved in the strictly p-adic realm and by p-adic smoothness these solutions are well defined also in as subset of rational points. The strictly p-adic points in a neighborhood of a given rational point correspond as real points to infinitely distant points of M^4. The possibility of p-adic pseudo constants means that for rational points of M^4 having sufficiently large p-adic norm, the values of CP_2 coordinates or induced spinor fields can be chosen more or less freely.

2. One can solve the p-adic field equations in any p-adic neighborhood $U_n(q) = \{x = q + p^n y\}$ of a rational point q of M^4, where y has a unit p-adic norm and select the values of fields at different points q_1 and q_2 freely as long as the spheres $U_n(q_1)$ and $U_n(q_2)$ are disjoint (these spheres are either identical or disjoint by p-adic ultra-metricity).

   The points in the p-adic continuum part of these solutions are at an infinite distance from q in M^4. The points which are well-defined in real sense form a discrete subset of rational points of M^4. The p-adic space-time surface constructed in this manner defines a discrete fractal hierarchy of rational space-time points besides the original points inside the p-adic spheres. In real sense the rational points have finite distances and could belong to disjoint real space-time sheets. The failure of the strict non-determinism for the field equations in the real sense gives hopes for gluing these sheets partially together (say in particle reactions with particles represented as 3-surfaces).

3. All rational points q of the p-adic space-time sheet can be interpreted as real rational points and one can solve the field equations in the real sense in the neighborhoods $U_n(q) = \{x = q + p^n y\}$ corresponding to real numbers in the the range $p^n \leq x \leq p^{n+1}$. Real smoothness and continuity fix the solutions at finite rational points inside $U_n(q)$ and by the phenomenon of p-adic pseudo constants these values can be consistent with p-adic field equations. Obviously one can can continue the construction process indefinitely.

p-Adic scalings act only in M^4 degrees of freedom

p-Adic fractality suggests that finite real space-time sheets around points $x + p^n$, $x = 0$, are obtained as by just scaling of the M^4 coordinates having origin at $x = 0$ by p^n of the solution defined in a neighborhood of x and leaving CP_2 coordinates as such. The known extremals of Kähler action indeed allow M^4 scalings as dynamical symmetries.

One can understand why no scaling should appear in CP_2 degrees of freedom. CP_2 is complex projective space for which points can be regarded as complex planes and for these p-adic scalings act trivially. It is worth of emphasizing that here could lie a further deep number theoretic reason for why the space S in $H = M^4 \times S$ must be a projective space.

What p-adic fractality for real space-time surfaces really means?

The identification of p-adic and real M^4 coordinates of rational points as such is crucial for p-adic fractality. On the other hand, the identification rational real and p-adic CP_2 coordinates as such would not be consistent with the idea that p-adic smoothness and continuity imply p-adic fractality manifested as long range correlations for real space-time sheets

The point is that p-adic fractality is not stable against small p-adic deformations of CP_2 coordinates as function of M^4 coordinates for solutions representable as maps $M^4 \rightarrow CP_2$. Indeed, if the rational valued p-adic CP_2 coordinates are mapped as such to real coordinates, the addition of large power p^n to CP_2 coordinate implies small modification in p-adic sense but large change in the real sense so that correlations of CP_2 at p-adically scaled M^4 points would be completely lost.

The situation changes if the map of p-adic CP_2 coordinates to real ones is continuous so that p-adically small deformations of the p-adic space-time points are mapped to small real deformations of the real space-time points.

1. Canonical identification $I : x = \sum x_n p^n \rightarrow \sum x_n p^{-n}$ satisfies continuity constraint but does not map rationals to rationals.

2. The modification of the canonical identification given by

$$I(q = p^k \times \frac{r}{s}) = p^k \times \frac{I(r)}{I(s)} \tag{1.3.2}$$

   is uniquely defined for rational points, maps rationals to rationals, has a symmetry under exchange of target and domain. This map reduces to a direct identification of rationals for $0 \leq r < p$ and $0 \leq s < p$.

3. The form of this map is not general coordinate invariant nor invariant under color isometries. The natural requirement is that the map should respect the symmetries of CP_2 maximally. Therefore the complex coordinates transforming linearly under U(2) subgroup of SU(3) defining the projective coordinates of CP_2 are a natural choice. The map in question would map the real components of complex coordinates to their p-adic variants and vice versa. The residual $U(2)$ symmetries correspond to rational unitary 2×2-matrices for which matrix elements are of form $U_{ij} = p^k r/s$, $r < p, s < p$. It would seem that these transformations must form a finite subgroup if they define a subgroup at all. In case of $U(1)$ Pythagorean phases define rational phases but sufficiently high powers fail to satisfy the conditions $r < p, s < p$. Also algebraic extensions of p-adic numbers can be considered.

4. The possibility of pseudo constant allows to modify canonical identification further so that it reduces to the direct identification of real and p-adic rationals if the highest powers of p in r and s ($q = p^n r/s$) are not higher than p^N. Write $x = \sum_{n\geq 0} x_n p^n = x^{N)} + p^{N+1}y$ with $x^{N)} = \sum_{n=0}^{N} x_n p^n$, $x_0 \neq 0$, $y_0 \neq 0$, and define $I_N(x) = x^{N)} + p^{N+1}I(y)$. For $q = p^n r/s$ define $I_N(q) = p^n I_N(r)/I_N(s)$. This map reduces to the direct identification of real and p-adic rationals for $y = 0$.

5. There is no need to introduce the imaginary unit explicitly. In case of spinors imaginary unit can be represented by the antisymmetric 2×2-matrix ϵ_{ij} satisfying $\epsilon_{12} = 1$. As a matter fact, the introduction of imaginary unit as number would lead to problems since for $p \ mod \ 4 = 3$ imaginary unit should be introduced as an algebraic extension and CP_2 in this sense would be an algebraic extension of RP_2. The fact that the algebraic extension of p-adic numbers by $\sqrt{-1}$ is equivalent with an extension introducing $\sqrt{p-1}$ supports the view that algebraic imaginary unit has nothing to do with the geometric imaginary unit defined by Kähler form of CP_2. For $p \ mod \ 4 = 1$ $\sqrt{-1}$ exists as a p-adic number but is infinite as a real number so that the notion of finite complex rational would not make sense.

Preferred CP_2 coordinates as a space-time correlate for the selection of quantization axis

Complex CP_2 coordinates are fixed only apart from the choice of the quantization directions of color isospin and hyper charge axis in SU(3) Lie algebra. Hence the selection of quantization axes seems to emerge at the level of the generalized space-time geometry as quantum classical correspondence indeed requires.

In a well-defined sense the choice of the quantization axis and a special coordinate system implies the breaking of color symmetry and general coordinate invariance. This breaking is induced by the presence of p-adic space-time sheets identified as correlates for cognition and intentionality. One could perhaps say that the cognition affects real physics via the imbedding space points shared by real and p-adic space-time sheets and that these common points define discrete coordinatization of the real space-time surface analogous to discretization resulting in any numerical computation.

Relationship between real and p-adic induced spinor fields

Besides imbedding space coordinates also induced spinor fields are fundamental variables in TGD. The free second quantized induced spinor fields define the fermionic oscillator operators in terms of which the gamma matrices giving rise to spinor structure of the "world of classical worlds" can be expressed.

p-Adic fractal long range correlations must hold true also for the induced spinor fields and they are in exactly the same role as CP_2 coordinates so that the variant of canonical identification mapping rationals to rationals should map the real and imaginary parts of real induced spinor fields to their p-adic counterparts and vice versa at the rational space-time points common to p-adic and real space-time sheets.

Could quantum jumps transforming intentions to actions really occur?

The idea that intentional action corresponds to a quantum jump in which p-adic space-time sheet is transformed to a real one traversing through rational points common to p-adic and real space-time sheet is consistent with the conservation laws since the sign of the conserved inertial energy can be also negative in TGD framework and the density of inertial energy vanishes in cosmological length scales [D5]. Also the non-diagonal transitions $p_1 \rightarrow p_2$ are in principle possible and would correspond to intersections of p-adic space-time sheets having a common subset of rational points. Kind of phase transitions changing the character of intention or cognition would be in question.

1. Realization of intention as a scattering process

The first question concerns the interpretation of this process and possibility to find some familiar counterpart for it in quantum field theory framework. The general framework of quantum TGD suggests that the points common to real and p-adic space-time sheets could perhaps be regarded as arguments of an n-point function determining the transition amplitudes for p-adic to real transition or $p_1 \rightarrow p_2$-adic transitions. The scattering event transforming an p-adic surface (infinitely distant real surface in real M^4) to a real finite sized surface (infinitely distant p-adic surface in p-adic M^4) would be in question.

2. Could S-matrix for realizations of intentions have the same general form as the ordinary S-matrix?

One might hope that the realization of intention as a number theoretic scattering process could be characterized by an S-matrix, which one might hope of being unitary in some sense. These S-matrix elements could be interpreted at fundamental level as probability amplitudes between intentions to prepare a define initial state and the state resulting in the process.

Super-conformal invariance is a basic symmetry of quantum TGD which suggests that the S-matrix in question should be constructible in terms of n-point functions of a conformal field theory restricted to a subset of rational points shared by real and p-adic space-time surfaces or their causal determinants. According to the general vision discussed in [C1], the construction of n-point functions effectively reduces to that at 2-dimensional sections of light-like causal determinants of space-time surfaces identified as partonic space-time sheets.

The idea that physics in various number fields results by algebraic continuation of rational physics serves as a valuable guideline and suggests that the form of the S-matrices between different number fields (call them non-diagonal S-matrices) could be essentially the same as that of diagonal S-matrices. If this picture is correct then the basic differences to ordinary real S-matrix would be following.

1. Intentional action could transform p-adic space-time surface to a real one only if the exponent of Kähler function for both is rational valued (or belongs to algebraic extension of rationals).

2. The points appearing as arguments of n-point function associated with the non-diagonal S-matrix are a subset of rational points of imbedding space whereas in the real case, where the integration over these points is well defined, all values of arguments can be allowed. Thus the difference between ordinary S-matrix and more general S-matrices would be that a continuous Fourier transform of n-point function in space-time domain is not possible in the latter case. The inherent nature of cognition would be that it favors localization in the position space.

3. Objection and its resolution

Exponent of Kähler function is the key piece of the configuration space spinor field. There is a strong counter argument against the existence of the Kähler function in the p-adic context. The basic problem is that the definite integral defining the Kähler action is not p-adically well-defined except in the special cases when it can be done algebraically. Algebraic integration is however very tricky and numerically completely unstable.

The definition of the exponent of Kähler function in terms of Dirac determinants or, perhaps equivalently, as a result of normal ordering of the modified Dirac action for second quantized induced spinors might however lead to an elegant resolution of this problem. This approach is discussed in detail in [A6, D1]. The idea is that Dirac determinant can be defined as a product of eigenvalues of the modified Dirac operator and one ends up to a hierarchy of theories based on the restriction of the eigenvalues to various algebraic extensions of rationals identified as a hierarchy associated with corresponding algebraic extensions of p-adic numbers. This hierarchy corresponds to a hierarchy of theories (and also physics!) based on varying values of Planck constant. The elegance of this approach is that no discretization at space-time level would be needed: everything reduces to the generalized eigenvalue spectrum of the modified Dirac operator.

4. A more detailed view

Consider the proposed approach in more detail.

1. Fermionic oscillator operators are assigned with the generalized eigenvectors of the modified Dirac operator defined at the light-like causal determinants:

$$\begin{aligned} \Psi &= \sum_n \Psi_n b_n \ , \\ D\Psi_n &= \Gamma^{\alpha} D_{\alpha}\Psi_n = \lambda_n O\Psi_n \ , \quad O \equiv n_{\alpha}\Gamma^{\alpha} \ . \end{aligned} \tag{1.3.3}$$

   Here $\Gamma^{\alpha} = T^{\alpha k}\Gamma_k$ denote so called modified gamma matrices expressible in terms of the energy momentum current $T^{\alpha k}$ assignable to Kähler action [A6]. The replacement of the ordinary gamma matrices with modified ones is forced by the requirement that the super-symmetries of the modified Dirac action are consistent with the property of being an extremal of Kähler action. n_{α} is a light like vector assignable to the light-like causal determinant and $O = n_{\alpha}\Gamma^{\alpha}$ must be rational and have the same value at real and p-adic side at rational points. The integer n labels the eigenvalues λ_n of the modified Dirac operator, and b_n corresponds to the corresponding fermionic oscillator operator.

2. The condition that the p-adic and real variants Ψ if the Ψ are identical at common rational points of real and p-adic space-time surface (the same applies to 4-surfaces corresponding to different p-adic number fields) poses a strong constraint on the algebraic continuation from rationals to p-adics and gives hopes of deriving implications of this approach.

3. Ordinary fermionic anti-commutation relations do not refer specifically to any number field. Super Virasoro (anti-)commutation relations involve only rationals. This suggest that fermionic Fock space spanned by the oscillator operators b_n is universal and same for reals and p-adic numbers and can be regarded as rational. Same would apply to Super Virasoro representations. Also the possibility to interpret configuration space spinor fields as quantum superpositions of Boolean statements supports this kind of universality. This gives good hopes that the contribution of the inner produces between Fock states to the S-matrix elements are number field independent.

4. Dirac determinant can be defined as the product of the eigenvalues λ_n restricted to a given algebraic extension of rationals. The solutions of the modified Dirac equation correspond to vanishing eigen values and define zero modes generating conformal super-symmetries and are not of course included.

5. Only those operators b_n for which λ_n belongs to the algebraic extension of rationals in question are used to construct physical states for a given algebraic extension of rationals. This might mean an enormous simplification of the formalism in accordance with the fact that configuration space Clifford algebra corresponds as a von Neumann algebra to a hyper-finite factor of type II_1 for which finite truncations by definition allow excellent approximations [C6]. One can even ask whether this hierarchy of algebraic extensions of rationals could in fact define a hierarchy of finite-dimensional Clifford algebras. If so then the general theory of hyper-finite factors of type II_1 would provide an extremely powerful tool.

1.3.3 Cognition, logic, and p-adicity

There seems to be a nice connection between logic aspects of cognition and p-adicity. In particular, p-valued logic for $p = 2^k - n$ has interpretation in terms of ordinary Boolean logic with n "taboos" so that p-valued logic does not conflict with common sense in this case. Also an interpretation of projections of p-adic space-time sheets to an integer lattice of real Minkowski space M^4 in terms of generalized Boolean functions emerges naturally so that M^4 projections of p-adic space-time would represent Boolean functions for a logic with n taboos.

2-adic valued functions of 2-adic variable and Boolean functions

The binary coefficients f_{nk} in the 2-adic expansions of terms $f_n x^n$ in the 2-adic Taylor expansion $f(x) = \sum_{n=0} f_n x^n$, assign a sequence of truth values to a 2-adic integer valued argument $x \in \{0, 1, ..., 2^N\}$ defining a sequence of N bits. Hence $f(x)$ assigns to each bit of this sequence a sequence of truth values which are ordered in the sense that the truth values corresponding to bits are not so important p-adically: much like higher decimals in decimal expansion. If a binary cutoff in N:th bit of $f(x)$ is introduced, B^M-valued function in B^N results, where B denotes Boolean algebra fo 2 elements. The formal generalization to p-adic case is trivial: 2 possible truth values are only replaced by p truth values representable as $0, ..., p-1$.

p-Adic valued functions of p-adic variable as generalized Boolean functions

One can speak of a generalized Boolean function mapping finite sequences of p-valued Boolean arguments to finite sequences of p-valued Boolean arguments. The restriction to a subset $x = kp^n$, $k = 0, ..., p-1$ and the replacement of the function $f(x)$ with its lowest pinary digit gives a generalized Boolean function of a single p-valued argument. If $f(x)$ is invariant under the scalings by powers of p^k, one obtains a hologram like representation of the generalized Boolean function with same function represented in infinitely many length scales. This guarantees the robustness of the representation.

The special role of 2-adicity explaining p-adic length scale hypothesis $p \simeq 2^k$, k integer, in terms of multi-p-acidic fractality would correlate with the special role of 2-valued logic in the world order. The fact that all generalizations of 2-valued logic ultimately involve 2-adic logic at the highest level, where the generalization is formulated would be analog of p-adic length scale hypothesis.

$p = 2^k - n$-adicity and Boolean functions with taboos

It is difficult to assign any reasonable interpretation to $p > 2$-valued logic. Also the generalization of logical connectives AND and OR is far from obvious. In the case $p = 2^k - n$ favored by the p-adic length scale hypothesis situation is however different. In this case one has interpretation in terms B^k with n Boolean statements dropped out so that one obtains what might be called $\hat{B^k}$. Since n is odd this set is not invariant under Boolean conjugation so that there is at least one statement, which is identically true and could be called taboo, axiom, or dogma: depending on taste. The allowed Boolean functions would be constructed in this case using standard Boolean functions AND and OR with the constraint that taboos are respected: in other words, both the inputs and values of functions belong to $\hat{B^k}$.

A unique manner to define the logic with taboos is to require that the number of taboos is maximal so that if statement is dropped its negation remains in the logic. This implies $n > B^k/2$.

The projections of p-adic space-time sheets to real imbedding space as representations of Boolean functions

Quantum classical correspondence suggests that generalized Boolean functions should have space-time correlates. Since Boolean cognition involves free will, it should be possible to construct space-time representations of arbitrary Boolean functions with finite number of arguments freely. The non-determinism of p-adic differential equations guarantees this freedom.

p-Adic space-time sheets and p-adic non-determinism make possible to represent generalization of Boolean functions of four Boolean variables obtained by replacing both argument and function with p-valued pinary digit instead of bit. These representations result as discrete projections of p-adic space-time sheets to integer valued points of real Minkowski space M^4. The interpretation would be in terms of 4 sequences of truth values of p-valued logic associated with a finite 4-D integer lattice whose lattice points can be identified as sequences of truth values of a p-valued logic with a set of p-valued truth value at each point so that in the 2-adic case one has map $B^{4M} \rightarrow B^{4N}$. Here the number of lattice points in a given coordinate direction of M^4 is M and N is the number of bits allowed by binary cutoff for CP_2 coordinates. For $p = 2^k - n$ representing Boolean algebra with n taboos, the maps can be interpreted as maps $\hat{B}^{4M} \rightarrow \hat{B}^{4N}$.

These lattices can be seen as subsets of rational shadows of p-adic space-time sheets to Minkowski space. The condensed matter analog would be a lattice with a a sequence of p-valued dynamical variables (sequence of bits/spins for $p = 2$) at each lattice point. At a fixed spatial point of M^4 the lowest bits define a time evolution of a generalized Boolean function: $B \rightarrow B$.

These observations support the view that intentionality and logic related cognition could perhaps be regarded as 2-adic aspects of consciousness. The special role of primes $p = 2^k - n$ could also be understood as special role of Boolean logic among p-valued logics and $p = 2^k - n$ logic would correspond to B^k with n axioms representing logic respecting a belief system with n beliefs. Recall that multi-p p-adic fractality involving 2-adic fractality is possible for the solutions of field equations and explains p-adic length scale hypothesis.

Most points of the p-adic space-time sheets correspond to real points which are literally infinite as real points. Therefore cognition would be in quite literal sense outside the real cosmos. Perhaps this is a direct correlate for the basic experience that mind is looking the material world from outside.

Connection with the theory of computational complexity?

There are interesting questions concerning the interpretation of four generalized Boolean arguments. TGD explains the number $D = 4$ for space-time dimensions and also the dimension of imbedding space. Could one also find explanation why $d = 4$ defines special value for the number of generalized Boolean inputs and outputs?

1. Could the general theory of computational complexity allow to understand $d = 4$ as a maximum number of inputs and outputs allowing the computation of something related to these functions in polynomial time? For instance, complexity theorist could probably immediately answer following questions. Could the computation of the 2-adic values of CP_2 coordinates as a function of 2-adic M^4 coordinates expressed in terms of fundamental logical connectives take a time which is polynomial as a function of the number of N^4 pinary digits of M^4 coordinates and N^4 pinary digits of CP_2 coordinates? Is this time non-polynomial for M^d and S_d, S_d d-dimensional internal space, $d > 4$. Unfortunately I do not possess the needed complexity theoretic knowhow to answer these questions.

2. The same question could make sense also for $p > 2$ if the notion of the logical connectives and functions generalizes as it indeed does for $p = 2^k - n$. Therefore the question would be whether p-adic length scale hypothesis and dimensions of imbedding space and space-time are implied by a polynomial computation time? This could be the case since essentially a restriction of values and arguments of Boolean functions to a subset of B^k is in question.

Some calculational details

In the following the details of p-adic non-determinism are described for a differential equation of single p-adic variable and some comments about the generalization to the realistic case are given.

1. One-dimensional case

To understand the essentials consider for simplicity a solution of a p-adic differential equation giving function $y = f(x)$ of one independent variable $x = \sum_{n \geq n_0} x_n p^n$.

1. p-Adic non-determinism means that the initial values $f(x)$ of the solution can be fixed arbitrarily up to $N + 1$:th pinary digit. In other words, $f(x_N)$, where $x_N = \sum_{n_0 \leq n \geq N} x_n p^n$ is a rational obtained by dropping all pinary digits higher than N in $x = \sum_{n \geq n_0} x_n p^n$ can be chosen arbitrarily.

2. Consider the projection of $f(x)$ to the set of rationals assumed to be common to reals and p-adics.

   i) Genuinely p-adic numbers have infinite number of positive pinary digits in their non-periodic expansion (non-periodicity guarantees non-rationality) and are strictly infinite as real numbers. In this regime p-adic differential equation fixes completely the solution. This is the case also at rational points $q = m/n$ having infinite number of pinary digits in their pinary expansion.

   ii) The projection of p-adic x-axis to real axis consists of rationals. The set in which solution of p-adic differential equations is non-vanishing can be chosen rather freely. For instance, p-adic ball of radius p^{-n} consisting of points $x = p^M y$, $y \neq 0$, $|y|_p \leq 1$, can be considered. Assume

$N > M$. p-Adic nondeterminism implies that $f(q)$ for $q = \sum_{M \leq n \leq N} x_n p^n$, can be chosen arbitrarily. For $M \geq 0$ q is always integer valued and the scaling of x by a suitable power of p always allows to get a finite integer lattice at x-axis.

iii) The lowest pinary digit in the expansion of $f(q)$ in powers of p in defines a pinary digit. These pinary digits would define a representation for a sequence of truth values of p-logic. $p = 2$ gives the ordinary Boolean logic. It is also interpret this pinary function as a function of pinary argument giving Boolean function of one variable in 2-adic case.

2. Generalization to the space-time level

This picture generalizes to space-time level in a rather straight forward manner. y is replaced with CP_2 coordinates, x is replaced with M^4 coordinates, and differential equation with field equations deducible from the Kähler action. The essential point is that p-adic space-time sheets have projection to real Minkowski space which consists of a discrete subset of integers when suitable scaling of M^4 coordinates is allowed. The restriction of 4 CP_2 coordinates to a finite integer lattice of M^4 defines 4 Boolean functions of four Boolean arguments or their generalizations for $p > 2$. Also the modes of the induce spinor field define a similar representation.

1.3.4 Fibonacci numbers, Golden Mean, and Jones inclusions

The picture discussed above does not apply in the case of Golden Mean since powers of Φ do not have any special role for the algebraic extension of rationals by $\sqrt{5}$. It is however possible to understand the emergence of Fibonacci numbers and Golden Mean using quantum classical correspondence and the fact that the Clifford algebra and its sub-algebras associated with configuration space spinors corresponds to the so called hyper-finite factor of type II_1 (configuration space refers to the "world of classical worlds").

Infinite braids as representations of Jones inclusions

The appearance of hyper-finite factor of type II_1 at the level of basic quantum tGD justifies the expectation that Jones inclusions $\mathcal{N} \subset \mathcal{M}$ of these factors play a key role in TGD Universe. For instance, subsystem system inclusions could induce Jones inclusions.

For the Jones inclusion $\mathcal{N} \subset \mathcal{M}$ $\mathcal{M}$ can be regarded as an $\mathcal{N}$-module with fractal dimension given by Beraha number $B_n = 4cos^2(\pi/n)$, $n \geq 3$ or equivalently by the quantum group phases $exp(i\pi/n)$. B_5 satisfies $B_5 = 4cos^2(\pi/5) = \Phi^2 = \Phi + 1$ so that the special role of $n = 5$ inclusion could explain the special role of Golden Mean in Nature.

Hecke algebras H_n, which are also characterized by quantum phase $q = exp(i\pi/n)$ or the corresponding Beraha number $B_n = 4cos^2(\pi/n)$, characterize the anyonic quantum statistics of n-braid system. Braids are understood as threads which can get linked and define in this manner braiding. Braid group describes these braidings. Like any algebra, Hecke algebra H_n can be decomposed into a direct sum of matrix algebras. Fibonacci numbers characterize the dimensions of these matrix algebras for $n = 5$. Interestinglyt, topological quantum computation is based on the idea that computer programs can be coded into braidings. What is remarkable is that $n = 5$ characterizes the simplest universal quantum computer so that Golden Mean could indeed have very deep roots to quantum information processing.

The so called Bratteli diagrams characterize the inclusions of various direct summands of H_k to direct summands H_{k+1} in the sequence $H_3 \subset H_4 \subset ... \subset H_k \subset ...$ of Hecke algebras. Essentially the reduction of the representations of H_{k+1} to those of H_k is in question. The same Bratteli diagrams characterize also the Jones inclusions $\mathcal{N} \subset \mathcal{M}$ of hyper-finite factors of type II_1 with index n as a limit of a finite-dimensional inclusion. Thus Jones inclusion can be visualized as a system consisting of infinite number of braids. In TGD framework the braids could be represented by magnetic flux lines or flux tubes.

Logarithmic spirals as representations of Jones inclusions

The inclusion sequence for Hecke algebras has a representations as a logarithmic spiral. The angle $\pi/5$ can be identified as a limit for angles ϕ_n with $cos(\phi_n) = F_{n+1}/2F_n$ assignable to orthogonal triangle with hypothenuse $2F_n$ and short side F_{n+1} and $\sqrt{4F_n^2 - F_{n+1}^2}$. Fibonacci sequence defines via this prescription a logarithmic spiral as a symbolic representation of the $n = 5$ Jones inclusion representable also in terms of infinite number of braids.

DNA as a topological quantum computer?

Quantum classical correspondence encourages to think that space-time geometry could define a correlate for Jones inclusions of hyper-finite factors of Clifford sub-algebras associated with Clifford algebra of configuration space spinors. The appearance of Fibonacci series in living systems could represent one example of this correspondence. The angle $\pi/10$ closely related to Golden Mean characterizes the winding of DNA double strand. Could this mean that DNA allows to realize topological quantum computer programs as braidings? A possible realization would be based on the notion of super-genes [L2], which are like pages of a book identified as magnetic flux sheets containing genomes of sequences of cell nuclei as text lines. These text lines would represent line through which magnetic flux lines traverse.

The braiding of magnetic flux lines (or possibly flux sheets regarded as flattened tubes) would define the braiding and the particles involved would be anyons obeying dynamics having quantum group $SU(2)_q$, $q = exp(i\pi/5)$, as its symmetries. The anyons could be assigned with DNA nucleotides or triplets.

TGD predicts also different kind of new physics to DNA double strand. So called H_N-atoms consist of ordinary proton an N dark electrons at space-time sheet which is λ-fold covering of space-time sheet of ordinary hydrogen atom. The effective charge of H_N-atom is $1 - N/\lambda$ since the fine structure constant for dark electrons is scaled down by $1/\lambda$. H_λ-atoms have full electron shell and are therefore exceptionally stable. The proposal is that H_λ-atoms could replace ordinary hydrogen atoms in hydrogen bonds [L2, N4]. Single base pair corresponds to 2 or 3 hydrogen bonds. The question is whether λ-hydrogen atom might somehow relate to the anyons involved with topological quantum computation.

Anyons could be dark protons resulting in the formation dark hydrogen bond in the fusion of H_N atom and its conjugate H_{N_c}, $N_c = \lambda - N$. Neutron scattering and electron diffraction suggest that 1/4:th of protons of water are in dark phase in attosecond time scale [54], and the model explains this number.

1.4 The recent view about p-adic coupling constant evolution

One of the basic problems of quantum TGD is the understanding of p-adic coupling constant evolution. This evolution is discrete by p-adic length scale hypothesis justified by zero energy ontology. Discreteness means that continuous mass scale is replaced by mass scales coming as half octaves of CP_2 mass. One key question has been whether it is Kähler coupling constant squared g_K^2, gravitational coupling constant or both, which remain invariant under p-adic coupling constant evolution. Second problem relates to the value of g_K^2.

The most important outcome is a formula for Kähler coupling strength in terms of a calculable and manifestly finite Dirac determinant without any need for zeta function regularization. The formula fixes completely the number theoretic anatomy of Kähler coupling strength and of other gauge coupling strengths. When the formula for the gravitational constant involving Kähler coupling strength and the exponent of Kähler action for CP_2 type vacuum extremal - which remains still a conjecture - is combined with the number theoretical results and with the constraints from the predictions of p-adic mass calculations, one ends up to an identification of Kähler coupling strength as fine structure constant at electron length scale characterized by p-adic prime M_{127}. Also the number theoretic anatomy of the ratio $R^2/\hbar G$, where R is CP_2 size, can be understood to high degree and a relationship between the p-adic evolutions of electromagnetic and color coupling strengths emerges.

1.4.1 The bosonic action defining Kähler function as the effective action associated with the induced spinor fields

One could *define* the classical action defining Kähler function as the bosonic action giving rise to the divergences of the isometry currents. In this manner bosonic action, especially the value of the Kähler coupling strength, would come out as prediction of the theory containing no free parameters.

Thus the Kähler action S_B of preferred extremal of Käction defining Kähler function could be *defined* by the functional integral over the Grassmann variables for the exponent of the massless Dirac action. Formally the functional integral is defined as

$$\begin{aligned} exp(S_B(X^4)) &= \int exp(S_F) D\Psi D\bar{\Psi} \ , \\ S_F &= \bar{\Psi} \left[\hat{\Gamma}^\alpha D_\alpha^{\rightarrow} - D_\alpha^{\leftarrow} \hat{\Gamma}^\alpha \right] \Psi \sqrt{g} \ . \end{aligned} \tag{1.4.1}$$

Formally the bosonic effective action is expressible as a logarithm of the fermionic functional determinant resulting from the functional integral over the Grassmann variables

$$\begin{aligned} S_B(X^4) &= log(det(D)) \ , \\ D &= \hat{\Gamma}^\alpha D_\alpha^{\rightarrow} \ . \end{aligned} \tag{1.4.2}$$

Can one do without zeta function regularization?

The rigorous definition of the fermionic determinant has been already discussed in [A6]. The best one hope that the formal definition of the determinant as the the product of the generalized eigenvalues of D_K works as such. This is the case if the number of eigenvalues is finite; if the eigenvalues approach to constant which can be chosen to be equal to unity; or if the eigenvalues have approximate symmetry $\lambda \rightarrow 1/\lambda$.

1. Somewhat surprisingly the detailed construction of the eigenvalue spectrum discussed in [A6] shows that the number of eigenvalues is indeed finite and that eigenvalues are bounded from above. The basic idea of the construction is following. The eigenvalues correspond to the generalized eigenvalues of the modified Dirac operator D_K for Kähler action at X^3_l.

2. Since modified Dirac equation for D_K is equivalent with the conservation of super current, the shock wave property means that the super current is restricted to X^3_l and thus has a vanishing normal component. In the case of wormhole throats the construction requires boundary conditions stating that there exist coordinates in which $J_{ni} = 0$ and $g_{ni} = 0$ at X^3_l [A6]. Therefore classical gravitational field is effectively static at X^3_l and the Maxwell field defined by the induced Kähler form has only the magnetic part in these coordinates.

3. The generalized eigenvalues of D_K appearing in Dirac determinant can be identified as eigenvalues of the transversal part of 3-D Dirac operator defined by the restriction of D_K to X^3_l describing fermions in the electro-weak magnetic field associated with X^3_l. The physical analog is energy spectrum for Dirac operator in external magnetic field. The effective metric appearing in the modified Dirac operator corresponds to

$$\hat{g}^{\alpha\beta} = \frac{\partial L_K}{\partial h^k_\alpha} \frac{\partial L_K}{\partial h^l_\beta} h_{kl} \ ,$$

and vanishes at the boundaries of regions carrying non-vanishing Kähler magnetic field. Hence the shock waves must be localized to regions $X^3_{l,i}$ containing a non-vanishing Kähler magnetic field. Cyclotron states in constant magnetic field serve as a good analog for the situation and only a finite number of cyclotron states are possible since for higher cyclotron states the wave function -essentially harmonic oscillator wave function- would concentrate outside $X^3_{l,i}$.

4. A more precise argument goes as follows. Assume that it is induced Kähler magnetic field B_K that matters. The vanishing of the effective contravariant metric near the boundary of $X^3_{l,i}$ corresponds to an infinite effective mass for massive particle in constant magnetic field so that the counterpart for the cyclotron frequency scale eB/m reduces to zero. The radius of the cyclotron orbit is proportional to $1/\sqrt{eB}$ and approaches to infinity. Hence the required localization is not possible only for cyclotron states for which the cyclotron radius is below that the transversal size scale of $X^3_{l,i}$.

5. The eigenvalues of the modified Dirac operator vanish for the vacuum extremals but the Dirac determinant equals to one in this case since zero eigenvalues do not correspond to localized solutions and by definition do not contribute to it.

Zeta function regularization

In the more general case regularization is needed. The sum over the logarithms of the eigen values in turn can be identified as the derivative of the logarithm of the generalized Zeta function

$$\begin{aligned}
\zeta_F(s) &\equiv \sum_n \lambda_n^{-s} \ , \\
D\Psi_n &= \lambda_n o \Psi_n \ , \\
o &= n^\alpha \gamma_\alpha \ , \ \ [D,0]=0 \ .
\end{aligned} \tag{1.4.3}$$

at $s=0$:

$$S_B(X^4) = log(det(D)) = \sum_n \log(\lambda_n) = -\frac{d}{ds} log(\zeta_F)(s,X^4) \ . \tag{1.4.4}$$

The vector n_α identified as the gradient of a coordinate x^N normal to X^3. As shown in [A6], the hermiticity of the modified Dirac operator is guaranteed if X^3 is minimal hyper-surface or if Kähler action density L_K vanishes at X^3.

The vanishing of the normal components T^{nk} of the conserved currents associated with the isometries of H is necessary in order to have effective 3-dimensionality in the sense that the modified Dirac equation contains only derivatives acting on X^3 coordinates. The reduction to the boundary and the dependence on the normal derivatives of the imbedding space coordinates realizes quantum gravitational holography.

The definition relying on the generalized Zeta function allows to circumvent the possible technical difficulties related to the precise definition of the Grassmannian functional integral and of the functional determinant since the possibly divergent sum over the logarithms of the eigenvalues can be identified as the derivative of Zeta function at $s=0$, which can be defined by analytically continuing the zeta function outside the domain where the definition in terms of the eigenvalues works.

Formula for the Kähler coupling strength

The identification of exponent of Kähler function as Dirac determinant leads to a formula relating Kähler action for the preferred extremal to the Dirac determinant. The eigenvalues are proportional to $1/\alpha_K$ since the matrices $\hat{\Gamma}^\alpha$ have this proportionality. This gives the formula

$$exp(\frac{S_K(X^4(X^3))}{8\pi\alpha_K}) = \prod_i \lambda_i = \frac{\prod_i \lambda_{0,i}}{\alpha_K^N} \ . \tag{1.4.5}$$

Here $\lambda_{0,i}$ corresponds to $\alpha_K = 1$. $S_K = \int J^* J$ is the reduced Kähler action.

For $S_K = 0$, which might correspond to so called massless extremals [D1] one obtains the formula

$$\alpha_K = (\prod_i \lambda_{0,i})^{1/N} \ . \tag{1.4.6}$$

Thus for $S_K = 0$ extremals one has an explicit formula for α_K having interpretation as the geometric mean of the eigenvalues $\lambda_{0,i}$. Several values of α_K are in principle possible.

p-Adicization suggests that $\lambda_{0,i}$ are rational or at most algebraic numbers. This would mean that α_K is N:th root of this kind of number. S_K in turn would be

$$S_K = 8\pi\alpha_K log(\frac{\prod_i \lambda_{0,i}}{\alpha_K^N}) \ . \tag{1.4.7}$$

so that S_K would be expressible as a product of the transcendental π, N:th root of rational, and logarithm of rational. This result would provide a general answer to the question about number theoretical anatomy of Kähler coupling strength and S_K. Note that S_K makes sense p-adically only if one adds π and its all powers to the extension of p-adic numbers. The exponent of Kähler function however makes sense also p-adically.

1.4.2 A revised view about coupling constant evolution

The development of the ideas related to number theoretic aspects has been rather tortuous and based on guess work since basic theory has been lacking.

1. The original hypothesis was that Kähler coupling strength is invariant under p-adic coupling constant evolution. Later I gave up this hypothesis and replaced it with the invariance of gravitational coupling since otherwise the prediction would have been that gravitational coupling strength is proportional to p-adic length scale squared. Second first guess was that Kähler coupling strength equals to the value of fine structure constant at electron length scale corresponding to Mersenne prime M_{127}. Later I replaced fine structure constant with electro-weak U(1) coupling strength at this length scale. The recent discussion returns back to the roots in both aspects.

2. The recent discussion relies on the progress made in the understanding of quantum TGD at partonic level [A6]. What comes out is an explicit formula for Kähler couplings strength in terms of Dirac determinant involving only a finite number of eigenvalues of the modified Dirac operator. This formula dictates the number theoretical anatomy of g_K^2 and also of other coupling constants: the most general option is that α_K is a root of rational. The requirement that the rationals involved are simple combined with simple experimental inputs leads to very powerful predictions for the coupling parameters.

3. A further simplification is due to the discreteness of p-adic coupling constant evolution allowing to consider only length scales coming as powers of $\sqrt{2}$. This kind of discretization is necessary also number theoretically since logarithms can be replaced with 2-adic logarithms for powers of 2 giving integers. This raises the question whether $p \simeq 2^k$ should be replaced with 2^k in all formulas as the recent view about quantum TGD suggests.

4. The prediction is that Kähler coupling strength α_K is invariant under p-adic coupling constant evolution and from the constraint coming from electron and top quark masses very near to fine structure constant so that the identification as fine structure constant is natural. Gravitational constant is predicted to be proportional to p-adic length scale squared and corresponds to the largest Mersenne prime (M_{127}), which does not correspond to a completely super-astronomical p-adic length scale. For the parameter R^2/G p-adicization program allows to consider two options: either this constant is of form e^q or 2^q: in both cases q is rational number. $R^2/G = exp(q)$ allows only M_{127} gravitons if number theory is taken completely seriously. $R^2/G = 2^q$ allows all p-adic length scales for gravitons and thus both strong and weak variants of ordinary gravitation.

5. A relationship between electromagnetic and color coupling constant evolutions based on the formula $1/\alpha_{em} + 1/\alpha_s = 1/\alpha_K$ is suggested by the induced gauge field concept, and would mean that the otherwise hard-to-calculate evolution of color coupling strength is fixed completely. The predicted value of α_s at intermediate boson length scale is correct.

It seems fair to conclude that the attempts to understand the implications of p-adicization for coupling constant evolution have begun to bear fruits.

Identifications of Kähler coupling strength and gravitational coupling strength

To construct an expression for gravitational constant one can use the following ingredients.

1. The exponent $exp(2S_K(CP_2))$ defining the value of Kähler function in terms of the Kähler action $S_K(CP_2)$ of CP_2 type extremal representing elementary particle expressible as

$$S_K(CP_2) \quad = \quad \frac{S_{K,R}(CP_2)}{8\pi\alpha_K} = \frac{\pi}{8\alpha_K} \quad . \tag{1.4.8}$$

Since CP_2 type extremals suffer topological condensation, one expects that the action is modified:

$$S_K(CP_2) \quad \rightarrow \quad a \times S_K(CP_2) \ . \tag{1.4.9}$$

$a < 1$ conforms with the idea that a piece of CP_2 type extremal defining a wormhole contact is in question. One must however keep mind open in this respect.

2. The p-adic length scale L_p assignable to the space-time sheet along which gravitational interactions are mediated. Since Mersenne primes seem to characterized elementary bosons and since the Mersenne prime $M_{127} = 2^{127} - 1$ defining electron length scale is the largest non-super-astronomical length scale it is natural to guess that M_{127} characterizes these space-time sheets.

1. The formula for the gravitational constant

A long standing basic conjecture has been that gravitational constant satisfies the following formula

$$\begin{aligned} \hbar G \quad &\equiv \quad r\hbar_0 G = L_p^2 \times exp(-2aS_K(CP_2)) \ , \\ L_p \quad &= \quad \sqrt{p}R \ . \end{aligned} \tag{1.4.10}$$

Here R is CP_2 radius defined by the length $2\pi R$ of the geodesic circle. What was noticed before is that this relationship allows even constant value of G if a has appropriate dependence on p.

This formula seems to be correct but the argument leading to it was based on two erratic assumptions compensating each other.

1. I assumed that modulus squared for vacuum functional is in question: hence the factor $2a$ in the exponent. The interpretation of zero energy state as a generalized Feynman diagram requires the use of vacuum functional so that the replacement $2a \rightarrow a$ is necessary.

2. Second wrong assumption was that graviton corresponds to CP_2 type vacuum extremal- that is wormhole contact in the recent picture. This does allow graviton to have spin 2. Rather, two wormhole contacts represented by CP_2 vacuum extremals and connected by fluxes associated with various charges at their throats are needed so that graviton is string like object. This saves the factor $2a$ in the exponent.

The highly non-trivial implication to be discussed later is that ordinary coupling constant strengths should be proportional to $exp(-aS_K(CP_2))$.

The basic constraint to the coupling constant evolution comes for the invariance of g_K^2 in p-adic coupling constant evolution:

$$\begin{aligned} g_K^2 \quad &= \quad \frac{a(p,r)\pi^2}{log(pK)} \ , \\ K \quad &= \quad \frac{R^2}{\hbar G(p)} = \frac{1}{r}\frac{R^2}{\hbar_0 G(p)} \equiv \frac{K_0(p)}{r} \quad . \end{aligned} \tag{1.4.11}$$

2. How to guarantee that g_K^2 is RG invariant and N:th root of rational?

Suppose that g_K^2 is N:th root of rational number and invariant under p-adic coupling constant evolution.

1. The most general manner to guarantee the expressibility of g_K^2 as N:th root of rational is guaranteed for both options by the condition

$$a(p,r) = \frac{g_K^2}{\pi^2} log(\frac{pK_0}{r}) \ . \tag{1.4.12}$$

   That a would depend logarithmically on p and $r = \hbar/\hbar_0$ looks rather natural. Even the invariance of G under p-adic coupling constant evolution can be considered.

2. The condition

$$\frac{r}{p} < K_0(p) \ . \tag{1.4.13}$$

   must hold true to guarantee the condition $a > 0$. Since the value of gravitational Planck constant is very large, also the value of corresponding p-adic prime must very large to guarantee this condition. The condition $a < 1$ is guaranteed by the condition

$$\frac{r}{p} > exp(-\frac{\pi^2}{g_K^2}) \times K_0(p) \ . \tag{1.4.14}$$

   The condition implies that for very large values of p the value of Planck constant must be larger than $\hbar_0$.

3. The two conditions are summarized by the formula

$$K_0(p) \times exp(-\frac{\pi^2}{g_K^2}) < \frac{r}{p} < K_0(p) \tag{1.4.15}$$

   characterizing the allowed interval for r/p. If G does not depend on p, the minimum value for r/p is constant. The factor $exp(-\frac{\pi^2}{g_K^2})$ equals to 1.8×10^{-47} for $\alpha_K = \alpha_{em}$ so that $r > 1$ is required for $p \geq 4.2 \times 10^{-40}$. $M_{127} \sim 10^{38}$ is near the upper bound for p allowing $r = 1$. The constraint on r would be roughly $r \geq 2^{k-131}$ and $p \simeq 2^{131}$ is the first p-adic prime for which $\hbar > 1$ is necessarily. The corresponding p-adic length scale is .1 Angstroms.

   This conclusion need not apply to elementary particles such as neutrinos but only to the space-time sheets mediating gravitational interaction so that in the minimal scenario it would be gravitons which must become dark above this scale. This would bring a new aspect to vision about the role of gravitation in quantum biology and consciousness.

   The upper bound for r behaves roughly as $r < 2.3 \times 10^7 p$. This condition becomes relevant for gravitational Planck constant GM_1M_2/v_0 having gigantic values. For Earth-Sun system and for $v_0 = 2^{-11}$ the condition gives the rough estimate $p > 6 \times 10^{63}$. The corresponding p-adic length scale would be of around $L(215) \sim 40$ meters.

4. p-Adic mass calculations predict the mass of electron as $m_e^2 = (5+Y_e)2^{-127}/R^2$ where $Y_e \in [0,1)$ parameterizes the not completely known second order contribution. Top quark mass favors a small value of Y_e (the original experimental estimates for m_t were above the range allowed by TGD but the recent estimates are consistent with small value Y_e [F4]). The range $[0,1)$ for Y_e restricts $K_0 = R^2/\hbar_0 G$ to the range $[2.3683, 2.5262] \times 10^7$.

5. The best value for the inverse of the fine structure constant is $1/\alpha_{em} = 137.035999070(98)$ and would correspond to $1/g_K^2 = 10.9050$ and to the range $(0.9757, 0.9763)$ for a for $\hbar = \hbar_0$ and $p = M_{127}$. Hence one can seriously consider the possibility that $\alpha_K = \alpha_{em}(M_{127}$ holds true. As a matter fact, this was the original hypothesis but was replaced later with the hypothesis that α_K corresponds to electro-weak $U(1)$ coupling strength in this length scale. The fact that M_{127} defines the largest Mersenne prime, which does not correspond to super-astrophysical length scale might relate to this co-incidence.

To sum up, the recent view about coupling constant evolution differs strongly from previous much more speculative scenarios. It implies that g_K^2 is root of rational number, possibly even rational, and can be assumed to be equal to e^2. Also $R^2/\hbar G$ could be rational. The new element is that G need not be proportional to p and can be even invariant under coupling constant evolution since the the parameter a can depend on both p and r. An unexpected constraint relating p and r for space-time sheets mediating gravitation emerges.

Are the color and electromagnetic coupling constant evolutions related?

Classical theory should be also able to say something non-trivial about color coupling strength α_s too at the general level. The basic observations are following.

1. Both classical color YM action and electro-weak U(1) action reduce to Kähler action.

2. Classical color holonomy is Abelian which is consistent also with the fact that the only signature of color that induced spinor fields carry is anomalous color hyper charge identifiable as an electro-weak hyper charge.

Suppose that α_K is a strict RG invariant. One can consider two options.

1. The original idea was that the sum of classical color action and electro-weak U(1) action is RG invariant and thus equals to its asymptotic value obtained for $\alpha_{U(1)} = \alpha_s = 2\alpha_K$. Asymptotically the couplings would approach to a fixed point defined by $2\alpha_K$ rather than to zero as in asymptotically free gauge theories.

   Thus one would have

$$\frac{1}{\alpha_{U(1)}} + \frac{1}{\alpha_s} = \frac{1}{\alpha_K} \quad . \tag{1.4.16}$$

   The relationship between $U(1)$ and em coupling strengths is

$$\begin{aligned} \alpha_{U(1)} &= \frac{\alpha_{em}}{cos^2(\theta_W)} \simeq \frac{1}{104.1867} \ , \\ sin^2(\theta_W)_{|10\ MeV} &\simeq 0.2397(13) \ , \\ \alpha_{em}(M_{127}) &= 0.00729735253327 \ . \end{aligned} \tag{1.4.17}$$

   Here Weinberg angle corresponds to 10 MeV energy is reasonably near to the value at electron mass scale. The value $sin^2(\theta_W) = 0.2397(13)$ corresponding to 10 MeV mass scale [32] is used. Note however that the previous argument implying $\alpha_K = \alpha_{em}(M_{127})$ excludes $\alpha = \alpha_{U(1)}(M_{127})$ option.

2. Second option is obtained by replacing $U(1)$ with electromagnetic gauge $U(1)_{em}$.

$$\frac{1}{\alpha_{em}} + \frac{1}{\alpha_s} = \frac{1}{\alpha_K}. \tag{1.4.18}$$

Possible justifications for this assumption are following. The notion of induced gauge field makes it possible to characterize the dynamics of classical electro-weak gauge fields using only the Kähler part of electro-weak action, and the induced Kähler form appears only in the electromagnetic part of the induced classical gauge field. A further justification is that em and color interactions correspond to unbroken gauge symmetries.

The following arguments are consistent with this conclusion.

1. In TGD framework coupling constant is discrete and comes as powers of $\sqrt{2}$ corresponding to p-adic primes $p \simeq 2^k$. Number theoretic considerations suggest that coupling constants g_i^2 are algebraic or perhaps even rational numbers, and that the logarithm of mass scale appearing as argument of the renormalized coupling constant is replaced with 2-based logarithm of the p-adic length scale so that one would have $g_i^2 = g_i^2(k)$. g_K^2 is predicted to be N:th root of rational but could also reduce to a rational. This would allow rational values for other coupling strengths too. This is possible if $sin(\theta_W)$ and $cos(\theta_W)$ are rational numbers which would mean that Weinberg angle corresponds to a Pythagorean triangle as proposed already earlier. This would mean the formulas $sin(\theta_W) = (r^2 - s^2)/(r^2 + s^2)$ and $cos(\theta_W) = 2rs(r^2 + s^2)$.

2. A very strong prediction is that the beta functions for color and $U(1)$ degrees of freedom are apart from sign identical and the increase of $U(1)$ coupling compensates the decrease of the color coupling. This allows to predict the hard-to-calculate evolution of QCD coupling constant strength completely.

3. $\alpha(M_{127}) = \alpha_K$ implies that M_{127} defines the confinement length scale in which the sign of α_s becomes negative. TGD predicts that also M_{127} copy of QCD should exist and that M_{127} quarks should play a key role in nuclear physics [F8, F9]. Hence one can argue that color coupling strength indeed diverges at M_{127} (the largest not completely super-astrophysical Mersenne prime) so that one would have $\alpha_K = \alpha(M_{127})$. Therefore the precise knowledge of $\alpha(M_{127})$ in principle fixes the value of parameter $K = R^2/G$ and thus also the second order contribution to the mass of electron.

4. $\alpha_s(M_{89})$ is predicted to be $1/\alpha_s(M_{89}) = 1/\alpha_K - 1/\alpha(M_{89})$. $sin^2(\theta_W) = .23120$, $\alpha_{em}(M_{89}) \simeq 1/127$, and $\alpha_{U(1)} = \alpha_{em}/cos^2(\theta_W)$ give $1/\alpha_{U(1)}(M_{89}) = 97.6374$. $\alpha = \alpha_{em}$ option gives $1/\alpha_s(M_{89}) \simeq 10$, which is consistent with experimental facts. $\alpha = \alpha_{U(1)}$ option gives $\alpha_s(M_{89}) = 0.1572$, which is larger than QCD value. Hence $\alpha = \alpha_{em}$ option is favored.

To sum up, the proposed formula would dictate the evolution of α_s from the evolution of the electro-weak parameters without any need for perturbative computations. Although the formula of proposed kind is encouraged by the strong constraints between classical gauge fields in TGD framework, it should be deduced in a rigorous manner from the basic assumptions of TGD before it can be taken seriously.

Can one deduce formulae for gauge couplings?

The improved physical picture behind gravitational constant allows also to consider a general formula for gauge couplings.

1. The natural guess for the general formula would be as

$$g^2(p,r) = kg_K^2 \times exp[-a_g(p,r) \times S_K(CP_2)] \ . \tag{1.4.19}$$

here k is a numerical constant.

2. The condition

$g_K^2 = e^2(M_{127})$ fixes the value of k if it's value does not depend on the character of gauge interaction:

$$k = exp[a_{gr}(M_{127}, r=1) \times S_K(CP_2)] \ . \tag{1.4.20}$$

Hence the general formula reads as

$$g^2(p,r) = g_K^2 \times exp[(-a_g(p,r) + a_{gr}(M_{127}), r=1)) \times S_K(CP_2)] \ . \tag{1.4.21}$$

The value of $a(M_{127}, r=1)$ is near to its maximum value so that the exponential factor tends to increase the value of g^2 from e^2. The formula can reproduce α_s and various electro-weak couplings although it is quite possibile that Weinberg angle corresponds to a group theoretic factor not representable in terms of $a_g(p,r)$. The volume of the CP_2 type vacuum extremal would characterize gauge bosons. Analogous formula should apply also in the case of Higgs.

3. α_{em} in very long length scales would correspond to

$$e^2(p \rightarrow \infty, r=1) = e^2 \times exp[(-1 + a(M_{127}), r=1)) \times S_K(CP_2)] = e^2 x \ , \tag{1.4.22}$$

where x is in the range $[0.6549, 0.6609]$.

Formula relating v_0 to α_K and R^2/G

The parameter $v_0 = 2^{-11}$ plays a key role in the formula for gravitational Planck constant and can be also seen as a fundamental constant in TGD framework. As a matter, factor v_0 has interpretation as velocity parameter and is dimensionless when $c=1$ is used.

If v_0 is identified as the rotation velocity of distant stars in galactic plane, one can use the Newtonian model for the motion of mass in the gravitational field of long straight string giving $v_0 = \sqrt{TG}$. String tension T can be expressed in terms of Kähler coupling strength as

$$T = \frac{b}{2\alpha_K R^2} \ ,$$

where R is the radius of geodesic circle. The factor $b \leq 1$ would explain reduction of string tension in topological condensation caused by the fact that not entire geodesic sphere contributes to the action.

This gives

$$\begin{aligned} v_0 &= \frac{b}{2\sqrt{\alpha_K K}} \ , \\ \alpha_K(p) &= \frac{a\pi}{4log(pK)} \ , \\ K &= \frac{R^2}{\hbar G} \ . \end{aligned} \tag{1.4.23}$$

The condition that α_K has the desired value for $p = M_{127} = 2^{127}-1$ defining the p-adic length scale of electron fixes the value of b for given value of a. The value of b should be smaller than 1 corresponding to the reduction of string tension in topological condensation.

The condition 1.4.23 for $v_0 = 2^{-m}$, say $m = 11$, allows to deduce the value of a/b as

$$\frac{a}{b} = \frac{4 * log(pK)}{\pi} \frac{2^{2m-1}}{K} \ . \tag{1.4.24}$$

For both $K = e^q$ with $q = 17$ and $K = 2^q$ option with $q = 24 + 1/2$ $m = 10$ is the smallest integer giving $b < 1$. $K = e^q$ option gives $b = .3302$ $(.0826)$ and $K = 2^q$ option gives $b = .3362$ $(.0841)$ for $m = 10$ $(m = 11)$.

$m = 10$ corresponds to one third of the action of free cosmic string. $m = 11$ corresponds to much smaller action smaller by a factor rather near 1/12. The interpretation would be that as m increases the action of the topologically condensed cosmic string decreases. This would correspond to a gradual transformation of the cosmic string to a magnetic flux tube.

Is the p-adic temperature proportional to the Chern-Simons coupling strength?

Chern-Simons coupling strength has the same spectrum as p-adic temperature T_p apart from a multiplicative factor. The identification $T_p = 1/k$ is indeed very natural since also $1/k$ is temperature like parameter. The simplest guess is

$$T_p = \frac{1}{k} . \tag{1.4.25}$$

α_K is also temperature like parameter and the original conjecture was that α_K and also other coupling strengths are expressible in terms of k. The recent view about how the information about Kähler action is feeded to the eigenvalue spectrum of the modified Dirac operator D_K associated with Kähler action [A6] does not encourage this conjecture.

For fermions one has $T_p = 1$ so that fermionic light-like wormhole throats would correspond to $k = 1$. Since photon, graviton, and gluons are massless in an excellent approximation, p-adic temperature $T_p = 1/k$ should be small for them. This holds true for intermediate gauge bosons too since ground state conformal weight gives the dominating contribution to their mass. Gauge bosons are identified as pairs of light-like wormhole throats associated with wormhole contacts, and one can consider the possibility that there is maximal p-adic temperature at which gauge boson wormhole contacts are stable against splitting to fermion-antifermion pair. Fermions and possible exotic bosons created by bosonic generators of super-symplectic algebra would correspond to single wormhole throat and could also naturally correspond to the maximal value of p-adic temperature since there is nothing to which they can decay.

What could go wrong with this picture? The different values of k for fermions and bosons make sense only if the 4-D space-time sheets associated with fermions and bosons can be regarded as disjoint space-time regions. Gauge bosons correspond to wormhole contacts connecting (deformed pieces of CP_2 type extremal) positive and negative energy space-time sheets whereas fermions would correspond to deformed CP_2 type extremal glued to single space-time sheet having either positive or negative energy. These space-time sheets should make contact only in interaction vertices of the generalized Feynman diagrams, where partonic 3-surfaces are glued together along their ends. If this gluing together occurs only in these vertices, fermionic and bosonic space-time sheets are disjoint. For stringy diagrams this picture would fail.

To sum up, the resulting overall vision seems to be internally consistent and is consistent with generalized Feynman graphics, predicts the value of g_K^2, suggests the identification of the inverse of p-adic temperature with k, allows to understand the differences between fermionic and bosonic massivation. One might hope that the additional objections (to be found sooner or later!) could allow to develop a more detailed picture.

1.5 The recent view about quantum TGD

Before detailed discussion of what p-adicization of quantum TGD could mean, it is good to have an overall view about what quantum TGD in real context is.

1.5.1 Basic notions

The notions of imbedding space, 3-surface (and 4-surface), and configuration space (world of classical worlds (WCW)) are central to quantum TGD. The original idea was that 3-surfaces are space-like 3-surfaces of $H = M^4 \times CP_2$ or $H = M^4_+ \times CP_2$, and WCW consists of all possible 3-surfaces in H. The basic idea was that the definition of Kähler metric of WCW assigns to each X^3 a unique space-time surface $X^4(X^3)$ allowing in this manner to realize general coordinate invariance. During years these notions have however evolved considerably. Therefore it seems better to begin directly from the recent picture.

The notion of imbedding space

Two generalizations of the notion of imbedding space were forced by number theoretical vision [E2, E3].

1. p-Adicization forced to generalize the notion of imbedding space by gluing real and p-adic variants of imbedding space together along rationals and common algebraic numbers. The generalized imbedding space has a book like structure with reals and various p-adic number fields (including their algebraic extensions) representing the pages of the book.

2. With the discovery of zero energy ontology [C1, A6] it became clear that the so called causal diamonds (CDs) interpreted as intersections $M^4_+ \cap M^4_-$ of future and past directed light-cones of $M^4 \times CP_2$ define correlates for the quantum states. The position of the "lower" tip of CD characterizes the position of CD in H. If the temporal distance between upper and lower tip of CD is quantized power of 2 multiples of CP_2 length, p-adic length scale hypothesis [E5] follows as a consequence. The upper *resp.* lower light-like boundary $\delta M^4_+ \times CP_2$ *resp.* $\delta M^4_- \times CP_2$ of CD can be regarded as the carrier of positive *resp.* negative energy part of the state. All net quantum numbers of states vanish so that everything is creatable from vacuum. Space-time surfaces assignable to zero energy states would would reside inside $CD \times CP_2$s and have their 3-D ends at the light-like boundaries of $CD \times CP_2$. Fractal structure is present in the sense that CDs can contains CDs within CDs, and measurement resolution dictates the length scale below which the sub-CDs are not visible.

3. The realization of the hierarchy of Planck constants [A9] led to a further generalization of the notion of imbedding space. Generalized imbedding space is obtained by gluing together Cartesian products of singular coverings and factor spaces of CD and CP_2 to form a book like structure. The particles at different pages of this book behave like dark matter relative to each other. This generalization also brings in the geometric correlate for the selection of quantization axes in the sense that the geometry of the sectors of the generalized imbedding space with non-standard value of Planck constant involves symmetry breaking reducing the isometries to Cartan subalgebra. Roughly speaking, each CD and CP_2 is replaced with a union of CDs and CP_2s corresponding to different choices of quantization axes so that no breaking of Poincare and color symmetries occurs at the level of entire WCW.

4. The construction of quantum theory at partonic level brings in very important delicacies related to the Kähler gauge potential of CP_2. Kähler gauge potential must have what one might call pure gauge parts in M^4 in order that the theory does not reduce to mere topological quantum field theory. Hence the strict Cartesian product structure $M^4 \times CP_2$ breaks down in a delicate manner. These additional gauge components -present also in CP_2- play key role in the model of anyons, charge fractionization, and quantum Hall effect [F12].

The notions of 3-surface and space-time surface

The question what one exactly means with 3-surface turned out to be non-trivial.

1. The original identification of 3-surfaces was as arbitrary space-like 3-surfaces subject to Equivalence implied by General Coordinate Invariance. There was a problem related to the realization of Equivalence Principle since it was not at all obvious why the preferred extremal $X^4(Y^3)$ for Y^3 at $X^4(X^3)$ and Diff^4 related X^3 should satisfy $X^4(Y^3) = X^4(X^3)$.

2. Much later it became clear that light-like 3-surfaces have unique properties for serving as basic dynamical objects, in particular for realizing the General Coordinate Invariance in 4-D sense (obviously the identification resolves the above mentioned problem) and understanding the conformal symmetries of the theory. On basis of these symmetries light-like 3-surfaces can be regarded as orbits of partonic 2-surfaces so that the theory is locally 2-dimensional. It is however important to emphasize that this indeed holds true only locally. At the level of WCW metric

this means that the components of the Kähler form and metric can be expressed in terms of data assignable to 2-D partonic surfaces. It is however essential that information about normal space of the 2-surface is needed.

3. Rather recently came the realization that light-like 3-surfaces can have singular topology in the sense that they are analogous to Feynman diagrams. This means that the light-like 3-surfaces representing lines of Feynman diagram can be glued along their 2-D ends playing the role of vertices to form what I call generalized Feynman diagrams. The ends of lines are located at boundaries of sub-CDs. This brings in also a hierarchy of time scales: the increase of the measurement resolution means introduction of sub-CDs containing sub-Feynman diagrams. As the resolution is improved, new sub-Feynman diagrams emerge so that effective 2-D character holds true in discretized sense and in given resolution scale only.

The basic vision has been that space-time surfaces correspond to preferred extremals $X^4(X^3)$ of Kähler action. Kähler function $K(X^3)$ defining the Kähler geometry of the world of classical worlds would correspond to the Kähler action for the preferred extremal. The precise identification of the preferred extremals actually has however remained open.

1. The obvious guess motivated by physical intuition was that preferred extremals correspond to the absolute minima of Kähler action for space-time surfaces containing X^3. This choice has some nice implications. For instance, one can develop an argument for the existence of an infinite number of conserved charges. If X^3 is light-like surface- either light-like boundary of X^4 or light-like 3-surface assignable to a wormhole throat at which the induced metric of X^4 changes its signature- this identification circumvents the obvious objections.

2. Much later number theoretical vision led to the conclusion that $X^4(X^3_{l,i})$, where $X^3_{l,i}$ denotes a connected component of the light-like 3-surfaces X^3_l, contain in their 4-D tangent space $T(X^4(X^3_{l,i}))$ a subspace $M^2_i \subset M^4$ having interpretation as the plane of non-physical polarizations. This means a close connection with super string models. Geometrically this would mean that the deformations of 3-surface in the plane of non-physical polarizations would not contribute to the line element of WCW. This is as it must be since complexification does not make sense in M^2 degrees of freedom.

   In number theoretical framework M^2_i has interpretation as a preferred hyper-complex sub-space of hyper-octonions defined as 8-D subspace of complexified octonions with the property that the metric defined by the octonionic inner product has signature of M^8. A stronger condition would be that the condition holds true at all points of $X^4(X^3)$ for a global choice M^2 but this is un-necessary and leads to strong un-proven conjectures. The condition $M^2_i \subset T(X^4(X^3_{l,i}))$ in principle fixes the tangent space at $X^3_{l,i}$, and one has good hopes that the boundary value problem is well-defined and fixes $X^4(X^3)$ uniquely as a preferred extremal of Kähler action. This picture is rather convincing since the choice $M^2_i \subset M^3$ plays also other important roles.

3. The next step [A6] was the realization that the construction of the configuration space geometry in terms of modified Dirac action strengthens the boundary conditions to the condition that there exists space-time coordinates in which the induced CP_2 Kähler form and induced metric satisfy the conditions $J_{ni} = 0$, $g_{ni} = 0$ hold at X^3_l. One could say that at X^3_l situation is static both metrically and for the Maxwell field defined by the induced Kähler form. There are reasons to hope that this is the final step in a long process.

4. The weakest form of number theoretic compactification [E2] states that light-like 3-surfaces $X^3 \subset X^4(X^3) \subset M^8$, where $X^4(X^3)$ hyper-quaternionic surface in hyper-octonionic M^8 can be mapped to light-like 3-surfaces $X^3 \subset X^4(X^3) \subset M^4 \times CP_2$, where $X^4(X^3)$ is now preferred extremum of Kähler action. The natural guess is that $X^4(X^3) \subset M^8$ is a preferred extremal of Kähler action associated with Kähler form of E^4 in the decomposition $M^8 = M^4 \times E^4$, where M^4 corresponds to hyper-quaternions. The conjecture would be that the value of the Kähler action in M^8 is same as in $M^4 \times CP_2$: in fact that 2-surface would have identical induced metric and Kähler form so that this conjecture would follow trivial. $M^8 - H$ duality would in this sense be Kähler isometry.

The notion of configuration space

From the beginning there was a problem related to the precise definition of the configuration space ("world of classical worlds" (WCW)). Should one regard CH as the space of 3-surfaces of $M^4 \times CP_2$ or $M^4_+ \times CP_2$ or perhaps something more delicate.

1. For a long time I believed that the question "M^4_+ or M^4?" had been settled in favor of M^4_+ by the fact that M^4_+ has interpretation as empty Roberson-Walker cosmology. The huge conformal symmetries assignable to $\delta M^4_+ \times CP_2$ were interpreted as cosmological rather than laboratory symmetries. The work with the conceptual problems related to the notions of energy and time, and with the symmetries of quantum TGD, however led gradually to the realization that there are strong reasons for considering M^4 instead of M^4_+.

2. With the discovery of zero energy ontology it became clear that the so called causal diamonds (CDs) define excellent candidates for the fundamental building blocks of the configuration space or "world of classical worlds" (WCW). The spaces $CD \times CP_2$ regarded as subsets of H defined the sectors of WCW.

3. This framework allows to realize the huge symmetries of $\delta M^4_\pm \times CP_2$ as isometries of WCW. The gigantic symmetries associated with the $\delta M^4_\pm \times CP_2$ are also laboratory symmetries. Poincare invariance fits very elegantly with the two types of super-conformal symmetries of TGD. The first conformal symmetry corresponds to the light-like surfaces $\delta M^4_\pm \times CP_2$ of the imbedding space representing the upper and lower boundaries of CD. Second conformal symmetry corresponds to light-like 3-surface X^3_l, which can be boundaries of X^4 and light-like surfaces separating space-time regions with different signatures of the induced metric. This symmetry is identifiable as the counterpart of the Kac Moody symmetry of string models.

A rather plausible conclusion is that configuration space (WCW) is a union of configuration spaces associated with the spaces $CD \times CP_2$. CDs can contain CDs within CDs so that a fractal like hierarchy having interpretation in terms of measurement resolution results. Since the complications due to p-adic sectors and hierarchy of Planck constants are not relevant for the basic construction, it reduces to a high degree to a study of a simple special case $\delta M^4_+ \times CP_2$.

A further piece of understanding emerged from the following observations.

1. The induced Kähler form at the partonic 2-surface X^2 - the basic dynamical object if holography is accepted- can be seen as a fundamental symplectic invariant so that the values of $\epsilon^{\alpha\beta}J_{\alpha\beta}$ at X^2 define local symplectic invariants not subject to quantum fluctuations in the sense that they would contribute to the configuration space metric. Hence only induced metric corresponds to quantum fluctuating degrees of freedom at configuration space level and TGD is a genuine theory of gravitation at this level.

2. Configuration space can be divided into slices for which the induced Kähler forms of CP_2 and $\delta M^4_\pm$ at the partonic 2-surfaces X^2 at the light-like boundaries of CDs are fixed. The symplectic group of $\delta M^4_\pm \times CP_2$ parameterizes quantum fluctuating degrees of freedom in given scale (recall the presence of hierarchy of CDs).

3. This leads to the identification of the coset space structure of the sub-configuration space associated with given CD in terms of the generalized coset construction for super-symplectic and super Kac-Moody type algebras (symmetries respecting light-likeness of light-like 3-surfaces). Configuration space in quantum fluctuating degrees of freedom for given values of zero modes can be regarded as being obtained by dividing symplectic group with Kac-Moody group. Equivalently, the local coset space $S^2 \times CP_2$ is in question: this was one of the first ideas about configuration space which I gave up as too naive!

4. Generalized coset construction and coset space structure have very deep physical meaning since they realize Equivalence Principle at quantum level: the identity of Super Virasoro generators for super-symplectic and super Kac-Moody algebras implies that inertial and gravitational four-momenta are identical.

1.5.2 The most recent vision about zero energy ontology

The generalization of the number concept obtained by fusing real and p-adics along rationals and common algebraics is the basic philosophy behind p-adicization. This however requires that it is possible to speak about rational points of the imbedding space and the basic objection against the notion of rational points of imbedding space common to real and various p-adic variants of the imbedding space is the necessity to fix some special coordinates in turn implying the loss of a manifest general coordinate invariance. The isometries of the imbedding space could save the situation provided one can identify some special coordinate system in which isometry group reduces to its discrete subgroup. The loss of the full isometry group could be compensated by assuming that WCW is union over sub-WCW:s obtained by applying isometries on basic sub-WCW with discrete subgroup of isometries.

The combination of zero energy ontology realized in terms of a hierarchy causal diamonds and hierarchy of Planck constants providing a description of dark matter and leading to a generalization of the notion of imbedding space suggests that it is possible to realize this dream. The article [16] provides a brief summary about recent state of quantum TGD helping to understand the big picture behind the following considerations.

Zero energy ontology briefly

1. The basic construct in the zero energy ontology is the space $CD \times CP_2$, where the causal diamond CD is defined as an intersection of future and past directed light-cones with time-like separation between their tips regarded as points of the underlying universal Minkowski space M^4. In zero energy ontology physical states correspond to pairs of positive and negative energy states located at the boundaries of the future and past directed light-cones of a particular CD. CD:s form a fractal hierarchy and one can glue smaller CD:s within larger CD along the upper light-cone boundary along a radial light-like ray: this construction recipe allows to understand the asymmetry between positive and negative energies and why the arrow of experienced time corresponds to the arrow of geometric time and also why the contents of sensory experience is located to so narrow interval of geometric time. One can imagine evolution to occur as quantum leaps in which the size of the largest CD in the hierarchy of personal CD:s increases in such a manner that it becomes sub-CD of a larger CD. p-Adic length scale hypothesis follows if the values of temporal distance T between tips of CD come in powers of 2^n: a weaker condition would be $T_p = pT_0$, p prime, and would assign all p-adic time scales to the size scale hierarchy of CDs. All conserved quantum numbers for zero energy states have vanishing net values. The interpretation of zero energy states in the framework of positive energy ontology is as physical events, say scattering events with positive and negative energy parts of the state interpreted as initial and final states of the event.

2. In the realization of the hierarchy of Planck constants $CD \times CP_2$ is replaced with a Cartesian product of book like structures formed by almost copies of CD:s and CP_2:s defined by singular coverings and factors spaces of CD and CP_2 with singularities corresponding to intersection $M^2 \cap CD$ and homologically trivial geodesic sphere S^2 of CP_2 for which the induced Kähler form vanishes. The coverings and factor spaces of CD:s are glued together along common $M^2 \cap CD$. The coverings and factors spaces of CP_2 are glued together along common homologically non-trivial geodesic sphere S^2. The choice of preferred M^2 as subspace of tangent space of X^4 at all its points and having interpretation as space of non-physical polarizations, brings M^2 into the theory also in different manner. S^2 in turn defines a subspace of the much larger space of vacuum extremals as surfaces inside $M^4 \times S^2$.

3. Configuration space (the world of classical worlds, WCW) decomposes into a union of sub-WCW:s corresponding to different choices of M^2 and S^2 and also to different choices of the quantization axes of spin and energy and and color isospin and hyper-charge for each choice of this kind. This means breaking down of the isometries to a subgroup. This can be compensated by the fact that the union can be taken over the different choices of this subgroup.

4. p-Adicization requires a further breakdown to discrete subgroups of the resulting sub-groups of the isometry groups but again a union over sub-WCW:s corresponding to different choices of the discrete subgroup can be assumed. Discretization relates also naturally to the notion of number theoretic braid.

Consider now the critical questions.

1. Very naively one could think that center of mass wave functions in the union of sectors could give rise to representations of Poincare group. This does not conform with zero energy ontology, where energy-momentum should be assignable to say positive energy part of the state and where these degrees of freedom are expected to be pure gauge degrees of freedom. If zero energy ontology makes sense, then the states in the union over the various copies corresponding to different choices of M^2 and S^2 would give rise to wave functions having no dynamical meaning. This would bring in nothing new so that one could fix the gauge by choosing preferred M^2 and S^2 without losing anything. This picture is favored by the interpretation of M^2 as the space of longitudinal polarizations.

2. The crucial question is whether it is really possible to speak about zero energy states for a given sector defined by generalized imbedding space with fixed M^2 and S^2. Classically this is possible and conserved quantities are well defined. In quantal situation the presence of the light-cone boundaries breaks full Poincare invariance although the infinitesimal version of this invariance is preserved. Note that the basic dynamical objects are 3-D light-like "legs" of the generalized Feynman diagrams.

Definition of energy in zero energy ontology

Can one then define the notion of energy for positive and negative energy parts of the state? There are two alternative approaches depending on whether one allows or does not allow wave-functions for the positions of tips of light-cones.

Consider first the naive option for which four momenta are assigned to the wave functions assigned to the tips of CD:s.

1. The condition that the tips are at time-like distance does not allow separation to a product but only following kind of wave functions

$$\Psi = exp[ip\cdot(m_+ - m_-)]\Theta(T^2)\Theta(m_+^0 - m_-^0)\Phi(p) \ \ , \ \ T^2 = (m_+ - m_-)^2 \ . \tag{1.5.1}$$

Here m_+ and m_- denote the positions of the light-cones and Θ denotes step function. Φ denotes configuration space spinor field in internal degrees of freedom of 3-surface. One can introduce also the decomposition into particles by introducing sub-CD:s glued to the upper light-cone boundary of CD.

2. The first criticism is that only a local eigen state of 4-momentum operators $p_\pm = \hbar\nabla/i$ is in question everywhere except at boundaries and at the tips of the CD with exact translational invariance broken by the two step functions having a natural classical interpretation. The second criticism is that the quantization of the temporal distance between the tips to $T = 2^kT_0$ is in conflict with translational invariance and reduces it to a discrete scaling invariance.

The less naive approach relying of super conformal structures of quantum TGD assumes fixed value of T and therefore allows the crucial quantization condition $T = 2^kT_0$.

1. Since light-like 3-surfaces assignable to incoming and outgoing legs of the generalized Feynman diagrams are the basic objects, can hope of having enough translational invariance to define the notion of energy. If translations are restricted to time-like translations acting in the direction of the future (past) then one has local translation invariance of dynamics for classical field equations inside $\delta M^4_\pm$ as a kind of semigroup. Also the M^4 translations leading to interior of X^4 from the light-like 2-surfaces surfaces act as translations. Classically these restrictions correspond to non-tachyonic momenta defining the allowed directions of translations realizable as particle motions. These two kinds of translations have been assigned to super-symplectic conformal symmetries at $\delta M^4_\pm \times CP_2$ and and super Kac-Moody type conformal symmetries at light-like 3-surfaces. Equivalence Principle in TGD framework states that these two conformal symmetries define a structure completely analogous to a coset representation of conformal algebras so that the four-momenta associated with the two representations are identical [C1].

2. The condition selecting preferred extremals of Kähler action is induced by a global selection of M^2 as a plane belonging to the tangent space of X^4 at all its points [C1]. The M^4 translations of X^4 as a whole in general respect the form of this condition in the interior. Furthermore, if M^4 translations are restricted to M^2, also the condition itself - rather than only its general form - is respected. This observation, the earlier experience with the p-adic mass calculations, and also the treatment of quarks and gluons in QCD encourage to consider the possibility that translational invariance should be restricted to M^2 translations so that mass squared, longitudinal momentum and transversal mass squared would be well defined quantum numbers. This would be enough to realize zero energy ontology. Encouragingly, M^2 appears also in the generalization of the causal diamond to a book-like structure forced by the realization of the hierarchy of Planck constant at the level of the imbedding space.

3. That the cm degrees of freedom for CD would be gauge like degrees of freedom sounds strange. The paradoxical feeling disappears as one realizes that this is not the case for sub-CD:s, which indeed can have non-trivial correlation functions with either upper or lower tip of the CD playing a role analogous to that of an argument of n-point function in QFT description. One can also say that largest CD in the hierarchy defines infrared cutoff.

1.5.3 Configuration space geometry

The reader not familiar with the basic ideas related to the construction of the configuration space geometry and spinor structure is warmly encouraged to read [B1, B2, B3, A6]. The number theoretic ideas as all other ideas have evolved through un-necessarily strong conjectures. One of them was the idea that conformal weights are complex and given by the zeros of Riemann zeta. Some numerical accidents motivated this idea but it soon lead to non-plausible conjectures about the number theoretic anatomy for the zeros of zeta and many of them turned out to be wrong. The idea about the role of zeta function was not however completely wrong. It turned out that one can assign to the eigenvalues of the modified Dirac operator what might be called Dirac zeta and ζ_D is expressible in terms of gamma functions and Rieman Zeta with shifted argument but do not satisfy Riemann Hypothesis.

Configuration space as a union of symmetric spaces

The idea about symmetric space is extremely beautiful but it took a long time and several false alarms before the time was ripe for identifying the precise form of the Cartan decomposition $g = t + h$ satisfying the defining conditions

$$g = t + h \ , \quad [t, t] \subset h \ , \quad [h, t] \subset t \ . \tag{1.5.2}$$

The ultimate solution of the puzzle turned out to be amazingly simple and came only after quantum TGD was understood well enough.

Configuration space geometry allows two super-conformal symmetries. The first one corresponds to super-symplectic transformations acting at the level of imbedding space. The second one corresponds to super Kac-Moody symmetry acting as deformations of light-like 3-surfaces respecting their light-likeness. Super Kac-Moody algebra can be regarded as sub-algebra of super-symplectic algebra, and quantum states correspond to the coset representations for these two algebras so that the differences of the corresponding super-Virasoro generators annihilate physical states. This obviously generalizes Goddard-Olive-Kent construction [33]. The physical interpretation is in terms of Equivalence Principle. After having realized this it took still some time to realize that this coset representation and therefore also Equivalence Principle also corresponds to the coset structure of the configuration space!

The conclusion would be that t corresponds to super-symplectic algebra made also local with respect to X^3 and h corresponds to super Kac-Moody algebra. The experience with finite-dimensional coset spaces would suggest that super Kac-Moody generators interpreted in terms of h leave the points of configuration space analogous to the origin of say CP_2 invariant and in fact vanish at this point. Therefore super Kac-Moody generators should vanish for those 3-surfaces X^3_l which correspond to the origin of coset space. The maxima of Kähler function could correspond to this kind of points and could play also an essential role in the integration over configuration space by generalizing the Gaussian integration of free quantum field theories.

The identification of the precise form of the coset space structure is however somewhat delicate.

1. The essential point is that both symplectic and Kac-Moody algebras allow representation in terms of X^3-local Hamiltonians. The general expression for the Hamilton of Kac-Moody algebra is

$$H = \sum \Phi_A(x) H^A \ . \tag{1.5.3}$$

Here H^A are Hamiltonians of $SO(3) \times SU(3)$ acting in $\delta X_l^3 \times CP_2$. For symplectic algebra any Hamiltonian is allowed. If x corresponds to any point of X_l^3, one must assume a slicing of the causal diamond CD by translates of $\delta M_\pm^4$.

2. The functions $\Phi(x)$ are not arbitrary but constrained by the condition that $J = \epsilon^{\alpha\beta} J_{\alpha\beta}\sqrt{g_2}$ remains invariant under to action of the algebra at X^2 at least. Let us assume that one can restrict the consideration to single Hamiltonian so that the transformation is generated by $\Phi(x) H_A$ and that to each $\Phi(x)$ there corresponds a diffeomorphism of X^2, which is a symplectic transformation of X^2 with respect to symplectic form $\epsilon^{\alpha\beta}$ and generated by Hamiltonian $\Psi(x)$. This transforms the invariance condition to

$$\{H^A, \Phi\} \equiv \partial_\alpha H^A \epsilon^{\alpha\beta} \partial_\beta \Phi = \partial_\alpha J \epsilon^{\alpha\beta} \partial_\beta \Psi_A = \{J, \Psi_A\} \ . \tag{1.5.4}$$

This condition can be solved identically by assuming that Φ_A and Ψ are proportional to arbitrary smooth function of J:

$$\Phi = f(J) \ , \quad \Psi_A = -f(J) H_A \ . \tag{1.5.5}$$

Therefore the X^2 local symplectomorphisms of H reduce to symplectic transformations of X^2 with Hamiltonians depending on single coordinate J of X^2. The analogy with conformal invariance for which transformations depend on single coordinate z is obvious. By effective metric 2-dimensionality these conditions can be formulated and satisfied at entire light-like 3-surface Y_l^3 since ϵ^α exists as a tensor also now. As far as the anti-commutation relations for induced spinor fields are considered this means that $J = consant$ curves behave as points points. For extrema of J appearing as candidates for points of number theoretic braids $J = constant$ curves reduce to points.

3. For symplectic generators the dependence of form on r^Δ on light-like coordinate of $\delta X_l^3 \times CP_2$ is allowed. Δ is complex parameter whose modulus squared is interpreted as conformal weight. Δ is identified as analogous quantum number labeling the modes of induced spinor field.

4. One can wonder whether the choices of the $r_M = constant$ sphere S^2 is the only choice. The Hamiltonin-Jacobi coordinate for $X^4_{X_l^3}$ suggest an alternative choice as E^2 in the decomposition of $M^4 = M^2(x) \times E^2(x)$ required by number theoretical compactification and present for known extremals of Kähler action with Minkowskian signature of induced metric. In this case $SO(3)$ would be replaced with $SO(2)$. It however seems that the radial light-like coordinate u of $X^4(X_l^3)$ would remain the same since any other curve along light-like boundary would be space-like.

5. The vector fields for representing Kac-Moody algebra must vanish at the partonic 2-surface $X^2 \subset \delta M_\pm^4 \times CP_2$. The corresponding vector field must vanish at each point of X^2:

$$j^k = \sum \Phi_A(x) J^{kl} H_l^A = 0 \ . \tag{1.5.6}$$

This means that the vector field corresponds to $SO(2) \times U(2)$ defining the isotropy group of the point of $S^2 \times CP_2$. This expression must be generalized to the case when Kac-Moody transformation is allowed to induced diffeomorphism of X^2.

This expression could be deduced from the idea that the surfaces X^2 are analogous to origin of CP_2 at which $U(2)$ vector fields vanish. Configuration space at X^2 could be also regarded as the analog of the origin of local $S^2 \times CP_2$. This interpretation is in accordance with the original idea which however was given up in the lack of proper realization. The same picture can be deduced from braiding in which case the Kac-Moody algebra corresponds to local $SO(2) \times U(2)$ for each point of the braid at X^2. The condition that Kac-Moody generators with positive conformal weight annihilate physical states could be interpreted by stating effective 2-dimensionality in the sense that the deformations of X^3_l preserving its light-likeness do not affect the physics. Note however that Kac-Moody type Virasoro generators do not annihilate physical states.

6. Kac-Moody algebra generators must leave induced Kähler form invariant at X^2 but this trivially true since they vanish at each point of X^2. Their commutators with symplectic generators do not however vanish.

7. The conditions of Cartan decomposition are satisfied. The commutators of the Kac-Moody vector fields with symplectic generators are non-vanishing since the action of symplectic generator on Kac-Moody generator restricted to X^2 gives a non-vanishing result belonging to the symplectic algebra. Also the commutators of Kac-Moody generators are Kac-Moody generators.

Zero modes

Zero modes are by definition those degrees of freedom which do not correspond to the complex coordinates of the configuration space contributing to the metric.

1. J as function of X^2 coordinates defines the fundamental collection of zero modes and its extrema at the points of braid defines subset of zero modes. There are also other zero modes labeled by symplectic invariants described in [B2]. The size and shape of the 3-surface and classical Kähler field correspond to these zero modes. In particular, the induced Kähler form is purely symplectic invariant from which one can deduce this kind of non-local invariants. Especially interesting local symplectic and diffeo-invariants are the extrema of $J = \epsilon^{\mu\nu} J_{\mu\nu}$. Both CP_2 and $\delta M^4_\pm$ Kähler form define this kind of invariants. These appear in the construction of symplectic fusion algebras [17].

2. Zero modes decompose to symplectic covariants and invariants. The symplectic transformations are generated by the function basis of $M^4_+ \times CP_2$ consist of complexified Hamiltonians labeled by the label -call it n - assignable to the functions $f_n(J)$ and by the labels of Hamiltonians of $\delta M^4_\pm \times CP_2$. If Hamiltonian is real it corresponds to zero mode. The most obvious candidates for zero modes are Hamiltonians which do not depend neither on the radial coordinate of $\delta M^4_\pm$ nor on J.

3. Since the values of the induced Kähler form represent local zero modes, the quantum fluctuating degrees of freedom are parameterized by the symplectic transformations of $\delta M^\pm \times CP_2$ [C2]. From the point of view of quantum theory configuration space decomposes into slices characterized by the induced Kähler form at partonic 2-surfaces and functional integral reduces to that over the symplectic group. Induced Kähler form is genuinely classical field and only the induced metric quantum fluctuates so that TGD in a well-defined sense reduces to quantum gravity in the quantum fluctuating configuration space degrees of freedom.

Kac-Moody algebra respecting the light-like character of 3-surface and leaving partonic surface X^2 invariant defines second candidate for a sub-space of zero modes. These zero modes correspond to the interior of space-like 3-surface X^3 or its light-like dual X^3_l. Zero mode is in question only if the configuration space metric remains invariant under Kac-Moody symmetries. The identification of Kähler function as Dirac determinant makes zero mode condition non-trivial.

1. If the eigenvalues correspond to the generalized eigenvalues of X^2 part $D(X^2)$ of $D(X^3_l)$ rather than those of $D(X^3_l)$, this independence is achieved. This implies also the effective finite-dimensionality of the configuration space. One can however argue that General Coordinate Invariance allows the replacement of X^2 with an arbitrary time=constant section $X^2(v)$ along X^3_l. The condition would be that the eigenvalues of $D(X^2(v))$ for X^3_l and its Kac-Moody

transforms differ by a multiplication by modulus squared of a holomorphic function of parameters characterizing Kac-Moody group. Also the replacement of X^3_l with Y^3_l parallel should be possible by General Coordinate Invariance and accompanied by the replacement $X^2 \rightarrow X^2(u)$. Obviously General Coordinate Invariance would pose immense constraints on configuration space metric.

2. In the presence of instanton term $D(X^3_l)$ could be used to define Dirac determinant. If the part x_k of eigenvalue $\lambda_k + \sqrt{n}x_k$ scales like λ_k in Kac-Moody transformations and if the scaling is as above, zero mode property is guaranteed.

3. The value of the Kähler function in principle varies and can have maximum for some values of deformation parameters. If one can define functional integral over zero modes (not possible in terms of the functional integral defined by configuration space metric), quantum classical correspondence realized in terms of stationary phase approximation of functional integral by utilizing a phase factor depending on quantum numbers assigned to the braid strands would provide the general gauge fixing procedure. On the other hand, conformal cutoff would reduce the integration to that over a finite-dimensional space so that stationary phase approximation could work. If there exist no functional integral of this kind, one could still select the preferred zero mode as by stationary phase criterion. This would be natural since genuinely classical degrees of freedom are in question. This option would be also p-adically very natural.

How to construct the super-symplectic algebra?

The configuration space of 3-surfaces Y^3 as a union of infinite-dimensional symmetric spaces labeled by zero modes obeying real topology and having metric and spinor structure determined solely by super-symmetry, is the basic intuitive picture about configuration space geometry.

Algebraic physics vision suggests that the representation of the generators of the symplectic transformations of the lightlike 7-surface $\delta M^4_\pm \times CP_2$ must be expressible in terms of rational functions. In the case that Hamiltonians correspond to irreducible representations of $SU(3)$, they are products of rational functions of preferred CP_2 coordinates with functions depending on coordinates of X^3_l. If the Hamiltonians transform according to an irreducible representation of the rotation group leaving $r_M = constant$ sphere S^2 invariant, they are rational functions of the complex coordinates of S^2. The remaining problems relate to the 3-integrals appearing in the definition of configuration space Hamiltonians. The solution of these problems comes in terms of (number theoretic) braids, which are now a basic notion of quantum TGD. Integrals are simply replaced by sums making sense also p-adically.

The modified Dirac action allows to deduce explicit expressions for the super generators. This allows to extend the formulas for the configuration space Hamiltonians in terms of the classical symplectic charges associated with the Kähler action to the formulas for super-symplectic charges. Configuration space metric, being numerically equal to the Kähler form in complex coordinates, in turn relates directly to the symplectic charges. A natural expectation is that gamma matrices can be related by an analogous formula to the expressions for the super-symplectic charges.

1.5.4 The identification of number theoretic braids

To specify number theoretical criticality one must specify some physically preferred coordinates for $M^4 \times CP_2$ or at least $\delta M^4_\pm \times CP_2$. Number theoretical criticality requires that braid belongs to the algebraic intersection of real and p-adic variants of the partonic 2-surface so that number theoretical criticality reduces to a finite number of conditions. This is however not strong enough condition and one must specify further physical conditions.

What are the preferred coordinates for H?

What are the preferred coordinates of M^4 and CP_2 in which algebraicity of the points is required is not completely clear. The isometries of these spaces must be involved in the identification as well as the choice of quantization axes for given CD. In [E4] I have discussed the natural preferred coordinates of M^4 and CP_2.

1. For M^4 linear M^4 coordinates chosen in such manner that $M^2 \times E^2$ decomposition fixing quantization axes is respected are very natural. This restricts the allowed Lorentz transformations to

Lorentz boosts in M^2 and rotations in E^2 and the identification of M^2 as hyper-complex plane fixes time coordinate uniquely. E^2 coordinates are fixed apart from the action of $SO(2)$ rotation. The rationalization of trigonometric functions of angle variables allows angles associated with Pythagorean triangles as number theoretically simplest ones.

2. The case of CP_2 is not so easy. The most obvious guess in the case of CP_2 the coordinates corresponds to complex coordinates of CP_2 transforming linearly under $U(2)$. The condition that color isospin rotations act as phase multiplications fixes the complex coordinates uniquely. Also the complex coordinates transforming linearly under $SO(3)$ rotations are natural choice for S^2 ($r_M = constant$ sphere at $\delta M^4_\pm$).

3. Another manner to deal with CP_2 is to apply number M^8-H duality. In M^8 CP_2 corresponds to E^4 and the situation reduces to linear one and $SO(4)$ isometries help to fix preferred coordinate axis by decomposing E^4 as $E^4 = E^2 \times E^2$. Coordinates are fixed apart the action of the commuting $SO(2)$ sub-groups acting in the planes E^2. It is not clear whether the images of algebraic points of E^4 at space-time surface are mapped to algebraic points of CP_2.

It took some years to end up with a unique identification of number theoretic braids [A6, F12]. As a matter fact, there are several alternative identifications and it seems that all of them are needed. Consider first just braids without the attribute 'number theoretical'.

Critical number theoretical braids

Quantum criticality with respect to phase transitions changing Planck constant would be one possible criterion of braid. The additional requirement that braid points at X^2 are algebraic would make the braid number theoretical.

1. Braids can be identified as lifts of the projections of X^3_l to the quantum critical sub-manifolds M^2 or S^2_I, $i = I, II$, and in the generic case consist of 1-dimensional strands in X^3_l These sub-manifolds are obviously in the same role as the plane to which the braid is projected to obtain a braid diagram.

2. Braid points are always quantum critical against the change of Planck constant so that TQFT like theory characterizes the freedom remaining intact at quantum criticality. Quantum criticality in this sense need not have anything to do with the quantum criticality in the sense that the second variation of Kähler action vanishes -at least for the variations representing dynamical symmetries in the sense that only the inner product $\int (\partial L_D/\partial h^k_\alpha)\delta h^k d^4x$ (L_D denotes modified Dirac Lagrangian) without the vanishing of the integrand. This criticality leads to a generalization of the conceptual framework of Thom's catastrophe theory [A6].

3. It is not clear whether these three braids form some kind of trinity so that one of them is enough to formulate the theory or whether all of them are needed. Note also that one has quantum superposition over CDs corresponding to different choices of M^2 and the pair formed by S^2_I and S^2_{II} (note that the spheres are not independent if both appear). Quantum measurement however selects one of these choices since it defines the choice of quantization axes.

What about symplectic contribution to number theoretic braids?

Also the symplectically invariant degrees of freedom representing zero modes must be treated and this leads to the notion of symplectic QFT. These braid points would not be critical with respect to phase transition changing Planck constant. The explicit construction of symplectic fusion rules has been discussed in [17]. These rules make sense only as a discretized version. Discreteness can be understood also as a manifestation of finite measurement resolution: at this time it is associated with the impossibility to know the induced Kähler form at each point of partonic 2-surface. What one can measure is the Kähler flux associated with a triangle and the density of triangulation determines the measurement accuracy. The discrete set of points associated with the symplectic algebra characterizes the measurement resolution and there is an infinite hierarchy of symplectic fusion algebras corresponding to gradually increasing measurement resolution in classical sense [17].

Second interesting question is whether the symplectic triangulation could be used to represent a hierarchy of cutoffs of super conformal algebras by introducing additional fermionic oscillators at the points of the triangulation. The M^4 coordinates at the points of symplectic triangulation of S^2_i, $i = I, II$ projection and CP_2 coordinates at the points of symplectic triangulation of S^2 could define discrete version of quantized conformal fields. The functional integral over symplectic group would mean integral over symplectic triangulations. Note that M^2 number theoretic braid is trivial as symplectic triangulation since the points are along light-like geodesic of $\delta M^4_\pm$.

In the original variant of symplectic triangulation [17] the exact form of triangulation was left free. It would be however nice if symplectic triangulation could be fixed purely physically by the properties of the induced Kähler form since also the number of fermionic oscillator modes and number theoretical braids is fixed by the dynamics of Kähler action.

1. A symplectically invariant manner to fix the nodes of the triangulation could be in terms of extrema of the symplectic invariant $J = \epsilon^{\alpha\beta} J_{\alpha\beta}\sqrt{g_2}$ (the dependence on metric is only apparent). Here the Kähler forms of both S^2 and CP_2 can be considered. The maxima for the magnitude of Kähler magnetic field are indeed natural observables as also the areas of projections of X^2 to S^2. The nodes are completely fixed by dynamics and the contribution to number theoretic braid involves no ad hoc elements. Physical intuition suggests that this is not enough: magnetic flux quantization is what strongly suggests itself as additional source of braid points.

2. $J = constant$ curves define the analogs of height curves surrounding the extrema of J. Inside each region where J has definite sign, the quantization of the Kähler magnetic flux defines a collection of height curves bounding disks for which Kähler magnetic flux is given by $Flux = \int_{J<J_q} JdS = q2\pi r$, where $r = \hbar/\hbar_0$ and q are rational.

3. Symplectic and Kac-Moody algebras [B2] algebras are local with respect to X^2 but the dependence is only through J. Hence the analogy with conformal field theory would suggest that the quantization of the fermionic oscillator operators should treat $J = constant$ curve more or less as a single point or at most as a disrete point set. Hence the addition of height curves would give additional "points" to the number theoretic braid.

4. Could one reduce the set of symplectic height curves to a discrete point set? The canonically conjugate coordinate Φ for J (analogous to canonical momentum) defined with respect to the symplectic form $\epsilon^{\mu\nu}$ of X^2 and by the condition $\{\Phi, J\} = 1$ defines an angle variable varying in the range $(0, 2\pi)$. The flux would be given in these coordinates simply as $Flux = \int_{J_q} Jd\Phi = 2\pi J = q \times 2\pi r$ so that $J = qr$ would be rational valued for rational values of magnetic flux. Rational values $\Phi = m2\pi/n$ would divide symplectic disks with quantized flux to quadrangles with quantized flux reduced by factor $1/n$. Symplectic transformations of $\delta M^4_\pm \times CP_2$ and of X^2 would leave the fluxes invariant. A discrete point set could be selected as the intersection of the coordinate curves associated with J and Φ and would define number theoretic braid, which can be used in the second quantization of the induced spinor fields.

5. If the precise specification of the edges of the triangulation [17] has any physical meaning, this meaning must come from the quantization of magnetic fluxes for symplectic triangles and from their unique specification. A possible definition of symplectic triangulation satisfying these criteria relies on the observation that $J = constant$ and $\Phi = constant$ coordinate curves divide the region surrounding given extremum of J to quadrangles. By connecting the vertices of quadrangles by straight lines in linear coordinates defined by J and Φ, one obtains unique symplectic triangulation with rationally quantized fluxes. Also sub-triangulations with the same property can be constructed.

To sum up, the symplectic contribution to all three types of number theoretic braids could be present and would differ from the above described contribution in that the points of the braid are not critical with respect to phase transitions changing Planck constant.

What makes the braid number theoretic?

Number theoretic braids would be braids which are number theoretically critical. This means that the points of braid in preferred coordinates are algebraic points so that they can be regarded as being shared by real partonic 2-surface and its p-adic counterpart obeying same algebraic equations. The phase transitions between number fields would mean leakage via these 2-surfaces playing the role of back of a book along which real and p-adic physics representing the pages of a book are glued together. The transformation of intention to action would represent basic example of this kind of leakage and number theoretic criticality could be decisive feature of living matter. For number theoretic braids at X_l^3 whose real and p-adic variants obey same algebraic equations, only subset of algebraic points is common to real and p-adic pages of the book so that discretization of braid strand is unavoidable.

1.5.5 Finite measurement resolution and reduced configuration space

Finite measurement resolution implies the notion of braid which is now central part of construction of M-matric [A6]. The notion of braid in turn leads to the notion of reduced configuration space.

1. 3-surface reduces effectively to a set of points defined by the intersection of $\delta M_\pm^4 \times CP_2$ projection of the partonic 2-surface X^2 with light-like radial geodesic or the intersection of its CP_2 projection with the geodesic sphere S_i^2, $i = I, II$.

2. Second kind of braid corresponds to the extrema of $J = \epsilon^{\alpha\beta} J_{\alpha\beta}\sqrt{g_2}$ at X^2. Here the induced Kähler forms of both $\delta M_\pm^4$ and CP_2 can be considered. Also this option defines the braid physically and the number of points is finite in the generic situation.

Number theoretic braids reduce the configuration space to a finite-dimensional space defined as a coset space of symplectic group of $\delta M_\pm^4 \times CP_2$ obtained by dividing with the sub-group of the symplectic group leaving the braid points invariant. The resulting space is $(\delta M_\pm^4 \times CP_2)^n/S_n$, where n is the number of braid points. If the proposed criteria define the braid, n and measurement resolution is characterized by the geometry of X^2.

This raises issues about the metric of the reduced configuration space as deduced from the spectrum of the modified Dirac operator.

1. Kac-Moody symmetry would suggest that the finite number of $n = 0$ modes determine the Kähler function and metric exactly. Also the metric of the coset space determined by measurement resolution could naturally determined as derivatives of the logarithms of the eigen values with respect to the complex coordinates of $(S^2 \times CP_2)^n$. In principle, it would be possible to deduced the metric numerically. If one allows arbitrary number of braid points then $n \rightarrow \infty$ limit could give rise to the continuum formulation of configuration space Hamiltonians and metric.

2. The simplest option would be that the metric reduces apart from a scaling factor to a direct sum of the metrics assignable to the factors of the Cartesian power. Even if this happens, the scaling factor must be non-trivial and carry dependence on the induced Kähler form which is constant along the symplectic orbit and defines the fundamental zero modes. This expectation is probably wrong. Kähler function codes correlations even between different components of partonic 2-surfaces and it would be surprising if there were no correlations between points of the same partonic 2-surface. A new element as compared to general relativity would be geometrization of n-particle system in terms of the metric of the reduced configuration space.

1.5.6 Does reduced configuration space allow TGD Universe to act as a universal math machine?

The title relates only the very loosely to the main topic of the chapter. The excuse for including this material is that TGD inspired theory of consciousness allows to interpret the notions of zero energy state and reduced configuration space in terms of mathematical cognition.

The questions which lead to the arguments represented below were represented in different context [E12] related to the TGD inspired ideas about number theoretic Langlands correspondence. TGD inspired theory of consciousness - in particular the question about the physical correlates of Boolean statements and conscious mathematical deductions- is second definer of context.

The questions are following. Could one find a representations of both Lie groups and their linear and non-linear representation spaces -and even more - of any manifold representable as a sub-manifold of some linear space in terms of braid points at partonic 2-surfaces X^2? What about various kinds of projective spaces and coset spaces? Can one construct representations of corresponding function spaces in terms of configuration space spinor fields? Can one build representations of parameter groups of Lie groups as braided representations defined by the orbits of braid points in X_l^3?

A professional mathematician - if she still continues reading - might regard the following argument as rather pathetic poor man's argument but I want to be honest and demonstrate my stupidity openly.

1. The n braid points represent points of $\delta H = \delta M^4_\pm \times CP_2$ so that braid points represent a point of $7n$-dimensional space $\delta H^n/S_n$. $\delta M^4_\pm$ corresponds to E^3 with origin removed but $E^{2n}/S_n = C^n/S_n$ can be represented as a sub-manifold of $\delta M^4_\pm$. This allows to almost-represent both real and complex linear spaces. E^2 has a unique identification based on $M^4 = M^2 \times E_2$ decomposition required by the choice of quantization axis. One can also represent the spaces $(CP_2)^n/S_n$ in this manner.

2. The first - and really serious - problem is caused by the identification of the points obtained by permuting the n coordinates: this is of course what makes possible the braiding since braid group is the fundamental group of $(X^2)^n$. Could the quantum numbers at the braid points act as markers distinguishing between them so that one would effectively have E^{2n}? Could the fact that the representing points are those of imbedding space rather than X^2 be of significance? Second - less serious - problem is that the finite size of CD allows to represent only a finite region of E^2. On the other hand, ideal mathematician is a non-existing species and even non-ideal mathematician can imagine the limit at which the size of CD becomes infinite.

3. Matrix groups can be represented as sub-manifolds of linear spaces defined by the general linear group $Gl(n,R)$ and $Gl(n,C)$. In the p-adic pages of the imbedding space one can realize also the p-adic variants of general linear groups. Hence it is possible to imbed any real (complex) Lie group to E^{2n} (C^n), if n is chosen large enough.

4. Configuration space spinor fields restricted to the linear representations spaces or to the group itself represented in this manner would allow to realize as a special case various function spaces, in particular groups algebras. If configuration space spinor fields satisfy additional symmetries, projective spaces and various coset spaces can be realized as effective spaces. For instance CP_2 could be realized effectively as $SU(3)/U(2)$ by requiring $U(2)$ invariance of the configuration space spinor fields in $SU(3)$ or as C^3/Z by requiring that configuration space spinor field is scale invariant. Projective spaces might be also realized more concretely as imbeddings to $(CP_2)^n$.

5. The action of group element $g = exp(Xt)$ belonging to a one-parameter sub-group of a non-compact linear group in a real (complex) linear representation space of dimension m could be realized in a subspace of E^{2n}, $m < 2n$ (C^n, $m \leq n$), as a flow in X^3_l taking the initial configuration of points of representation space to the final configuration. Braid strands - the orbits of points p_i defining the point p of the representation manifold under the action of one-parameter subgroup- would correspond to the points $exp(Xu)(p)$, $0 \leq u \leq t$. Similar representation would work also in the group itself represented in a similar manner.

6. Braiding in X^3_l would induce a braided representation for the action of the one parameter subgroup. This representation is not quite the same thing as the automorphic representation since braiding is involved. Also trivial braid group representation is possible if the representation can be selected freely rather than being determined by the transformation properties of fermionic oscillator operator basis in the braiding.

7. An important prerequisite for math machine property is that the wave function in the space of light-like 3-surfaces with fixed ends can be chosen freely. This is the case since the degrees of freedom associate with the interior of light-like 3-surface X^3_l correspond to zero modes assignable to Kac-Moody symmetries [B2]. Dicretization seems however necessary since functional integral in these degrees of freedom is not-well defined even in the real sense and even less so p-adically. This conforms with the fact that real world mathematical representations are always discrete. Quantum classical correspondence suggests the dynamics represented by X^3_l correlates with the quantum numbers assigned with X^2 so that Boolean statements represented in terms of Fermionic Fock states would be in one-one correspondence with these wave functions.

Besides representing mathematical structures this kind of math machine would be able to perform mathematical deductions. The fermionic part of the state zero energy state could be interpreted as a quantum super-position of Boolean statement $A_i \rightarrow B_i$ representing various instances of the general rule $A \rightarrow B$. Only the statements consistent with fundamental conservation laws would be possible. Quantum measurements performed for both positive and negative energy parts of the state would produce statements. Performing the measurement of the observable $O(A \rightarrow B)$ would produce from a given state a zero energy state representing statement $A \rightarrow B$. If the measurement of observable $O(C \rightarrow D)$ affects this state then the statement $(A \rightarrow B) \rightarrow (C \rightarrow D)$ cannot hold true. For $A = B$ the situation reduces to simpler logic where one tests truth value of statements of form $A \rightarrow B$. By increasing the number of instances in the quantum states generalizations of the rule can be tested.

1.5.7 Configuration space Kähler function as Dirac determinant

The recent progress in the understanding of how the information about preferred extremal of Kähler action is feeded to the eigenvalue spectrum of modified Dirac operator [A6] provides additional insights and suggests that p-adic variant of configuration space might make sense in very general sense.

The basic conjecture is that the exponent of Kähler function is identifiable as Dirac determinant. The basic problem is which modified Dirac action should one choose. The four-dimensional modified Dirac action associated with Kähler action or the 3-D modified Dirac action associated with $C-S$ action? Or something else?

1. The first guess inspired by TGD as almost-TQFT was that $C-S$ action is enough. The problems are encountered when one tries to define Dirac determinant. The eigenvalues of the modified Dirac equation are functions rather than constants and this leads to difficulties in the definition of the Dirac determinant. The proposal was that Dirac determinant could be defined as product of the the values of generalized eigenvalues in the set of points defined by the number theoretic braid. This kind of definition is however questionable since it does not have obvious connection with the standard definition.

2. Second guess was that also 4-D modified Dirac action is needed. The physical picture would be that the induced spinor fields restricted to the light-like 3-surfaces are singular solutions of 4-D Dirac operator. Since the modified Dirac equation can be written as a conservation law for super current this idea translates to the condition that the "normal" component of the super current vanishes at X^43_l and tangential component satisfies current conservation meaning that 3-D variant of modified Dirac equation results. There is a unique function of the light-like coordinate r defining the time coordinate and eigenmodes of transversal part of modified Dirac operator define the spectrum of also the modified Dirac operator associated with $C-S$ action naturally. The system is 2-dimensional and if the modes of spinor fields are localized in regions of strong induced electro-weak magnetic field, their number is finite and the Dirac determinant defined in the standard manner is finite. A close connection with anyonic systems emerges. One can indeed define the action of D_K also at the limit when the light-like 3-surface associated with a wormhole throat is approached. This limit is singular since $det(g_4) = 0$ and $det(g_3) = 0$ hold true at this limit. As a consequence the normal component of Kähler electric field typically diverges in accordance with the idea that at short distances $U(1)$ gauge charges approach to infinity. Also the modified Gamma matrices diverge like $1/det(g_4)^3$. One of the problems is that only light-like 3-surfaces with 2-D CP_2 projection are allowed since D_{C-S} reduces to 1-D operator only for these.

3. The third guess inspired by the results relating to the number theoretic compactification was that D_{C-S} is not needed at all! Number theoretical compactification strongly suggets dual slicings of X^4 to string word sheets Y^2 and partonic 2-surfaces X^2, and the generalized eigenvalues can be identified as those associated with the longitudinal part $D_K(Y^2)$ or transverse part $D_K(X^2)$ of the modified Dirac operator D_K. The outcome is exactly the same as for D_{C-S} except that one avoids the problems associated with it. There is also an additional symmetry: the eigenvalue spectra associated with transversal slices must be such that Kähler action gives rise to the same Kähler metric.

4. The fourth guess was the inclusion of instanton term to the action meaning complexification of Kähler action. This does not affect configuration space metric at all but brings in CP breaking and also makes possible construction of generalized Feynman diagrammatics.

This identification led to a considerable increase in the understanding of quantum TGD at fundamental level.

1. A fermion in 2-D magnetic field provides the physical analog system. If CP breaking term is absent the zero modes are restricted to regions inside which the induced Kähler form is non-vanishing and are analogous to cyclotron states in a magnetic field restricted to a finite region of 3-D space-time. Hence the number of zero modes and therefore also the number of generalized eigenvalues of the modified Dirac operator is finite. Second quantization therefore requires selection of finite subset of points of X^2 and this leads to the notion of number theoretic braid.

2. With finite number of zero eigenvalues Dirac determinant can be defined as the product of the eigenvalues without any regularization procedure. Dirac determinant reduces to a product of determinants associated with regions of X^3_l inside which the induced Kähler form- having interpretation as magnetic field - is non-vanishing.

3. If CP breaking instanton term complexifying Kähler action is allowed, the situation becomes more intricate since infinite number of additional labeled by conformal weights is present. Since the localization of symplectic allows only functions of X^2 coordinates depending on $J = \epsilon^{\alpha\beta}J_{\alpha\beta}\sqrt{g_2}$, the situation is effectively 1-dimensional and anti-commutations of induced spinor fields are 1-dimensional since $J = constant$ curves are effectively points in accordance with the fact that conformal excitations are labeled by an integer. Zeta function regularization reduces to that using zeta function and exponent of Kähler function identified as Dirac determinant is infinite powers series in eigenvalues and it would be a miracle if it would reduce to an algebraic function of configuration space coordinates. If one accepts number theoretic braids as primary objects and identified in the proposed purely physical manner, one must introduce cutoff in conformal weights and the number of eigenvalues contributing to the Dirac determinant is finite.

4. One cannot exclude renormalization group invariance in these sense that configuration metric is independent of the cutoff for the conformal modes. This does not mean RG invariance of Kähler function.

1.6 p-Adicization at the level of imbedding space and space-time

In this section p-adicization program at the level if imbedding space and space-time is discussed. The general problems of p-adicization, namely the selection of preferred coordinates and the problems caused by the non-existence of p-adic definite integral and algebraic continuation a solution of these problems has been discussed in the introduction.

1.6.1 p-Adic variants of the imbedding space

Consider now the construction of p-adic variants of the imbedding space.

1. Rational values of p-adic coordinates are non-negative so that light-cone proper time $a_{4,+} = \sqrt{t^2 - z^2 - x^2 - y^2}$ is the unique Lorentz invariant choice for the p-adic time coordinate near the lower tip of CD. For the upper tip the identification of a_4 would be $a_{4,-} = \sqrt{(t-T)^2 - z^2 - x^2 - y^2}$. In the p-adic context the simultaneous existence of both square roots would pose additional conditions on T. For 2-adic numbers $T = 2^nT_0$, $n \geq 0$ (or more generally $T = \sum_{k\geq n_0} b_k 2^k$), would allow to satisfy these conditions and this would be one additional reason for $T = 2^nT_0$ implying p-adic length scale hypothesis. Note however that also $T_p = pT_0$, p prime, can be considered. The remaining coordinates of CD are naturally hyperbolic cosines and sines of the hyperbolic angle $\eta_{\pm,4}$ and cosines and sines of the spherical coordinates θ and ϕ.

2. The existence of the preferred plane M^2 of un-physical polarizations would suggest that the 2-D light-cone proper times $a_{2,+} = \sqrt{t^2 - z^2}$ $a_{2,-} = \sqrt{(t-T)^2 - z^2}$ can be also considered. The remaining coordinates would be naturally $\eta_{\pm,2}$ and cylindrical coordinates (ρ, ϕ).

3. The transcendental values of a_4 and a_2 are literally infinite as real numbers and could be visualized as points in infinitely distant geometric future so that the arrow of time might be said to emerge number theoretically. For M^2 option p-adic transcendental values of ρ are infinite as real numbers so that also spatial infinity could be said to emerge p-adically.

4. The selection of the preferred quantization axes of energy and angular momentum unique apart from a Lorentz transformation of M^2 would have purely number theoretic meaning in both cases. One must allow a union over sub-WCWs labeled by points of $SO(1,1)$. This suggests a deep connection between number theory, quantum theory, quantum measurement theory, and even quantum theory of mathematical consciousness.

5. In the case of CP_2 there are three real coordinate patches involved [Appendix]. The compactness of CP_2 allows to use cosines and sines of the preferred angle variable for a given coordinate patch.

$$\begin{aligned} \xi^1 &= tan(u)exp(i\frac{(\Psi+\Phi)}{2})cos(\frac{\Theta}{2}) \ , \\ \xi^2 &= tan(u)exp(i\frac{(\Psi-\Phi)}{2})sin(\frac{\Theta}{2}) \ . \end{aligned} \tag{1.6.1}$$

The ranges of the variables u, Θ, Φ, Ψ are $[0, \pi/2], [0, \pi], [0, 4\pi], [0, 2\pi]$ respectively. Note that u has naturally only the positive values in the allowed range. S^2 corresponds to the values $\Phi = \Psi = 0$ of the angle coordinates.

6. The rational values of the (hyperbolic) cosine and sine correspond to Pythagorean triangles having sides of integer length and thus satisfying $m^2 = n^2 + r^2$ ($m^2 = n^2 - r^2$). These conditions are equivalent and allow the well-known explicit solution [40]. One can construct a p-adic completion for the set of Pythagorean triangles by allowing p-adic integers which are infinite as real integers as solutions of the conditions $m^2 = r^2 \pm s^2$. These angles correspond to genuinely p-adic directions having no real counterpart. Hence one obtains p-adic continuum also in the angle degrees of freedom. Algebraic extensions of the p-adic numbers bringing in cosines and sines of the angles π/n lead to a hierarchy increasingly refined algebraic extensions of the generalized imbedding space. Since the different sectors of WCW directly correspond to correlates of selves this means direct correlation with the evolution of the mathematical consciousness. Trigonometric identities allow to construct points which in the real context correspond to sums and differences of angles.

7. Negative rational values of the cosines and sines correspond as p-adic integers to infinite real numbers and it seems that one use several coordinate patches obtained as copies of the octant $(x \geq 0, y \geq 0, z \geq 0,)$. An analogous picture applies in CP_2 degrees of freedom.

8. The expression of the metric tensor and spinor connection of the imbedding in the proposed coordinates makes sense as a p-adic numbers in the algebraic extension considered. The induction of the metric and spinor connection and curvature makes sense provided that the gradients of coordinates with respect to the internal coordinates of the space-time surface belong to the extensions. The most natural choice of the space-time coordinates is as subset of imbedding space-coordinates in a given coordinate patch. If the remaining imbedding space coordinates can be chosen to be rational functions of these preferred coordinates with coefficients in the algebraic extension of p-adic numbers considered for the preferred extremals of Kähler action, then also the gradients satisfy this condition. This is highly non-trivial condition on the extremals and if it works might fix completely the space of exact solutions of field equations. Space-time surfaces are also conjectured to be hyper-quaternionic [E2], this condition might relate to the simultaneous hyper-quaternionicity and Kähler extremal property. Note also that this picture would provide a partial explanation for the decomposition of the imbedding space to sectors dictated also by quantum measurement theory and hierarchy of Planck constants.

1.6.2 p-Adicization at the level of space-time

Number theoretical Universality in weak sense does not seem to pose problems. The field equations defining the preferred extremals of Kähler action make sense also p-adically if the preferred extremals correspond to critical space-time sheets for which the second variation of Kähler action vanishes [A6]: this guarantees that the Noether currents associated with the modified Dirac action are conserved. A weaker condition that the matrix determined by second variations has rank which is not maximal. The interpretation is in terms of a generalized catastrophe theory: space-time surfaces are critical with respect to the variation of Kähler action. These conditions are algebraic and make sense also p-adically. Also the conditions implied by number theoretical compactification make sense p-adically. Therefore one can construct the p-adic variants of preferred extremals of Kähler action. The new element is the possibility of p-adic pseudo constants depending on finite number of pinary digits only.

At number theoretical criticality it should be possible to assign to the real partonic 2-surfacea unique p-adic counterpart. This might be true also for X^3_l and even for the space-time sheet $X^4(X^3_l)$. This is possible if the objects in question are defined by algebraic equations making sense also p-adically. Also trigonometric functions and exponential functions can be considered. Obviously p-adic pseudo constants are genuine constants for the geometric objects being shared in algebraic sense by the worlds defined by different number fields.

1. The starting point are the algebraic equations defining light-like partonic 3-surfaces X^3_l via the condition that the determinant of the induced metric vanishes. If the coordinate functions appearing in the determinant are algebraic functions with algebraic coefficients, p-adicization should make sense.

2. General Coordinate Invariance would suggest that this true also for the light-like 3-surfaces parallel to X^3_l appearing in the slicing of $X^4(X^3_l)$ assumed in the quantization of induced spinor fields and suggested by the properties of known extremals.

3. If the 4-dimensional real space-time sheet is expressible as a hyper-quaternionic surface of hyper-octonionic variant M^8 of the imbedding space as number-theoretic vision suggests [E2], it might be possible to construct also the p-adic variant of the space-time sheet by algebraic continuation in the case that the functions appearing in the definition of the space-time sheet are algebraic.

Some preferred space-time coordinates are necessary.

1. Standard Minkowski coordinates associated with $M^2 \times E^2$ decomposition are implied by the selection of quantization axes also also preferred CP_2 coordinates and preferred coordinates for geodesic sphere S^2_i , $i = I$ or II. These coordinates could be used to define coordinates also for X^4. Which combination of coordinate variables is good would be determined by the dimensions of projections to M^4 and CP_2.

2. The construction of solutions of field equations leads to the so called Hamilton-Jacobi coordinates for M^4, when the induced metric has Minkowski signature [D1]. These coordinates define a slicing of $X^4(X^3_l)$ by string world sheets and their partonic duals required also by the number theoretic compactification. For 4-D M^4 projection these coordinates could be used also as X^4 coordinates. The light-like coordinates u, v assigned with the string world sheets *resp.* complex coordinate w associated with the partonic 2-surface would give a candidate for preferred coordinates fixed apart from hyper-conformal *resp.* conformal transformations.

3. A good candidate for preferred coordinates for $X^2(v)$ is defined by the fluxes $J = \epsilon^{\alpha\beta} J_{\alpha\beta}\sqrt{g_2}$ and their canonical conjugates assignable to partonic 2-surfaces X^2 and their translates $X^2(v)$ along $X^3_l(X^2)$. Here J could correspond to either S^2 or CP_2 Kähler form. These coordinates are discussed in detail in the section about number theoretic braids.

4. For u, v coordinates the basic condition is that v varies along $X^3_l(u)$ and u labels these slices. This condition allows only scalings as hyper-complex analytic transformations and one might hope of fixing this scaling uniquely.

1.6.3 p-Adicization of second quantized induced spinor fields

Induction procedure makes it possible to geometrize the concept of a classical gauge field and also of the spinor field with internal quantum numbers. In the case of the electro-weak gauge fields induction means the projection of the H-spinor connection to a spinor connection on the space-time surface.

In the most recent formulation induced spinor fields appear only at light-like 3-surfaces and satisfy modified Dirac action associated with Kähler action possibly complexified by addition imaginary CP breaking instanton term. The modified Dirac equation makes sense also p-adically as also the anti-commutation relations of the induced spinor fields at different points of the (number theoretic) braid. Here discreteness is essential since delta functions are not easy to define in p-adic context. Also the notion of generalized eigenvalues makes sense and in terms of them one can construct p-adic variant of Dirac determinant and therefore of configuration space metric.

Possible difficulties relate to the definition of p-adic variants of plane wave factors appearing in the construction and being defined with respect to the variable u labeling the slices in the slicing of $X^4(X_l^3)$ by light-like 3-surfaces $X_l^3(v)$ "parallel" to X_l^3. Exponent function as such is well-defined in p-adic context if the argument has p-adic norm smaller than one. It however fails to have the basic properties of its real variant failing to be periodic and having fixed unit p-adic norm for all values of its argument. Periodicity does not however seem to be essential for the formulation of quantum TGD in its recent form. The exponential functions involved are of form $exp(i\sqrt{n}u)$, and are not periodic even in real sense. The p-adic existence requires $u\ mod\ p = 0$ unless one introduces e and possibly also some roots of e to the extension of p-adics used (e^p exists so that the extension would be finite-dimensional).

These observations raise the hope that the continuation of the second quantized induced spinor fields to various p-adic number fields is a straightforward procedure at the level of principle.

1.7 p-Adicization at the level of configuration space

This section is not a distilled final answer to the challenges involved with the p-adicization of the configuration space geometry and spinor structure. There are several questions. What is the precise meaning of concepts like number theoretical universality and criticality? What does p-adicization mean and is it needed/possible? Is algebraic continuation the manner to achieve it?

The notion of reduced configuration space implied by the notion of finite measurement resolution is what gives hopes about performing this continuation in practice.

1. The weaker notion of reduced configuration space emerges from finite measurement resolution and for given induced Kähler form at partonic 2-surfaces reduces configuration space to a finite-dimensional space $(\delta M^4_\pm \times CP_2)^n/S_n$ for given number of points of number theoretic braid. The metric and Kähler structure of this space is determined dynamically in terms of the spectrum of the modified Dirac operator.

2. The stronger notion of reduced configuration space identified as the space of the maxima of Kähler function in quantum fluctuating degrees of freedom labeled by symplectic group is second key notion and suggests strongly discretization. The points of reduced configuration space with rational of algebraic coordinates would correspond to those 3-surfaces through which leakage between different sectors of configuration space is possible. Reduced configuration space in this sense is the direct counterpart of the spin glass landscape known to obey ultrametric topology naturally. This approach is reasonably concrete and relies heavily on the most recent, admittedly still speculative, view about quantum TGD.

1.7.1 Generalizing the construction of the configuration space geometry to the p-adic context

A problematics analogous to that related with the entanglement between real and p-adic number fields is encountered also in the construction of the configuration space geometry. The original construction was performed in the real context. What is needed are Kähler geometry and spinor structure for the configuration space of 3-surfaces, and a construction of the configuration space spinor fields.

What might solve these immense architectural challenges are the equally immense symmetries of the configuration space and algebraic continuation as the method of p-adicization.

What one can hope that everything of physical interest reduces to the level of algebra (rational or algebraic numbers) and that topology (be it real or p-adic) disappears totally at the level of the matrix elements of the metric and of U-matrix mediating transitions between sectors of configuration space corresponding to different number fields. It is not necessary to require this to happen for M-matrix identified as time-like entanglement coefficients between positive and negative energy parts of zero energy states.

The notions of number theoretical universality and number theoretical criticality

An essential question is however what one means with the notions of number theoretical universality and criticality.

1. The weak form of the number theoretical universality means that there are sub-configuration spaces which can be regarded as real, those which are genuinely p-adic, and those which are algebraic in the sense that the representation of partonic 2-surface, perhaps also 3-surface, and perhaps even space-time surface is in terms of rational/algebraic functions allows the interpretation in terms of both real and p-adic numbers. These surfaces would be like rational and algebraic numbers common for the continua formed by reals and p-adics. This poses conditions on the representations of surfaces and typically rational functions with rational coefficients would represent these surfaces.

   For these surfaces - and only for these- physics should be expressible in terms of algebraic numbers and define as a completion the physics in real and p-adic number fields. This would allow p-adic non-determinism. Book analogy is convenient here: the physics corresponding to various number fields would be like pages of books glued together along rational and algebraic physics. If the transitions between states in different number field taking place via a leakage between different pages of the book are allowed, one can regard the algebraic sectors of the configuration space as critical. This number theoretic criticality could be interpreted in terms of intentionality and cognition, and living matter would represent a school example about number theoretically critical phase. For this option it is not at all obvious whether it makes sense to speak about configuration space geometry. The construction of configuration space spinor structure reducing exponent of Kähler function to determinant is what gives some hopes.

2. A much stronger condition - which I adopted originally - is that all 3-surfaces allow interpretation as as both real and p-adic surfaces: in this case p-adic non-determinism would be excluded. The objection is that this kind of number theoretical universality might reduce to a purely algebraic physics. This condition has interpretation in terms of number theoretical criticality if the weaker notion of universality is adopted.

Generalizing the construction for configuration space metric

It is not enough to generalize this construction to the p-adic context. 3-surfaces contain both real and p-adic regions and should be able to perform the construction for this kind of objects.

1. Very naively, one could start from the Riemannian construction of the line element which tells the length squared between infinitesimally close points at each point of the Riemann manifold. The notion of line element involves the notion of nearness and one obviously cannot do without topology here. The line element makes formally sense sense for real and p-adic contexts but since p-adic definite integral does not exist, the notions of p-adic length and volume do not exist naturally. Of course, p-adic norm defines very rough measure of distance in number theoretic sense. The notion of line-element is not needed in the quantum theory at configuration space level since only the matrix elements of the configuration space metric matter.

2. Configuration space metric can be constructed in terms if Dirac determinant identified as exponent of Kähler function and the formula for matrix elements is expressible in terms of derivatives of logarithms of the eigen values of the modified Dirac operator with respect to complex coordinates of the configuration space. This means enormous simplication if the number of eigenvalues

is finite as implied by finite measurement resolution realized in terms of braids defined by physical conditions. If eigenvalues are algebraic functions of complex coordinates of configuration space then also the exponent of Kähler function and configuration space covariant metric defining as its inverse as propagator in configuration space degrees of freedom are algebraic functions.

I have also proposed a formula for the matrix elements of configuration space metric and Kähler form between the Killing vector fields of isometry generators. Isometries are identified as X^2 local symplectic symmetries. These expressions can be given also in terms of configuration space Hamiltonians as "half Poisson brackets" in complex coordinates. Also the construction of quantum states involves configuration space Hamiltonians and their super counterparts.

1. The definition of configuration spaces Hamiltonians involves definite integrals of corresponding complexified Hamiltonians of $(\delta M^4_\pm \times CP_2)^n$ over X^2. Definite integrals are problematic in the p-adic context, as is clear from the fact that in-numerable number of definitions of definite integral have been proposed.

2. Finite measurement resolution would reduce integrals to sums since configuration space reduces to $(\delta M^4_\pm \times CP_2)^n/S_n$ for given CD. Furthermore, only the Hamiltonians corresponding to triplet *resp.* octet representations of $SO(3)$ *resp.* $SU(3)$ would be needed to coordinatize $S^2 \times CP_2$ part of the reduced configuration space.

3. Without number theoretic braids the definition of these integrals seems really difficult in p-adic context. Residue calculus might give some hopes but One might however hope that one could reduce the construction in the real case to that for the representations of super-conformal and symplectic symmetries, and analytically continue the construction from the real context to the p-adic contexts by *defining* the matrix elements of the metric to be what the symmetry respecting analytical continuation gives.

Configuration space integration should be also continued algebraically to the p-adic context. Quantum criticality realized as the vanishing of loop corrections associated with the configuration space integral, would reduce configuration space integration to purely algebraic process much like in free field theory and this would give could hopes about p-adicization. Matrix elements would be proportional to the exponent of Kähler function at its maximum plus matrix elements expressible as correlation functions of conformal field theory: the recent state of construction is considered in [C2]. This encourages further the hopes about complete algebraization of the theory so that the independence of the basic formulation on number field could be raised to a principle analogous to general coordinate invariance.

Is the exponential of the Kähler function rational function?

The simplest possibility that one can imagine are that the exponent e^{2K} of Kähler function appearing in the configuration space inner products is a rational or at most a simple algebraic function existing in a finite-dimensional algebraic extension of p-adic numbers.

The exponent of the CP_2 Kähler function is a rational function of the standard complex coordinates and thus rational-valued for all rational values of complex CP_2 coordinates. Therefore one is lead to ask whether this property holds true quite generally for symmetric spaces and even in the infinite-dimensional context. If so, then the continuation of the vacuum functional to the p-adic sectors of the configuration space would be possible in the entire configuration space. Also the spherical harmonics of CP_2 are rational functions containing square roots in normalization constants. That also configuration space spinor fields could use rational functions containing square roots as normalization constant as basic building blocks would conform with general number theoretical ideas as well as with the general features of harmonic oscillator wave functions.

The most obvious manner to realize this idea relies on the restriction of light-like 3-surfaces X^3_l to those representable in terms of polynomials or rational functions with rational or at most algebraic coefficients serving as natural preferred coordinates of the configuration space. This of course requires identification of preferred coordinates also for H. This would lead to a hierarchy of inclusions for sub-configuration spaces induced by algebraic extensions of rationals.

The presence of cutoffs for the degrees of polynomials involved makes the situation finite-dimensional and give rise to a hierarchy of inclusions also now. These inclusion hierarchies would relate naturally also to hierarchies of inclusions for hyperfinite factors of type II_1 since the spinor spaces associated with these finite-D versions of WCW would be finite-dimensional. Hyper-finiteness means that this kind of cutoff can give arbitrarily precise approximate representation of the infinite-D situation.

This vision is supported by the recent understanding related to the definition of exponent of Kähler function as Dirac determinant [A6]. The number of eigenvalues involved is necessarily finite, and if the eigenvalues of D_K are algebraic numbers for 3-surfaces X_l^3 for which the coefficients characterizing the rational functions defining X_l^3 are algebraic numbers, the exponent of Kähler function is algebraic number.

The general number theoretical conjectures implied by p-adic physics and physics of cognition and intention support also this conjecture. Although one must take these arguments with a big grain of salt, the general idea might be correct. Also the elements of the configuration space metric would be rational functions as is clear from the fact that one can express the second derivatives of the Kähler function in terms of $F = exp(K)$ as

$$\partial_K \partial_{\overline{L}} K = \frac{\partial_K \partial_{\overline{L}} F}{F} - \frac{\partial_K F \partial_{\overline{L}} F}{F^2} \quad . \tag{1.7.1}$$

An expression of same form but with sum over eigenvalues of the modified Dirac operator with F replaced with eigenvalue results if exponent of Kähler function is expressible as Dirac determinant:

$$\partial_K \partial_{\overline{L}} K = \frac{\partial_K \partial_{\overline{L}} \lambda_k}{\lambda_k} - \frac{\partial_K \lambda \partial_{\overline{L}} \lambda_k}{\lambda_k^2} \quad . \tag{1.7.2}$$

What is important that this formula in principles relates configuration space geometry directly to quantum physics as represented by the modified Dirac operator.

Generalizing the notion of configuration space spinor field

One must also construct spinor structure. Also this construction relies crucially super Kac-Moody and super-symplectic symmetries. Spinors at a given point of the configuration space correspond to the Fock space spanned by fermionic oscillator operators and again one might hope that super-symmetries would allow algebraization of the whole procedure.

The identification of configuration space gamma matrices as super Hamiltonians of configuration space. The generators of various super-algebras are also needed in order to construction configuration space spinors at given point of configuration space. In ideal measurement resolution these algebra elements are expressible as integrals of Hamiltonians and super-Hamiltonians of $\delta M^4_\pm \times CP_2$ and this leads to difficulties in p-adic context. It might be that finite measurement resolution which seems to be coded by the classical dynamics provides the only possible solution of these difficulties. In the case of reduced configuration space the construction of orthonormalized based of configuration space spinor fields looks a rather reasonable challenge and the continuation of this procedure to p-adic context might make sense.

1.7.2 Configuration space functional integral

One can make some general statements about configuration space functional integral.

1. If only braid points are specified, there is a functional integral over a huge number of 2-surfaces meaning sum of perturbative contributions from very large number of partonic 2-surfaces selected as maxima of Kähler function or by stationary phase approximation. This kind of non-perturbative contribution makes it very difficult to understand what is involved so that it seems that some restrictions must be posed. Also all information about crucial vacuum degeneracy of Kähler action would be lost as a non-local information.

2. Induced Kähler form represents perhaps the most fundamental zero modes since it remains invariant under symplectic transformations acting as isometries of the configuration space. Therefore it seems natural organize configuration space integral in such a manner that each choice of the induced Kähler form represents its own quantized theory and functional integral is only over

deformations leaving induced Kähler form invariant. The deformations of the partonic 2-surfaces would leave invariant both the induced areas and magnetic fluxes. The symplectic orbits of the partonic 2-surfaces (and 3-surfaces) would therefore define a slicing of the configuration space with separate quantization for each slice.

3. The functional integral would be over the symplectic group of CP_2 and over M^4 degrees of freedom -perhaps also in this case over the symplectic group of δM^4_+ - a rather well-defined mathematical structure. Symplectic transformations of CP_2 affect only the CP_2 part of the induced metric so that a nice separation of degrees of freedom results and the functional integral can be assigned solely to the gravitational degrees of freedom in accordance with the idea that fundamental quantum fluctuating bosonic degrees of freedom are gravitational.

4. Configuration space integration around a partonic 2-surface for which the Kähler function is maximum with respect to quantum fluctuating degrees of freedom should give only tree diagrams with propagator factors proportional to g_K^2 if loop corrections to the configuration space integral vanish. One could hope that there exist preferred S^2 and CP_2 coordinates such that vertex factors involving finite polynomials of S^2 and CP_2 coordinates reduce to a finite number of diagrams just as in free field theory.

If the configuration space functional integral algebraizes by the vanishing of loop corrections, one has hopes that even p-adic variant of configuration space functional integral might make sense. The exponent of Kähler function appears and if given by the Dirac determinant it would reduce to a finite product of eigenvalues of modified Dirac operator which makes sense also p-adically.

Algebraization of the configuration space functional integral

Configuration space is a union of infinite-dimensional symmetric spaces labeled by zero modes. One can hope that the functional integral could be performed perturbatively around the maxima of the Kähler function. In the case of CP_2 Kähler function has only single maximum and is a monotonically decreasing function of the radial variable r of CP_2 and thus defines a Morse function. This suggests that a similar situation is true for all symmetric spaces and this might indeed be the case.

1. The point is that the presence of several maxima implies also saddle points at which the matrix defined by the second derivatives of the Kähler function is not positive definite. If the derivatives of type $\partial_K \partial_L K$ and $\partial_{\overline{K}} \partial_{\overline{L}} K$ vanish at the saddle point (this is the crucial assumption) in some complex coordinates holomorphically related to those in which the same holds true at maximum, the Kähler metric is not positive definite at this point. On the other hand, by symmetric space property the metric should be isometric with the positive define metric at maxima so that a contradiction results.

2. If this argument holds true, for given values of zero modes Kähler function has only one maximum, whose value depends on the values zero modes. Staying in the optimistic mood, one could go on to guess that the Duistermaat-Heckman theorem generalizes and the functional integral is simply the exponent of the Kähler function at the maximum (due to the compensation of Gaussian and metric determinants). Even more, one could bravely guess that for configuration space spinor fields belonging to the representations of symmetries the inner products reduces to the generalization of correlation functions of Gaussian free field theory. Each configuration space spinor field would define a vertex from which lines representing the propagators defined by the contravariant configuration space metric in isometry basis emanate.

If this optimistic line of reasoning makes sense, the definition of the p-adic configuration space integral reduces to a purely algebraic one. What is needed is that the contravariant Kähler metric fixed by the symmetric space-property exists and that the exponent of the maximum of the Kähler function exists for rational values of zero modes or subset of them if finite-dimensional algebraic extension is allowed. This would give could hopes that the U-matrix elements resulting from the configuration space integrals would exist also in the p-adic sense.

Should one p-adicize only the reduced configuration space?

An attractive approach to p-adicization might be characterized as minimalism and would involve geometrization of only the reduced configuration space consisting of the maxima of Kähler function in quantum fluctuating degrees of freedom. A further reduction results from the finite measurement resolution replacing configuration space effectively with $(\delta M^4_\pm \times CP_2)^n/S_n$. In zero modes discretization realizing quantum classical correspondence is attractive possibility.

1. If Duistermaat-Heckman theorem [20] holds true in TGD context, one could express real configuration space functional integral in terms of exactly calculable Gaussian integrals around the maxima of the Kähler function in quantum fluctuating degrees of freedom defining what might be called reduced configuration space CH_{red}. The exponent of Kähler function and propagator identified as contravariant metric of configuration space could be deduced from the spectrum of the modified Dirac operator.

2. The huge super-conformal symmetries raise the hope that the rest of M-matrix elements could be deduced using group theoretical considerations so that everything would become algebraic. If this optimistic scenario is realized, the p-adicization of CH_{red} might be enough to p-adicize all operations needed to construct the p-adic variant of M-matrix.

3. A possible problem of this reduction is that the number of degrees of freedom in functional integral is still infinite, which might mean problems in terms of algebraization. For instance, the inverse of covariant metric identified as algebraic function need not represent algebraic object. Finite measurement resolution improves the situation in this respect. Finite measurement resolution realized in terms of number theoretic braids would reduce configuration space to $(\delta M^4_\pm \times CP_2)^n/S_n$ for given CD and this would reduce the situation to a finite dimensional one and maxima of Kähler function would form a discrete set, possibly only single point of $(\delta M^4_\pm \times CP_2)^n/S_n$. Also in this case exponent of Kähler function and the spectrum of modified Dirac operator are needed. Also the values of $J = \epsilon^{\alpha\beta}J_{\alpha\beta}\sqrt{g_2}$ at the points of number theoretic braids labeled by $\delta M^4_\pm \times CP_2/S_n$ are needed.

Zero modes pose a further problem.

1. The absence of functional integral measure in zero modes would suggest that states depend on finite number of zero modes only and that there is localization in this degrees of freedom. Finite measurement resolution suggests the same. The extrema of the quantity $J = \epsilon^{\alpha\beta}J_{\alpha\beta}\sqrt{g_2}$ at the points of number theoretic represent finite set of values of fundamental zero modes assignable to X^2 forming a finite-dimensional space naturally. Non-local isometry invariants can be defined as Kähler magnetic fluxes if it is possible to define symplectic triangulation of X^2 with vertices identifiable naturally as points of number theoretic braid corresponding to the extrema of J. The notion of symplectic fusion algebra based on this kind of triangulation is discussed in [17].

2. Kac-Moody group parameterizes zero modes assignable to X^3_l and a correlation between these zero modes and the quantum numbers of quantum state is natural and could result by stationary phase approximation if finite-dimensional variant of functional integral can be defined. If there is localization in zero modes, this correspondence could be discrete and implied by classical equations of motion for braid points. A unique selection of preferred quantization axis would be made possible by the hierarchy of Planck constants selecting $M^2 \subset M^4$ and $S^2_i \subset CP_2$ as critical manifolds with respect to the change of Planck constant.

What other difficulties can one imagine?

1. The optimal situation would be that M-matrix elements in real case are algebraic functions or at least functions continuable to the p-adic context in a form having sensible physical interpretation.

2. If one starts directly from Fourier transforms in p-adic context, difficulties are caused by trigonometric functions and exponent function whose p-adic counterparts do not behave in physically acceptable manner. It seems that it is phase factors defined by plane waves which should should restricted to roots of unity and continued to the p-adic realm as such. In p-adic context either momentum or position makes sense as p-adic number unless one introduces infinite-dimensional

extension containing logarithms and π. Maybe the only manner to avoid problems is to accept discretization and algebraization of the phase factors.

Concerning number field changing transitions at number theoretical criticality possibly relevant for U-matrix some comments are in order. For real$\leftrightarrow$ p-adic transitions only the algebraic points of number theoretic braid common to both real and p-adic variant of partonic 2-surface are relevant and situation reduces to algebraic braid points in $(\delta M^4_\pm \times CP_2)/S_n$. Algebraic points in a given extension of rationals would be common to real and p-adic surfaces. It could happen that there are very few common algebraic points. For instance, Fermat's theorem says that the surface $x^n + y^n = z^n$ has no rational points for $n > 2$. The integral over reduced configuration space should reduce to a sum over possible values of coordinates for these points. If only maxima of Kähler function an analytic continuation of real M-matrix to p-adic-real M-matri could make sense.

If this picture is correct, the p-adicization of the configuration space would mean p-adicization of CH_{red} consisting of the maxima of the Kähler function with respect to both fiber degrees of freedom and zero modes acting effectively as control parameters of the quantum dynamics. Finite measurement resolution simplifies the situation dramatically. If CH_{red} is a discrete subset of CH or its finite-dimensional variant, ultrametric topology induced from finite-p p-adic norm is indeed natural for it. 'Discrete set in CH' need not mean a discrete set in the usual sense and the reduced configuration space could be even finite-dimensional continuum. p-Adicization as a cognitive model would suggest that p-adicization in given point of CH_{red} is possible for all p-adic primes associated with the corresponding space-time surface (maximum of Kähler function) and represents a particular cognitive representation about CH_{red}.

1.7.3 Number theoretic constraints on M-matrix

Assume that U-matrix assignable to quantum jump between zero energy states exists simultaneously in all number fields and perhaps even between different number fields at number theoretical quantum criticality (allowing finite-dimensional extensions of p-adics). If so the immediate question is whether also the construction procedure of the M-matrix defined as time-like entanglement coefficients between positive and negative energy parts of zero energy state could have a p-adic counterpart for each p, and whether the mere requirement that this is the case could provide non-trivial intuitions about the general structure of the theory. The identification of M-matrices as building blocks of U-matrix in the manner discussed in [C2] supports affirmative answer to the first question. Not only the configuration space but also Kähler function and its exponent, Kähler metric, and configuration space functional integral should have p-adic variants. In the following this challenge is discussed in a rather optimistic number theoretic mood using the ideas stimulated by the connections between number theory and cognition.

Number theoretical Universality and M-matrix

Number theoretic constraints on M-matrix are non-trivial even for the weaker notion of number theoretical universality. Number theoretical criticality (or number theoretical universality in strong sense) requires that M-matrix elements are algebraic numbers. This is achieved naturally if the definition of M-matrix elements involves only the data associated with the number theoretic braid. Note that this data is non-local since it involves information about tangent space of X^4 at the point so that discretization happens in geometric sense but not in information theoretic sense. Note also that for algebraic surfaces finite number of points of surface allows to deduce the parameters of the polynomials involved and thus to deduce the entire surface.

If quantum version of configuration space is adopted one must perform quantization for $E^2 \subset M^4$ coordinates of points S^2_i braid and CP_2 coordinates of M^2 braid. In this kind of situation it becomes unclear whether one can speak about braiding anymore. This might make sense if each braid topology corresponds to its own quantization containing information about the fact that deformations of X^3_l respect the braiding topology.

The partonic vertices appearing in M-matrix elements should be expressible in terms of N-point functions of some rational super-conformal field theory but with the p-adically questionable N-fold integrals over string appearing in the definition of n-point functions. The most elegant manner to proceed is to replace them with their explicit expressions if they are algebraic functions- quite generally or at number theoretical criticality. Spin chain type string discretization is an alternative, not so elegant option.

Propagators, that is correlations between partonic 2-surfaces, would be due to the interior dynamics of space-time sheets which means a deviation from super string theory. Another function of interior degrees of freedom is to provide zero modes of metric of WCW identifiable as classical degrees of freedom of quantum measurement theory entangling with quantal degrees of freedom at partonic 3-surfaces.

Number theoretical criticality and M-matrix

Number theoretical criticality poses very strong conditions on the theory.

1. The p-adic variants of 4-D field equations associated with Kähler action make sense. Also the notion of preferred extremal makes sense in p-adic context if it corresponds to quantum criticality in the sense that second variation of Kähler action vanishes for dynamical symmetries. A natural further condition is that the surface is representable in terms of algebraic equations involving only rational or algebraic coefficients and thus making sense both in real and p-adic sense. In this case also Kähler action and classical charges could exist in some algebraic extension of p-adic numbers.

2. Also modified Dirac equation makes sense p-adically. The exponent of Kähler function defining vacuum functional is well-defined notion p-adically if the identification as product of finite number of eigenvalues of the modified Dirac operator is accepted and eigenvalues are algebraic. Also the notion of configuration space metric expressible in terms of derivatives of the eigenvalues with respect to complex coordinates of configuration space makes sense.

3. The functional integral over configuration space can be defined only as an algebraic extension of real functional integral around maximum of Kähler function if the theory is integrable and gives as a result an algebraic number. One might hope that algebraic p-adicization makes sense for the vacuum function at points corresponding to the maxima of Kähler function with respect to quantum fluctuating degrees of freedom (assuming they exist) and with respect to zero modes. As discussed already earlier, in the case of zero modes quantum classical correspondence allows to select preferred value of zero modes even if functional integral in zero modes does not make sense. The basic requirement is that the inverse of the matrix defined by the Kähler metric defining propagator is algebraic function of the complex coordinate of configuration space. If the eigen-values of the modified Dirac operator satisfy this condition this is indeed the case.

4. Ordinary perturbation series based on Feynman diagrams makes sense also in p-adic sense since the presence of cutoff for the size of CD implies that the number of terms if finite. One must be however cautious with momentum integrations which should reduce to finite sum due to the presence of both IR and UV cutoff implied by the finite size of CD. The formulation in terms of number theoretic braids whose intersections with partonic 2-surfaces consist of finite number of points supports the possibility of number theoretic universality.

There are hopes that M-matrix make sense p-adically. As far M-matrix is considered, The most plausible interpretation relies on the weaker form of number theoretic universality so that genuinely p-adic M-matrices should exist.

1. Dirac determinant exists for any p-adic 3-surfaces since the eigenvalues of modified Dirac operator represent a purely local notion sensible also in p-adic context. The reason is that finite measurement resolution - now deducible from the vacuum degeneracy of Kähler action- implies that the number of eigenvalues is finite. Preferred extremals of Kähler action obey quantum criticality condition meaning that the second variation of Kähler action vanishes. This condition makes sense also p-adically.

2. If loops vanish, configuration space integration gives only contractions with propagator expressible as the contravariant configuration space Kähler metric expressible in terms of derivatives of the Kähler function with respect to the preferred complex coordinates of configuration space. If this function is algebraic function, it allows algebraic continuation to p-adic context and all that

is needed for calculation of M-matrix elements makes sense p-adically. The crucial question is whether the Kähler metric is algebraic function in preferred coordinates.

3. N-point functions involve also symplectically invariant multiplicative factors discussed in [17] in terms of symplectic fusion algebras. For them algebraic universality holds true. N-point functions of conformal field theory associated with the generalized vertices should also be algebraic functions.

4. Finite measurement resolution realized in terms of braids for given $J = \epsilon^{\alpha\beta}J_{\alpha\beta}$ means a reduction of a given sector of the configuration space in quantum fluctuating degrees of freedom to finite-dimensional space $\delta M^4_\pm \times CP_2/S_n$ associated with the boundaries of CD. For instance, configuration space Hamiltonians reduce apart from J factor to those assignable naturally to the reduced configuration space. Finite-dimensionality gives hopes of algebraic continuation of M-matrix defined in terms of general Feynman diagrams in real context using finite purely algebraic operations due to the cutoff in the size of CDs. In zero modes the simplest option would be that quantum states correspond to sums over different values of zero modes, in particular J as function in X^2.

Also number theoretical criticality is consistent with this picture.

1. If partonic 2-surface X^2 is determined by algebraic equations involving only rational coefficients, same equations define real and p-adic variants of X^2.

2. Number theoretic criticality for braids means that their points are algebraic and common to real and p-adic partonic 2-surfaces. The extrema of J -determined by algebraic conditions- must be algebraic numbers.

3. At quantum criticality Dirac determinant is algebraic number if the number of eigenvalues is finite (implied by finite measurement resolution) and if they are algebraic numbers. If the p-adic counterpart of X^3_l exists, this allows to assign to the p-adic counterpart of X^3_l the exponent of Kähler function as Dirac determinant although Kähler action remains ill-defined p-adically.

1.8 Appendix: Basic facts about algebraic numbers, quaternions and octonions

To understand the detailed connection between infinite primes, polynomial primes and Fock states, some basic concepts of algebraic number theory related to the generalization of prime and prime factorization [26, 26, 24] (the first reference is warmly recommended for a physicist because it teaches the basic facts through exercises; also second book is highly enjoyable reading because of its non-Bourbakian style of representation).

1.8.1 Generalizing the notion of prime

Algebraic numbers are defined as roots of polynomial equations with rational coefficients. Algebraic integers are identified as roots of monic polynomials (highest coefficient equals to one) with integer coefficients. Algebraic number fields correspond to algebraic extensions of rationals and can have any dimension as linear spaces over rationals. The notion of prime is extremely general and involves rather actract mathematics in general case.

Quite generally, commutative ring R called integral domain, if the product ab vanishes only if a or b vanishes. To a given integral domain one can assign a number field by essentially the same construction by which one assigns the field of rationals to ordinary integers. The integer valued function $a \rightarrow N(a)$ in R is called norm if it has the properties $N(ab) = N(a)N(b)$ and $N(1) = 1$. For instance, for the algebraic extension $Q(\sqrt{-D})$ of rationals consisting of points $z = r + \sqrt{-D}s$, the function $N(z) = r^2 + Ds^2$ defines norm. More generally, the determinant of the linear map defined by the action of z in algebraic number field defines norm function. This determinant reduces to the product of all conjugates of z in K and is n:th order polynomial with respect to the components of z when K is n-dimensional.

Irreducible elements (almost the counterparts of primes) can be defined as elements P of integral domain having the property that if one has $P = bc$, then either b or c has unit norm. Elements with unit norm are called units and elements differing by a multiplication with unit are called associates. Note that in the case of p-adics all p-adic numbers with unit norm are units.

1.8.2 UFDs, PIDs and EDs

If the elements of R allow a unique factorization to irreducible elements, R is said to be unique factorization domain (UFD). Ordinary integers are obviously UFD. The field $Z(\sqrt{-5})$ is not UFD: for instance, one has $6 = 2 \times 3 = (1 + \sqrt{-5})(1 - \sqrt{-5})$. The fact that prime factorization is not unique forces to generalize the notion of primeness such that ideals in the ring of algebraic integers take the role of integers. The counterparts of primes can be identified as irreducible elements, which generate prime ideals containing one and only one rational prime. Irreducible elements, such as $1 \pm \sqrt{-5}$ in $Z(\sqrt{-5})$, are not primes in this sense.

Principal ideal domain (PID) is defined as an integral domain for which all ideals are principal, that is are generated as powers of single element. In the case of ordinary integers powers of integers define PID.

Euclidian domain (ED) is integral domain with the property that for any pair a and b one can find pair (q, r) such that $a = bq + r$ with $N(r) < N(a)$. This guarantees that the Euclidian algorithm used in the division of rationals converges. Integers form an Euclidian domain but polynomials with integer coefficients do not (elements 2 and x do not allow decomposition $2 = q(x)x + r$). It can be shown that EDs are PIDs in turn are UFDs. For instance, for complex quadratic extensions of integers $Z(\sqrt{-d})$ there are only 9 UFDs and they correspond to $d = 1, 2, 3, 7, 11, 19, 43, 67, 163$. For extensions of type $Z(\sqrt{d})$ the number of UFD:s is infinite. There are not too many quadratic extensions which are ED:s and the possible values of d are $d = -1, \pm 2, \pm 3, 5, 6, \pm 7, \pm 11, 13, 17, 19, 21, 29, 33, 37, 41, 57, 73$.

Any algebraic number field K is representable always as a polynomial ring $Q[\theta]$ obtained from the polynomial ring $Q[x]$ by replacing x with an algebraic number θ, which is a root of an irreducible polynomial with rational coefficients. This field has dimension n over rationals, where n is the degree of the polynomial in question.

1.8.3 The notion of prime ideal

As already noticed, a general algebraic number field K does not allow a unique factorization into irreducibles and one must generalize the notion of prime number and integer in order to achieve a unique factorization. The ideals of the ring O_K of algebraic integers in K take the role of integers whereas prime ideals take the role of primes. The factorization of an ideal to a product of prime ideals is unique and each prime ideal contains single rational prime characterizing it. One can assign to an ideal norm which orders the ideals: $N(a) < N(b) \leftrightarrow b \subset a$. The smaller the integer generating ideal, the larger the ideal is and the ideals generated by primes are maximal ones in PID. The equivalence classes of the ideals of O_K under equivalence defined by integer multiplication form a group. The number of classes is a characteristic of an algebraic number field. For class-one algebraic number fields prime factorization of ideals is equivalent with the factorization to irreducibles in K. $Z(\sqrt{-5})$, which is not UFD, allows two classes of prime ideals. Cyclotomic number fields $Q(\zeta_m)$, where ζ_m is m:th root of unity have class number one for $3 \leq m \leq 10$. In particular, the four-dimensional algebraic number fields $Q(\zeta_8)$ and $Q(\zeta_5) = Q(\zeta_{10})$ are ED and thus UFD.

Basic facts about primality for polynomial rings

The notion of primality can be abstracted to the level of polynomial algebras in field K and these polynomial algebras seem to be more or less identical with the algebra formed by infinite integers. The following two results are crucial for the argument demonstrating that this is indeed the case.

Polynomial ring associated with any number field is UFD

The elements in the ring $K[x_1, ..., x_n]$ formed by the polynomials having coefficients in *any* field K and x_i having values in K, allow a unique decomposition into prime factors. This means that things are much simpler at the next abstraction level, since there is no need for refined class theories needed in the case of algebraic number fields.

The number field K appearing as a coefficient field of polynomials could correspond to finite fields (Galois fields), rationals, any algebraic number field obtained as an extension of rational, p-adic numbers, reals or complex numbers. For $Q[x]$, where Q denotes rationals, the simplest prime factors are monomials of form $x-q$, q rational number. More complicated prime factors correspond to minimal polynomials having algebraic number α and its conjugates as their roots. In the case of complex number field only monomomials $x-z$, z complex number are the only prime polynomials. Clearly, the primes at the higher level of abstraction are generalized rationals of previous level plus numbers which are algebraic with respect to the generalized rationals.

The polynomial rings associated with any UFD are UFD

If R is a unique factorization domain (UFD), then also $R[x]$ is UFD: this holds also for $R[x_1,...,x_n]$. Hence one obtains an infinite hierarchy of UFDs by a repeated abstraction process by starting from a given algebraic number field K. At the first step one obtains the ring $K[x]$ of polynomials in K. At the next step one obtains the ring of polynomials $K^{2)}[y]$ having as coefficient ring the ring $K[x]\equiv K^{1)}[x]$ of polynomials. At the next step one obtains $K^{2)}[z]$, etc.. Note that $O_K[x]$ is not ED in general and need not be UFD neither unless O_K is UFD. $O_K[x]$ is not however interesting from the viewpoint of TGD.

An element of $K^{2)}(y)$ corresponds to a polynomial $P(y,x)$ of y such that its coefficients are K-rational functions of x. A polynomial in $K^{3)}(z)$ corresponds to a polynomial of $P(z,y,z)$ such that the coefficients of z are K-rational functions of functions of y with coefficients which are K-rational functions of z. Note that as a special case, polynomials of all n variables result. Note also the hierarchical ordering of the variables. Thus the hierarchy of polynomials gives rise to a hierarchy of functions having increasingly number of independent variables.

1.8.4 Examples of two-dimensional algebraic number fields

The general two-dimensional (in algebraic sense) algebraic extension of rationals corresponds to $K(\theta)$, where $\theta=(-b\pm\sqrt{b^2-4c})/2$ is root of second order irreducible polynomial x^2+bx+c. Depending on whether the discriminant $D=b^2-4c$ is positive or negative, one obtains real and complex extensions. θ and its conjugate generate equivalent extensions and all extensions can be obtained as extensions of form $Q(\sqrt{\pm d})$.

For $Q(\sqrt{d})$, d square-free integer, units correspond to powers of $x=\pm(p_{n-1}+q_{n-1}\sqrt{d})$, where n defines the period of the continued fraction expansion of $\sqrt{d}$ and p_k/q_k defines k:th convergent in the continued fraction expansion. For $Q(\sqrt{-d})$, $d>1$ units form group Z_2. For $d=1$ the group is Z_2^2 and for $Q(w)$ where $w=-1/2+\sqrt{3}/2$ is the third root of unity ($w^3=1$), this group is $Z_2\times Z^3$ (note that in this case the minimal polynomial is $(x^3-1)/(x-1)$.

$Z(w)$ and $Z(i)$ are exceptional in the sense that the group of the roots of unity is exceptionally large. $Z(i)$ and $Z(w)$ allow a unique factorization of their elements into products of irreducibles. The primes π of $Z(w)$ consist of rational primes p, $p\ mod\ 4=3$ and complex Gaussian primes satisfying $N(\pi)=\pi\overline{\pi}=p$, $p\ mod\ 4=1$. Squares of the Gaussian primes generate as their product complex numbers giving rise to Pythagorean phases. The primes π of $Z(w)$ consist of rational primes p, $p\ mod\ 3=2$ and complex Eisenstein primes satisfing $N(\pi)=\pi\overline{\pi}=p$, $p\ mod\ 3=1$.

1.8.5 Cyclotomic number fields as examples of four-dimensional algebraic number fields

By the 'theorem of primitive element' all algebraic number fields are obtained by replacing the polynomial algebra $Q[x]$, by $Q[\theta]$, where θ is a root of an irreducible minimal polynomial which is of fourth order. One can readily calculate the extensions associated with a given irreducible polynomial by using quadratures for 4:th order polynomials. These polynomials are of general form $P_4(x)=x^4+a_3x^3+a_2x^2+a_1x+a_0$ and by a substitution $x=y-a_3/4)$ which does not change the nature of algbebraic number field, they can be reduced to a canonical form $P_4(x)=x^4+a_2x^2+a_1x+a_0$. Thus a very rough view is that three rationals parametrize the 4-dimensional algebraic number fields.

A second manner to represent extensions is in form $K(\theta_1,\theta,..)$ such that the units θ_i have no common factors different from one. In this case the dimension of the extension is 2^n, where n is the number of units. Examples of four-dimensional extensions are the algebraic extensions

$Q(\sqrt{\pm d_1}, \sqrt{\pm d_2})$ of rationals, where d_i are square-free integers, reduce to form $Q(\theta)$. The cyclic extension of rationals by the powers of the m:th root of unity with $m = 5, 8, 12$ are four-dimensional extensions called cyclotomic number fields. Also the extensions $Q((\pm)d)^{1/4})$ are simple four-dimensional extensions. These extensions allow completion to a corresponding p-adic algebraic extension for some p-adic primes.

Quite generally, cyclotomic number fields $Q(\zeta_m)$ are obtained from polynomial algebra $Q[x]$ by replacing x with the m:th primitive root of unity denoted by ζ_m and thus satisfying $\zeta_m^m = 1$. There are three cyclic extensions of dimension 4 and they correspond to $Q(\zeta_5) = Q(\zeta_{10})$, $Q(\zeta_8)$ and $Q(\zeta_{12})$. Cyclotomic extensions are higly symmetric since the roots of unity act as symmetries of the norm.

The units of cyclotomic field $Q(\zeta_m)$ form group $Z_2 \times Z_m \times Z$. Z corresponds to the powers of units for $Q(\zeta_m + 1/\zeta_m)$. These powers have unit norm only with respect to the norm of $Q(\zeta_m)$ whereas with respect to the ordinary complex norm they correspond to fractal scalings. What looks fractal obtained by repeated scalings of the same structure with respect to the real norm looks like a lattice when algebraic norm is used.

1. $Q(\zeta_8)$

The cyclotomic number field $Q(\zeta_8)$, $\zeta_8 = exp(i\pi/4)$ satisfying $\zeta_8^8 = 1$, consists of numbers of form $k = m + in + \sqrt{i}(r + is)$. All roots ($\pm i^{1/2}$ and $\pm i^{3/2}$) are complex. The group of units is $Z_2^4 \times Z$. Z corresponds in real topology to the fractal scalings generated by $L = 1 + \sqrt{2}$. The integer multiples of $log(L)$ could be interpreted as a quantized momentum. $Q(\zeta_8)$ can be generated by $\pm\zeta_8$ and $\pm i\zeta_8$. This means additional Z_2^2 Galois symmetry which does not define multiplicative quantum number.

2. $Q(\zeta_{12})$

The extension $Q(\sqrt{-1}, w)$, $w = \zeta_3$, can be regarded as a cyclic extension $Q(iw) = Q(\zeta_{12})$ as is clear from the fact that the six lowest powers of iw come as $iw, -w^2, -i, w = -1 - w^2, iw^2 = -iw - i, -1$. $Z(iw)$ is especially interesting because it contains $Q(i)$ and $Q(w)$ for which primes correspond to Gaussian and Eisenstein primes. A unique factorization to a product of irreducibles is possible only for $Q(\zeta_m)$ $m \leq 10$: thus the algebraic integers in $Z(iw)$ do not always allow a unique decomposition into irreducibles. The most obvious candidates for primes not allowing unique factorization are primes satisfying simultaneously the conditions $p \ mod\ 4 = 3 = 1$ implying decomposition into a product of Gaussian prime and its conjugate and $p \ mod\ 3 = 1$ guaranteing the decomposition into a product of Eisestein prime and its conjugate.

The group of units reduces to $Z_2^2 \times Z_3 \times Z$ might have something to do with the group of discrete quantum numbers C,P and $SU(3)$ triality telling the number of quarks modulo 3 in the state. For the extensions $Q(\sqrt{-1}, \sqrt{d})$ the roots of unity form the group Z_2^2: these extensions could correspond to gauge bosons and the quantum numbers would correspond to C and P. For real extensions the group of the roots of unity reduces to Z_2: in this case the interpretation inters of parity suggests itself.

The lattice defined by Z corresponds to the scalings by powers of $\sqrt{3} + 2$. It could be also interpreted also as the lattice of longitudinal momenta for hadronic quarks which move collinearly inside space-time sheet which might be identified as a massless extremal (ME) for which longitudinal direction is a preferred spatial direction.

$Q(\zeta_{12})$ can be generated by $\pm iw, \pm iw^2$ and the replacement of iw with these alternatives generates Z_2^2 symmetry not realizable as a multiplication with units.

3. $Q(\zeta_5)$ and biology

$Q(\zeta_5)$ indeed gives 4-dimensional extension of rationals since one has $1+\zeta_5+...\zeta_5^4 = 0$ implying that $\zeta_5^4 = 1/\zeta_5$ is expressible as rational combination of other units. Both $Q(\zeta_5)$ and $Q(\zeta_8)$ allows a unique decomposition of rational integers into prime factors. The primes in $Q(\zeta_5)$ allow decomposition to a product of $r = 1, 2$ or 4 primes of $Q(\zeta_5)$ [26]. The value of r for a given p is fixed by the requirement that $f = 4/r$ is the smallest natural number for which $p^f - 1 \ mod\ p = 0$ holds true. For instance, $p = 2, 3$ correspond to $f = 4$ and are primes of $Q(\zeta_5)$, $p = 11$ has decomposition into a product of four primes of $Q(\zeta_5)$, and $p = 19$ has decomposition into two primes of $Q(\zeta_5)$).

What makes this extension interesting is that the phase angle associated with ζ_5 corresponds to the angle of 72 degrees closely related with Golden Mean $\tau = (1 + \sqrt{5})/2$ satisfying the equation $\tau^2 - \tau - 1 = 0$. The phase of the fifth root is given by $\zeta_5 = (\tau - 1 + i\sqrt{2 + \tau})2$. The group of units is $Z_2 \times Z_5 \times Z$. Z corresponds to the factal scalings by $\tau = (1 + \sqrt{5})/2$. The conjugations $\zeta_5 \rightarrow \zeta_5^k$, $k = 1, 2, 3, 4$ leave the norm invariant and generate group Z_2^5.

Fractal scalings by Golden Mean and the closely related Fibonacci numbers are closely related with the fractal structures associated with living systems (botany is full of logarithmic spirals involving Golden Mean and the phase angle 36 is involved even with DNA). It has been suggested that Golden Mean might be even a fundamental constant of physics [53]. Of course, the very fact that Golden Mean emerges in biological length scales provides strongest evidence for its dynamical origin in algebraic framework.

$Q(\zeta_5)$ cannot be realized as an algebraic extension $K(\theta,i)$ naturally associated with the transversal part of quaternionic primes but can appear only as a subfield of the 8-dimensional extension $K(i,cos(2\pi/5),sin(2\pi/5)$ containing also 20:th root of unity as $\zeta_{20}=i\zeta_5$. In [E9] it is indeed found that Golden Mean plays a fundamental role in topological quantum computation and is indeed a fundamental constant in TGD Universe.

Fractal scalings

By Dirichlet's unit theorem the group of units quite generally reduces to $Z_m \times Z^r$, where Z_m is cyclic group of roots of unity and Z^r can be regarded as an r-dimensional lattice with latticed units determined by the extension. For real extensions Z_m reduces to Z_2 since the only real roots of unity are $\{\pm 1\}$. All components of four-momentum represented by a quaternionic prime can be multiplied by separate real units of $Q(\theta)$. For a given quaternionic prime, one can always factor out the common factor of the units of $Q(\theta)$ or $Q(\theta,i)$.

The units generate nontrivial transformations at the level of single quaternionic prime. If the dimension of the real extension is n, the transformations form an $n-1$-dimensional lattice of scalings. Alternative but less plausible interpretation is that the logarithms of the scalings represent $n-1$-dimensional momentum lattice. Particle would be like a part of an algebraic hologram carrying information about external world in accordance with the ideas about fractality. Of course, units represent fractal scalings only with respect to ordinary real norm, with respect to number theoretical norm they act like phase factors.

For instance, in the case of $Q(\sqrt{5})$ the units correspond to scalings by powers of Golden Mean $\tau=(1+\sqrt{5})/2$ having number theoretic norm equal to one. Bio-systems are indeed full of fractals with scaling symmetry. For $K=Q(\sqrt{3})$ the scalings correspond to powers of $L=2+\sqrt{3}$. An interesting possibility is that hadron physics might reveal fractality in powers of L. More generally, for $Q(\sqrt{d})$, d square-free integer, the basic fractal scaling is $L=p_{n-1}+q_{n-1}\sqrt{d}$, where n defines the period of the continued fraction expansion of $\sqrt{d}$ and p_k/q_k defines k:th convergent in the continued fraction expansion.

Four-dimensional algebraic extensions are very interesting for several reasons. First, algebraic dimension four is a borderline in complexity in the sense that for higher-dimensional irreducible algebraic extensions there is no general quadratures analogous to the formulas associated with second order polynomials giving the roots of the polynomial. Secondly, in transversal degrees of freedom the minimal dimension for $K(\theta,i)$ is four. The units of K which are algebraic integers having a unit norm in K. Quite generally, the group of units is a product $Z_{2k}\times Z_r$ of two groups. $Z_{2k}=Z_2\times Z_k$ is the cyclic group generated by k:th root of unity. For real extensions one has $k=1$. In transversal degrees of freedom one can have $k>1$ since extension is $Q(\theta,i)$. The roots of unity possible in four-dimensional case correspond to $k=2,4,6,8,10,12$. Corresponding cyclic groups are products of Z_2^i, Z_3 and Z_5. Z_2, Z_2 and Z_3 and act as symmetries of the root lattices of Cartan algebras.

Z_3 gives rise to the Cartan algebra of $SU(3)$ and an interesting question is whether color symmetry is generated dynamically or whether it can be regarded as a basic symmetry with the lattice of integer quaternions providing scaled-up version for the root lattice of color group. Note that in TGD quark color is not spin like quantum number but corresponds to CP_2 partial waves for quark like spinors.

Permutations of the real roots of the minimal polynomial of θ

The replacements of the primitive element θ of $K(\theta)$ with a new one obtained by acting in it with the elements of Galois group of the minimal polynomial of θ generate different internal states of number theoretic fermions and bosons. The subgroup G_1 of Galois group permuting the real roots of the minimal polynomial with each other acts also as a symmetry. The number of equivalent primitive elements is $n_1=n-2r_1$, where r_2 is the number of complex root pairs. For instance, for 2-dimensional extensions these symmetries permute the real roots of a second order polynomial irreducible in the set of rationals. Since the entire polynomial has rational coefficients, kind of G_1-confinement is realized. One could say that kind of algebraically confined n-color is in question.

1.8.6 Quaternionic primes

Primeness makes sense for quaternions and octonions. The following considerations are however restricted to quaternionic primes but can be easily generalized to the octonionic case. Quaternionic primes have Euclidian norm squared equal to a rational prime. The number $N(p)$ of primes associated with a given rational p depends on p and each p allows at least two primes. Quaternionic primes correspond to points of 3-sphere with prime-valued radius squared. Prime-valued radius squared is consistent with p-adic length scale hypothesis, and one can indeed reduce p-adic length scale hypothesis to the assumption that the Euclidian region associated with CP_2 type extremal has prime-valued radius squared.

It is interesting to count the number of quaternionic primes with same prime valued length squared.

1. In the case of algebraic extensions the first definition of quaternionic norm is by using number theoretic norm either for entire quaternion squared or for each component of quaternion separately. The construction of infinite primes suggests that the first definition is more appropriate. Both definitions of norm are natural for four-momentum squared since they give integer valued mass squared spectrum associated with super-conformally invariant systems. One could also decompose quaternion to two parts as $q = (q_0 + Iq_1) + J(q_2 + Iq_3)$ and define number theoretic norm with respect to the algebraic extension $Q(\theta, I)$.

2. Quaternionic primes with the same norm are related by $SO(4)$ rotation plus a change of sign of the real component of quaternion. The components of integer quaternion are analogous to components of four-momentum.

3. There are 2^4 quaternionic $\pm E_i$ and multiplication by these units defines symmetries. Non-commutativity of the quaternionic multiplication makes the interpretation of units as parity like quantum numbers somewhat problematic since the net parity associated with a product of primes representing physical particles associated with the infinite primes depends on the order of quaternionic primes. For real algebraic extensions $K = Q(\theta)$ there is also the units defining a 'momentum' lattice with dimension $n - 1$, where n is the degree of the minimal polynomial $P(\theta)$.

4. Quaternionic primes cannot be real so that a given quaternionic prime with $k \geq 2$ components has 2^k conjugates obtained by changing the signs of the components of quaternion. Basic conjugation changes the signs of imagy components of quaternion. This corresponds to group $Z_2^k \subset Z_2^4$, $2 \leq k \leq 4$.

5. The group S_4 of $4! = 24$ permutations of four objects preserves the norm of a prime quaternion: these permutations are representable as a multiplication with non-prime quaternion and thus identifiable as subgroup of $SO(4)$ and also as a subgroup of $SO(3)$ (invariance group of tedrahedron). In degenerate cases (say when some components of q are identical), some subgroup of S_4 leaves quaternionic prime invariant and the rotational degeneracy reduces from $D = 24$ to some smaller number which is some factor of 24 and equals to $4, 6$ or 12 as is easy to see. There are 16 quaternionic conjugations corresponding to change of sign of any quaternion unit but all these conjugations are obtained from single quaternionic conjugation changing the sign of the imaginary part of quaternion by combining them with a multiplication with unit and its inverse. Thus the restricted group of symmetries is $S_4 \times Z_2$.

6. It is possible to find for every prime p at least two quaternionic (primes with norm squared equal to p. For a given prime p there are in general several quaternionic primes not obtainable from each other by transformations of S_4. There must exist some discrete subgroup of $SO(4)$ relating these quaternionic primes to each other.

7. The maximal number of quaternionic primes generated by $S_4 \times Z_2$ is 24×2. In noncommutative situation it is not clear whether units can be regarded as parity type quantum numbers. In any case, one can divide the entire group with Z_2^4 to obtain Z_3. This group corresponds to cyclic permutations of imaginary quaternion units.

$D = 24$ is the number of physical dimensions in bosonic string model. In TGD framework a possible interpretation is based on the observation that infinite primes constructed from rational primes the product of all primes contains the first power of each prime having interpretation as a representation for a single filled state of the fermionic sea. In the case of quaternions the Fock vacuum defined as a product of all quaternionic primes gives rise to a vacuum state

$$X = \prod_p p^{N(p)/2} \ ,$$

since each prime and its quaternionic conjugate contribute one power of p.

1.8.7 Imbedding space metric and vielbein must involve only rational functions

Algebraization requires that imbedding space exists in the algebraic sense containing only points for which preferred coordinate variables have values in some algebraic extension of rationals. Imbedding space metric at the algebraic level can be defined as a quadratic form without any reference to metric concepts like line element or distance. The metric tensors of both M^4_+ and CP_2 are indeed represented by algebraic functions in the preferred coordinates dictated by the symmetries of these spaces.

One should also construct spinor structure and this requires the introduction of an algebraic extension containing square roots since vielbein vectors appearing in the definition of the gamma matrices involve square roots of the components of the metric. In CP_2 degrees of freedom this forces the introduction of square root function, and thus all square roots, unless one restricts the values of the radial CP_2 coordinate appearing in the vielbein in such a manner that rationals result. What is interesting is that all components of spinor curvature and Kähler form of CP_2 are quadratic with respect to vierbein and algebraic functions of CP_2 complex coordinates. Also the square root of the determinant of the induce metric appears only as a multiplicative factor in the Euler-Lagrange equations so that one can get rid of the square roots.

Induced spinor structure and Dirac equation relies on the notion of the induced gamma matrices and here the projections of the vierbein of CP_2 containing square roots are unavoidable. In complex coordinates the components of CP_2 vielbein in complex coordinates ξ_1, ξ_2, in which the action of $U(2)$ is linear holomorphic transformation, involve the square roots $r = \sqrt{|\xi|^2 + |\xi_2|^2}$ and $\sqrt{1+r^2}$ (for detailed formulas see Appendix at the end of the book). If one has $r = m/n$, the requirement that $\sqrt{1+r^2}$ is rational, implies $m^2 + n^2 = k^2$ so that (m,n) defines Pythagorean square. Thus induced Dirac equation is rationalized if the allowed values of r correspond to Pythagorean phases. The notion of the phase preserving canonical identification [E6], crucial for the earlier formulation of TGD, is consistent with this assumption. The metric of $S^2 = CP_1$ is a simplified example of what happens. One can write the metric as $g_{z\bar{z}=r^2} = \frac{1}{1+r^2}$ and vielbein component is proportional to $1/\sqrt{1+r^2}$, this exists for $r = m/n$ as rational number if one has $m^2 + n^2 = k^2$, which indeed defines Pythagorean triangle.

The restriction of the phases associated with the CP_2 coordinates to Pythagorean ones has deeper coordinate-invariant meaning. Rational CP_2 can be defined as a coset space $SU_Q(3)/U_Q(2)$ of rational groups $SU_Q(3)$ and $U_Q(2)$: rationality is required in the linear matrix representation of these groups.

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Chapter 2

TGD as a Generalized Number Theory II: Quaternions, Octonions, and their Hyper Counterparts

2.1 Introduction

This chapter is second part of the multi-chapter devoted to the vision about TGD as a generalized number theory. The basic theme of the chapter is what I called originally number theoretical compactification.

After the discovery of the basic idea of number theoretic compactification -as I called it then- I inflated it with a bundle of un-necessarily strong conjectures based on duality thinking. This uncritical playing with dualities was indeed the highest fashion in M-theory at the time of writing the first version of this chapter and the saying that if five minutes is not enough to prove that the conjectured duality is wrong it must hold true describes well the rigor of theoretical thinking this period which achieved its climax in landscape crisis meaning a total loss of connection with the experimental reality.

After the realization that light-like 3-surfaces are the fundamental dynamical objects of quantum TGD and the emergence of the notion of zero energy ontology leading finally to the understanding of how induced spinor fields at light-like 3-surfaces X_l^3 code for the theory, it became clear that situation is much simpler than I had thought. This realization led to a merciless process of throwing out obsolete speculations from the chapters of various books (accompanied by a feeling of disgust and shame!) and the recent chapter represents what survived this process. I can however defend myself: the only way to make progress with a really difficult problem is to generate as many ideas as possible and do the best that one can to kill them.

2.1.1 Hyper-octonions and hyper-quaternions

The discussions for years ago with Tony Smith [20] stimulated very general ideas about space-time surface as an associative, quaternionic sub-manifold of octonionic 8-space. Also the observation that quaternionic and octonionic primes have norm squared equal to prime in complete accordance with p-adic length scale hypothesis, led to suspect that the notion of primeness for quaternions, and perhaps even for octonions, might be fundamental for the formulation of quantum TGD. The original idea was that space-time surfaces could be regarded as four-surfaces in 8-D imbedding space with the property that the tangent spaces of these spaces can be locally regarded as 4- *resp.* 8-dimensional quaternions and octonions.

It took some years to realize that the difficulties related to the realization of Lorentz invariance might be overcome by replacing quaternions and octonions with hyper-quaternions and hyper-octonions. Hyper-quaternions *resp.* -octonions is obtained from the algebra of ordinary quaternions and octonions by multiplying the imaginary part with $\sqrt{-1}$ and can be regarded as a sub-space of complexified quaternions *resp.* octonions. The transition is the number theoretical counterpart of the transition from Riemannian to pseudo-Riemannin geometry performed already in Special Relativity.

The loss of number field and even sub-algebra property is not fatal and has a clear physical meaning. The notion of primeness is inherited from that for complexified quaternions *resp.* octonions.

Note that hyper-variants of number fields make also sense p-adically unlike the notions of number fields themselves unless restricted to be algebraic extensions of rational variants of number fields. What deserves separate emphasis is that the basic structure of the standard model would reduce to number theory.

2.1.2 Number theoretical compactification and $M^8 - H$ duality

The notion of hyper-quaternionic and octonionic manifold makes sense but it not plausible that $H = M^4 \times CP_2$ could be endowed with a hyper-octonionic manifold structure. Situation changes if H is replaced with hyper-octonionic M^8. Suppose that $X^4 \subset M^8$ consists of hyper-quaternionic and co-hyper-quaternionic regions. The basic observation is that the hyper-quaternionic sub-spaces of M^8 with a fixed hyper-complex structure (containing in their tangent space a fixed hyper-complex subspace M^2 or at least one of the light-like lines of M^2) are labeled by points of CP_2. Hence each hyper-quaternionic and co-hyper-quaternionic four-surface of M^8 defines a 4-surface of $M^4 \times CP_2$. One can loosely say that the number-theoretic analog of spontaneous compactification occurs: this of course has nothing to do with dynamics.

This picture was still too naive and it became clear that not all known extremals of Kähler action contain fixed $M^2 \subset M^4$ or light-like line of M^2 in their tangent space.

1. The first option represents the minimal form of number theoretical compactification. M^8 is interpreted as the tangent space of H. Only the 4-D tangent spaces of light-like 3-surfaces X_l^3 (wormhole throats or boundaries) are assumed to be hyper-quaternionic or co-hyper-quaternionic and contain fixed M^2 or its light-like line in their tangent space. Hyper-quaternionic regions would naturally correspond to space-time regions with Minkowskian signature of the induced metric and their co-counterparts to the regions for which the signature is Euclidian. What is of special importance is that this assumption solves the problem of identifying the boundary conditions fixing the preferred extremals of Kähler action since in the generic case the intersection of M^2 with the 3-D tangent space of X_l^3 is 1-dimensional. The surfaces $X^4(X_l^3) \subset M^8$ would be hyper-quaternionic or co-hyper-quaternionic but would not allow a local mapping between the 4-surfaces of M^8 and H.

2. One can also consider a more local map of $X^4(X_l^3) \subset H$ to $X^4(X_l^3) \subset M^8$. The idea is to allow $M^2 \subset M^4 \subset M^8$ to vary from point to point so that $S^2 = SO(3)/SO(2)$ characterizes the local choice of M^2 in the interior of X^4. This leads to a quite nice view about strong geometric form of $M^8 - H$ duality in which M^8 is interpreted as tangent space of H and $X^4(X_l^3) \subset M^8$ has interpretation as tangent for a curve defined by light-like 3-surfaces at X_l^3 and represented by $X^4(X_l^3) \subset H$. Space-time surfaces $X^4(X_l^3) \subset M^8$ consisting of hyper-quaternionic and co-hyper-quaternionic regions would naturally represent a preferred extremal of E^4 Kähler action. The value of the action would be same as CP_2 Kähler action. $M^8 - H$ duality would apply also at the induced spinor field and at the level of configuration space.

3. Strong form of $M^8 - H$ duality satisfies all the needed constraints if it represents Kähler isometry between $X^4(X_l^3) \subset M^8$ and $X^4(X_l^3) \subset H$. This implies that light-like 3-surface is mapped to light-like 3-surface and induced metrics and Kähler forms are identical so that also Kähler action and field equations are identical. The only differences appear at the level of induced spinor fields at the light-like boundaries since due to the fact that gauge potentials are not identical.

4. The map of $X_l^3 \subset H \to X_l^3 \subset M^8$ would be crucial for the realization of the number theoretical universality. $M^8 = M^4 \times E^4$ allows linear coordinates as those preferred coordinates in which the points of imbedding space are rational/algebraic. Thus the point of $X^4 \subset H$ is algebraic if it is mapped to algebraic point of M^8 in number theoretic compactification. This of course restricts the symmetry groups to their rational/algebraic variants but this does not have practical meaning. Number theoretical compactication could thus be motivated by the number theoretical universality.

5. The possibility to use either M^8 or H picture might be extremely useful for calculational purposes. In particular, M^8 picture based on $SO(4)$ gluons rather than $SU(3)$ gluons could perturbative description of low energy hadron physics. The strong $SO(4)$ symmetry of low energy hadron physics can be indeed seen direct experimental support for the $M^8 - H$ duality.

2.1.3 Romantic stuff

Octonions and quaternions have generated a lot of romantic speculations and my only defence is that I did not know! Combined with free speculation about dualities this generated a lot of non-sense which has been dropped from this version of the chapter.

1. A long standing romantic speculation was that conformal invariance could somehow extend to 4-D context. Conformal invariance indeed extends to 3-D situation in the case of light-like 3-surfaces and they indeed are the basic dynamical objects of quantum TGD. It seems however un-necessary to extend the conformal invariance to 4-D context except by slicing $X^4(X^3_l)$ by 3-D light-like slices possessing the 3-D conformal invariance.

2. The triality between 8-D spinors, their conjugates, and vectors has generated a lot of speculative literature and this triality is indeed important in super string models. If M^8 has hyper-octonionic structure, one can ask whether also the spinors of M^8 could be regarded as complexified octonions. Complexified octonions provide also a representation of 8-D gamma matrices which is not a matrix representation. In this framework the Clifford algebra defined by gamma matrices degenerates to algebra of complexified octonions identifiable as the algebra of octonionic spinors and coordinates of M^8_c.

3. The $1+\overline{1}+3+\overline{3}$ decomposition of complexified octonion units with respect to group $SU(3) \subset G_2$ acting as automorphisms of octonions inspired the idea that hyper-octonion spinor field could represent leptons, antileptons, quarks and antiquarks. This proposal is problematic. Hyper-octonionic coordinates would carry color and generic hyper-octonionic spinor is superposition of spinor components which correspond to quarks, leptons and and their antifermions and a lot of super-selection rules would be needed. The motivations behind these speculations was that in H picture color would correspond to CP_2 partial waves and spin and ew quantum numbers to spin like quantum numbers whereas in M^8 picture color would correspond to spin like quantum number and spin and electro-weak quantum numbers to E^4 partial waves.

4. There was an idea that hyper-octonion analyticity and hyper-octonionic spinors might somehow allow to understand how to construct the preferred extremals of Kähler action. The idea was to map of hyper-octonionic spinor field to an element of local $SU(3)$ Lie algebra, whose (unfortunately non-unique!) exponentiation gives rise to $SU(3)$ element in turn allowing a projection to local CP_2. Hence the points of M^8 could have been mapped to those of H by the correspondence $(m, e) \to (m, g(\psi(m, e))$, where $\psi(m, e)$ would be hyper-octonionic spinor field.

5. In hyper-octonionic context the speculations related to triality are also unavoidable. One can make all kinds of questions. For instance, could it be that hyper-octonionic triality for hyper-octonionic spinor fields could allow construction of N-point functions in interaction vertices? One cannot exclude the possibility that trialities are important but the recent formulation of M-matrix elements does quite well without them.

2.1.4 Notations

Some notational conventions are in order before continuing. The fields of quaternions *resp.* octonions having dimension 4 *resp.* 8 and will be denoted by Q and O. Their complexified variants will be denoted by Q_C and O_C. The sub-spaces of hyper-quaternions HQ and hyper-octonions HO are obtained by multiplying the quaternionic and octonionic imaginary units by $\sqrt{-1}$. These sub-spaces are very intimately related with the corresponding algebras, and can be seen as Euclidian and Minkowkian variants of the same basic structure. Also the Abelianized versions of the hyper-quaternionic and -octonionic sub-spaces can be considered: these algebras have a representation in the space of spinors of imbedding space $H = M^4 \times CP_2$.

2.2 Quaternion and octonion structures and their hyper counterparts

In this introductory section the notions of quaternion and octonion structures and their hyper counterparts are introduced with strong emphasis on the physical interpretation. Literature contains several variants of these structures (Hyper Kähler structure and quaternion Kähler structure [42]). The notion introduced here is inspired by the physical motivations coming from TGD and involves in an essential manner the notions of (hyper-)quaternion and (hyper-)octonion analyticity. Only in later stuff the real applications are discussed.

2.2.1 Motivations and basic ideas

Before going to details it is useful to make clear the constraints on the concept of the hyper-octonionic structure implied by TGD view about physics.

$M^4 \times CP_2$ cannot certainly be regarded as having any global octonionic structure (for instance in the sense that it could be regarded as a coset space associated with some exceptional group). There are however clear indications for the importance of the hyper-quaternionic and -octonionic structures.

1. $SU(3)$ is the only simple 8-dimensional Lie-group and acts as the group of isometries of CP_2: if $SU(3)$ had some kind of octonionic structure, CP_2 would become unique candidate for the space S. The decomposition $SU(3) = h + t$ to $U(2)$ subalgebra and its complement corresponds rather closely to the decomposition of (hyper-)octonions to (hyper-)quaternionic sub-space and its complement. The electro-weak $U(2)$ algebra has a natural 1+3 decomposition and generators allow natural hyper-quaternionic structure. Hyper Kähler structure with three covariantly constant quaternionic imaginary units represented Kähler forms is however not possible. The components of the Weyl tensor of CP_2 behave with respect to multiplication like quaternionic imaginary units but only one of them is covariantly constant so that hyper-Kähler structure is not possible.

2. M^4_+ has a natural 1+3 decomposition and a unique cosmic time coordinate defined as the light cone proper time. Hyper-quaternionic structure is consistent with the Minkowskian signature of the inner product and hyper quaternion units have a natural representation in terms of covariantly constant self-dual symplectic forms and their contractions with sigma matrices.

In the following only (hyper-)octonion structure is considered: the generalization to the (hyper-)quaternion case is trivial. One can imagine two approaches to the definition of (hyper)-octonion structure.

1. (Hyper-)octonionic manifolds are obtained by gluing together coordinate patches using (hyper-)octonion analytic functions with real Laurent coefficients (this guarantees associativity and commutativity). This definition does not yet involve metric or any other structures (such as Kähler structure). This structure seems to be a necessary ingredient of any definition confirming in spirit with TGD.

2. If the manifold is endowed with metric, octonionic structure should be defined as a local tangent space structure analogous to eight-bein structure and local gauge algebra structures. This can be achieved by contracting octo-bein vectors with the standard octonionic basis to get octonion form I_k. Each vector field a^k defines naturally octonion field $A = a^k I_k$. The product of two vector fields can be defined by the octonionic multiplication and this leads to the introduction of a tensor field d_{klm} of these structure constants obtained as the contraction of the octobein vectors with the octonionic structure constants d_{abc}. Hyper-octonion structure can defined in a completely analogous manner.

A possibly relevant notion is the induction of (hyper-)octonion structure.

1. It is possible to induce octonionic structure to any 4-dimensional space-time surface by forming the projection of I_k to the space-time surface and redefining the products of I_k:s by dropping away that part of the product, which is orthogonal to the space-time surface. This means that

the structure constants of the new 4-dimensional algebra are the projections of d_{klm} to the space-time surface. One can also define similar induced algebra in the 4-dimensional normal space of the space-time surface.

2. The projection is not absolutely necessary and its is possible to have quaternionic associative tangent spaces without this assumption. As a matter fact, this option seems to be the physically favored one, and leads naturally to the hyper-quaternionicity constraint on space-time surfaces. An attractive hypothesis is that the induced tangential or normal space algebra is associative or hyper-quaternionic algebra. Also co-associativity defined as associativity of the normal space algebra is possible. This property would give for the 4-dimensionality of the space-time surface quite special algebraic meaning.

2.2.2 Octonions and quaternions

In the following only the basic definitions relating to octonions and quaterions are given. There is an excellent article by John Baez [29] describing octonions and their relations to the rest of mathematics and physics.

Octonions can be expressed as real linear combinations $\sum_k x^k I_k$ of the octonionic real unit $I_0 = 1$ (counterpart of the unit matrix) and imaginary units I_a, $a = 1, ..., 7$ satisfying

$$\begin{aligned} I_0^2 &= I_0 \equiv 1 \ , \\ I_a^2 &= -I_0 = -1 \ , \\ I_0 I_a &= I_a \ . \end{aligned} \tag{2.2.1}$$

Octonions are closed with respect to the ordinary sum of the 8-dimensional vector space and with respect to the octonionic multiplication, which is neither commutative ($ab \neq ba$ in general) nor associative ($a(bc) \neq (ab)c$ in general).

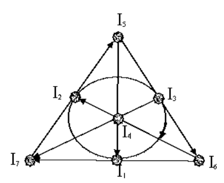

Figure 2.1: Octonionic triangle: the six lines and one circle containing three vertices define the seven associative triplets for which the multiplication rules of the ordinary quaternion imaginary units hold true. The arrow defines the orientation for each associative triplet. Note that the product for the units of each associative triplets equals to real unit apart from sign factor.

A concise manner to summarize octonionic multiplication is by using octonionic triangle. Each line (6 altogether) containing 3 octonionic imaginary units forms an associative triple which together with $I_0 = 1$ generate a division algebra of quaternions. Also the circle spanned by the 3 imaginary units at the middle of the sides of the triangle is associative triple. The multiplication rules for each associative triple are simple:

$$I_a I_b = \epsilon_{abc} I_c \ , \tag{2.2.2}$$

where ϵ_{abc} is 3-dimensional permutation symbol. $\epsilon_{abc} = 1$ for the clockwise sequence of vertices (the direction of the arrow along the circumference of the triangle and circle). As a special case this rule gives the multiplication table of quaternions. A crucial observation for what follows is that any pair of imaginary units belongs to one associative triple.

The non-vanishing structure constants $d_{ab}{}^c$ of the octonionic algebra can be read directly from the octonionic triangle. For a given pair I_a, I_b one has

$$\begin{aligned} I_a I_b &= d_{ab}{}^c I_c \ , \\ d_{ab} \quad c &= \epsilon_{ab}{}^c \ , \\ I_a^2 &= d_{aa}{}^0 I_0 = -I_0 \ , \\ I_0^2 &= d_{00}{}^0 I_0 \ , \\ I_0 I_a &= d_{0a}{}^a I_a = I_a \ . \end{aligned} \tag{2.2.3}$$

For ϵ_{abc} c belongs to the same associative triple as ab.

Non-associativity means that is not possible to represent octonions as matrices since matrix product is associative. Quaternions can be represented and the structure constants provide the defining representation as $I_a \to d_{abc}$, where b and c are regarded as matrix indices of 4×4 matrix. The algebra automorphisms of octonions form 14-dimensional group G_2, one of the so called exceptional Lie-groups. The isotropy group of imaginary octonion unit is the group $SU(3)$. The Euclidian inner product of the two octonions is defined as the real part of the product $\overline{x}y$

$$\begin{aligned} (x,y) &= Re(\overline{x}y) = \sum_{k=0,..7} x_k y_k \ , \\ \overline{x} &= x^0 I_0 - \sum_{i=1,..,7} x^k I_k \ , \end{aligned} \tag{2.2.4}$$

and is just the Euclidian norm of the 8-dimensional space.

2.2.3 Hyper-octonions and hyper-quaternions

The Euclidicity of the quaternion norm suggests that octonions are not a sensible concept in TGD context. One can imagine two manners to circumvent this conclusion.

1. M^4 metric as real part of product...

Minkowskian metric for octonions and quaternions is obtained by identifying Minkowski inner product xy as the real counterpart of the product

$$x \cdot y \equiv Re(xy) = x^0 y^0 - \sum_k x^k y^k \ . \tag{2.2.5}$$

$SO(1,7)$ ($SO(1,3)$ in quaternionic case) Lorentz invariance appears completely naturally as the symmetry of the real part of the octonion (quaternion) product and hence of octonions/quaternions and there is no need to perform the complexification of the octonion algebra. Furthermore, only the signature $(1,7)$ ($(1,3)$ in the quaternionic case) is possible and this would raise $M^4_+ \times CP_2$ in a preferred position.

This norm does not give rise to a number theoretic norm defining a homomorphism to real numbers. Indeed, the number theoretic norm defined by the determinant of the linear mapping defined by the multiplication with quaternion or octonion, is inherently Euclidian. This is in conflict with the idea that quaternionic and octonionic primes and their infinite variants should have key role in TGD [E3].

2. ....or hyper-octonions and -quaternions?

Hyper-octonions and hyper-quaternions provide a possible solution to these problems. These are obtained by multiplying imaginary units by $\sqrt{-1}$. These numbers form a sub-space of complexified octonions/quaternions and the cross product of imaginary parts leads out from this sub-space. In this case number theoretic norm induced from Q_C/O_C gives the fourth/eighth power of Minkowski length and Lorentz group acts as its symmetries. Light-like hyper-quaternions and -octonions causing the failure of the number field property have also a clear physical interpretation.

A criticism against the notion of hyper-quaternionic and octonionic primeness is that the tangent space as an algebra property is lost and the notion of primeness is inherited from Q_C/O_C. Also non-commutativity and non-associativity could cause difficulties. The proposed representation of hyper-quaternionic sub-manifolds in terms of real-analytic hyper-octonion analytic maps is equivalent with the the version based on maps using Abelian version of hyper-octonions for which the products of different imaginary units give zero. This observation allows to understand why the potential difficulties associated with non-commutativity and non-associativity can be circumvented.

2.2.4 p-Adic length scale hypothesis and quaternionic and hyper-quaternionic primes

p-Adic length scale hypothesis [E5] states that fundamental length scales correspond to the p-adic length scales proportional to $\sqrt{p}$, p prime. Even more: the p-adic primes $p \simeq 2^k$, k prime or possibly power of prime, are especially interesting physically. The so called elementary particle-blackhole analogy gives a partial theoretical justification for this hypothesis [E5]. A strong empirical support for the hypothesis comes from p-adic mass calculations [F2, F3, F4, F5].

Elementary particles correspond to the so called CP_2 type extremals in TGD Universe [D1, E5]. Elementary particle horizon can be defined as a surface at which the Euclidian signature of the metric of the space-time surface containing topologically condensed CP_2 type extremal, changes to Minkowskian signature. The generalization of the Hawking-Bekenstein formula relates the real counterpart of the p-adic entropy associated with the elementary particle to the area of the elementary particle horizon. If one requires that the radius of the elementary particle horizon corresponds to a p-adic length scale: $R = L(k)$ or $k^{n/2}L(k)$ where k is prime, then p is automatically near to 2^{k^n} and p-adic length scale hypothesis is reproduced! The proportionality of length scale to $\sqrt{p}$, rather than p, follows from p-adic thermodynamics for mass squared (!) operator and from Uncertainty Principle.

What Tony Smith [20] suggested, was a beautiful connection with number theory based on the generalization of the concept of a prime number. In the so called D^4 lattice regarded as consisting of integer quaternions, one could identify prime quaternions as the generators of the multiplicative algebra of the integer quaternions. From the basic properties of the quaternion norm it follows directly that prime quaternions correspond to the 3-dimensional spheres $R^2 = p$, p prime, with integer value E^4 coordinates. The worries are of course raised by the Euclidian signature of the number theoretical norm of quaternions.

Hyper-quaternionic and -octonionic primes and effective 2-dimensionality

The notion of prime generalizes to hyper-quaternionic and -octonionic case. The factorization $n_0^2 - n_3^2 = (n_0 + n_3)(n_0 - n_3)$ implies that any hyper-quaternionic and -octonionic primes can be represented as $(n_0, n_3, 0, ...) = (n_3 + 1, n_3, 0, ...)$, $n_3 = (p-1)/2$ for $p > 2$. $p = 2$ is exceptional: a representation with minimal number of components is given by $(2, 1, 1, 0, ...)$. The interpretation of hyper-quaternionic primes (or integers) as four-momenta suggests itself. Note that it is not possible to find a rest system for a massive particle unless the energy is allowed to be a square root of integer.

The notion of "irreducible" (see Appendix of [E1]) is defined as the equivalence class of primes related by a multiplication with a unit (integer with unit norm) and is more fundamental than that of prime. All Lorentz boosts of a hyper prime obtained by multiplication with units labelling $SO(D-1)$ cosets of $SO(D-1,1)$, $D = 4, 8$ to a hyper prime, combine to form a hyper irreducible. Note that the units cannot correspond to real particles in the arithmetic quantum field theory in which primes correspond to D-momenta labelling the physical states.

If the situation for $p > 2$ is effectively 2-dimensional in the sense that it is always possible to transform the hyper prime to a 2-component form by multiplying it by a suitable unit representing Lorentz boost, the theory for time-like hyper primes effectively reduces to the hyper-complex case. This hypothesis is physically highly attractive since it would imply number theoretic universality and conform with the effective 2-dimensionality.

Hyper-complex numbers H_2 define the maximal sub-algebra of HQ and HO. In the case of H_2 the failure of the number field property is due to the existence of light-like hyper-complex numbers with vanishing norm. The light-likeness of hyper-quaternions and -octonions is expected to have a deep physical significance and could define a number theoretic analog of propagator pole and light-like 3-D and 7-D causal determinants.

Also the rigorous notion of hyper primeness seems to require effective 2-dimensionality. If effective 2-dimensionality holds true, hyper integers have a decomposition to a product of hyper primes multiplied by a suitable unit. The representation is obtained by Lorentz boosting the hyper integer first to a 2-component form and then decomposing it to a product of hyper-complex primes. Note that the hyper-octonionic primes related by $SO(7,1)$ boosts need not represent physically equivalent states.

The situation becomes more complex if also space-like hyper primes with negative norm squared $n_0^2 - n_1^2 - ... = -p$ are allowed. Gaussian primes with $p \ mod 4 \ = 1$ would be representable as primes of form $(0, n_1, n_2, 0)$: $n_1^2 + n_2^2 = p$. If all quaternionic primes allow a representation as a quaternionic integer with three non-vanishing components, they can be identified as space-like hyper-quaternionic primes. Space-like primes with $p \ mod \ 4 = 3$ have at least 3 non-vanishing components which are odd integers. By their tachyonity space-like primes are not physically favored.

Hyper-quaternionic hyperboloids and p-adic length scale hypothesis

In the hyper-quaternionic case the 3-dimensional sphere $R^2 = p$ is replaced with Lobatchevski space (hyperboloid of M^4 with points having integer valued M^4 coordinates. Hence integer valued hyper-quaternions allow interpretation as quantized four-momenta.

Prime mass hyperboloids correspond to $n = p$. It is not possible to multiply hyperboloids since the cross product leads out of hyper sub-space. It is however possible to multiply the 2-dimensional hyperboloids and act on these by units to get full 3-D hyperboloids. The powers of hyperboloid p correspond to a multiplicatively closed structure consisting of powers p^n of the hyperboloid p. At space-time level the hyper-quaternionic lattice gives rise to a one-dimensional lattices of hyperboloidal lattices labelled by powers p^n, and the values of light-cone proper time $a \propto \sqrt{p}$ are expected to define fundamental p-adic time scales.

Also the space-like hyperboloids $R^2 = -n$ are possible and the notion of primeness makes sense also in this case. The space-like hyperboloids define one-dimensional lattice of space-like hyper-quaternionic lattices and an explanation for the spatial variant of the p-adic length scale hypothesis stating that p-adic length scales are proportional to $\sqrt{p}$ emerges in this manner naturally.

Euclidian version of the p-adic length scale hypothesis

Hyper-octonionic integers have a decomposition into hyper-quaternion and a product of $\sqrt{-1}K$ with quaternion so that quaternionic primes can be identified as hyper-octonionic space-like primes. The Euclidian version of the p-adic length scale hypothesis follows if one assumes that the Euclidian piece of the space-time surrounding the topologically condensed CP_2 type extremal can be approximated with a quaternion integer lattice with radius squared equal to $r^2 = k^n$, k prime. One manner to understand the finiteness in the time direction is that topological sum contacts of CP_2 type extremal are not static 3-dimensional topological sum contacts but genuinely four-dimensional: 3-dimensional contact is created, expands to a maximum size and is gradually reduced to point. The Euclidian space-time volume containing the contact would correspond to an Euclidian region $R^2 = k^n$ of space-time. The distances of the lattice points would be measured using the induced metric. These contacts could have arbitrarily long duration from the point of view of external observer since classical gravitational fields give rise to strong time dilation effects (strongest on the boundary of the Euclidian region where the metric becomes degenerate with the emergence of a light like direction).

Lattice structure is essential for the argument. Lattice structures of type D^4 indeed emerge naturally in the construction of the p-adic counterparts of the space-time surfaces as p-adically analytic surfaces. The essential idea is to construct the p-adic surface by first discretizing space-time surface using a p-adic cutoff in k:th pinary digit and mapping this surface to its p-adic counterpart and complete this to a unique smooth p-adically analytic surface.

This leads to a fractal construction in which a given interval is decomposed to p smaller intervals, when the resolution is increased. In the 4-dimensional case one naturally obtains a fractal hierarchy of nested D^4 lattices. The interior of the elementary particle horizon with Euclidian signature corresponds to some subset of the quaternionic integer lattice D^4: an attractive possibility is that the absolute minimization of the Kähler action and the maximization of the Kähler function force this set to be a ball $R^2 \leq k^n$, k prime.

2.2.5 Manifolds with (hyper-)octonion and (hyper-)quaternion structure

The definition of the notions of (hyper-)octonionic and (hyper-)quaternionic manifolds is straightforward. Since vielbein structure determines the geometry of the imbedding space completely, it seems natural to relate (hyper-)octonionic structure to the vielbein structure so that (hyper-)octonion structure becomes essentially metric concept. In the following only the Minkowskian case is considered in detail with restriction to hyper-quaternionic/octonionic case.

The notion of hyper-quaternionic/-octonionic analyticity

The crucial observation is that hyper-analytic series with real coefficients does not lead out from the hyper-subspace. Hence coordinate atlases based on hyper-analytic coordinate maps are possible and the notions of hyper-quaternionic and -octonionic manifolds are well-defined.

Since cross product terms in the (hyper-)octonionic Laurent series with real coefficients vanish, the real-analytic (hyper-)quaternionic and (hyper)-octonionic power series are expressible as

$$h_0 + \overline{h} \quad \rightarrow \quad ah_0 + b\overline{h} \ , \tag{2.2.6}$$

where the coefficients a and b depend only on h_0 and $|\overline{h}|^2$. This means that the result is linear in the imaginary part of h and in this case non-commutativity and non-associativity do not cause difficulties in the definition of derivatives. Hence the notion (hyper-)octonionic analytic map of HO to itself is well-defined and the notion of (hyper-)octonionic manifold makes sense since coordinate maps relating different coordinate patches can be (hyper-)quaternionic.

A more general HQ/HO analytic map results by allowing a global rotation of $\overline{h}$ induced by an automorphism of (hyper-)quaternions or (hyper-)octonions. Since a and b depend on automorphism invariants only, these automorphisms commute with HQ/HO analytic maps. Even more general notion of hyper-analyticity results when this rotation is allowed to be local.

The sub-group of the automorphism group $G_2 \subset SO(7)$ of octonions leaving a given imaginary octonion unit, say e_7 invariant, is $SU(3)$ and with respect to this group octonions decompose to two color singlets plus triplet and anti-triplet. The tensor product of triplets gives rise to a color octet defining an element of $SU(3)$ Lie algebra playing a crucial role in the proposed representation of space-time surfaces as hyper-quaternionic 4-surfaces of HO defined by hyper-octonion analytic maps.

Metric and vielbein

The ordinary inner product $Re(x\overline{y})$ can be used with conjugation acting on the hyper-octonionic/-quaternionic imaginary units but leaving $\sqrt{-1}$ invariant. This inner product can be lifted to the ordinary inner product for vector fields expressible as $a = a^k I_k$ in terms of the hyper vector fields related to the standard hyper basis I_a by a multiplication with hyper vielbein e^a_k,

$$I_k = e^a_k I_a \ , \tag{2.2.7}$$

where I_a, $a \neq 0$, is multiplied with $\sqrt{-1}$ in hyper-case. Each local vielbein $SO(D-1,1)$ rotation gives rise to a new basis at each point of the M^D $(D = 4, 8)$ but respects hyper inner product. Hence one can say that hyper structure is consistent with local $SO(D-1,1)$ gauge invariance.

One cannot perform arbitrary vierbein rotations of the quaternion units as is clear from the fact that I_0, which appears in a special role in the inner product, must be invariant under the automorphisms. In the case of the (hyper-)quaternions the automorphism group is $SO(3)$. In the case of the future light cone, the invariance of I_0 is natural if it corresponds to the Lorentz invariant proper time coordinate. In the case of hyper-octonions the allowed transformations must respect octonionic multiplication table and correspond to the group G_2.

The notions of (hyper-)octonion and (-)quaternion Hermitian manifolds

The notion of Hermitian metric is a crucial element of conformal invariance and it would be highly desirable to generalize this notion. The generalization of the notion of Hermitian metric forces naturally the selection of preferred quaternionic and complex planes in a manifold possessing octonion Hermitian structure.

1. Quaternionic case

For quaternions the line element can be expressed a a bilinear $dqd\overline{q}$. Thus q and its Hermitian conjugate resulting as anti-automorph define the first pair of coordinates. In order to obtain the second pair, the introduction of a preferred imaginary unit, call it e_1, is needed. The automorphic conjugate $q_1 = q_0 - q_1e_1 + q_2e_2 - q_3e_1e_2$ and its Hermitian conjugate define the second coordinate pair, and the line element can be expressed as

$$ds^2 = \frac{1}{2}\left[dqd\overline{q} + dq_1d\overline{q}_1\right] \quad .$$

The first guess is that for a general 4-manifold with quaternion Hermitian structure the generalization of the metric would read as

$$ds^2 = Fdqd\overline{q} + Gdq_1d\overline{q}_1 \quad .$$

Here F and G are functions of quaternion coordinates. The requirement that real quaternion analyticity provides a general solution to the Laplacian equation

$$\partial_\alpha(g^{\alpha\beta}g^{1/4}\partial\beta)\Psi \ = \ 0 \tag{2.2.8}$$

associated with a half density (spinor field most naturally) requires that the metric disappears from the equation. This implies a stronger condition

$$ds^2 \ = \ F\left[dqd\overline{q} + dq_1d\overline{q}_1\right] \quad . \tag{2.2.9}$$

The condition is so strong that space-time surfaces in $M^4 \times CP_2$ are not expected to satisfy it. The condition might however hold true for the hyper-quaternionic 4-surfaces of HO.

Real-analytic quaternion transformations are expected to induce a mere scaling of the metric determinant. For a general manifold with quaternion Hermitian structure the choice of the complex sub-space of the tangent space of quaternions is expected to depend on the point of the manifold and defines a map from the manifold to the sphere S^2 labelling the complex tangent planes of Q. The argument generalizes in a trivial manner to the case of HQ. In this case a $SO(3)$ connection is needed in order to define the parallel translation.

2. Octonionic case

In the octonionic case quaternionic sub-space of octonions is needed in order to define the Hermitian structure. The four automorphic quaternion conjugates induce three automorphic conjugates o_i, $i = 2, 3, 4$ of the octonion variable $o_1 = q_1 + e_3q_2$. The variables o_i and their octonionic Hermitian conjugates define 8 octonionic variables. The line element of octonionic manifold in the general case has the same form as in the quaternionic case. Half densities as natural real-analytic solutions of Laplace equation are replaced with 1/4-densities in this case.

In the general case the local quaternionic tangent sub-space depends on the point of the octonionic manifold. Hence the introduction of octonion Hermitian structure automatically forces the selection of a local quaternion sub-space and the Hermitian structure for the latter forces the selection of a local complex sub-space.

These considerations generalize in a trivial manner to the hyper-octonionic case. The generalization of the concept of Hermiticity provides support for the idea that HO is foliated by space-time surfaces defined by an integrable distribution of hyper-quaternionic planes of the tangent space of HO. Also the local selection of the preferred imaginary unit emerges naturally if the space-time surfaces are required to have a quaternion Hermitian structure.

Can one regard CP_2 and M^4_+ as Euclidian and Minkowskian variants of hyper-quaternionic projective space?

The notion of projective space generalizes also to the hyper-quaternionic case and one can ask whether it is possible to interpret future light-cone M^4_+ and CP_2 as hyper-quaternionic projective spaces.

The points of a 1-dimensional hyper-quaternionic projective space HP_1 would be pairs of points (h_1, h_2) with the equivalence relation $(h_1, h_2) \equiv \lambda(h_1, h_2)$, $\lambda \neq 0$. The two projective coordinate charts can be defined in the standard manner as $(h_a = h_1/h_2, 1)$ or as $(1, h_b = h_2/h_1)$. The generalization to the case of HP_n is obvious.

In the case of hyper-quaternionic numbers the failure of the number field property implies that the coordinate singularities corresponding to $q_1 = 0$ *resp.* $q_2 = 0$ are replaced by coordinate singularities corresponding to all light-like values of h_1 *resp.* h_2. Thus the space in question can be interpreted as the intersection of future and past light-cones. The boundaries of the cones intersect at points where both h_1 and h_2 are light-like. This brings in mind the the fact that S-matrix involves in the minimal situation future and past directed light-cones with partonic 2-surfaces representing incoming and outgoing particles located at the boundaries of these light-cones.

This observation supports the view that $M^4_+(a_1) \cap M^4_-(a_2)$ and CP_2 emerge naturally as Minkowskian and Euclidian variants of the hyper-quaternionic projective space.

1. If the metric of the hyper-quaternionic projective space has Minkowskian signature then the natural identification of HP_1 is as $M^4_+(a_1) \cap M^4_-(a_2)$. The boundary of HP_1 is metrically 2-dimensional but topologically 3-dimensional. Light-cone boundary is hyper-quaternionic space itself since scalings respect the light-likeness of the projective coordinates. It is possible to construct several projective spaces by posing conditions on projective scalings such as $\lambda_0 > 0$ and selecting regions of M^4 properly by posing conditions on the sign of M^4 time coordinate. For instance, M^4 with light-cone boundary excluded is possible and becomes full M^4 when the boundary is added.

2. If the metric of the hyper-quaternionic projective space has an Euclidian signature, metric 2-dimensionality requires topological 2-dimensionality, and it is necessary to identify the points having different values of the light-like radial coordinate and the boundary becomes sphere S^2 attached to E^4. The resulting space would be nothing but CP_2. Thus CP_2 and M^4 are very closely related.

One can of course argue that Euclidian signature means that hyper-quaternions are replaced by quaternions. It is indeed known that CP_2 allows quaternion Kähler structure [42] which is weaker structure than Hyper Kähler structure. Even in Kähler metric making CP_2 symmetric space the components of Weyl tensor obey quaternionic multiplication table but only one component of the Weyl tensor is covariantly constant. In fact, the breaking of the quaternion structure to a unique complex structure is what extends holonomy group from $SU(2)$ forced by the Hyper Kähler structure to $U(2)$ and brings in the missing $U(1)$ factor of the electro-weak gauge group. The result would mean that $M^4 \times CP_2$ can be regarded as product of hyper-quaternionic and quaternion Kähler manifolds.

The key question is whether $M^4 \times CP_2$ could be regarded as hyper-octonionic manifold in some sense. It is highly improbable that the topology of $M^4 \times CP_2$ would allow hyper-octonio-analytic coordinate maps between different coordinate patches since complex analytic coordinate maps allow much more structure than hyper-octonion-analytic coordinate maps. The basic 8-dimensional hyper-octonionic space is just M^8 and the most natural assumption is that hyper-octonionic structure is realized in the tangent space M^8 of $M^4 \times CP_2$. A more refined structure is obtained by allowing preferred hyper-quaternionic plane at each point of M^8 implying decomposition $M^8 = M^4 \times E^4$.

2.2.6 Light-like causal determinants, number theoretic light-likeness, and generalization of residue calculus

The poles and cuts of complex functions correspond in hyper-quaternionic *resp.* octonionic framework to 3- *resp.* 7-dimensional surfaces at which hyper-quaternionic *resp.* hyper-octonionic variable is light-like. This raises obvious questions. How the number-theoretic light-likeness in HO relates to the metric light-likeness in $M^4 \times CP_2$? Does the residue calculus generalize to the hyper analytic context and provide a generalization of the basic formulas of conformal field theory?

Is there a relationship between metric light-likeness and hyper-quaternionic light-likeness?

In the case of $HQ = M^4$ and $HO = M^8$ the metric light-cones correspond to the light-likeness of the hyper counterpart h of Minkowski coordinate. For HQ- and HO-analytic functions the image of point h is given by $h = ah_0 + bO_h(\overline{h})$, where O_h corresponds to a local $G_2 \subset SO(7)$ rotation, and a and b are $SO(7)$ invariants. Light-likeness condition reads as $a^2h_0^2 - b^2|\overline{h}|^2 = 0$. The question is whether this condition could correspond to the metric light-likeness in the metric induced from Minkowski metric. For the map $w = h^2$ the light-likeness corresponds to that for h and thus to light-cone as is easy to see. By the multiplicative property of the number theoretical norm this is the case also for h^n and for any real-analytic power series which vanishes at $h = 0$. Thus HQ and HO hyper-analytic map seem to respect causality in a well-defined sense.

This and the central role of 3-D and 7-D light like causal determinants in the formulation of quantum TGD inspire some questions.

1. Could the number theoretic light-likeness in HQ and HO quite generally correspond to metric light-likeness in the induced metric.

2. Could the metric light-likeness of 3-D causal determinants $X^3_l \subset X^4 \subset M^4 \times CP_2$ in the induced metric be equivalent with the light-likeness with respect to the metric induced from OH. This would be a natural condition on the correspondence between HO and $M^4 \times CP_2$ representations of the X^4.

3. Is the hyper-quaternionic counterpart of Kähler structure possible. In other words, does the metric of space-time surface induced from HO possess only non-diagonal components in hyper-quaternionic coordinates? If this were the case, hyper-quaternion analytic transformations of $X^4 \subset HO$ would induce an analog of conformal scaling of the metric determinant, and could be interpreted as active transformations of space-time surface modifying its shape. Metric determinant of HQ Hermitian metric would transformed by the hyper-quaternionic norm of df/dh to the product of its all conjugates. Thus these map would preserve the character of light-like causal determinants with $\sqrt{g} = 0$.

Singularities of hyper analytic maps

In ordinary complex analysis the singularities of analytic maps are important. The map $z \rightarrow w = \sqrt{z}$ is the basic example. It creates two-fold covering of complex plane having singularity at origin. The hyper-elliptic Riemann surfaces in C^2 provide a more interesting example: in this case double covering of D^2 is in question except in points which correspond to degenerate roots of second degree polynomial. The singularities of hyper-quaternion analytic maps $h \rightarrow f(h)$ are expected to correspond to the light-likeness of df/dh.

Hyper-quaternionic 4-surfaces of HO with coordinate $H = h_1 + e_3h_2$ are represented as solutions of system of form $F_i(h_1 + e_3h_2) = 0$, $i = 1, ..., 4$. This gives $h_2 = f(h_1)$ and h_2. One might hope that f is hyper-quaternion analytic function with real Laurent coefficients. This function is in general multi-valued and when some roots co-inside $df/dh_1 = 0$ holds true. By $df/dh_1 = -(dF/dh_1)/(dF/dh_2)$ this corresponds to the vanishing of either dF/dh_1 or dF/dh_2 and to discrete points of the space-time surface. Something singular would happens also at the 3-D surfaces at which dF/dh_1 or dF/dh_2 is light-like.

Does the hyper variant of the residue calculus exist?

Residue calculus is in a key role in the complex analysis and thus in the formulation of conformal field theories. One might wonder whether its generalization to (hyper-)quaternionic and (-)octonionic case might exist and be useful in quantum TGD context. The fact that hyper-quaternion/-octonion analytic functions with real Laurent coefficients are linear in the imaginary part $\overline{h}$ of the argument implies effective commutativity and associativity and could make the notion of integral function and even definite integral well defined.

As a matter fact, the same notion of analyticity results if it is assumed that quaternionic units annihilate each other as in the induced Abelian algebra obtained by regarding hyper-quaternions as sub-space of complexified quaternions and projecting normal component from the product.

The physical intuition serves as a guideline in attempts to guess what the generalization of integrals $\int f(z)dz$ over curves of complex plane might mean.

The construction of configuration space geometry and of physical states reduces to the data given at two-dimensional partonic surfaces, which have co-dimension two as have also the poles of an analytic function. The hyper-quaternionic counterparts of residue integrals correspond to integrals over codimension 1 surfaces X^3 in X^4. Thus it would seem that 3-D light-like causal determinants are more like cuts than poles. These integrals should reduce to integrals over partonic two-surfaces X^2 defined by the intersections $X^3 \cap X^3_l$, perhaps defined by the value of the integrand at these surfaces serving as end points of integration curve.

A good guess is that admissible integration paths X^3 correspond to light-like 3-surfaces X^3_l having interpretation as lines of generalized Feynman diagrams. By taking one integration variable to be h they would reduce to sum of 2-dimensional integrals over partonic 2-surfaces X^2. Hyper-quaternion analyticity requires that the determinant of the induced metric, which is certainly non-analytic function, does not appear in the admissible integrands. Hence these integrals could define conformal (or hyper-conformal) invariants. These kind of invariants would naturally appear in the definition of S-matrix elements using generalized Feynman diagrams for which by definition diagrams with loops are equivalent to tree diagrams.

Let us see whether these ideas survive more quantitative inspection. For hyper-quaternionic function $1/h$ in $HQ = M^4$ 3-dimensional light-cone $t^2 - x^2 - y^2 - z^2$ defines the singularity, and could be also seen as the analog of a cut rather than pole of an analytic function. For $HO = M^8$ 7-dimensional light-like cone takes the same role.

The idea can be tested in the case of H_2 by calculating the integral $\int dh/h$ around closed curve intersecting light-cone $a^2 = t^2 - z^2 = 0$ twice. The integral function is $log(h)$, with $h = \pm\sqrt(|a^2|)exp(e_1\eta)$ using the hyperbolic analog of polar coordinates. The modulus of h has now both signs and is discontinuous along the 2-D light-cone boundary. The integral reduces to the sum of the discontinuities at points where the curve intersects the 1-D light-cone. The discontinuity is given by $log(|a^2|/-|a^2|)$ at the limit $a^2 \rightarrow 0$, and equals to $log(-1)$, which can be identified as $\pm i\pi$. The only natural definition is based on same sign of discontinuity so that the integral over a closed curve vanishes and one avoids the introduction of the imaginary unit highly un-natural in hyper-complex context. Note however that there is no obvious objection against complex extension of hyper-complex numbers.

In the case of HQ the pole corresponds to $t^2 - x^2 - y^2 - z^2 = 0$ and it is clear that the only sensible option is the one for which residue integrals over closed curves vanish. This conforms with the physically motivated definition of residue integrals as kind of conformal or hyper-conformal invariants assignable to light-like surfaces X^3_l having boundaries at light-like 3-surfaces X^7 of $H = M4 \times CP_2$.

2.2.7 Induction of the (hyper-)octonionic structure

The induction of (hyper-)octonionic structure corresponds to the projection of (hyper-)octonion basis to space-time surface. The normal component of the algebra product could be projected out.

Two manners to induce (hyper-)octonionic structure

The induction of the (hyper-)octonion structure to the space-time surface means that (hyper-)octonionic units $I_k = e^A_k I_A$, where I_A are (hyper-)octonion units multiplied, are projected to the space-time surface

$$I_\alpha = I_k \partial_\alpha h^k \quad . \tag{2.2.10}$$

If the product of tangent space (hyper-)octonions is defined using the original inner product (no conjugation for $\sqrt{-1}$), the inner product gives induced metric

$$\langle I_\alpha I_\beta \rangle = g_{\alpha\beta} \quad , \tag{2.2.11}$$

This result is nice but the problem is is that the components of the induced (hyper-)octonion field do *not* generate 4-dimensional (complexified) sub-algebra since the product contains components belonging to the normal space of the space-time surface.

The requirement that the product is automatically tangential to the surface, gives stringent conditions for the space-time surface but is possible to satisfy at least in the case of (hyper-)quaternionic manifolds since the (hyper-)quaternionic sub-spaces of (hyper-)octonions are labelled by CP_2. The assumption that the tangent space of X^4 closes algebraically to (complexified) quaternions makes sense and would assign to each point of resulting 4-surface a point of CP_2.

One can imagine also a second alternative. A four-dimensional algebra property is achieved quite generally if one redefines the (hyper-)octonion product by projecting away the component normal to the space-time surface. This projection operation means that one defines the structure constants of the induced algebra as projections of the structure constants of the octonionic algebra:

$$\begin{aligned} I_\alpha I_\beta &= d_{\alpha\beta}{}^{\gamma} I_\gamma \quad , \\ d_{\alpha\beta\gamma} &= d_{klm} \partial_\alpha h^k \partial_\beta h^k \partial_\gamma h^m \quad . \end{aligned} \tag{2.2.12}$$

One can also induce the algebra to the normal space of the space-time surface and basic formulas are very similar to those encountered in the case of the tangent space induction.

Is the induced (hyper)-octonion structure always associative or co-associative?

The basic motivation behind the entire construction is the idea that either the tangent space of the space-time surface or its normal space could be regarded as an associative algebra. The explicit form of the tangent space associativity conditions

$$I_\alpha (I_\beta I_\gamma) = (I_\alpha I_\beta) I_\gamma \quad , \tag{2.2.13}$$

reads explicitly as

$$d_{\alpha\beta}{}^{\mu} d_{\mu\gamma}{}^{\delta} = d_{\beta\gamma}{}^{\mu} d_{\alpha\mu}{}^{\delta} \quad . \tag{2.2.14}$$

In the case of the normal space induction, the conditions are of the similar form. It is convenient to say that space-time surface is co-associative if its normal space possesses associative induced algebra. The situation for the hyper-octonionic induction is essentially the same since only extension by $\sqrt{-1}$ is involved.

The following arguments suggest that associativity/coassociativity indeed holds true. The idea is to use general coordinate invariance to reduce the problem at a given point of the space-time surface to the study of the orthogonal 4+4 decompositions of the standard octonion basis and then explicitly study the induced algebra for various decompositions.

1. Reduction of the problem to the study of the 4+4 orthogonal decompositions of the standard octonion basis

Since a manifestly general coordinate invariant tangent space structure is in question, it seems obvious that it is always possible to find such coordinates that, at a given point of the space-time surface, the components of the octonionic form of H reduce to the standard form having standard multiplication rules of the octonionic generators. This is achieved if at a given point of X^4 one can choose orthonormal coordinates in H such that four coordinate curves are orthogonal to space-time surface and four are parallel to it. The second half of the H-coordinates serves as orthogonal coordinates for the space-time surface. Under these assumptions the algebra of the octonionic components I_k at the point of X^4 is of the standard form and one must only study different 4+4 decompositions of the octonion basis to orthogonal 4-dimensional subspaces to find whether associativity or co-associativity holds true.

In the standard basis, the induction procedure means that one drops away orthogonal components from the product of two octonion units belonging to the tangent space of X^4. Similarly in the case of normal space induction. This means that one can readily look what kind of 4-dimensional algebras are obtained by this procedure and whether they are associative or co-associative.

2. Various 4+4 orthogonal decompositions of the octonionic algebra

There are two cases to be considered according to whether I_0 belongs to the quadruple or not. The crucial observation in what follows is that *any* two imaginary octonion units belong to some of the seven associative triples.

Case A: I_0 belongs to the quadruple

There are two cases to be considered.

i) All three I_k:s belong to same associative triple. In this case, space-time surface has quaternionic structure.

ii) If the third I_k does *not* belong in same triple then all products of I_k lead out from the tangent space. These products vanish in the induced algebra. Thus I_k annihilate each other in the induced algebra and their squares are equal to $-I_0$. The defining relations of the 4-dimensional algebra

$$I_k^2 = -I_0 \ , \quad I_k I_l = 0 \ , \quad k \neq l \ . \tag{2.2.15}$$

This is an *associative* algebra representable by 4x4 unit matrix and 3 imaginary matrices with one non-vanishing element i at the diagonal.

There are no other possibilities. These subspaces are *associative* as expected. The result means also that the complements of these spaces are automatically co-associative.

Case B: I_0 does not belong to the quadruple

There are two possibilities also now.

i) There is full associative triple plus one outsider. All products of the outsider with the triple vanish as also vanish the squares of each I_k in the induced algebra structure.

$$I_i I_j = e_{ijk} I_k \ , \quad I_k^2 = 0 \ , \quad I_4 I_k = 0 \ , \quad I_4^2 = 0 \ . \tag{2.2.16}$$

This algebra is nothing but the algebra generated by the original associative triple endowed with the 3-dimensional cross product and by the fourth element with vanishing square and annihilating the elements of the triple. Since cross product is *non-associative*, also the entire algebra is non-associative.

ii) There is no full associative triple. In this case all products lead out of the system and each algebra generator annihilates itself and others in the induced algebra.

$$I_j I_k \quad = \quad 0 \quad \text{for all } j \text{ and } k \ . \tag{2.2.17}$$

This algebra is obviously *associative*. The matrix realization is obtained by taking the four diagonal elements of 4x4 matrix and by replacing them by a nilponent 2x2 matrix.

To conclude, if the assumptions about reducibility of the octonion basis to the standard form are correct, then M^4_+ and CP_2 as a sub-manifolds of $M^4 \times CP_2$ are both associative and co-associative. Same holds also true for the local fiber-base decomposition of $SU(3)$ regarded as a $U(2)$ bundle over CP_2. An example of a non-associative space-time surface is provided by the surface $E^3 \times$

S^1, where E^3 is space-like hyperplane of M^4 and S^1 is geodesic circle of CP_2. It seems that non-associative space-time surfaces are not physically interesting in TGD context. One can also consider the induced quaternion structure at 2-dimensional surfaces of a 4-dimensional manifold. The local algebra associated with a given 2-surface is either the algebra of the complex numbers or the algebra generated by two nilpotent elements annihilating each other. For 3-dimensional sub-manifolds one obtains the non-associative algebra defined by the ordinary cross product.

2.3 Quantum TGD in nutshell

This section provides a summary about quantum TGD, which is essential for understanding the recent developments related to $M^8 - H$ duality. The discussions are based on the general vision that quantum states of the Universe correspond to the modes of classical spinor fields in the "world of the classical worlds" identified as the infinite-dimensional configuration space of light-like 3-surfaces of $H = M^4 \times CP_2$ (more or less-equivalently, the corresponding 4-surfaces defining generalized Bohr orbits).

2.3.1 Geometric ideas

TGD relies heavily on geometric ideas, which have gradually generalized during the years. Symmetries play a key role as one might expect on basis of general definition of geometry as a structure characterized by a given symmetry.

Physics as infinite-dimensional Kähler geometry

1. The basic idea is that it is possible to reduce quantum theory to configuration space geometry and spinor structure. The geometrization of loop spaces inspires the idea that the mere existence of Riemann connection fixes configuration space Kähler geometry uniquely. Accordingly, configuration space can be regarded as a union of infinite-dimensional symmetric spaces labelled by zero modes labelling classical non-quantum fluctuating degrees of freedom.

   The huge symmetries of the configuration space geometry deriving from the light-likeness of 3-surfaces and from the special conformal properties of the boundary of 4-D light-cone would guarantee the maximal isometry group necessary for the symmetric space property. Quantum criticality is the fundamental hypothesis allowing to fix the Kähler function and thus dynamics of TGD uniquely. Quantum criticality leads to surprisingly strong predictions about the evolution of coupling constants.

2. Configuration space spinors correspond to Fock states and anti-commutation relations for fermionic oscillator operators correspond to anti-commutation relations for the gamma matrices of the configuration space. Configuration space gamma matrices contracted with Killing vector fields give rise to a super-algebra which together with Hamiltonians of the configuration space forms what I have used to called super-symplectic algebra.

   Super-symplectic degrees of freedom represent completely new degrees of freedom and have no electroweak couplings. In the case of hadrons super-symplectic quanta correspond to what has been identified as non-perturbative sector of QCD: they define TGD correlate for the degrees of freedom assignable to hadronic strings. They are responsible for the most of the mass of hadron and resolve spin puzzle of proton.

   Besides super-symplectic symmetries there are Super-Kac Moody symmetries assignable to light-like 3-surfaces and together these algebras extend the conformal symmetries of string models to dynamical conformal symmetries instead of mere gauge symmetries. The construction of the representations of these symmetries is one of the main challenges of quantum TGD. The assumption that the commutator algebra of these super-symplectic and super Kac-Moody algebras annihilates physical states gives rise to Super Virasoro conditions which could be regarded as analogs of configuration space Dirac equation.

   Modular invariance is one aspect of conformal symmetries and plays a key role in the understanding of elementary particle vacuum functionals and the description of family replication phenomenon in terms of the topology of partonic 2-surfaces.

3. Configuration space spinors define a von Neumann algebra known as hyper-finite factor of type II_1 (HFFs). This realization has led also to a profound generalization of quantum TGD through a generalization of the notion of imbedding space to characterize quantum criticality. The resulting space has a book like structure with various almost-copies of imbedding space representing the pages of the book meeting at quantum critical sub-manifolds. The outcome of this approach is that the exponents of Kähler function and Chern-Simons action are not fundamental objects but reduce to the Dirac determinant associated with the modified Dirac operator assigned to the light-like 3-surfaces.

p-Adic physics as physics of cognition and intentionality

p-Adic mass calculations relying on p-adic length scale hypothesis led to an understanding of elementary particle masses using only super-conformal symmetries and p-adic thermodynamics. The need to fuse real physics and various p-adic physics to single coherent whole led to a generalization of the notion of number obtained by gluing together reals and p-adics together along common rationals and algebraics. The interpretation of p-adic space-time sheets is as correlates for cognition and intentionality. p-Adic and real space-time sheets intersect along common rationals and algebraics and the subset of these points defines what I call number theoretic braid in terms of which both configuration space geometry and S-matrix elements should be expressible. Thus one would obtain number theoretical discretization which involves no adhoc elements and is inherent to the physics of TGD.

Perhaps the most dramatic implication relates to the fact that points, which are p-adically infinitesimally close to each other, are infinitely distant in the real sense (recall that real and p-adic imbedding spaces are glued together along rational imbedding space points). This means that any open set of p-adic space-time sheet is discrete and of infinite extension in the real sense. This means that cognition is a cosmic phenomenon and involves always discretization from the point of view of the real topology. The testable physical implication of effective p-adic topology of real space-time sheets is p-adic fractality meaning characteristic long range correlations combined with short range chaos.

Also a given real space-time sheets should correspond to a well-defined prime or possibly several of them. The classical non-determinism of Kähler action should correspond to p-adic non-determinism for some prime(s) p in the sense that the effective topology of the real space-time sheet is p-adic in some length scale range. p-Adic space-time sheets with same prime should have many common rational points with the real space-time and be easily transformable to the real space-time sheet in quantum jump representing intention-to-action transformation. The concrete model for the transformation of intention to action leads to a series of highly non-trivial number theoretical conjectures assuming that the extensions of p-adics involved are finite-dimensional and can contain also transcendentals.

An ideal realization of the space-time sheet as a cognitive representation results if the CP_2 coordinates as functions of M^4_+ coordinates have the same functional form for reals and various p-adic number fields and that these surfaces have discrete subset of rational numbers with upper and lower length scale cutoffs as common. The hierarchical structure of cognition inspires the idea that S-matrices form a hierarchy labelled by primes p and the dimensions of algebraic extensions.

The number-theoretic hierarchy of extensions of rationals appears also at the level of configuration space spinor fields and allows to replace the notion of entanglement entropy based on Shannon entropy with its number theoretic counterpart having also negative values in which case one can speak about genuine information. In this case case entanglement is stable against Negentropy Maximization Principle stating that entanglement entropy is minimized in the self measurement and can be regarded as bound state entanglement. Bound state entanglement makes possible macro-temporal quantum coherence. One can say that rationals and their finite-dimensional extensions define islands of order in the chaos of continua and that life and intelligence correspond to these islands.

TGD inspired theory of consciousness and number theoretic considerations inspired for years ago the notion of infinite primes [E3]. It came as a surprise, that this notion might have direct relevance for the understanding of mathematical cognition. The ideas is very simple. There is infinite hierarchy of infinite rationals having real norm one but different but finite p-adic norms. Thus single real number (complex number, (hyper-)quaternion, (hyper-)octonion) corresponds to an algebraically infinite-dimensional space of numbers equivalent in the sense of real topology. Space-time and imbedding space points ((hyper-)quaternions, (hyper-)octonions) become infinitely structured and single space-time point would represent the Platonia of mathematical ideas. This structure would be completely invisible at the level of real physics but would be crucial for mathematical cognition and explain why we are able to imagine also those mathematical structures which do not exist physically. Space-time could be also regarded as an algebraic hologram. The connection with Brahman=Atman idea is also obvious.

Hierarchy of Planck constants and dark matter hierarchy

The work with hyper-finite factors of type II_1 (HFFs) combined with experimental input led to the notion of hierarchy of Planck constants interpreted in terms of dark matter [A9]. The hierarchy is realized via a generalization of the notion of imbedding space obtained by gluing infinite number of its variants along common lower-dimensional quantum critical sub-manifolds. These variants of imbedding space are characterized by discrete subgroups of SU(2) acting in M^4 and CP_2 degrees of freedom as either symmetry groups or homotopy groups of covering. Among other things this picture implies a general model of fractional quantum Hall effect.

This framework also leads to the identification of number theoretical braids as points of partonic 2-surface which correspond to the minima of a generalized eigenvalue of Dirac operator, a scalar field to which Higgs vacuum expectation is proportional to. Higgs vacuum expectation has thus a purely geometric interpretation. The outcome is an explicit formula for the Dirac determinant consistent with the vacuum degeneracy of Kähler action and its finiteness and algebraic number property required by p-adicization requiring number theoretic universality. The zeta function associated with the eigenvalues (rather than Riemann Zeta as believed originally) in turn defines the super-symplectic conformal weights as its zeros so that a highly coherent picture result.

What is especially remarkable is that the construction gives also the 4-D space-time sheets associated with the light-like orbits of the partonic 2-surfaces: it remains to be shown whether they correspond to preferred extremals of Kähler action. It is clear that the hierarchy of Planck constants has become an essential part of the construction of quantum TGD and of mathematical realization of the notion of quantum criticality rather than a possible generalization of TGD.

Number theoretical symmetries

TGD as a generalized number theory vision leads to the idea that also number theoretical symmetries are important for physics.

1. There are good reasons to believe that the strands of number theoretical braids can be assigned with the roots of a polynomial with suggests the interpretation corresponding Galois groups as purely number theoretical symmetries of quantum TGD. Galois groups are subgroups of the permutation group S_∞ of infinitely manner objects acting as the Galois group of algebraic numbers. The group algebra of S_∞ is HFF which can be mapped to the HFF defined by configuration space spinors. This picture suggest a number theoretical gauge invariance stating that S_∞ acts as a gauge group of the theory and that global gauge transformations in its completion correspond to the elements of finite Galois groups represented as diagonal groups of $G\times G\times ....$ of the completion of S_∞. The groups G should relate closely to finite groups defining inclusions of HFFs.

2. HFFs inspire also an idea about how entire TGD emerges from classical number fields, actually their complexifications. In particular, SU(3) acts as subgroup of octonion automorphisms leaving invariant preferred imaginary unit and $M^4 \times CP_2$ can be interpreted as a structure related to hyper-octonions which is a subspace of complexified octonions for which metric has naturally Minkowski signature. This would mean that TGD could be seen also as a generalized number theory. This conjecture predicts the existence of two dual formulations of TGD based on the identification space-times as 4-surfaces in hyper-octonionic space M^8 *resp.* $M^4 \times CP_2$.

3. The vision about TGD as a generalized number theory involves also the notion of infinite primes. This notion leads to a further generalization of the ideas about geometry: this time the notion of space-time point generalizes so that it has an infinitely complex number theoretical anatomy not visible in real topology.

2.3.2 The notions of imbedding space, 3-surface, and configuration space

The notions of imbedding space, 3-surface (and 4-surface), and configuration space (world of classical worlds (WCW)) are central to quantum TGD. The original idea was that 3-surfaces are space-like 3-surfaces of $H = M^4 \times CP_2$ or $H = M^4_+ \times CP_2$, and WCW consists of all possible 3-surfaces in H. The basic idea was that the definition of Kähler metric of WCW assigns to each X^3 a unique space-time surface $X^4(X^3)$ allowing in this manner to realize general coordinate invariance. During years these notions have however evolved considerably.

The notion of imbedding space

Two generalizations of the notion of imbedding space were forced by number theoretical vision [E1, E2, E3].

1. p-Adicization forced to generalize the notion of imbedding space by gluing real and p-adic variants of imbedding space together along rationals and common algebraic numbers. The generalized imbedding space has a book like structure with reals and various p-adic number fields (including their algebraic extensions) representing the pages of the book.

2. With the discovery of zero energy ontology [C1, A6] it became clear that the so called causal diamonds (CDs) interpreted as intersections $M^4_+ \cap M^4_-$ of future and past directed light-cones of $M^4 \times CP_2$ define correlates for the quantum states. The position of the "lower" tip of CD characterizes the position of CD in H. If the temporal distance between upper and lower tip of CD is quantized in power-of-two multiples of CP_2 length, p-adic length scale hypothesis [E5] follows as a consequence. The upper *resp.* lower light-like boundary $\delta M^4_+ \times CP_2$ *resp.* $\delta M^4_- \times CP_2$ of CD can be regarded as the carrier of positive *resp.* negative energy part of the state. All net quantum numbers of states vanish so that everything is creatable from vacuum. Space-time surfaces assignable to zero energy states would would reside inside $CD \times CP_2$s and have their 3-D ends at the light-like boundaries of $CD \times CP_2$. Fractal structure is present in the sense that CDs can contains CDs within CDs, and measurement resolution dictates the length scale below which the sub-CDs are not visible.

3. The realization of the hierarchy of Planck constants [A9] led to a further generalization of the notion of imbedding space. Generalized imbedding space is obtained by gluing together Cartesian products of singular coverings and factor spaces of CD and CP_2 to form a book like structure. The particles at different pages of this book behave like dark matter relative to each other. This generalization also brings in the geometric correlate for the selection of quantization axes in the sense that the geometry of the sectors of the generalized imbedding space with non-standard value of Planck constant involves symmetry breaking reducing the isometries to Cartan subalgebra. Roughly speaking, each CD and CP_2 is replaced with a union of CDs and CP_2s corresponding to different choices of quantization axes so that no breaking of Poincare and color symmetries occurs at the level of entire WCW.

4. The construction of quantum theory at partonic level brings in very important delicacies related to the Kähler gauge potential of CP_2. Kähler gauge potential must have what one might call pure gauge parts in M^4 in order that the theory does not reduce to mere topological quantum field theory. Hence the strict Cartesian product structure $M^4 \times CP_2$ breaks down in a delicate manner. These additional gauge components -present also in CP_2- play key role in the model of anyons, charge fractionization, and quantum Hall effect [F12].

The notions of 3-surface and space-time surface

The question what one exactly means with 3-surface turned out to be non-trivial.

1. The original identification of 3-surfaces was as arbitrary space-like 3-surfaces subject to Equivalence implied by General Coordinate Invariance. There was a problem related to the realization of Equivalence Principle since it was not at all obvious why the absolute minimum $X^4(Y^3)$ for Y^3 at $X^4(X^3)$ and Diff^4 related X^3 should satisfy $X^4(Y^3) = X^4(X^3)$.

2. Much later it became clear that light-like 3-surfaces have unique properties for serving as basic dynamical objects, in particular for realizing the General Coordinate Invariance in 4-D sense (obviously the identification resolves the above mentioned problem) and understanding the conformal symmetries of the theory. On basis of these symmetries light-like 3-surfaces can be regarded as orbits of partonic 2-surfaces so that the theory is locally 2-dimensional. It is however important to emphasize that this indeed holds true only locally. At the level of WCW metric this means that the components of the Kähler form and metric can be expressed in terms of data assignable to 2-D partonic surfaces. It is however essential that information about normal space of the 2-surface is needed.

3. Rather recently came the realization that light-like 3-surfaces can have singular topology in the sense that they are analogous to Feynman diagrams. This means that the light-like 3-surfaces representing lines of Feynman diagram can be glued along their 2-D ends playing the role of vertices to form what I call generalized Feynman diagrams. The ends of lines are located at boundaries of sub-CDs. This brings in also a hierarchy of time scales: the increase of the measurement resolution means introduction of sub-CDs containing sub-Feynman diagrams. As the resolution is improved, new sub-Feynman diagrams emerge so that effective 2-D character holds true in discretized sense and in given resolution scale only.

The basic vision has been that space-time surfaces correspond to preferred extremals $X^4(X^3)$ of Kähler action. Kähler function $K(X^3)$ defining the Kähler geometry of the world of classical worlds would correspond to the Kähler action for the preferred extremal. The precise identification of the preferred extremals actually has however remained open.

1. The obvious guess motivated by physical intuition was that preferred extremals correspond to the absolute minima of Kähler action for space-time surfaces containing X^3. This choice has some nice implications. For instance, one can develop an argument for the existence of an infinite number of conserved charges. If X^3 is light-like surface- either light-like boundary of X^4 or light-like 3-surface assignable to a wormhole throat at which the induced metric of X^4 changes its signature- this identification circumvents the obvious objections.

2. Much later number theoretical vision led to the conclusion that $X^4(X^3_{l,i})$, where $X^3_{l,i}$ denotes a connected component of the light-like 3-surfaces X^3_l, contain in their 4-D tangent space $T(X^4(X^3_{l,i}))$ a subspace $M^2_i \subset M^4$ having interpretation as the plane of non-physical polarizations. This means a close connection with super string models. Geometrically this would mean that the deformations of 3-surface in the plane of non-physical polarizations would not contribute to the line element of WCW. This is as it must be since complexification does not make sense in M^2 degrees of freedom.

   In number theoretical framework M^2_i has interpretation as a preferred hyper-complex sub-space of hyper-octonions defined as 8-D subspace of complexified octonions with the property that the metric defined by the octonionic inner product has signature of M^8. A stronger condition would be that the condition holds true at all points of $X^4(X^3)$ for a global choice M^2 but this is un-necessary and leads to strong un-proven conjectures. The condition $M^2_i \subset T(X^4(X^3_{l,i}))$ in principle fixes the tangent space at $X^3_{l,i}$, and one has good hopes that the boundary value problem is well-defined and fixes $X^4(X^3)$ uniquely as a preferred extremal of Kähler action. This picture is rather convincing since the choice $M^2_i \subset M^3$ plays also other important roles.

3. The next step [A6] was the realization that the construction of the configuration space geometry in terms of modified Dirac action strengthens the boundary conditions to the condition that there exists space-time coordinates in which the induced CP_2 Kähler form and induced metric satisfy the conditions $J_{ni} = 0$, $g_{ni} = 0$ hold at X^3_l. One could say that at X^3_l situation is static both metrically and for the Maxwell field defined by the induced Kähler form. There are reasons to hope that this is the final step in a long process.

4. The weakest form of number theoretic compactification states that light-like 3-surfaces $X^3 \subset X^4(X^3) \subset M^8$, where $X^4(X^3)$ hyper-quaternionic surface in hyper-octonionic M^8 can be mapped to light-like 3-surfaces $X^3 \subset X^4(X^3) \subset M^4 \times CP_2$, where $X^4(X^3)$ is now preferred extremum of Kähler action. The natural guess is that $X^4(X^3) \subset M^8$ is a preferred extremal of

Kähler action associated with Kähler form of E^4 in the decomposition $M^8 = M^4 \times E^4$, where M^4 corresponds to hyper-quaternions. The conjecture would be that the value of the Kähler action in M^8 is same as in $M^4 \times CP_2$. A second interesting conjecture is that the hyper-quaternionic surfaces correspond to Kähler calibrations giving rise to absolute minima or maxima of Kähler action for M^8.

The notion of configuration space

From the beginning there was a problem related to the precise definition of the configuration space ("world of classical worlds" (WCW)). Should one regard CH as the space of 3-surfaces of $M^4 \times CP_2$ or $M^4_+ \times CP_2$ or perhaps something more delicate.

1. For a long time I believed that the question "M^4_+ or M^4?" had been settled in favor of M^4_+ by the fact that M^4_+ has interpretation as empty Roberson-Walker cosmology. The huge conformal symmetries assignable to $\delta M^4_+ \times CP_2$ were interpreted as cosmological rather than laboratory symmetries. The work with the conceptual problems related to the notions of energy and time, and with the symmetries of quantum TGD, however led gradually to the realization that there are strong reasons for considering M^4 instead of M^4_+.

2. With the discovery of zero energy ontology it became clear that the so called causal diamonds (CDs) define excellent candidates for the fundamental building blocks of the configuration space or "world of classical worlds" (WCW). The spaces $CD \times CP_2$ regarded as subsets of H defined the sectors of WCW.

3. This framework allows to realize the huge symmetries of $\delta M^4_\pm \times CP_2$ as isometries of WCW. The gigantic symmetries associated with the $\delta M^4_\pm \times CP_2$ are also laboratory symmetries. Poincare invariance fits very elegantly with the two types of super-conformal symmetries of TGD. The first conformal symmetry corresponds to the light-like surfaces $\delta M^4_\pm \times CP_2$ of the imbedding space representing the upper and lower boundaries of CD. Second conformal symmetry corresponds to light-like 3-surface X^3_l, which can be boundaries of X^4 and light-like surfaces separating space-time regions with different signatures of the induced metric. This symmetry is identifiable as the counterpart of the Kac Moody symmetry of string models.

A rather plausible conclusion is that configuration space (WCW) is a union of configuration spaces associated with the spaces $CD \times CP_2$. CDs can contain CDs within CDs so that a fractal like hierarchy having interpretation in terms of measurement resolution results. Since the complications due to p-adic sectors and hierarchy of Planck constants are not relevant for the basic construction, it reduces to a high degree to a study of a simple special case $\delta M^4_+ \times CP_2$.

2.3.3 The construction of M-matrix

The construction of S-matrix involves several ideas that have emerged during last years and involve symmetries in an essential manner.

Zero energy ontology

Zero energy ontology motivated originally by TGD inspired cosmology means that physical states have vanishing conserved net quantum numbers and are decomposable to positive and negative energy parts separated by a temporal distance characterizing the system as a space-time sheet of finite size in time direction. The particle physics interpretation is as initial and final states of a particle reaction. Obviously a profound modification of existing views about realization of symmetries is in question.

S-matrix and density matrix are unified to the notion of M-matrix defining time-like entanglement and expressible as a product of square root of density matrix and of unitary S-matrix. Thermodynamics becomes therefore a part of quantum theory. One must distinguish M-matrix from U-matrix defined between zero energy states and analogous to S-matrix and characterizing the unitary process associated with quantum jump. U-matrix is most naturally related to the description of intentional action since in a well-defined sense it has elements between physical systems corresponding to different number fields.

Quantum TGD as almost topological QFT

Light-likeness of the basic fundamental objects implies that TGD is almost topological QFT so that the formulation in terms of category theoretical notions is expected to work. M-matrices form in a natural manner a functor from the category of cobordisms to the category of pairs of Hilbert spaces and this gives additional strong constraints on the theory. Super-conformal symmetries implied by the light-likeness pose very strong constraints on both state construction and on M-matrix and U-matrix. The notions of n-category and n-groupoid which represents a generalization of the notion of group could be very relevant to this view about M-matrix.

Quantum measurement theory with finite measurement resolution

The notion of measurement resolution represented in terms of inclusions $\mathcal{N} \subset \mathcal{M}$ of HFFs is an essential element of the picture. Measurement resolution corresponds to the action of the included sub-algebra creating zero energy states in time scales shorter than the cutoff scale. This means that complex rays of state space are effectively replaced with $\mathcal{N}$ rays. The condition that the action of $\mathcal{N}$ commutes with the M-matrix is a powerful symmetry and implies that the time-like entanglement characterized by M-matrix corresponds to Connes tensor product. Together with super-conformal symmetries this symmetry should fix possible M-matrices to a very high degree.

The notion of number theoretical braid realizes the notion of finite measurement resolution at space-time level and gives a direct connection to topological QFTs describing braids. The connection with quantum groups is highly suggestive since already the inclusions of HFFs involve these groups. Effective non-commutative geometry for the quantum critical sub-manifolds $M^2 \subset M^4$ and $S^2 \subset CP_2$ might provide an alternative notion for the reduction of stringy anti-commutation relations for induced spinor fields to anti-commutations at the points of braids.

Generalization of Feynman diagrams

An essential difference between TGD and string models is the replacement of stringy diagrams with generalized Feynman diagrams obtained by gluing 3-D light-like surfaces (instead of lines) together at their ends represented as partonic 2-surfaces. This makes the construction of vertices very simple. The notion of number theoretic braid in turn implies discretization having also interpretation in terms of non-commutativity due to finite measurement resolution replacing anti-commutativity along stringy curves with anti-commutativity at points of braids. Braids can replicate at vertices which suggests an interpretation in terms of topological quantum computation combined with non-faithful copying and communication of information. The analogs of stringy diagrams have quite different interpretation in TGD: for instance, photons travelling via two different paths in double slit experiment are represented in terms of stringy branching of the photonic 2-surface.

Symplectic variant of QFT as basic building block of construction

The latest discovery related to the construction of M-matrix was the realization that a symplectic variant of conformal field theories might be a further key element in the concrete construction of n-point functions and M-matrix in zero energy ontology. Although I have known super-symplectic (super-symplectic) symmetries to be fundamental symmetries of quantum TGD for almost two decades, I failed for some reason to realize the existence of symplectic QFT, and discovered it while trying to understand quite different problem - the fluctuations of cosmic microwave background! The symplectic contribution to the n-point function satisfies fusion rules and involves only variables which are symplectic invariants constructed using geodesic polygons assignable to the sub-polygons of n-polygon defined by the arguments of n-point function. Fusion rules lead to a concrete recursive formula for n-point functions and M-matrix in contrast to the iterative construction of n-point functions used in perturbative QFT.

2.4 Number theoretic compactification and $M^8 - H$ duality

This section summarizes the basic vision about number theoretic compactification reducing the classical dynamics to number theory. In strong form $M^8 - H$ duality boils down to the assumption that space-time surfaces can be regarded either as surfaces of H or as surfaces of M^8 composed of hyper-quaternionic and co-hyper-quaternionic regions identifiable as regions of space-time possessing Minkowskian *resp.* Euclidian signature of the induced metric.

2.4.1 Basic idea behind $M^8 - M^4 \times CP_2$ duality

The hopes of giving $M^4 \times CP_2$ hyper-octonionic structure are meager. This circumstance forces to ask whether four-surfaces $X^4 \subset M^8$ could under some conditions define 4-surfaces in $M^4 \times CP_2$ indirectly so that the spontaneous compactification of super string models would correspond in TGD to two different manners to interpret the space-time surface. The following arguments suggest that this is indeed the case.

The hard mathematical fact behind number theoretical compactification is that the quaternionic sub-algebras of octonions with fixed complex structure (that is complex sub-space) are parameterized by CP_2 just as the complex planes of quaternion space are parameterized by $CP_1 = S^2$. Same applies to hyper-quaternionic sub-spaces of hyper-octonions. $SU(3)$ would thus have an interpretation as the isometry group of CP_2, as the automorphism sub-group of octonions, and as color group.

1. The space of complex structures of the octonion space is parameterized by S^6. The subgroup $SU(3)$ of the full automorphism group G_2 respects the a priori selected complex structure and thus leaves invariant one octonionic imaginary unit, call it e_1. Hyper-quaternions can be identified as $U(2)$ Lie-algebra but it is obvious that hyper-octonions do not allow an identification as $SU(3)$ Lie algebra. Rather, octonions decompose as $1 \oplus 1 \oplus 3 \oplus \overline{3}$ to the irreducible representations of $SU(3)$.

2. Geometrically the choice of a preferred complex (quaternionic) structure means fixing of complex (quaternionic) sub-space of octonions. The fixing of a hyper-quaternionic structure of hyper-octonionic M^8 means a selection of a fixed hyper-quaternionic sub-space $M^4 \subset M^8$ implying the decomposition $M^8 = M^4 \times E^4$. If M^8 is identified as the tangent space of $H = M^4 \times CP_2$, this decomposition results naturally. It is also possible to select a fixed hyper-complex structure, which means a further decomposition $M^4 = M^2 \times E^2$.

3. The basic result behind number theoretic compactification and $M^8 - H$ duality is that hyper-quaternionic sub-spaces $M^4 \subset M^8$ containing a fixed hyper-complex sub-space $M^2 \subset M^4$ or its light-like line $M_\pm$ are parameterized by CP_2. The choices of a fixed hyper-quaternionic basis $1, e_1, e_2, e_3$ with a fixed complex sub-space (choice of e_1) are labeled by $U(2) \subset SU(3)$. The choice of e_2 and e_3 amounts to fixing $e_2 \pm \sqrt{-1}e_3$, which selects the $U(2) = SU(2) \times U(1)$ subgroup of $SU(3)$. $U(1)$ leaves 1 invariant and induced a phase multiplication of e_1 and $e_2 \pm e_3$. $SU(2)$ induces rotations of the spinor having e_2 and e_3 components. Hence all possible completions of $1, e_1$ by adding e_2, e_3 doublet are labeled by $SU(3)/U(2) = CP_2$.

4. Space-time surface $X^4 \subset M^8$ is by definition hyper-quaternionic if the tangent spaces of X^4 are hyper-quaternionic planes. Co-hyper-quaternionictity means the same for normal spaces. The presence of fixed hyper-complex structure means at space-time level that the tangent space of X^4 contains fixed M^2 at each point. Under this assumption one can map the points $(m, e) \in M^8$ to points $(m, s) \in H$ by assigning to the point (m, e) of X^4 the point (m, s), where $s \in CP_2$ characterize $T(X^4)$ as hyper-quaternionic plane.

5. The choice of M^2 can be made also local in the sense that one has $T(X^4) \supset M^2(x) \subset M^4 \subset H$. It turns out that strong form of number theoretic compactification requires this kind of generalization. In this case one must be able to fix the convention how the point of CP_2 is assigned to a hyper-quaternionic plane so that it applies to all possible choices of $M^2 \subset M^4$. Since $SO(3)$ hyper-quaternionic rotation relates the hyper-quaternionic planes to each other, the natural assumption is hyper-quaternionic planes related by $SO(3)$ rotation correspond to the same point of CP_2. Under this assumption it is possible to map hyper-quaternionic surfaces of M^8 for which $M^2 \subset M^4$ depends on point of X^4 to H.

2.4.2 Minimal form of $M^8 - H$ duality

The basic problem in the construction of quantum TGD has been the identification of the preferred extremals of Kähler action playing a key role in the definition of the theory. The most elegant manner to do this is by fixing the 4-D tangent space $T(X^4(X^3_l))$ of $X^4(X^3_l)$ at each point of X^3_l so that the boundary value problem is well defined. What I called number theoretical compactification allows to achieve just this although I did not fully realize this in the original vision. The minimal picture is following.

1. The basic observations are following. Let M^8 be endowed with hyper-octonionic structure. For hyper-quaternionic space-time surfaces in M^8 tangent spaces are by definition hyper-quaternionic. If they contain a preferred plane $M^2 \subset M^4 \subset M^8$ in their tangent space, they can be mapped to 4-surfaces in $M^4 \times CP_2$. The reason is that the hyper-quaternionic planes containing preferred the hyper-complex plane M^2 of $M_\pm \subset M^2$ are parameterized by points of CP_2. The map is simply $(m,e) \rightarrow (m, s(m,e))$, where m is point of M^4, e is point of E^4, and $s(m,2)$ is point of CP_2 representing the hyperquaternionic tangent plane. The inverse map assigns to each point (m,s) in $M^4 \times CP_2$ point m of M^4, undetermined point e of E^4 and 4-D plane. The requirement that the distribution of planes containing the preferred M^2 or $M_\pm$ corresponds to a distribution of planes for 4-D surface is expected to fix the points e. The physical interpretation of M^2 is in terms of plane of non-physical polarizations so that gauge conditions have purely number theoretical interpretation.

2. In principle, the condition that $T(X^4)$ contains M^2 can be replaced with a weaker condition that either of the two light-like vectors of M^2 is contained in it since already this condition assigns to $T(X^4)$ M^2 and the map $H \rightarrow M^8$ becomes possible. Only this weaker form applies in the case of massless extremals [D1] as will be found.

3. The original idea was that hyper-quaternionic 4-surfaces in M^8 containing $M^2 \subset M^4$ in their tangent space could correspond to preferred extremals of Kähler action. This condition does not seem to be consistent with what is known about the extremals of Kähler action. The weaker form of the hypothesis is that hyper-quaternionicity holds only for 4-D tangent spaces of $X^3_l \subset H = M^4 \times CP_2$ identified as wormhole throats or boundary components lifted to 3-surfaces in 8-D tangent space M^8 of H. The minimal hypothesis would be that only $T(X^4(X^3_l))$ at X^3_l is associative that is hyper-quaternionic for fixed M^2. $X^3_l \subset M^8$ and $T(X^4(X^3_l))$ at X^3_l can be mapped to $X^3_l \subset H$ if tangent space contains also $M_\pm \subset M^2$ or $M^2 \subset M^4 \subset M^8$ itself having interpretation as preferred hyper-complex plane. This condition is not satisfied by all surfaces X^3_l as is clear from the fact that the inverse map involves local E^4 translation. The requirements that the distribution of hyper-quaternionic planes containing M^2 corresponds to a distribution of 4-D tangent planes should fix the E^4 translation to a high degree.

4. A natural requirement is that the image of $X^3_l \subset H$ in M^8 is light-like. The condition that the determinant of induced metric vanishes gives an additional condition reducing the number of free parameters by one. This condition cannot be formulated as a condition on CP_2 coordinate characterizing the hyper-quaternionic tangent plane. Since M^4 projections are same for the two representations, this condition is satisfied if the contributions from CP_2 and E^4 and projections to the induced metric are identical: $s_{kl}\partial_\alpha s^k \partial_\beta s^l = e_{kl}\partial_\alpha e^k \partial_\beta e^l$. This condition means that only a subset of light-like surfaces of M^8 are realized physically. One might argue that this is as it must be since the volume of E^4 is infinite and that of CP_2 finite: only an infinitesimal portion of all possible light-like 3-surfaces in M^8 can can have H counterparts. The conclusion would be that number theoretical compactification is 4-D isometry between $X^4 \subset H$ and $X^4 \subset M^8$ at X^3_l. This unproven conjecture is unavoidable.

5. $M^2 \subset T(X^4(X^3_l))$ condition fixes $T(X^4(X^3_l))$ in the generic case by extending the tangent space of X^3_l, and the construction of configuration space spinor structure fixes boundary conditions completely by additional conditions necessary when X^3_l corresponds to a light-like 3 surfaces defining wormhole throat at which the signature of induced metric changes. What is especially beautiful that only the data in $T(X^4(X^3_l))$ at X^3_l is needed to calculate the vacuum functional of the theory as Dirac determinant: the only remaining conjecture (strictly speaking un-necessary

but realistic looking) is that this determinant gives exponent of Kähler action for the preferred extremal and there are excellent hopes for this by the structure of the basic construction.

The basic criticism relates to the condition that light-like 3-surfaces are mapped to light-like 3-surfaces guaranteed by the condition that $M^8 - H$ duality is isometry at X_l^3.

2.4.3 Strong form of $M^8 - H$ duality

The proposed picture is the minimal one. One can of course ask whether the original much stronger conjecture that the preferred extrema of Kähler action correspond to hyper-quaternionic surfaces could make sense in some form. One can also wonder whether one could allow the choice of the plane M^2 of non-physical polarization to be local so that one would have $M^2(x) \subset M^4 \subset M^4 \times E^4$, where M^4 is fixed hyper-quaternionic sub-space of M^8 and identifiable as M^4 factor of H.

1. If M^2 is same for all points of X_l^3, the inverse map $X_l^3 \subset H \to X_l^3 \subset M^8$ is fixed apart from possible non-uniquencess related to the local translation in E^4 from the condition that hyper-quaternionic planes represent light-like tangent 4-planes of light-like 3-surfaces. The question is whether not only X_l^3 but entire four-surface $X^4(X_l^3)$ could be mapped to the tangent space of M^8. By selecting suitably the local E^4 translation one might hope of achieving the achieving this. The conjecture would be that the preferred extrema of Kähler action are those for which the distribution integrates to a distribution of tangent planes.

2. There is however a problem. What is known about extremals of Kähler action is not consistent with the assumption that fixed M^2 of $M_\pm \subset M^2$is contained in the tangent space of X^4. This suggests that one should relax the condition that $M^2 \subset M^4 \subset M^8$ is a fixed hyper-complex plane associated with the tangent space or normal space X^4 and allow M^2 to vary from point to point so that one would have $M^2 = M^2(x)$. In $M^8 \to H$ direction the justification comes from the observation (to be discussed below) that it is possible to uniquely fix the convention assigning CP_2 point to a hyper-quaternionic plane containing varying hyper-complex plane $M^2(x) \subset M^4$.

   Number theoretic compactification fixes naturally $M^4 \subset M^8$ so that it applies to any $M^2(x) \subset M^4$. Under this condition the selection is parameterized by an element of $SO(3)/SO(2) = S^2$. Note that M^4 projection of X^4 would be at least 2-dimensional in hyper-quaternionic case. In co-hyper-quaternionic case E^4 projection would be at least 2-D. $SO(2)$ would act as a number theoretic gauge symmetry and the $SO(3)$ valued chiral field would approach to constant at X_l^3 invariant under global $SO(2)$ in the case that one keeps the assumption that M^2 is fixed ad X_l^3.

3. This picture requires a generalization of the map assigning to hyper-quaternionic plane a point of CP_2 so that this map is defined for all possible choices of $M^2 \subset M^4$. Since the $SO(3)$ rotation of the hyper-quaternionic unit defining M^2 rotates different choices parameterized by S^2 to each other, a natural assumption is that the hyper-quaternionic planes related by $SO(3)$ rotation correspond to the same point of CP_2. Denoting by M^2 the standard representative of M^2, this means that for the map $M^8 \to H$ one must perform $SO(3)$ rotation of hyper-quaternionic plane taking $M^2(x)$ to M^2 and map the rotated tangent plane to CP_2 point. In $M^8 \to H$ case one must first map the point of CP_2 to hyper-quaternionic plane and rotate this plane by a rotation taking $M^2(x)$ to M^2.

4. In this framework local M^2 can vary also at the surfaces X_l^3, which considerably relaxes the boundary conditions at wormhole throats and light-like boundaries and allows much more general variety of light-like 3-surfaces since the basic requirement is that M^4 projection is at least 1-dimensional. The physical interpretation would be that a local choice of the plane of non-physical polarizations is possible everywhere in $X^4(X_l^3)$. This does not seem to be in any obvious conflict with physical intuition.

These observation provide support for the conjecture that (classical) $S^2 = SO(3)/SO(2)$ conformal field theory might be relevant for (classical) TGD.

1. General coordinate invariance suggests that the theory should allow a formulation using any light-like 3-surface X^3 inside $X^4(X_l^3)$ besides X_l^3 identified as union of wormhole throats and boundary components. For these surfaces the element $g(x) \in SO(3)$ would vary also at partonic

2-surfaces X^2 defined as intersections of $\delta CD \times CP_2$ and X^3 (here CD denotes causal diamond defined as intersection of future and past directed light-cones). Hence one could have $S^2 = SO(3)/SO(2)$ conformal field theory at X^2 (regarded as quantum fluctuating so that also $g(x)$ varies) generalizing to WZW model for light-like surfaces X^3.

2. The presence of E^4 factor would extend this theory to a classical $E^4 \times S^2$ WZW model bringing in mind string model with 6-D Euclidian target space extended to a model of light-like 3-surfaces. A further extension to X^4 would be needed to integrate the WZW models associated with 3-surfaces to a full 4-D description. General Coordinate Invariance however suggests that X^3_l description is enough for practical purposes.

3. The choices of $M^2(x)$ in the interior of X^3_l is dictated by dynamics and the first optimistic conjecture is that a classical solution of $SO(3)/SO(2)$ Wess-Zumino-Witten model obtained by coupling $SO(3)$ valued field to a covariantly constant $SO(2)$ gauge potential characterizes the choice of $M^2(x)$ in the interior of $M^8 \supset X^4(X^3_l) \subset H$ and thus also partially the structure of the preferred extremal. Second optimistic conjecture is that the Kähler action involving also E^4 degrees of freedom allows to assign light-like 3-surface to light-like 3-surface.

4. The best that one can hope is that $M^8 - H$ duality could allow to transform the extremely non-linear classical dynamics of TGD to a generalization of WZW-type model. The basic problem is to understand how to characterize the dynamics of CP_2 projection at each point.

In H picture there are two basic types of vacuum extremals: CP_2 type extremals representing elementary particles and vacuum extremals having CP_2 projection which is at most 2-dimensional Lagrange manifold and representing say hadron. Vacuum extremals can appear only as limiting cases of preferred extremals which are non-vacuum extremals. Since vacuum extremals have so decisive role in TGD, it is natural to requires that this notion makes sense also in M^8 picture. In particular, the notion of vacuum extremal makes sense in M^8.

This requires that Kähler form exist in M^8. E^4 indeed allows full S^2 of covariantly constant Kähler forms representing quaternionic imaginary units so that one can identify Kähler form and construct Kähler action. The obvious conjecture is that hyper-quaternionic space-time surface is extremal of this Kähler action and that the values of Kähler actions in M^8 and H are identical. The elegant manner to achieve this, as well as the mapping of vacuum extremals to vacuum extremals and the mapping of light-like 3-surfaces to light-like 3-surfaces is to assume that $M^8 - H$ duality is Kähler isometry so that induced Kähler forms are identical.

This picture contains many speculative elements and some words of warning are in order.

1. Light-likeness conjecture would boil down to the hypothesis that $M^8 - H$ correspondence is Kähler isometry so that the metric and Kähler form of X^4 induced from M^8 and H would be identical. This would guarantee also that Kähler actions for the preferred extremal are identical. This conjecture is beautiful but strong.

2. The slicing of $X^4(X^3_l)$ by light-like 3-surfaces is very strong condition on the classical dynamics of Kähler action and does not make sense for pieces of CP_2 type vacuum extremals.

Minkowskian-Euclidian $\leftrightarrow$ associative–co-associative

The 8-dimensionality of M^8 allows to consider both associativity (hyper-quaternionicity) of the tangent space and associativity of the normal space- let us call this co-assosiativity of tangent space- as alternative options. Both options are needed as has been already found. Since space-time surface decomposes into regions whose induced metric possesses either Minkowskian or Euclidian signature, there is a strong temptation to propose that Minkowskian regions correspond to associative and Euclidian regions to co-associative regions so that space-time itself would provide both the description and its dual. The electric-magnetic duality encountered in the construction of the configuration space geometry (one can use either electric or magnetic Hamiltonians of configuration space) could be also equivalent with these two dualities.

The proposed interpretation of conjectured associative-co-associative duality relates in an interesting manner to p-adic length scale hypothesis selecting the primes $p \simeq 2^k$, k positive integer as preferred p-adic length scales. $L_p \propto \sqrt{p}$ corresponds to the p-adic length scale defining the size of the space-time sheet at which elementary particle represented as CP_2 type extremal is topologically condensed and is of order Compton length. $L_k \propto \sqrt{k}$ represents the p-adic length scale of the wormhole contacts associated with the CP_2 type extremal and CP_2 size is the natural length unit now. Obviously the quantitative formulation for associative-co-associative duality would be in terms $p \to k$ duality.

Are the known extremals of Kähler action consistent with the strong form of $M^8 - H$ duality

It is interesting to check whether the known extremals of Kähler action [D1] are consistent with strong form of $M^8 - H$ duality assuming that M^2 or its light-like ray is contained in $T(X^4)$ or normal space.

1. CP_2 type vacuum extremals correspond cannot be hyper-quaternionic surfaces but co-hyper-quaternionicity is natural for them. In the same manner canonically imbedded M^4 can be only hyper-quaternionic.

2. String like objects are associative since tangent space obviously contains $M^2(x)$. Objects of form $M^1 \times X^3 \subset M^4 \times CP_2$ do not have M^2 either in their tangent space or normal space in H. So that the map from $H \to M^8$ is not well defined. There are no known extremals of Kähler action of this type. The replacement of M^1 random light-like curve however gives vacuum extremal with vanishing volume, which need not mean physical triviality since fundamental objects of the theory are light-like 3-surfaces.

3. For canonically imbedded CP_2 the assignment of $M^2(x)$ to normal space is possible but the choice of $M^2(x) \subset N(CP_2)$ is completely arbitrary. For a generic CP_2 type vacuum extremals M^4 projection is a random light-like curve in $M^4 = M^1 \times E^3$ and $M^2(x)$ can be defined uniquely by the normal vector $n \in E^3$ for the local plane defined by the tangent vector dx^μ/dt and acceleration vector d^2x^μ/dt^2 assignable to the orbit.

4. Consider next massless extremals. Let us fix the coordinates of X^4 as $(t, z, x, y) = (m^0, m^2, m^1, m^2)$. For simplest massless extremals CP_2 coordinates are arbitrary functions of variables $u = k \cdot m = t - z$ and $v = \epsilon \cdot m = x$, where $k = (1, 1, 0, 0)$ is light-like vector of M^4 and $\epsilon = (0, 0, 1, 0)$ a polarization vector orthogonal to it. Obviously, the extremals defines a decomposition $M^4 = M^2 \times E^2$. Tangent space is spanned by the four H-vectors $\nabla_\alpha h^k$ with M^4 part given by $\nabla_\alpha m^k = \delta^k_\alpha$ and CP_2 part by $\nabla_\alpha s^k = \partial_u s^k k_\alpha + \partial_v s^k \epsilon_\alpha$.

   The normal space cannot contain M^4 vectors since the M^4 projection of the extremal is M^4. To realize hyper-quaternionic representation one should be able to from these vector two vectors of M^2, which means linear combinations of tangent vectors for which CP_2 part vanishes. The vector $\partial_t h^k - \partial_z h^k$ has vanishing CP_2 part and corresponds to M^4 vector $(1, -1, 0, 0)$ fix assigns to each point the plane M^2. To obtain M^2 one would need $(1, 1, 0, 0)$ too but this is not possible. The vector $\partial_y h^k$ is M^4 vector orthogonal to ϵ but M^2 would require also $(1, 0, 0, 0)$. The proposed generalization of massless extremals allows the light-like line $M_\pm$ to depend on point of M^4 [D1], and leads to the introduction of Hamilton-Jacobi coordinates involving a local decomposition of M^4 to $M^2(x)$ and its orthogonal complement with light-like coordinate lines having interpretation as curved light rays. $M^2(x) \subset T(X^4)$ assumption fails fails also for vacuum extremals of form $X^1 \times X^3 \subset M^4 \times CP_2$, where X^1 is light-like random curve. In the latter case, vacuum property follows from the vanishing of the determinant of the induced metric.

5. The deformations of string like objects to magnetic flux quanta are basic conjectural extremals of Kähler action and the proposed picture supports this conjecture. In hyper-quaternionic case the assumption that local 4-D tangent plane of X^3 contains $M^2(x)$ but that $T(X^3)$ does not contain it, is very strong. It states that $T(X^4)$ at each point can be regarded as a product $M^2(x) \times T^2$, $T^2 \subset T(CP_2)$, so that hyper-quaternionic X^4 would be a collection of Cartesian products of infinitesimal 2-D planes $M^2(x) \subset M^4$ and $T^2(x) \subset CP_2$. The extremals in question could be seen as local variants of string like objects $X^2 \times Y^2 \subset M^4 \times CP_2$, where X^2 is minimal surface and Y^2 holomorphic surface of CP_2. One can say that X^2 is replaced by a collection of infinitesimal pieces of $M^2(x)$ and Y^2 with similar pieces of homologically non-trivial geodesic

sphere $S^2(x)$ of CP_2, and the Cartesian products of these pieces are glued together to form a continuous surface defining an extremal of Kähler action. Field equations would pose conditions on how $M^2(x)$ and $S^2(x)$ can depend on x. This description applies to magnetic flux quanta, which are the most important must-be extremals of Kähler action.

Geometric interpretation of strong $M^8 - H$ duality

In the proposed framework $M^8 - H$ duality would have a purely geometric meaning and there would nothing magical in it.

1. $X^4(X_l^3) \subset H$ could be seen a curve representing the orbit of a light-like 3-surface defining a 4-D surface. The question is how to determine the notion of tangent vector for the orbit of X_l^3. Intuitively tangent vector is a one-dimensional arrow tangential to the curve at point X_l^3. The identification of the hyper-quaternionic surface $X^4(X_l^3) \subset M^8$ as tangent vector conforms with this intuition.

2. One could argue that M^8 representation of space-time surface is kind of chart of the real space-time surface obtained by replacing real curve by its tangent line. If so, one cannot avoid the question under which conditions this kind of chart is faithful. An alternative interpretation is that a representation making possible to realize number theoretical universality is in question.

3. An interesting question is whether $X^4(X_l^3)$ as orbit of light-like 3-surface is analogous to a geodesic line -possibly light-like- so that its tangent vector would be parallel translated in the sense that $X^4(X^3)$ for any light-like surface at the orbit is same as $X^4(X_l^3)$. This would give justification for the possibility to interpret space-time surfaces as a geodesic of configuration space: this is one of the first -and practically forgotten- speculations inspired by the construction of configuration space geometry. The light-likeness of the geodesic could correspond at the level of X^4 the possibility to decompose the tangent space to a direct sum of two light-like spaces and 2-D transversal space producing the foliation of X^4 to light-like 3-surfaces X_l^3 along light-like curves.

4. M^8-H duality would assign to X_l^3 classical orbit and its tangent vector at X_l^3 as a generalization of Bohr orbit. This picture differs from the wave particle duality of wave mechanics stating that once the position of particle is known its momentum is completely unknown. The outcome is however the same: for X_l^3 corresponding to wormhole throats and light-like boundaries of X^4, canonical momentum densities in the normal direction vanish identically by conservation laws and one can say that the the analog of (q,p) phase space as the space carrying wave functions is replaced with the analog of subspace consisting of points $(q,0)$. The dual description in M^8 would not be analogous to wave functions in momentum space space but to those in the space of unique tangents of curves at their initial points.

The Kähler and spinor structures of M^8

If one introduces M^8 as dual of H, one cannot avoid the idea that hyper-quaternionic surfaces obtained as images of the preferred extremals of Kähler action in H are also extremals of M^8 Kähler action with same value of Kähler action. As found, this leads to the conclusion that the$M^8 - H$ duality is Kähler isometry. Coupling of spinors to Kähler potential is the next step and this in turn leads to the introduction of spinor structure so that quantum TGD in H should have full M^8 dual.

There are strong physical constraints on M^8 dual and they could kill the hypothesis. The basic constraint to the spinor structure of M^8 is that it reproduces basic facts about electro-weak interactions. This includes neutral electro-weak couplings to quarks and leptons identified as different H-chiralities and parity breaking.

1. By the flatness of the metric of E^4 its spinor connection is trivial. E^4 however allows full S^2 of covariantly constant Kähler forms so that one can accommodate free independent Abelian gauge fields assuming that the independent gauge fields are orthogonal to each other when interpreted as realizations of quaternionic imaginary units.

2. One should be able to distinguish between quarks and leptons also in M^8, which suggests that one introduce spinor structure and Kähler structure in E^4. The Kähler structure of E^4 is unique apart form $SO(3)$ rotation since all three quaternionic imaginary units and the unit vectors formed from them allow a representation as an antisymmetric tensor. Hence one must select one preferred Kähler structure, that is fix a point of S^2 representing the selected imaginary unit. It is natural to assume different couplings of the Kähler gauge potential to spinor chiralities representing quarks and leptons: these couplings can be assumed to be same as in case of H.

3. Electro-weak gauge potential has vectorial and axial parts. Em part is vectorial involving coupling to Kähler form and Z^0 contains both axial and vector parts. The free Kähler forms could thus allow to produce M^8 counterparts of these gauge potentials possessing same couplings as their H counterparts. This picture would produce parity breaking in M^8 picture correctly.

4. Only the charged parts of classical electro-weak gauge fields would be absent. This would conform with the standard thinking that charged classical fields are not important. The predicted classical W fields is one of the basic distinctions between TGD and standard model and in this framework. A further prediction is that this distinction becomes visible only in situations, where H picture is necessary. This is the case at high energies, where the description of quarks in terms of $SU(3)$ color is convenient whereas $SO(4)$ QCD would require large number of E^4 partial waves. At low energies large number of $SU(3)$ color partial waves are needed and the convenient description would be in terms of $SO(4)$ QCD. Proton spin crisis might relate to this.

5. Also super-symmetries of quantum TGD crucial for the construction of configuration space geometry force this picture. In the absence of coupling to Kähler gauge potential all constant spinor fields and their conjugates would generate super-symmetries so that M^8 would allow $N = 8$ super-symmetry. The introduction of the coupling to Kähler gauge potential in turn means that all covariantly constant spinor fields are lost. Only the representation of all three neutral parts of electro-weak gauge potentials in terms of three independent Kähler gauge potentials allows right-handed neutrino as the only super-symmetry generator as in the case of H.

6. The $SO(3)$ element characterizing $M^2(x)$ is fixed apart from a local $SO(2)$ transformation, which suggests an additional $U(1)$ gauge field associated with $SO(2)$ gauge invariance and representable as Kähler form corresponding to a quaternionic unit of E^4. A possible identification of this gauge field would be as a part of electro-weak gauge field.

M^8 dual of configuration space geometry and spinor structure?

If one introduces M^8 spinor structure and preferred extremals of M^8 Kähler action, one cannot avoid the question whether it is possible or useful to formulate the notion of configuration space geometry and spinor structure for light-like 3-surfaces in M^8 using the exponent of Kähler action as vacuum functional.

1. The isometries of the configuration space in M^8 and H formulations would correspond to symplectic transformation of $\delta M^4_\pm \times E^4$ and $\delta M^4_\pm \times CP_2$ and the Hamiltonians involved would belong to the representations of $SO(4)$ and $SU(3)$ with 2-dimensional Cartan sub-algebras. In H picture color group would be the familiar $SU(3)$ but in M^8 picture it would be $SO(4)$. Color confinement in both $SU(3)$ and $SO(4)$ sense could allow these two pictures without any inconsistency.

2. For $M^4 \times CP_2$ the two spin states of covariantly constant right handed neutrino and antineutrino spinors generate super-symmetries. This super-symmetry plays an important role in the proposed construction of configuration space geometry. As found, this symmetry would be present also in M^8 formulation so that the construction of M^8 geometry should reduce more or less to the replacement of CP_2 Hamiltonians in representations of $SU(3)$ with E^4 Hamiltonians in representations of $SO(4)$. These Hamiltonians can be taken to be proportional to functions of E^4 radius which is $SO(4)$ invariant and these functions bring in additional degree of freedom.

3. The construction of Dirac determinant identified as a vacuum functional can be done also in M^8 picture and the conjecture is that the result is same as in the case of H. In this framework the

construction is much simpler due to the flatness of E^4. In particular, the generalized eigen modes of the Chern-Simons Dirac operator D_{C-S} identified as zero modes of 4-D Dirac operator D_K restricted to the X^3_l correspond to a situation in which one has fermion in induced Maxwell field mimicking the neutral part of electro-weak gauge field in H as far as couplings are considered. Induced Kähler field would be same as in H. Eigen modes are localized to regions inside which the Kähler magnetic field is non-vanishing and apart from the fact that the metric is the effective metric defined in terms of canonical momentum densities via the formula $\hat{\Gamma}^\alpha = \partial L_K/\partial h^k_\alpha \Gamma_k$ for effective gamma matrices. This in fact, forces the localization of modes implying that their number is finite so that Dirac determinant is a product over finite number eigenvalues. It is clear that M^8 picture could dramatically simplify the construction of configuration space geometry.

4. The eigenvalue spectra of the transversal parts of D_K operators in M^8 and H should identical. This motivates the question whether it is possible to achieve a complete correspondence between H and M^8 pictures also at the level of spinor fields at X^3 by performing a gauge transformation eliminating the classical W gauge boson field altogether at X^3_l and whether this allows to transform the modified Dirac equation in H to that in M^8 when restricted to X^3_l. That something like this might be achieved is supported by the fact that in Coulombic gauge the component of gauge potential in the light-like direction vanishes so that the situation is effectively 2-dimensional and holonomy group is Abelian.

Why $M^8 - H$ duality is useful?

Skeptic could of course argue that $M^8 - H$ duality produces only an inflation of unproven conjectures. There are however strong reasons for $M^8 - H$ duality: both theoretical and physical.

1. The map of $X^3_l \subset H \rightarrow X^3_l \subset M^8$ and corresponding map of space-time surfaces would allow to realize number theoretical universality. $M^8 = M^4 \times E^4$ allows linear coordinates as natural coordinates in which one can say what it means that the point of imbedding space is rational/algebraic. The point of $X^4 \subset H$ is algebraic if it is mapped to an algebraic point of M^8 in number theoretic compactification. This of course restricts the symmetry groups to their rational/algebraic variants but this does not have practical meaning. Number theoretical compactication could in fact be motivated by the number theoretical universality.

2. $M^8 - H$ duality could provide much simpler description of preferred extremals of Kähler action since the Kähler form in E^4 has constant components. If the spinor connection in E^4 is combination of the three Kähler forms mimicking neutral part of electro-weak gauge potential, the eigenvalue spectrum for the modified Dirac operator would correspond to that for a fermion in $U(1)$ magnetic field defined by an Abelian magnetic field whereas in $M^4 \times CP_2$ picture $U(2)_{ew}$ magnetic fields would be present.

3. $M^8 - H$ duality provides insights to low energy hadron physics. M^8 description might work when H-description fails. For instance, perturbative QCD which corresponds to H-description fails at low energies whereas M^8 description might become perturbative description at this limit. Strong $SO(4) = SU(2)_L \times SU(2)_R$ invariance is the basic symmetry of the phenomenological low energy hadron models based on conserved vector current hypothesis (CVC) and partially conserved axial current hypothesis (PCAC). Strong $SO(4) = SU(2)_L \times SU(2)_R$ relates closely also to electro-weak gauge group $SU(2)_L \times U(1)$ and this connection is not well understood in QCD description. $M^8 - H$ duality could provide this connection. Strong $SO(4)$ symmetry would emerge as a low energy dual of the color symmetry. Orbital $SO(4)$ would correspond to strong $SU(2)_L \times SU(2)_R$ and by flatness of E^4 spin like $SO(4)$ would correspond to electro-weak group $SU(2)_L \times U(1)_R \subset SO(4)$. Note that the inclusion of coupling to Kähler gauge potential is necessary to achieve respectable spinor structure in CP_2. One could say that the orbital angular momentum in $SO(4)$ corresponds to strong isospin and spin part of angular momentum to the weak isospin.

2.4.4 $M^8 - H$ duality and low energy hadron physics

The description of $M^8 - H$ at the configuration space level can be applied to gain a view about color confinement and its dual for electro-weak interactions at short distance limit. The basic idea is that

$SO(4)$ and $SU(3)$ provide provide dual descriptions of quark color using E^4 and CP_2 partial waves and low energy hadron physics corresponds to a situation in which M^8 picture provides the perturbative approach whereas H picture works at high energies. The basic prediction is that $SO(4)$ should appear as dynamical symmetry group of low energy hadron physics and this is indeed the case.

Consider color confinement at the long length scale limit in terms of $M^8 - H$ duality.

1. At high energy limit only lowest color triplet color partial waves for quarks dominate so that QCD description becomes appropriate whereas very higher color partial waves for quarks and gluons are expected to appear at the confinement limit. Since configuration space degrees of freedom begin to dominate, color confinement limit transcends the descriptive power of QCD.

2. The success of $SO(4)$ sigma model in the description of low lying hadrons would directly relate to the fact that this group labels also the E^4 Hamiltonians in M^8 picture. Strong $SO(4)$ quantum numbers can be identified as orbital counterparts of right and left handed electro-weak isospin coinciding with strong isospin for lowest quarks. In sigma model pion and sigma boson form the components of E^4 valued vector field or equivalently collection of four E^4 Hamiltonians corresponding to spherical E^4 coordinates. Pion corresponds to S^3 valued unit vector field with charge states of pion identifiable as three Hamiltonians defined by the coordinate components. Sigma is mapped to the Hamiltonian defined by the E^4 radial coordinate. Excited mesons corresponding to more complex Hamiltonians are predicted.

3. The generalization of sigma model would assign to quarks E^4 partial waves belonging to the representations of $SO(4)$. The model would involve also 6 $SO(4)$ gluons and their $SO(4)$ partial waves. At the low energy limit only lowest representations would be be important whereas at higher energies higher partial waves would be excited and the description based on CP_2 partial waves would become more appropriate.

4. The low energy quark model would rely on quarks moving $SO(4)$ color partial waves. Left *resp.* right handed quarks could correspond to $SU(2)_L$ *resp.* $SU(2)_R$ triplets so that spin statistics problem would be solved in the same manner as in the standard quark model.

5. Family replication phenomenon is described in TGD framework the same manner in both cases so that quantum numbers like strangeness and charm are not fundamental. Indeed, p-adic mass calculations allowing fractally scaled up versions of various quarks allow to replace Gell-Mann mass formula with highly successful predictions for hadron masses [F4].

To my opinion these observations are intriguing enough to motivate a concrete attempt to construct low energy hadron physics in terms of $SO(4)$ gauge theory.

2.5 Number theoretic compactification and $M^8 - H$ duality

This section summarizes the basic vision about number theoretic compactification reducing the classical dynamics to number theory. In strong form $M^8 - H$ duality boils down to the assumption that space-time surfaces can be regarded either as surfaces of H or as surfaces of M^8 composed of hyper-quaternionic and co-hyper-quaternionic regions identifiable as regions of space-time possessing Minkowskian *resp.* Euclidian signature of the induced metric.

2.5.1 Basic idea behind $M^8 - M^4 \times CP_2$ duality

The hopes of giving $M^4 \times CP_2$ hyper-octonionic structure are meager. This circumstance forces to ask whether four-surfaces $X^4 \subset M^8$ could under some conditions define 4-surfaces in $M^4 \times CP_2$ indirectly so that the spontaneous compactification of super string models would correspond in TGD to two different manners to interpret the space-time surface. The following arguments suggest that this is indeed the case.

The hard mathematical fact behind number theoretical compactification is that the quaternionic sub-algebras of octonions with fixed complex structure (that is complex sub-space) are parameterized by CP_2 just as the complex planes of quaternion space are parameterized by $CP_1 = S^2$. Same applies to hyper-quaternionic sub-spaces of hyper-octonions. $SU(3)$ would thus have an interpretation as the isometry group of CP_2, as the automorphism sub-group of octonions, and as color group.

1. The space of complex structures of the octonion space is parameterized by S^6. The subgroup $SU(3)$ of the full automorphism group G_2 respects the a priori selected complex structure and thus leaves invariant one octonionic imaginary unit, call it e_1. Hyper-quaternions can be identified as $U(2)$ Lie-algebra but it is obvious that hyper-octonions do not allow an identification as $SU(3)$ Lie algebra. Rather, octonions decompose as $1 \oplus 1 \oplus 3 \oplus \overline{3}$ to the irreducible representations of $SU(3)$.

2. Geometrically the choice of a preferred complex (quaternionic) structure means fixing of complex (quaternionic) sub-space of octonions. The fixing of a hyper-quaternionic structure of hyper-octonionic M^8 means a selection of a fixed hyper-quaternionic sub-space $M^4 \subset M^8$ implying the decomposition $M^8 = M^4 \times E^4$. If M^8 is identified as the tangent space of $H = M^4 \times CP_2$, this decomposition results naturally. It is also possible to select a fixed hyper-complex structure, which means a further decomposition $M^4 = M^2 \times E^2$.

3. The basic result behind number theoretic compactification and $M^8 - H$ duality is that hyper-quaternionic sub-spaces $M^4 \subset M^8$ containing a fixed hyper-complex sub-space $M^2 \subset M^4$ or its light-like line $M_\pm$ are parameterized by CP_2. The choices of a fixed hyper-quaternionic basis $1, e_1, e_2, e_3$ with a fixed complex sub-space (choice of e_1) are labeled by $U(2) \subset SU(3)$. The choice of e_2 and e_3 amounts to fixing $e_2 \pm \sqrt{-1}e_3$, which selects the $U(2) = SU(2) \times U(1)$ subgroup of $SU(3)$. $U(1)$ leaves 1 invariant and induced a phase multiplication of e_1 and $e_2 \pm e_3$. $SU(2)$ induces rotations of the spinor having e_2 and e_3 components. Hence all possible completions of $1, e_1$ by adding e_2, e_3 doublet are labeled by $SU(3)/U(2) = CP_2$.

4. Space-time surface $X^4 \subset M^8$ is by definition hyper-quaternionic if the tangent spaces of X^4 are hyper-quaternionic planes. Co-hyper-quaternionictity means the same for normal spaces. The presence of fixed hyper-complex structure means at space-time level that the tangent space of X^4 contains fixed M^2 at each point. Under this assumption one can map the points $(m, e) \in M^8$ to points $(m, s) \in H$ by assigning to the point (m, e) of X^4 the point (m, s), where $s \in CP_2$ characterize $T(X^4)$ as hyper-quaternionic plane.

5. The choice of M^2 can be made also local in the sense that one has $T(X^4) \supset M^2(x) \subset M^4 \subset H$. It turns out that strong form of number theoretic compactification requires this kind of generalization. In this case one must be able to fix the convention how the point of CP_2 is assigned to a hyper-quaternionic plane so that it applies to all possible choices of $M^2 \subset M^4$. Since $SO(3)$ hyper-quaternionic rotation relates the hyper-quaternionic planes to each other, the natural assumption is hyper-quaternionic planes related by $SO(3)$ rotation correspond to the same point of CP_2. Under this assumption it is possible to map hyper-quaternionic surfaces of M^8 for which $M^2 \subset M^4$ depends on point of X^4 to H.

2.5.2 Minimal form of $M^8 - H$ duality

The basic problem in the construction of quantum TGD has been the identification of the preferred extremals of Kähler action playing a key role in the definition of the theory. The most elegant manner to do this is by fixing the 4-D tangent space $T(X^4(X^3_l))$ of $X^4(X^3_l)$ at each point of X^3_l so that the boundary value problem is well defined. What I called number theoretical compactification allows to achieve just this although I did not fully realize this in the original vision. The minimal picture is following.

1. The basic observations are following. Let M^8 be endowed with hyper-octonionic structure. For hyper-quaternionic space-time surfaces in M^8 tangent spaces are by definition hyper-quaternionic. If they contain a preferred plane $M^2 \subset M^4 \subset M^8$ in their tangent space, they can be mapped to 4-surfaces in $M^4 \times CP_2$. The reason is that the hyper-quaternionic planes containing preferred the hyper-complex plane M^2 of $M_\pm \subset M^2$ are parameterized by points of CP_2. The map is simply $(m, e) \rightarrow (m, s(m, e))$, where m is point of M^4, e is point of E^4, and $s(m, 2)$ is point of CP_2 representing the hyperquaternionic tangent plane. The inverse map assigns to each point (m, s) in $M^4 \times CP_2$ point m of M^4, undetermined point e of E^4 and 4-D plane. The requirement that the distribution of planes containing the preferred M^2 or $M_\pm$ corresponds to a distribution of planes for 4-D surface is expected to fix the points e. The physical interpretation of M^2 is

in terms of plane of non-physical polarizations so that gauge conditions have purely number theoretical interpretation.

2. In principle, the condition that $T(X^4)$ contains M^2 can be replaced with a weaker condition that either of the two light-like vectors of M^2 is contained in it since already this condition assigns to $T(X^4)$ M^2 and the map $H \rightarrow M^8$ becomes possible. Only this weaker form applies in the case of massless extremals [D1] as will be found.

3. The original idea was that hyper-quaternionic 4-surfaces in M^8 containing $M^2 \subset M^4$ in their tangent space could correspond to preferred extremals of Kähler action. This condition does not seem to be consistent with what is known about the extremals of Kähler action. The weaker form of the hypothesis is that hyper-quaternionicity holds only for 4-D tangent spaces of $X^3_l \subset H = M^4 \times CP_2$ identified as wormhole throats or boundary components lifted to 3-surfaces in 8-D tangent space M^8 of H. The minimal hypothesis would be that only $T(X^4(X^3_l))$ at X^3_l is associative that is hyper-quaternionic for fixed M^2. $X^3_l \subset M^8$ and $T(X^4(X^3_l))$ at X^3_l can be mapped to $X^3_l \subset H$ if tangent space contains also $M_\pm \subset M^2$ or $M^2 \subset M^4 \subset M^8$ itself having interpretation as preferred hyper-complex plane. This condition is not satisfied by all surfaces X^3_l as is clear from the fact that the inverse map involves local E^4 translation. The requirements that the distribution of hyper-quaternionic planes containing M^2 corresponds to a distribution of 4-D tangent planes should fix the E^4 translation to a high degree.

4. A natural requirement is that the image of $X^3_l \subset H$ in M^8 is light-like. The condition that the determinant of induced metric vanishes gives an additional condition reducing the number of free parameters by one. This condition cannot be formulated as a condition on CP_2 coordinate characterizing the hyper-quaternionic tangent plane. Since M^4 projections are same for the two representations, this condition is satisfied if the contributions from CP_2 and E^4 and projections to the induced metric are identical: $s_{kl}\partial_\alpha s^k \partial_\beta s^l = e_{kl}\partial_\alpha e^k \partial_\beta e^l$. This condition means that only a subset of light-like surfaces of M^8 are realized physically. One might argue that this is as it must be since the volume of E^4 is infinite and that of CP_2 finite: only an infinitesimal portion of all possible light-like 3-surfaces in M^8 can can have H counterparts. The conclusion would be that number theoretical compactification is 4-D isometry between $X^4 \subset H$ and $X^4 \subset M^8$ at X^3_l. This unproven conjecture is unavoidable.

5. $M^2 \subset T(X^4(X^3_l))$ condition fixes $T(X^4(X^3_l))$ in the generic case by extending the tangent space of X^3_l, and the construction of configuration space spinor structure fixes boundary conditions completely by additional conditions necessary when X^3_l corresponds to a light-like 3 surfaces defining wormhole throat at which the signature of induced metric changes. What is especially beautiful that only the data in $T(X^4(X^3_l))$ at X^3_l is needed to calculate the vacuum functional of the theory as Dirac determinant: the only remaining conjecture (strictly speaking un-necessary but realistic looking) is that this determinant gives exponent of Kähler action for the preferred extremal and there are excellent hopes for this by the structure of the basic construction.

The basic criticism relates to the condition that light-like 3-surfaces are mapped to light-like 3-surfaces guaranteed by the condition that $M^8 - H$ duality is isometry at X^3_l.

2.5.3 Strong form of $M^8 - H$ duality

The proposed picture is the minimal one. One can of course ask whether the original much stronger conjecture that the preferred extrema of Kähler action correspond to hyper-quaternionic surfaces could make sense in some form. One can also wonder whether one could allow the choice of the plane M^2 of non-physical polarization to be local so that one would have $M^2(x) \subset M^4 \subset M^4 \times E^4$, where M^4 is fixed hyper-quaternionic sub-space of M^8 and identifiable as M^4 factor of H.

1. If M^2 is same for all points of X^3_l, the inverse map $X^3_l \subset H \rightarrow X^3_l \subset M^8$ is fixed apart from possible non-uniquencess related to the local translation in E^4 from the condition that hyper-quaternionic planes represent light-like tangent 4-planes of light-like 3-surfaces. The question is whether not only X^3_l but entire four-surface $X^4(X^3_l)$ could be mapped to the tangent space of M^8. By selecting suitably the local E^4 translation one might hope of achieving the achieving this. The conjecture would be that the preferred extrema of Kähler action are those for which the distribution integrates to a distribution of tangent planes.

2. There is however a problem. What is known about extremals of Kähler action is not consistent with the assumption that fixed M^2 of $M_\pm \subset M^2$is contained in the tangent space of X^4. This suggests that one should relax the condition that $M^2 \subset M^4 \subset M^8$ is a fixed hyper-complex plane associated with the tangent space or normal space X^4 and allow M^2 to vary from point to point so that one would have $M^2 = M^2(x)$. In $M^8 \to H$ direction the justification comes from the observation (to be discussed below) that it is possible to uniquely fix the convention assigning CP_2 point to a hyper-quaternionic plane containing varying hyper-complex plane $M^2(x) \subset M^4$.

   Number theoretic compactification fixes naturally $M^4 \subset M^8$ so that it applies to any $M^2(x) \subset M^4$. Under this condition the selection is parameterized by an element of $SO(3)/SO(2) = S^2$. Note that M^4 projection of X^4 would be at least 2-dimensional in hyper-quaternionic case. In co-hyper-quaternionic case E^4 projection would be at least 2-D. $SO(2)$ would act as a number theoretic gauge symmetry and the $SO(3)$ valued chiral field would approach to constant at X^3_l invariant under global $SO(2)$ in the case that one keeps the assumption that M^2 is fixed ad X^3_l.

3. This picture requires a generalization of the map assigning to hyper-quaternionic plane a point of CP_2 so that this map is defined for all possible choices of $M^2 \subset M^4$. Since the $SO(3)$ rotation of the hyper-quaternionic unit defining M^2 rotates different choices parameterized by S^2 to each other, a natural assumption is that the hyper-quaternionic planes related by $SO(3)$ rotation correspond to the same point of CP_2. Denoting by M^2 the standard representative of M^2, this means that for the map $M^8 \to H$ one must perform $SO(3)$ rotation of hyper-quaternionic plane taking $M^2(x)$ to M^2 and map the rotated tangent plane to CP_2 point. In $M^8 \to H$ case one must first map the point of CP_2 to hyper-quaternionic plane and rotate this plane by a rotation taking $M^2(x)$ to M^2.

4. In this framework local M^2 can vary also at the surfaces X^3_l, which considerably relaxes the boundary conditions at wormhole throats and light-like boundaries and allows much more general variety of light-like 3-surfaces since the basic requirement is that M^4 projection is at least 1-dimensional. The physical interpretation would be that a local choice of the plane of non-physical polarizations is possible everywhere in $X^4(X^3_l)$. This does not seem to be in any obvious conflict with physical intuition.

These observation provide support for the conjecture that (classical) $S^2 = SO(3)/SO(2)$ conformal field theory might be relevant for (classical) TGD.

1. General coordinate invariance suggests that the theory should allow a formulation using any light-like 3-surface X^3 inside $X^4(X^3_l)$ besides X^3_l identified as union of wormhole throats and boundary components. For these surfaces the element $g(x) \in SO(3)$ would vary also at partonic 2-surfaces X^2 defined as intersections of $\delta CD \times CP_2$ and X^3 (here CD denotes causal diamond defined as intersection of future and past directed light-cones). Hence one could have $S^2 = SO(3)/SO(2)$ conformal field theory at X^2 (regarded as quantum fluctuating so that also $g(x)$ varies) generalizing to WZW model for light-like surfaces X^3.

2. The presence of E^4 factor would extend this theory to a classical $E^4 \times S^2$ WZW model bringing in mind string model with 6-D Euclidian target space extended to a model of light-like 3-surfaces. A further extension to X^4 would be needed to integrate the WZW models associated with 3-surfaces to a full 4-D description. General Coordinate Invariance however suggests that X^3_l description is enough for practical purposes.

3. The choices of $M^2(x)$ in the interior of X^3_l is dictated by dynamics and the first optimistic conjecture is that a classical solution of $SO(3)/SO(2)$ Wess-Zumino-Witten model obtained by coupling $SO(3)$ valued field to a covariantly constant $SO(2)$ gauge potential characterizes the choice of $M^2(x)$ in the interior of $M^8 \supset X^4(X^3_l) \subset H$ and thus also partially the structure of the preferred extremal. Second optimistic conjecture is that the Kähler action involving also E^4 degrees of freedom allows to assign light-like 3-surface to light-like 3-surface.

4. The best that one can hope is that $M^8 - H$ duality could allow to transform the extremely non-linear classical dynamics of TGD to a generalization of WZW-type model. The basic problem is to understand how to characterize the dynamics of CP_2 projection at each point.

In H picture there are two basic types of vacuum extremals: CP_2 type extremals representing elementary particles and vacuum extremals having CP_2 projection which is at most 2-dimensional Lagrange manifold and representing say hadron. Vacuum extremals can appear only as limiting cases of preferred extremals which are non-vacuum extremals. Since vacuum extremals have so decisive role in TGD, it is natural to requires that this notion makes sense also in M^8 picture. In particular, the notion of vacuum extremal makes sense in M^8.

This requires that Kähler form exist in M^8. E^4 indeed allows full S^2 of covariantly constant Kähler forms representing quaternionic imaginary units so that one can identify Kähler form and construct Kähler action. The obvious conjecture is that hyper-quaternionic space-time surface is extremal of this Kähler action and that the values of Kähler actions in M^8 and H are identical. The elegant manner to achieve this, as well as the mapping of vacuum extremals to vacuum extremals and the mapping of light-like 3-surfaces to light-like 3-surfaces is to assume that $M^8 - H$ duality is Kähler isometry so that induced Kähler forms are identical.

This picture contains many speculative elements and some words of warning are in order.

1. Light-likeness conjecture would boil down to the hypothesis that $M^8 - H$ correspondence is Kähler isometry so that the metric and Kähler form of X^4 induced from M^8 and H would be identical. This would guarantee also that Kähler actions for the preferred extremal are identical. This conjecture is beautiful but strong.

2. The slicing of $X^4(X_l^3)$ by light-like 3-surfaces is very strong condition on the classical dynamics of Kähler action and does not make sense for pieces of CP_2 type vacuum extremals.

Minkowskian-Euclidian $\leftrightarrow$ associative–co-associative

The 8-dimensionality of M^8 allows to consider both associativity (hyper-quaternionicity) of the tangent space and associativity of the normal space- let us call this co-assosiativity of tangent space- as alternative options. Both options are needed as has been already found. Since space-time surface decomposes into regions whose induced metric possesses either Minkowskian or Euclidian signature, there is a strong temptation to propose that Minkowskian regions correspond to associative and Euclidian regions to co-associative regions so that space-time itself would provide both the description and its dual.

The proposed interpretation of conjectured associative-co-associative duality relates in an interesting manner to p-adic length scale hypothesis selecting the primes $p \simeq 2^k$, k positive integer as preferred p-adic length scales. $L_p \propto \sqrt{p}$ corresponds to the p-adic length scale defining the size of the space-time sheet at which elementary particle represented as CP_2 type extremal is topologically condensed and is of order Compton length. $L_k \propto \sqrt{k}$ represents the p-adic length scale of the wormhole contacts associated with the CP_2 type extremal and CP_2 size is the natural length unit now. Obviously the quantitative formulation for associative-co-associative duality would be in terms $p \rightarrow k$ duality.

Are the known extremals of Kähler action consistent with the strong form of $M^8 - H$ duality

It is interesting to check whether the known extremals of Kähler action [D1] are consistent with strong form of $M^8 - H$ duality assuming that M^2 or its light-like ray is contained in $T(X^4)$ or normal space.

1. CP_2 type vacuum extremals correspond cannot be hyper-quaternionic surfaces but co-hyper-quaternionicity is natural for them. In the same manner canonically imbedded M^4 can be only hyper-quaternionic.

2. String like objects are associative since tangent space obviously contains $M^2(x)$. Objects of form $M^1 \times X^3 \subset M^4 \times CP_2$ do not have M^2 either in their tangent space or normal space in H. So that the map from $H \rightarrow M^8$ is not well defined. There are no known extremals of Kähler action of this type. The replacement of M^1 random light-like curve however gives vacuum extremal with vanishing volume, which need not mean physical triviality since fundamental objects of the theory are light-like 3-surfaces.

3. For canonically imbedded CP_2 the assignment of $M^2(x)$ to normal space is possible but the choice of $M^2(x) \subset N(CP_2)$ is completely arbitrary. For a generic CP_2 type vacuum extremals M^4 projection is a random light-like curve in $M^4 = M^1 \times E^3$ and $M^2(x)$ can be defined uniquely by the normal vector $n \in E^3$ for the local plane defined by the tangent vector dx^μ/dt and acceleration vector d^2x^μ/dt^2 assignable to the orbit.

4. Consider next massless extremals. Let us fix the coordinates of X^4 as $(t,z,x,y) = (m^0,m^2,m^1,m^2)$. For simplest massless extremals CP_2 coordinates are arbitrary functions of variables $u = k \cdot m = t - z$ and $v = \epsilon \cdot m = x$, where $k = (1,1,0,0)$ is light-like vector of M^4 and $\epsilon = (0,0,1,0)$ a polarization vector orthogonal to it. Obviously, the extremals defines a decomposition $M^4 = M^2 \times E^2$. Tangent space is spanned by the four H-vectors $\nabla_\alpha h^k$ with M^4 part given by $\nabla_\alpha m^k = \delta^k_\alpha$ and CP_2 part by $\nabla_\alpha s^k = \partial_u s^k k_\alpha + \partial_v s^k \epsilon_\alpha$.

   The normal space cannot contain M^4 vectors since the M^4 projection of the extremal is M^4. To realize hyper-quaternionic representation one should be able to from these vector two vectors of M^2, which means linear combinations of tangent vectors for which CP_2 part vanishes. The vector $\partial_t h^k - \partial_z h^k$ has vanishing CP_2 part and corresponds to M^4 vector $(1,-1,0,0)$ fix assigns to each point the plane M^2. To obtain M^2 one would need $(1,1,0,0)$ too but this is not possible. The vector $\partial_y h^k$ is M^4 vector orthogonal to ϵ but M^2 would require also $(1,0,0,0)$. The proposed generalization of massless extremals allows the light-like line $M_\pm$ to depend on point of M^4 [D1], and leads to the introduction of Hamilton-Jacobi coordinates involving a local decomposition of M^4 to $M^2(x)$ and its orthogonal complement with light-like coordinate lines having interpretation as curved light rays. $M^2(x) \subset T(X^4)$ assumption fails fails also for vacuum extremals of form $X^1 \times X^3 \subset M^4 \times CP_2$, where X^1 is light-like random curve. In the latter case, vacuum property follows from the vanishing of the determinant of the induced metric.

5. The deformations of string like objects to magnetic flux quanta are basic conjectural extremals of Kähler action and the proposed picture supports this conjecture. In hyper-quaternionic case the assumption that local 4-D tangent plane of X^3 contains $M^2(x)$ but that $T(X^3)$ does not contain it, is very strong. It states that $T(X^4)$ at each point can be regarded as a product $M^2(x) \times T^2$, $T^2 \subset T(CP_2)$, so that hyper-quaternionic X^4 would be a collection of Cartesian products of infinitesimal 2-D planes $M^2(x) \subset M^4$ and $T^2(x) \subset CP_2$. The extremals in question could be seen as local variants of string like objects $X^2 \times Y^2 \subset M^4 \times CP_2$, where X^2 is minimal surface and Y^2 holomorphic surface of CP_2. One can say that X^2 is replaced by a collection of infinitesimal pieces of $M^2(x)$ and Y^2 with similar pieces of homologically non-trivial geodesic sphere $S^2(x)$ of CP_2, and the Cartesian products of these pieces are glued together to form a continuous surface defining an extremal of Kähler action. Field equations would pose conditions on how $M^2(x)$ and $S^2(x)$ can depend on x. This description applies to magnetic flux quanta, which are the most important must-be extremals of Kähler action.

Geometric interpretation of strong $M^8 - H$ duality

In the proposed framework $M^8 - H$ duality would have a purely geometric meaning and there would nothing magical in it.

1. $X^4(X^3_l) \subset H$ could be seen a curve representing the orbit of a light-like 3-surface defining a 4-D surface. The question is how to determine the notion of tangent vector for the orbit of X^3_l. Intuitively tangent vector is a one-dimensional arrow tangential to the curve at point X^3_l. The identification of the hyper-quaternionic surface $X^4(X^3_l) \subset M^8$ as tangent vector conforms with this intuition.

2. One could argue that M^8 representation of space-time surface is kind of chart of the real space-time surface obtained by replacing real curve by its tangent line. If so, one cannot avoid the question under which conditions this kind of chart is faithful. An alternative interpretation is that a representation making possible to realize number theoretical universality is in question.

3. An interesting question is whether $X^4(X^3_l)$ as orbit of light-like 3-surface is analogous to a geodesic line -possibly light-like- so that its tangent vector would be parallel translated in the

sense that $X^4(X^3)$ for any light-like surface at the orbit is same as $X^4(X_l^3)$. This would give justification for the possibility to interpret space-time surfaces as a geodesic of configuration space: this is one of the first -and practically forgotten- speculations inspired by the construction of configuration space geometry. The light-likeness of the geodesic could correspond at the level of X^4 the possibility to decompose the tangent space to a direct sum of two light-like spaces and 2-D transversal space producing the foliation of X^4 to light-like 3-surfaces X_l^3 along light-like curves.

4. M^8-H duality would assign to X_l^3 classical orbit and its tangent vector at X_l^3 as a generalization of Bohr orbit. This picture differs from the wave particle duality of wave mechanics stating that once the position of particle is known its momentum is completely unknown. The outcome is however the same: for X_l^3 corresponding to wormhole throats and light-like boundaries of X^4, canonical momentum densities in the normal direction vanish identically by conservation laws and one can say that the the analog of (q,p) phase space as the space carrying wave functions is replaced with the analog of subspace consisting of points $(q,0)$. The dual description in M^8 would not be analogous to wave functions in momentum space space but to those in the space of unique tangents of curves at their initial points.

The Kähler and spinor structures of M^8

If one introduces M^8 as dual of H, one cannot avoid the idea that hyper-quaternionic surfaces obtained as images of the preferred extremals of Kähler action in H are also extremals of M^8 Kähler action with same value of Kähler action. As found, this leads to the conclusion that the$M^8 - H$ duality is Kähler isometry. Coupling of spinors to Kähler potential is the next step and this in turn leads to the introduction of spinor structure so that quantum TGD in H should have full M^8 dual.

There are strong physical constraints on M^8 dual and they could kill the hypothesis. The basic constraint to the spinor structure of M^8 is that it reproduces basic facts about electro-weak interactions. This includes neutral electro-weak couplings to quarks and leptons identified as different H-chiralities and parity breaking.

1. By the flatness of the metric of E^4 its spinor connection is trivial. E^4 however allows full S^2 of covariantly constant Kähler forms so that one can accommodate free independent Abelian gauge fields assuming that the independent gauge fields are orthogonal to each other when interpreted as realizations of quaternionic imaginary units.

2. One should be able to distinguish between quarks and leptons also in M^8, which suggests that one introduce spinor structure and Kähler structure in E^4. The Kähler structure of E^4 is unique apart form $SO(3)$ rotation since all three quaternionic imaginary units and the unit vectors formed from them allow a representation as an antisymmetric tensor. Hence one must select one preferred Kähler structure, that is fix a point of S^2 representing the selected imaginary unit. It is natural to assume different couplings of the Kähler gauge potential to spinor chiralities representing quarks and leptons: these couplings can be assumed to be same as in case of H.

3. Electro-weak gauge potential has vectorial and axial parts. Em part is vectorial involving coupling to Kähler form and Z^0 contains both axial and vector parts. The free Kähler forms could thus allow to produce M^8 counterparts of these gauge potentials possessing same couplings as their H counterparts. This picture would produce parity breaking in M^8 picture correctly.

4. Only the charged parts of classical electro-weak gauge fields would be absent. This would conform with the standard thinking that charged classical fields are not important. The predicted classical W fields is one of the basic distinctions between TGD and standard model and in this framework. A further prediction is that this distinction becomes visible only in situations, where H picture is necessary. This is the case at high energies, where the description of quarks in terms of $SU(3)$ color is convenient whereas $SO(4)$ QCD would require large number of E^4 partial waves. At low energies large number of $SU(3)$ color partial waves are needed and the convenient description would be in terms of $SO(4)$ QCD. Proton spin crisis might relate to this.

5. Also super-symmetries of quantum TGD crucial for the construction of configuration space geometry force this picture. In the absence of coupling to Kähler gauge potential all constant

spinor fields and their conjugates would generate super-symmetries so that M^8 would allow $N =$ 8 super-symmetry. The introduction of the coupling to Kähler gauge potential in turn means that all covariantly constant spinor fields are lost. Only the representation of all three neutral parts of electro-weak gauge potentials in terms of three independent Kähler gauge potentials allows right-handed neutrino as the only super-symmetry generator as in the case of H.

6. The $SO(3)$ element characterizing $M^2(x)$ is fixed apart from a local $SO(2)$ transformation, which suggests an additional $U(1)$ gauge field associated with $SO(2)$ gauge invariance and representable as Kähler form corresponding to a quaternionic unit of E^4. A possible identification of this gauge field would be as a part of electro-weak gauge field.

M^8 dual of configuration space geometry and spinor structure?

If one introduces M^8 spinor structure and preferred extremals of M^8 Kähler action, one cannot avoid the question whether it is possible or useful to formulate the notion of configuration space geometry and spinor structure for light-like 3-surfaces in M^8 using the exponent of Kähler action as vacuum functional.

1. The isometries of the configuration space in M^8 and H formulations would correspond to symplectic transformation of $\delta M^4_\pm \times E^4$ and $\delta M^4_\pm \times CP_2$ and the Hamiltonians involved would belong to the representations of $SO(4)$ and $SU(3)$ with 2-dimensional Cartan sub-algebras. In H picture color group would be the familiar $SU(3)$ but in M^8 picture it would be $SO(4)$. Color confinement in both $SU(3)$ and $SO(4)$ sense could allow these two pictures without any inconsistency.

2. For $M^4 \times CP_2$ the two spin states of covariantly constant right handed neutrino and antineutrino spinors generate super-symmetries. This super-symmetry plays an important role in the proposed construction of configuration space geometry. As found, this symmetry would be present also in M^8 formulation so that the construction of M^8 geometry should reduce more or less to the replacement of CP_2 Hamiltonians in representations of $SU(3)$ with E^4 Hamiltonians in representations of $SO(4)$. These Hamiltonians can be taken to be proportional to functions of E^4 radius which is $SO(4)$ invariant and these functions bring in additional degree of freedom.

3. The construction of Dirac determinant identified as a vacuum functional can be done also in M^8 picture and the conjecture is that the result is same as in the case of H. In this framework the construction is much simpler due to the flatness of E^4. In particular, the generalized eigen modes of the Chern-Simons Dirac operator D_{C-S} identified as zero modes of 4-D Dirac operator D_K restricted to the X^3_l correspond to a situation in which one has fermion in induced Maxwell field mimicking the neutral part of electro-weak gauge field in H as far as couplings are considered. Induced Kähler field would be same as in H. Eigen modes are localized to regions inside which the Kähler magnetic field is non-vanishing and apart from the fact that the metric is the effective metric defined in terms of canonical momentum densities via the formula $\hat{\Gamma}^\alpha = \partial L_K/\partial h^k_\alpha \Gamma_k$ for effective gamma matrices. This in fact, forces the localization of modes implying that their number is finite so that Dirac determinant is a product over finite number eigenvalues. It is clear that M^8 picture could dramatically simplify the construction of configuration space geometry.

4. The eigenvalue spectra of the transversal parts of D_K operators in M^8 and H should identical. This motivates the question whether it is possible to achieve a complete correspondence between H and M^8 pictures also at the level of spinor fields at X^3 by performing a gauge transformation eliminating the classical W gauge boson field altogether at X^3_l and whether this allows to transform the modified Dirac equation in H to that in M^8 when restricted to X^3_l. That something like this might be achieved is supported by the fact that in Coulombic gauge the component of gauge potential in the light-like direction vanishes so that the situation is effectively 2-dimensional and holonomy group is Abelian.

Why $M^8 - H$ duality is useful?

Skeptic could of course argue that $M^8 - H$ duality produces only an inflation of unproven conjectures. There are however strong reasons for $M^8 - H$ duality: both theoretical and physical.

1. The map of $X_l^3 \subset H \rightarrow X_l^3 \subset M^8$ and corresponding map of space-time surfaces would allow to realize number theoretical universality. $M^8 = M^4 \times E^4$ allows linear coordinates as natural coordinates in which one can say what it means that the point of imbedding space is rational/algebraic. The point of $X^4 \subset H$ is algebraic if it is mapped to an algebraic point of M^8 in number theoretic compactification. This of course restricts the symmetry groups to their rational/algebraic variants but this does not have practical meaning. Number theoretical compactication could in fact be motivated by the number theoretical universality.

2. $M^8 - H$ duality could provide much simpler description of preferred extremals of Kähler action since the Kähler form in E^4 has constant components. If the spinor connection in E^4 is combination of the three Kähler forms mimicking neutral part of electro-weak gauge potential, the eigenvalue spectrum for the modified Dirac operator would correspond to that for a fermion in $U(1)$ magnetic field defined by an Abelian magnetic field whereas in $M^4 \times CP_2$ picture $U(2)_{ew}$ magnetic fields would be present.

3. $M^8 - H$ duality provides insights to low energy hadron physics. M^8 description might work when H-description fails. For instance, perturbative QCD which corresponds to H-description fails at low energies whereas M^8 description might become perturbative description at this limit. Strong $SO(4) = SU(2)_L \times SU(2)_R$ invariance is the basic symmetry of the phenomenological low energy hadron models based on conserved vector current hypothesis (CVC) and partially conserved axial current hypothesis (PCAC). Strong $SO(4) = SU(2)_L \times SU(2)_R$ relates closely also to electro-weak gauge group $SU(2)_L \times U(1)$ and this connection is not well understood in QCD description. $M^8 - H$ duality could provide this connection. Strong $SO(4)$ symmetry would emerge as a low energy dual of the color symmetry. Orbital $SO(4)$ would correspond to strong $SU(2)_L \times SU(2)_R$ and by flatness of E^4 spin like $SO(4)$ would correspond to electro-weak group $SU(2)_L \times U(1)_R \subset SO(4)$. Note that the inclusion of coupling to Kähler gauge potential is necessary to achieve respectable spinor structure in CP_2. One could say that the orbital angular momentum in $SO(4)$ corresponds to strong isospin and spin part of angular momentum to the weak isospin.

2.5.4 $M^8 - H$ duality and low energy hadron physics

The description of $M^8 - H$ at the configuration space level can be applied to gain a view about color confinement and its dual for electro-weak interactions at short distance limit. The basic idea is that $SO(4)$ and $SU(3)$ provide provide dual descriptions of quark color using E^4 and CP_2 partial waves and low energy hadron physics corresponds to a situation in which M^8 picture provides the perturbative approach whereas H picture works at high energies. The basic prediction is that $SO(4)$ should appear as dynamical symmetry group of low energy hadron physics and this is indeed the case.

Consider color confinement at the long length scale limit in terms of $M^8 - H$ duality.

1. At high energy limit only lowest color triplet color partial waves for quarks dominate so that QCD description becomes appropriate whereas very higher color partial waves for quarks and gluons are expected to appear at the confinement limit. Since configuration space degrees of freedom begin to dominate, color confinement limit transcends the descriptive power of QCD.

2. The success of $SO(4)$ sigma model in the description of low lying hadrons would directly relate to the fact that this group labels also the E^4 Hamiltonians in M^8 picture. Strong $SO(4)$ quantum numbers can be identified as orbital counterparts of right and left handed electro-weak isospin coinciding with strong isospin for lowest quarks. In sigma model pion and sigma boson form the components of E^4 valued vector field or equivalently collection of four E^4 Hamiltonians corresponding to spherical E^4 coordinates. Pion corresponds to S^3 valued unit vector field with charge states of pion identifiable as three Hamiltonians defined by the coordinate components. Sigma is mapped to the Hamiltonian defined by the E^4 radial coordinate. Excited mesons corresponding to more complex Hamiltonians are predicted.

3. The generalization of sigma model would assign to quarks E^4 partial waves belonging to the representations of $SO(4)$. The model would involve also 6 $SO(4)$ gluons and their $SO(4)$ partial waves. At the low energy limit only lowest representations would be be important whereas at

higher energies higher partial waves would be excited and the description based on CP_2 partial waves would become more appropriate.

4. The low energy quark model would rely on quarks moving $SO(4)$ color partial waves. Left *resp.* right handed quarks could correspond to $SU(2)_L$ *resp.* $SU(2)_R$ triplets so that spin statistics problem would be solved in the same manner as in the standard quark model.

5. Family replication phenomenon is described in TGD framework the same manner in both cases so that quantum numbers like strangeness and charm are not fundamental. Indeed, p-adic mass calculations allowing fractally scaled up versions of various quarks allow to replace Gell-Mann mass formula with highly successful predictions for hadron masses [F4].

To my opinion these observations are intriguing enough to motivate a concrete attempt to construct low energy hadron physics in terms of $SO(4)$ gauge theory.

2.5.5 The notion of number theoretical braid

The notion of number theoretic braid is essential for the view about quantum TGD as almost topological quantum field theory. It also realization discretization as a space-time correlate for the finite measurement resolution. Number theoretical universality leads to this notion also and requires that the points in the intersection of the number theoretic braid with partonic 2-surface correspond to rational or at most algebraic points of H in preferred coordinates fixed by symmetry considerations. The challenge has been to find a unique identification of the number theoretic braid. Number theoretic vision indeed makes this possible.

The core element of number theoretic vision is that the laws of physics could be reduced to associativity conditions. One realization for associativity conditions is the level of M^8 endowed with hyper-octonionic structure as a condition that the points sets possible as arguments of N-point function in X^4 are associative and thus belong to hyper-quaternionic subspace $M^4 \subset M^8$. This decomposition must be consistent with the $M^4 \times E^4$ decomposition implied by $M^4 \times CP_2$ decomposition of H. What comes first in mind is that partonic 2-surfaces X^2 belong to $\delta M^4_\pm \subset M^8$ defining the ends of the causal diamond and are thus associative. This boundary condition however freezes E^4 degrees of freedom completely so that M^8 configuration space geometry trivializes.

One can also consider the commutativity condition by requiring that arguments belong to a preferred commutative hyper-complex sub-space M^2 of M^8 which can be regarded as a curve in complex plane. Fixing preferred real and imaginary units means a choice of M^2 interpreted as a partial choice of quantization axes at the level of M^8. One must distinguish this choice from the hyper-quaternionicity of space-time surfaces and from the condition that each tangent space of X^4 contains $M^2(x) \subset M^4$ in its tangent space or normal space. Commutativity condition indeed implies the notion of number theoretic braid and fixes it uniquely once a global selection of $M^2 \subset M^8$ is made. There is also an alternative identification of number theoretic braid based on the assumption that braids are light-like curves with tangent vector in $M^2(x)$.

1. The strong form of commutativity condition would require that the arguments of the n-point function at partonic 2-surface belong to the intersection $X^2 \cap M_\pm$. This however allows quite too few points since an intersection of 2-D and 1-D objects in 7-D space would be in question. Associativity condition would reduce cure the problem but would trivialize configuration space geometry.

2. The weaker condition that only $\delta M^4_\pm$ projections for the points of X^2 commute is however sensible since the intersection of 1-D and 2-D surfaces of 3-D space results. This condition is also invariant under number theoretical duality. In the generic case this gives a discrete set of points as intersection of light-like radial geodesic and the projection $P_{\delta M^4_\pm}(X^2)$. This set is naturally identifiable in terms of points in the intersection of number theoretic braids with $\delta CD \times E^4$. One should show that this set of points consists of rational or at most algebraic points. Here the possibility to choose X^2 to some degree could be essential. Any radial light ray from the tip of light-cone allows commutativity and one can consider the possibility of integrating over n-point functions with arguments at light ray to obtain maximal information.

3. For the pre-images of light-like 3-surfaces commutativity of the points in $\delta M^4_\pm$ projection would allow the projections to be one-dimensional curves of M^2 having thus interpretation as braid strands. M^2 would play exactly the same role as the plane into which braid strands are projected in the construction of braid invariants. Therefore the plane of non-physical polarizations in gauge theories corresponds to the plane to which braids and knots are projected in braid and knot theories. A further constraint is that the braid strand connects algebraic points of M^8 to algebraic points of M^8. It seems that this can be guaranteed only by posing some additional conditions to the light-like 3-surfaces themselves which is of course possible since they are in the role of fundamental dynamical objects.

4. An alternative identification of the number theoretic braid would give up commutativity condition for M^4 projection and assume braid strand to be as a light-like curve having light-like tangent belonging to the local hyper-complex tangent sub-space $M^2(x)$ at point x. This definition would apply both in $X^3 \subset \delta M^4_\pm \times CP_2$ and in X^3_l. Also now one would have a continuous distribution of number theoretic braids, with one braid assignable to each light-like curve with tangent $\delta M^4_+ \supset M_+(x) \subset M^2(x)$. In this case each light-like curve at δM^4_+ with tangent in $M_+(x)$ would define a number theoretic braid so that the only difference would be the replacement of light-like ray with a more general light-like curve.

There are reasons why the identification of the number theoretic braid strand as a curve having hyper-complex light-like tangent looks more attractive.

1. The preferred plane $M^2(x)$ can be interpreted as the local plane of non-physical polarizations so that the interpretation as a number theoretic analog of gauge conditions posed in both quantum field theories and string models is possible. In TGD framework this would mean that super-conformal degrees of freedom are restricted to the orthogonal complement of $M^2(x)$ and $M^2(x)$ does not contribute to the configuration space metric. In Hamilton-Jacobi coordinates the pairs of light-like curves associated with coordinate lines can be interpreted as curved light rays. Hence the partonic planes $M^2(x_i)$ associated with the points of the number theoretic braid could be also regarded as carriers four-momenta of fermions associated with the braid strands so that the standard gauge conditions $\epsilon \cdot p = 0$ for polarization vector and four-momentum would be realized geometrically. The possibility of M^2 to depend on point of X^3_l would be essential to have non-collinear momenta and for a classical description of interactions between braid strands.

2. One could also define analogs of string world sheets as sub-manifolds of $P_{M^4_+}(X^4)$ having $M^2(x) \subset M^4$ as their tangent space or being assignable to their tangent containing $M_+(x)$ in the case that the distribution defined by the planes $M^2(x)$ exists and is integrable. It must be emphasized that in the case of massless extremals one can assign only $M_+(x) \subset M^4$ to $T(X^4(x))$ so that only a foliation of X^4 by light-like curves in M^4 is possible. For $P_{M^4_+}(X^4)$ however a foliation by 2-D stringy surfaces is obtained. Integrability of this distribution and thus the duality with stringy description has been suggested to be a basic feature of the preferred extremals and is equivalent with the existence of Hamilton-Jacobi coordinates for a large class of extremals of Kähler action [D1].

3. The possibility of dual descriptions based on integrable distribution of planes $M^2(x)$ allowing identification as 2-dimensional stringy sub-manifolds of $X^4(X^3)$ and the flexibility provided by the hyper-complex conformal invariance raise the hopes of achieving the lifting of super-symplectic algebra SS and super Kac-Moody algebra SKM to H. At the light-cone boundary the light-like radial coordinate could be lifted to a hyper-complex coordinate defining coordinate for M^2. At X^3_l one could fix the light-like coordinate varying along the braid strands and it can can be lifted to a light-like hyper-complex coordinate in M^4 by requiring that the tangent to the coordinate curve is light-like line of $M^2(x)$ at point x. The total four-momenta and color quantum numbers assignable to SS and SKM degrees of freedom are naturally identical since they can be identified as the four-momentum of the partonic 2-surface $X^2 \subset X^3 \cap \delta M^4_\pm \times CP_2$. Equivalence Principle would emerge as an identity.

2.6 Configuration gamma matrices as hyper-octonionic conformal fields and number theoretic braids

The fact that the Clifford algebra generated by configuration space gamma matrices forms a canonical representation for hyper-finite factor of type II_1 (HFFs) and led to a breakthrough in the understanding of quantum TGD. The inclusions of hyper-finite factors of type II_1 led to a realization of finite quantum measurement resolution as a basic principle govering dynamics and together with zero energy ontology this approach led to the generalization of S-matrix to M-matrix identified as time like entanglement coefficients between positive and negative energy parts of zero energy state and its identification as Connes tensor product. HFFs generated also ideas about how quantum TGD might be reducible to a generalization of HFFs to its local variant which is necessarily complex-octonionic as also to a construction of quantum variant of gamma matrix algebra leading to identification of quantum counterparts of hyper-octonions and hyper-quaternions as unique structures.

2.6.1 Only the quantum variants of M^4 and M^8 emerge from local hyper-finite II_1 factors

The fantastic properties of hyperfinite factors of type II_1 (HFFs) inspire the idea that a localized hyper-octonionic version of Clifford algebra of configuration space might allow to see space-time, embedding space, and configuration space as structures emerging from a hyper-octonionic version of HFF. Surprisingly, commutativity and associativity imply most of the speculative "must-be-true's" of quantum TGD.

Configuration space gamma matrices act only in vibrational degrees of freedom of 3-surface. One must also include center of mass degrees of freedom which appear as zero modes. The natural idea is that the resulting local gamma matrices define a local version of HFF of type II_1 as a generalization of conformal field of gamma matrices appearing super string models obtained by replacing complex numbers with hyper-octonions identified as a subspace of complexified octonions.

As a matter fact, one can generalize octonions to quantum octonions for which quantum commutativity means restriction to a hyper-octonionic subspace of quantum octonions. Non-associativity is essential for obtaining something non-trivial: otherwise this algebra reduces to HFF of type II_1 since matrix algebra as a tensor factor would give an algebra isomorphic with the original one. The octonionic variant of conformal invariance fixes the dependence of local gamma matrix field on the coordinate of HO. The coefficients of Laurent expansion of this field must commute with octonions. !

Super-symmetry suggests that the representations of CH Clifford algebra $\mathcal{M}$ as $\mathcal{N}$ module $\mathcal{M}/\mathcal{N}$ should have bosonic counterpart in the sense that the coordinate for M^8 representable as a particular $M^2(Q)$ element should have quantum counterpart. Same would apply to M^4 coordinate representable as $M^2(C)$ element. Quantum matrix representation of $\mathcal{M}/\mathcal{N}$ as $SL_q(2,F)$ matrix, $F=C,H$ is the natural candidate for this representation. As a matter fact, this guess is not quite correct. It is the interpretation of $M_2(C)$ as a quaternionic quantum algebra whose generalization to the octonionic quantum algebra works.

Quantum variants of M^D exist for all dimensions but only spaces M^4 and M^8 and their linear sub-spaces emerge from hyper-finite factors of type II_1. This is due to the non-associativity of the octonionic representation of the gamma matrices making it impossible to absorb the powers of the octonionic coordinate to the Clifford algebra element so that the local algebra character would disappear. Even more: quantum coordinates for these spaces are commutative operators so that their spectra define ordinary M^4 and M^8 which are thus already quantal concepts.

Consider first hyper-quaternions and the emergence of M^4.

1. The commutation relations for $M_{2,q}(C)$ matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} , \tag{2.6.1}$$

read as

$$\begin{array}{llll} ab = qba \ , & ac = qac \ , & bd = qdb \ , cd = qdc \ , \\ [a,d] = (q-q^{-1})bc \ , & bc = cb \ . \end{array} \tag{2.6.2}$$

2. These relations could be extended by postulating complex conjugates of these relations for complex conjugates $a^\dagger, b^\dagger, c^\dagger, d^\dagger$ plus the following non-vanishing commutators of type $[x, y^\dagger]$:

$$[a, a^\dagger] = [b, b^\dagger] = [c, c^\dagger] = [d, d^\dagger] = 1 \ . \tag{2.6.3}$$

   This extension is not necessary for what comes.

3. The matrices representing M^4 point must be expressible as sums of Pauli spin matrices. This can be represented as following conditions on physical states

$$\begin{aligned} O|phys\rangle &= 0 \ , \\ O &\in \{a-a^\dagger, d-d^\dagger, b-c^\dagger, c-b^\dagger\} \ . \end{aligned} \tag{2.6.4}$$

   For instance, the first two conditions follow from the reality of Pauli sigma matrices $\sigma_x, \sigma_y, \sigma_z$. These conditions are compatible only if the operators O commute. These conditions need not be consistent with the commutation relations between a,b,c,d and their Hermitian conjugates. This is easy to see by noticing that the difference of $J_+ - J_-$ acts apart from imaginary unit like J_y and annihilates $j_y = 0$ state for every representation of rotation group diagonalized with respect to J_y.

4. What is essential is that the operators of O are of form $A - A^\dagger$ and their commutators are also of the same form that the commutativity conditions reduce the condition that the Lie-algebra like structure generated by these operators annihilates the physical state. Hence it is possible to define quantum states for which M^4 coordinates have well-defined eigenvalues so that ordinary M^4 emerges purely quantally from quaternions whose real coefficients are made non-Hermitian operators to obtain operator complexificiation of quaternions. Also the quantum states in which M^4 coordinates are emerge naturally.

5. $M_{2,q}(C)$ matrices define the quantum analog of C^4 and one can wonder whether also other linear sub-spaces can be defined consistently or whether M^4_q and thus Minkowski signature is unique. This seems to be not the case. For instance, the replacement $a - a^\dagger \to a + a^\dagger$ making also time variable Euclidian is impossible since $[a + a^\dagger, d - d^\dagger] = 2(q - q^{-1})(bc + b^\dagger c^\dagger$ is not proportional to a difference of operator and its hermitian conjugate and one does not obtain closed algebra.

What about M^8: does it have analogous description in terms of physical states annihilated by the Lie algebra generated by the differences $a_i - a_i^\dagger$, $i = 0, ..7$?

1. The representation of M^4 point as $M_2(C)$ matrix can be interpreted a combination of 4-D gamma matrices defining hyper-quaternionic units. Hyper-octonionic units indeed have anticommutation relations of gamma matrices of M^8 and would give classical representation of M^8. The counterpart of $M_{2,q}(C)$ would thus be obtained by replacing the coefficients of hyper-octonionic units with operators satisfying the generalization of $M_{2,q}(C)$ commutation relations. One should identify the reality conditions and find whether they are mutually consistent.

2. In quaternionic case basis for matrix algebra is formed by the sigma matrices and M^4 point is represented by a hermitian matrix expressible as linear combination of hermitian sigma matrices with coefficients which act on physical states like hermitian operators. In the hyper-octonionic case would expect that real octonion unit and octonionic imaginary units multiplied by commuting imaginary unit to define the counterparts of sigma matrices and that the physically

representable sub-space of complex quantum octonions corresponds to operator valued coordinates which act like hermitian matrices. The restriction to complex quaternionic sub-space must give hyper-quaternions and M^4 so that the only sensible generalization is that M^8 holds quite generally. This is also required by SO^7 invariance allowing to choose the sub-space M^4 freely. Again the key point should be that the conditions giving rise to real eigenvalues give rise to a Lie-algebra which must annihilate the physical state. For other signatures one would not obtain Lie algebra.

3. One can also make guess for the concrete realization of the algebra. Introduce the coefficients of E^4 gamma matrices having interpretation as quaternionic units as

$$\begin{array}{ll} a_0 = ix(a+d) \ , & a_3 = x(a-d) \ , \\ a_1 = x(ib+c) \ , & a_2 = x(ib-c) \ , \\ x = \frac{1}{\sqrt{2}} \ , & \end{array}$$

and write the commutations relations for them to see how the generalization should be performed.

4. The selections of complex and quaternionic sub-algebras of octonions are fundamental for TGD and quantum octonionic algebra should reflect these selections in its structure. In the case of hyper-quaternions the selection of commutative sub-algebra implies the breaking of 4-D Lorentz symmetry. In the case of hyper-octonions the selection of hyper-quaternion sub-algebra should induce the breaking of 8-D Lorentz symmetry. Hyper-quaternionic sub-algebra obeys the commutations of $M_q(2,C)$ whereas the coefficients in the complement commute mutually and quantum commute with the complex sub-algebra. This nails down the commutation relations completely:

$$\begin{array}{rcl} [a_0, a_3] & = & \frac{i}{2}(q-q^{-1})(a_1^2 - a_2^2) \ , \\ [a_i, a_j] & = & 0 \ , \ i,j \neq 0,3 \ , \\ a_0 a_i & = & q a_i a_0 \ , \ i \neq 0,3 \ , \\ a_3 a_i & = & q a_i a_3 \ , \ i \neq 0,3 \ . \end{array} \tag{2.6.5}$$

Note that there is symmetry breaking in the sense that the commutation relations for sub-algebras relating to both M^4 and M^2 are in distinguished role.

Dimensions $D = 4$ and $D = 8$ are indeed unique if one takes this argument seriously.

1. For dimensions other than $D = 4$ and $D = 8$ a representation of the point of M^D as element of Clifford algebra of M^D is needed. The coefficients should be real for the signatures and this requires that the elements of Clifford algebra are Hermitian. Gamma matrices are the only natural candidates and when Majorana conditions can be satisfied one obtains quantum representation of M^D. 10-D Minkowski space of super-string models would represent one example of this kind of situation.

2. For other dimensions $D \geq 8$ but now octonionic units must be replaced by gamma matrices and an explicit matrix representation can be introduced. These gamma matrices can be included as a tensor factor to the infinite-dimensional Clifford algebra so that the local Clifford algebra reduces to a mere Clifford algebra. The units of quantum octonions which are just ordinary octonion units do not however allow matrix representation so that this reduction is not possible and imbedding space and space-time indeed emerge genuinely. The non-associativity of octonions would determine the laws of physics in TGD Universe!

2.6.2 Configuration space spinor fields as hyper-octonionic conformal fields

A further proposed application of this picture is to the construction of configuration space spinor fields as generalizations of conformal fields. The basic problem is to treat center of mass degrees of freedom properly, and the idea that conformal invariance generalizes to hyper-octonionic - or at least hyper-quaternionic - conformal invariance is attractive. If so, the usual expansion in powers of complex coordinate z would be replaced in powers of hyper-octonionic coordinate h and the coefficients would be elements of Clifford algebra for sub-configuration space consisting of light-like 3-surfaces with frozen center of mass degrees of freedom. This is possible if one can map the points of H to those of M^8 and $M^8 - H$ duality allows to achieve this.

The natural condition would be that N-point functions defined by configuration space spinor fields for which M^8 coordinate labels the position of the tip of the causal diamond containing the zero energy state involve only those points which are mutually associative and would thus belong to a hyper-quaternionic sub-space $M^4 \subset M^8$ would be in question and the outcome would be the analog of M^4 quantum field theory.

Commutativity would restrict the points to $M^2 \subset M^4 \subset M^8$ and hyper-complex variant conformal field theory would result: this theory would be analogous with integrable models known as factorizing quantum field theories in M^2 in which particle scattering is almost trivial (interactions generate only phase lag).

2.7 E_8 theory of Garrett Lisi and TGD

Recently (towards end of the year 2007) there has been a lot of fuss about the E_8 theory proposed by Garrett Lisi [51] in physics blogs, in media, and even New Scientist [52] wrote about the topic. There are serious objections against Lisi's theory and it is interesting to find whether the theory could be modified so that it would survive the basic objections. Although it seems that Lisi's theory cannot be saved, one achieves further insights about HO-H duality. Number theoretical spontaneous compactification can be formulated in terms of the Kac-Moody algebra assignable to Poincare group and standard model gauge group having also rank 8. The representation can be constructed in standard manner using quantized M^8 coordinates at partonic 2-surfaces. Also E_8 representations are in principle possible and the question concerns their physical interpretation.

2.7.1 Objections against Lisi's theory

The basic claim of Lisi is that one can understand the particle spectrum of standard model in terms of the adjoint representation of a noncompact version E_8 group [53]. There are several objections against E_8 gauge theory interpretation of Lisi.

1. Statistics does not allow to put fermions and bosons in the same gauge multiplet. Also the identification of graviton as a part of a gauge multiplet seems very strange if not wrong since there are no roots corresponding to a spin 2 two state.

2. Gauge couplings come out wrong for fermions and one must replace YM action with an ad hoc action.

3. Poincare invariance is a problem. There is no clear relationship with the space-time geometry so that the interpretation of spin as E_8 quantum numbers is not really justified.

4. Finite-dimensional representations of non-compact E_8 are non-unitary. Non-compact gauge groups are however not possible since one would need unitary infinite-dimensional representations which would change the physical interpretation completely. Note that also Lorentz group has only infinite-D unitary representations and only the extension to Poincare group allows to have fields transforming according to finite-D representations.

5. The prediction of three fermion families is nice but one can question the whole idea of putting particles with mass scales differing by a factor of order 10^{12} (top and neutrinos) into same multiplet. For some reason colleagues stubbornly continue to see fundamental gauge symmetries where there seems to be no such symmetry. Accepting the existence of a hierarchy of mass scales

seems to be impossible for a theoretical physicist in main main stream although fractals have been here for decades.

6. Also some exotic particles not present in standard model are predicted: these carry weak hyper charge and color (6-plet representation) and are arranged in three families.

2.7.2 Three attempts to save Lisi's theory

To my opinion, the shortcomings of E_8 theory as a gauge theory are fatal but the possibility to put gauge bosons and fermions of the standard model to E_8 multiplets is intriguing and motivatse the question whether the model could be somehow saved by replacing gauge theory with a theory based on extended fundamental objects possessing conformal invariance.

1. In TGD framework H-HO duality allows to consider Super-Kac Moody algebra with rank 8 with Cartan algebra assigned with the quantized coordinates of partonic 2-surface in 8-D Minkowski space M^8 (identifiable as hyper-octonions HO). The standard construction for the representations of simply laced Kac-Moody algebras allows quite a number of possibilities concerning the choice of Kac-Moody algebra and the non-compact E_8 would be the maximal choice.

2. The first attempt to rescue the situation would be the identification of the weird spin 1/2 bosons in terms of supersymmetry involving addition of righthanded neutrino to the state giving it spin 1. This options does not seem to work.

3. The construction of representations of non-simply laced Kac-Moody algebras (performed by Goddard and Olive at eighties [47]) leads naturally to the introduction of fermionic fields for algebras of type B, C, and F: I do not know whether the construction has been made for G_2. E_6, E_7, and E_8 are however simply laced Lie groups with single root length 2 so that one does not obtain fermions in this manner.

4. The third resuscitation attempt is based on fractional statistics. Since the partonic 2-surfaces are 2-dimensional and because one has a hierarchy of Planck constants, one can have also fractional statistics. Spin 1/2 gauge bosons could perhaps be interpreted as anyonic gauge bosons meaning that particle exchange as permutation is replaced with braiding homotopy. If so, E_8 would not describe standard model particles and the possibility of states transforming according to its representations would reflect the ability of TGD to emulate any gauge or Kac-Moody symmetry.

The standard construction for simply laced Kac-Moody algebras might be generalized considerably to allow also more general algebras and fractionization of spin and other quantum numbers would suggest fractionization of roots. In stringy picture the symmetry group would be reduced considerably since longitudinal degrees of freedom (time and one spatial direction) are non-physical. This would suggest a symmetry breaking to $SO(1,1) \times E_6$ representations with ground states created by tachyonic Lie allebra generators and carrying mass squared 2 in suitable units. In TGD framework the tachyonic conformal weight can be compensated by super-canonical conformal weight so that massless states getting their masses via Higgs mechanism and p-adic thermodynamics would be obtained.

2.7.3 Could super-symmetry rescue the situation?

E_8 is unique among Lie algebras in that its adjoint rather than fundamental representation has the smallest dimension. One can decompose the 240 roots of E_8 to 112 roots for which two components of SO(7,1) root vector are $\pm$ 1 and to 128 vectors for which all components are $\pm$ 1/2 such that the sum of components is even. The latter roots Lisi assigns to fermionic states. This is not consistent with spin and statistics although SO(3,1) spin is half-integer in M^8 picture.

The first idea which comes in mind is that these states correspond to super-partners of the ordinary fermions. In TGD framework they might be obtained by just adding covariantly constant right-handed neutrino or antineutrino state to a given particle state. The simplest option is that fermionic super-partners are complex scalar fields and sbosons are spin 1/2 fermions. It however seems that the super-conformal symmetries associated with the right-handed neutrino are strictly local in the sense that global super-generators vanish. This would mean that super-conformal super-symmetries change the color and angular momentum quantum numbers of states. This is a pity if indeed true since super-symmetry could be broken by different p-adic mass scale for super partners so that no explicit breaking would be needed.

2.7.4 Could Kac Moody variant of E_8 make sense in TGD?

One can leave gauge theory framework and consider stringy picture and its generalization in TGD framework obtained by replacing string orbits with 3-D light-like surfaces allowing a generalization of conformal symmetries.

H-HO duality is one of the speculative aspects of TGD. The duality states that one can either regard imbedding space as $H = M^4 \times CP_2$ or as 8-D Minkowski space M^8 identifiable as the space HO of hyper-octonions which is a subspace of complexified octonions. Spontaneous compactification for M^8 described as a phenomenon occurring at the level of Kac-Moody algebra would relate HO-picture to H-picture which is definitely the fundamental picture. For instance, standard model symmetries have purely number theoretic meaning in the resulting picture.

The question is whether the non-compact E_8 could be replaced with the corresponding Kac Moody algebra and act as a stringy symmetry. Note that this would be by no means anything new. The Kac-Moody analogs of E_{10} and E_{11} algebras appear in M-theory speculations. Very little is known about these algebras. Already $E < sub > n < /sub >$, $n > 8$ is infinite-dimensional as an analog of Lie algebra. The following argument shows that E_8 representations do not work in TGD context unless one allows anyonic statistics.

1. In TGD framework space-time dimension is D=8. The speculative hypothesis of HO-H duality inspired by string model dualities states that the descriptions based on the two choices of imbedding space are dual. One can start from 8-D Cartan algebra defined by quantized M^8 coordinates regarded as fields at string orbit just as in string model. A natural constraint is that the symmetries act as isometries or holonomies of the effectively compactified M^8. The article "The Octonions" [29] of John Baez discusses exceptional Lie groups and shows that compact form of E_8 appears as isometry group of 16-dimensional octo-octonionic projective plane $E_8/(Spin(16)/Z_2)$: the analog of CP_2 for complexified octonions. There is no 8-D space allowing E_8 as an isometry group. Only SO(1,7) can be realized as the maximal Lorentz group with 8-D translational invariance.

2. In HO picture some Kac Moody algebra with rank 8 acting on quantized M^8 coordinates defining stringy fields is natural. The charged generators of this algebra are constructible using the standard recipe involving operators creating coherent states and their conjugates obtained as operator counterparts of plane waves with momenta replaced by roots of the simply laced algebra in question and by normal ordering.

3. Poincare group has 4-D maximal Cartan algebra and this means that only 4 Euclidian dimensions remain. Lorentz generators can be constructed in standard manner in terms of Kac-Moody generators as Noether currents.

4. The natural Kac-Moody counterpart for spontaneous compactification to CP_2 would be that these dimensions give rise to the generators of electro-weak gauge group identifiable as a product of isometry and holonomy groups of CP_2 in the dual H-picture based on $M^4 \times CP_2$. Note that in this picture electro-weak symmetries would act geometrically in E^4 whereas in CP_2 picture they would act only as holonomies.

Could one weaken the assumption that Kac-Moody generators act as symmetries and that spin-statistics relation would be satisfied?

1. The hierarchy of Planck constants relying on the generalization of the notion of imbedding space breaks Poincare symmetry to Lorentz symmetry for a given sector of the world of classical worlds for which one considers light-like 3-surfaces inside future and past directed light cones. Translational invariance is obtained from the wave function for the position of the tip of the light cone in M^4. In this kind of situation one could consider even E_8 symmetry as a dynamical symmetry.

2. The hierarchy of Planck constants involves a hierarchy of groups and fractional statistics at the partonic 2-surface with rotations interpreted as braiding homotopies. The fractionization of spin allows anyonic statistics and could allow bosons with anyonic half-odd integer spin. Also more general fractional spins are possible so that one can consider also more general algebras than Kac-Moody algebras by allowing roots to have more general values. Quantum versions of Kac-Moody algebras would be in question. This picture would be consistent with the view that TGD can emulate any gauge algebra with 8-D Cartan algebra and Kac-Moody algebra dynamically. This vision was originally inspired by the study of the inclusions of hyper-finite factors of type II¡sub¿1¡/sub¿. Even higher dimensional Kac-Moody algebras are predicted to be possible.

3. It must be emphasized that these considerations relate in TGD framework to Super-Kac Moody algebra only. The so called super-canonical algebra is the second quintessential part of the story. In particular, color is not spin-like quantum number for quarks and quark color corresponds to color partial waves in the world of classical worlds or more concretely, to the rotational degrees of freedom in CP_2 analogous to ordinary rotational degrees of freedom of rigid body. Arbitrarily high color partial waves are possible and also leptons can move in triality zero color partial waves and there is a considerable experimental evidence for color octet excitations of electron and muon but put under the rug.

2.7.5 Can one interpret three fermion families in terms of E_8 in TGD framework?

The prediction of three fermion generations by E_8 picture must be taken very seriously. In TGD three fermion generations correspond to three lowest genera $g = 0, 1, 2$ (handle number) for which all 2-surfaces have Z_2 as global conformal symmetry (hyper-ellipticity [F1, F2]). One can assign to the three genera a dynamical SU(3) symmetry. Theye are related by SU(3) triality which brings in mind the triality symmetry acting on fermion generations in E_8 model. SU(3) octet and singlet bosons correspond to pairs of light-like 3-surfaces defining the throats of a wormhole contact and since their genera can be different one has color singlet and octet bosons. Singlet corresponds to ordinary bosons. Color octet bosons must be heavy since they define neutral currents between fermion families.

The three E_8 anyonic boson families cannot represent family replication since these symmetries are not local conformal symmetries: it obviously does not make sense to assign a handle number to a given point of partonic 2-surface! Also bosonic octet would be missing in E_8 picture.

One could of course say that in E_8 picture based on fractional statistics, anyonic gauge bosons can mimic the dynamical symmetry associated with the family replication. This is in spirit with the idea that TGD Universe is able to emulate practically any gauge - or Kac-Moody symmetry and that TGD Universe is busily mimicking also itself.

To sum up, the rank 8 Kac-Moody algebra - emerging naturally if one takes HO-H duality seriously - corresponds very naturally to Kac-Moody representations in terms of free stringy fields for Poincare-, color-, and electro-weak symmetries. One can however consider the possibility of anyonic symmetries and the emergence of non-compact version of E_8 as a dynamical symmetry, and TGD suggests much more general dynamical symmetries if TGD Universe is able to act as the physics analog of the Universal Turing machine.

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Chapter 3

TGD as a Generalized Number Theory III: Infinite Primes

3.1 Introduction

The third part of the multi-chapter discussing the idea about physics as a generalized number theory is devoted to the possible role of infinite primes in TGD.

The notion of prime seems to capture something very essential about what it is to be elementary building block of matter and has become a fundamental conceptual element of TGD. The notion of prime gains it generality from its reducibility to the notion of prime ideal of an algebra. Thus the notion of prime is well-defined, not only in case of quaternions and octonions, but also in the case of hyper-quaternions and -octonions, which are especially natural physically and for which numbers having zero norm correspond physically to light-like 8-vectors. Many interpretations for infinite primes have been competing for survival but it seems that the recent state of TGD allows to exclude some of them from consideration.

3.1.1 The notion of infinite prime

p-Adic unitarity implies that each quantum jump involves unitarity evolution U followed by a quantum jump to some sector D_p of the configuration space labeled by a p-adic prime. Simple arguments show that the p-adic prime characterizing the 3-surface representing the entire universe increases in a statistical sense. This leads to a peculiar paradox: if the number of quantum jumps already occurred is infinite, this prime is most naturally infinite. On the other hand, if one assumes that only finite number of quantum jumps have occurred, one encounters the problem of understanding why the initial quantum history was what it was. Furthermore, since the size of the 3-surface representing the entire Universe is infinite, p-adic length scale hypothesis suggest also that the p-adic prime associated with the entire universe is infinite.

These arguments motivate the attempt to construct a theory of infinite primes and to extend quantum TGD so that also infinite primes are possible. Rather surprisingly, one can construct infinite primes by repeating a procedure analogous to a quantization of a super symmetric quantum field theory. At given level of hierarchy one can identify the decomposition of space-time surface to p-adic regions representing selves with the corresponding decomposition of the infinite prime to primes at lower level of infinity: at the basic level are finite primes for which one cannot find any formula.

This and other observations suggest that the Universe of quantum TGD might basically provide a physical representation of number theory allowing also infinite primes. The proposal suggests also a possible generalization of real numbers to a number system akin to hyper-reals introduced by Robinson in his non-standard calculus [40] providing rigorous mathematical basis for calculus. In fact, some rather natural requirements lead to a unique generalization for the concepts of integer, rational and real. Somewhat surprisingly, infinite integers and reals can be regarded as infinite-dimensional vector spaces with integer and real valued coefficients respectively and this raises the question whether the tangent space for the configuration space of 3-surfaces could be regarded as the space of generalized 8-D hyper-octonionic numbers.

3.1.2 Generalization of ordinary number fields

The introduction of infinite primes, integers, and rationals leads also to a generalization of real numbers since an infinite algebra of real units defined by finite ratios of infinite rationals multiplied by ordinary rationals which are their inverses becomes possible. These units are not units in the p-adic sense and have a finite p-adic norm which can be differ from one. This construction generalizes also to the case of hyper- quaternions and -octonions although non-commutativity and in case of hyper-octonions also non-associativity pose technical problems. Obviously this approach differs from the standard introduction of infinitesimals in the sense that sum is replaced by multiplication meaning that the set of real units becomes infinitely degenerate.

3.1.3 Infinite primes and physics in TGD Universe

Several different views about how infinite primes, integers, and rationals might be relevant in TGD Universe have emerged.

Infinite primes, cognition, and intentionality

The correlation of infinite primes with cognition is the first fascinating possibility and this possibility has stimulated several ideas.

1. The hierarchy of infinite primes associated with algebraic extensions of rationals leading gradually towards algebraic closure of rationals would in turn define cognitive hierarchy corresponding to algebraic extensions of p-adic numbers.

2. Infinite primes form an infinite hierarchy so that the points of space-time and imbedding space can be seen as infinitely structured and able to represent all imaginable algebraic structures. Certainly counter-intuitively, single space-time point might be even capable of representing the quantum state of the entire physical Universe in its structure. For instance, in the real sense surfaces in the space of units correspond to the same real number 1, and single point, which is structure-less in the real sense could represent arbitrarily high-dimensional spaces as unions of real units. For real physics this structure is completely invisible and is relevant only for the physics of cognition. One can say that Universe is an algebraic hologram, and there is an obvious connection both with Brahman=Atman identity of Eastern philosophies and Leibniz's notion of monad.

3. One can assign to infinite primes at n^{th} level of hierarchy rational functions of n rational arguments which form a natural hierarchical structure in that highest level corresponds to a polynomial with coefficients which are rational functions of the arguments at the lower level. One can solve one of the arguments in terms of lower ones to get a hierarchy of algebraic extensions. At the lowest level algebraic extensions of rationals emerge, at the next level algebraic extensions of space of rational functions of single variable, etc... This would suggest that infinite primes code for the correlation between quantum states and the algebraic extensions appearing in their their physical description and characterizing their cognitive correlates. The hierarchy of infinite primes would also correlate with a hierarchy of logics of various orders (hierarchy of statements about statements about...).

Infinite primes and super-symmetric quantum field theory

Consider next the physical interpretation.

1. The discovery of infinite primes suggested strongly the possibility to reduce physics to number theory. The construction of infinite primes can be regarded as a repeated second quantization of a super-symmetric arithmetic quantum field theory. This suggests that configuration space spinor fields or at least the ground states of associated super-conformal representations could be mapped to infinite primes in both bosonic and fermionic degrees of freedom. The process might generalize so that it applies in the case of quaternionic and octonionic primes and their hyper counterparts. This hierarchy of second quantizations means enormous generalization of physics to what might be regarded a physical counterpart for a hierarchy of abstractions about

abstractions about.... The ordinary second quantized quantum physics corresponds only to the lowest level infinite primes.

2. The ordinary primes appearing as building blocks of infinite primes at the first level of the hierarchy could be identified as coding for p-adic primes assignable to fermionic and bosonic partons identified as 2-surfaces of a given space-time sheet. The hierarchy of infinite primes would correspond to hierarchy of space-time sheets defined by the topological condensate. This leads also to a precise identification of p-adic and real variants of bosonic partonic 2-surfaces as correlates of intention and action and pairs of p-adic and real fermionic partons as correlates for cognitive representations.

3. The idea that infinite primes characterize quantum states of the entire Universe, perhaps ground states of super-conformal representations, if not all states, could be taken further. Could 8-D hyper-octonions correspond to 8-momenta in the description of TGD in terms of 8-D hyper-octonion space M^8? Could 4-D hyper-quaternions have an interpretation as four-momenta? The problems caused by non-associativity and non-commutativity however suggests that it is perhaps wiser to restrict the consideration to infinite primes associated with rationals and their algebraic extensions.

Here however emerges the idea about the number theoretic analog of color confinement. Rational (infinite) primes allow not only a decomposition to (infinite) primes of algebraic extensions of rationals but also to algebraic extensions of quaternionic and octonionic (infinite) primes. The physical analog is the decomposition of a particle to its more elementary constituents. This fits nicely with the idea about number theoretic resolution represented as a hierarchy of Galois groups defined by the extensions of rationals and realized at the level of physics in terms of Jones inclusions [C6] defined by these groups having a natural action on space-time surfaces, induced spinor fields, and on configuration space spinor fields representing physical states [C1].

Infinite primes and physics as number theory

The hierarchy of algebraic extensions of rationals implying corresponding extensions of p-adic numbers suggests that Galois groups, which are the basic symmetry groups of number theory, should have concrete physical representations using induced spinor fields and configuration space spinor fields and also infinite primes and real units formed as infinite rationals. These groups permute zeros of polynomials and thus have a concrete physical interpretation both at the level of partonic 2-surfaces dictated by algebraic equations and at the level of braid hierarchy. The vision about the role of hyperfinite factors of II_1 and of Jones inclusions as descriptions of quantum measurements with finite measurement resolution leads to concrete ideas about how these groups are realized.

Space-time correlates of infinite primes

One can assign to infinite primes at the n^{th} level of hierarchy rational functions of n arguments with arguments ordered in a hierarchical manner. It would be nice to assign some concrete interpretation to the polynomials of n arguments in the extension of field of rationals.

1. Do infinite primes code for space-time surfaces?

Infinite primes code naturally for Fock states in a hierarchy of super-symmetric arithmetic quantum field theories. Quantum classical correspondence leads to ask whether infinite primes could also code for the space-time surfaces serving as symbolic representations of quantum states. This would a generalization of algebraic geometry would emerge and could reduce the dynamics of Kähler action to algebraic geometry and organize 4-surfaces to a physical hierarchy according to their algebraic complexity. Note that this conjecture which should be consistent with several conjectures about the dynamics of space-time surfaces (space-time surfaces as preferred extrema of Kähler action, as Kähler calibrations, as quaternionic or co-quaternionic (as associative or co-associative) 4-surfaces of hyper-octonion space M^8.

The most promising variant of this idea is based on the conjecture that hyper-octonion real-analytic maps define foliations of $HO = M^8$ by hyper-quaternionic space-time surfaces providing in turn preferred extremals of Kähler action. This would mean that lowest level infinite primes would define hyper-analytic maps $HO \rightarrow HO$ as polynomials. The intuitive expectation is that higher levels should give rise to more complex HO analytic maps.

The basic objections against the idea is the failure of associativity. The only manner to guarantee associativity is to assume that the arguments oh_n in the polynomial are not independent but that one has $h_i = f_i(h_{i-1},\ i = 2, ..., n$ where f_i is hyper-octonion real-analytic function. This assumption means that one indeed obtains foliation of M^8 by hyper-quaternionic surfaces also now and that these foliations become increasingly complex as n increases. One could of course consider also the possibility that the hierarchy of infinite primes is directly mapped to functions of single hyper-octonionic argument $h_n = ... = h_1 = h$.

2. What about the interpretation of zeros and poles of rational functions associated with infinite primes

If one accepts this interpretation of infinite primes, one must reconsider the interpretation of the zeros and also poles of the functions $f(o)$ defined by the infinite primes. The set of zeros and poles consists of discrete points and this suggests an interpretation in terms of preferred points of M^8, which appear naturally in the quantization of quantum TGD [C1] if one accepts the ideas about hyper-finite factors of type II_1 [C6] and the generalization of the notion of imbedding space and quantization of Planck constant [A9].

The M^4 projection of the preferred point would code for the position tip of future or past light-cone $\delta M^4_\pm$ whereas E^4 projection would choose preferred origin for coordinates transforming linearly under $SO(4)$. At the level of CP_2 the preferred point would correspond to the origin of coordinates transforming linearly under $U(2) \subset SU(3)$. These preferred points would have interpretation as arguments of n-point function in the construction of S-matrix and theory would assign to each argument of n-point function (not necessarily so) "big bang" or "big crunch".

Also configuration space CH (the world of classical worlds) would decompose to a union CH_h of the classical world consisting of 3-surfaces inside $\delta M^4_\pm \times CP_2$ with CP_2 possessing also a preferred point. The necessity of this decomposition in M^4 degrees of freedom became clear long time ago.

3. Why effective 1-dimensionality in algebraic sense?

The identification of arguments (via hyper-octonion real-analytic map in the most general case) means that one consider essentially functions of single variable in the algebraic sense of the word. Rational functions of single variable defined on curve define the simplest function fields having many resemblances with ordinary number fields, and it is known that the dimension $D = 1$ is completely exceptional in algebraic sense [22].

1. Langlands program [21] is based on the idea that the representations of Galois groups can be constructed in terms of so called automorphic functions to which zeta functions relate via Mellin transform. The zeta functions associated with 1-dimensional algebraic curve on finite field F_q, $q = p^n$, code the numbers of solutions to the equations defining algebraic curve in extensions of F_q which form a hierarchy of finite fields F_{q^m} with $m = kn$ [27]: these conjectures have been proven. Algebraic 1-dimensionality is also responsible for the deep results related to the number theoretic Langlands program as far as 1-dimensional function fields on finite fields are considered [27, 21]. In fact, Langlands program is formulated only for algebraic extensions of 1-dimensional function fields.

2. The exceptional character of algebraically 1-dimensional surfaces is responsible the successes of conformal field theory inspired approach to the realization of the geometric Langlands program [22]. It is also responsible for the successes of string models.

3. Effective 1-dimensionality in the sense that the induced spinor fields anti-commute only along 1-D curve of partonic 2-surface is also crucial for the stringy aspects of quantum TGD [C1].

4. Associativity is a key axiom of conformal field theories and would dictate both classical and quantum dynamics of TGD in the approach based on hyper-finite factors of type II_1[C6]. Hence it is rather satisfactory outcome that the mere associativity for octonionic polynomials forces algebraic 1-dimensionality.

3.1.4 About literature

The reader not familiar with the basic algebra of quaternions and octonions is encouraged to study some background material: the home page of Tony Smith provides among other things an excellent introduction to quaternions and octonions [20]. String model builders are beginning to grasp the potential importance of octonions and quaternions and the articles about possible applications of octonions [21, 22, 23] provide an introduction to octonions using the language of physicist.

Personally I found quite frustrating to realize that I had neglected totally learning of the basic ideas of algebraic geometry, despite its obvious potential importance for TGD and its applications in string models. This kind of losses are the price one must pay for working outside the scientific community. It is not easy for a physicist to find readable texts about algebraic geometry and algebraic number theory from the bookshelves of mathematical libraries. The book "Algebraic Geometry for Scientists and Engineers" by Abhyankar [24], which is not so elementary as the name would suggest, introduces in enjoyable manner the basic concepts of algebraic geometry and binds the basic ideas with the more recent developments in the field. "Problems in Algebraic Number Theory" by Esmonde and Murty [26] in turn teaches algebraic number theory through exercises which concretize the abstract ideas. The book "Invitation to Algebraic Geometry" by K. E. Smith. L. Kahanpää, P. Kekäläinen and W. Traves is perhaps the easiest and most enjoyable introduction to the topic for a novice. It also contains references to the latest physics inspired work in the field.

3.2 Infinite primes, integers, and rationals

By the arguments of introduction p-adic evolution leads to a gradual increase of the p-adic prime p and at the limit $p \rightarrow \infty$ Omega Point is reached in the sense that the negentropy gain associated with quantum jump can become arbitrarily large. There several interesting questions to be answered. Does the topology R_P at the limit of infinite P indeed approximate real topology? Is it possible to generalize the concept of prime number and p-adic number field to include infinite primes? This is is possible is suggested by the fact that sheets of 3-surface are expected to have infinite size and thus to correspond to infinite p-adic length scale. Do p-adic numbers R_P for sufficiently large P give rise to reals by canonical identification? Do the number fields R_P provide an alternative formulation/generalization of the non-standard analysis based on the hyper-real numbers of Robinson [40]? Is it possible to generalize the adelic formula [E4] so that one could generalize quantum TGD so that it allows effective p-adic topology for infinite values of p-adic prime? It must be emphasized that the consideration of infinite primes need not be a purely academic exercise: for infinite values of p p-adic perturbation series contains only two terms and this limit, when properly formulated, could give excellent approximation of the finite p theory for large p.

It turns out that there is not any unique infinite prime nor even smallest infinite prime and that there is an entire hierarchy of infinite primes. Somewhat surprisingly, R_P is not mapped to entire set of reals nor even rationals in canonical identification: the image however forms a dense subset of reals. Furthermore, by introducing the corresponding p-adic number fields R_P, one necessarily obtains something more than reals: one might hope that for sufficiently large infinite values of P this something might be regarded as a generalization of real numbers to a number field containing both infinite numbers and infinitesimals.

The pleasant surprise is that one can find a general construction recipe for infinite primes and that this recipe can be characterized as a repeated second quantization procedure in which the many boson states of the previous level become single boson states of the next level of the hierarchy: this realizes Cantor's definition 'Set as Many allowing to regard itself as One' in terms of the basic concepts of quantum physics. Infinite prime allows decomposition to primes at lower level of infinity and these primes can be identified as primes labeling various space-time sheets which are in turn geometric correlates of selves in TGD inspired theory of consciousness. Furthermore, each infinite prime defines decomposition of a fictive many particle state to a purely bosonic part and to part for which fermion number is one in each mode. This decomposition corresponds to the decomposition of the space-time surface to cognitive and material space-time sheets. Thus the concept of infinite prime suggests completely unexpected connection between quantum field theory, TGD based theory of consciousness and number theory by providing in its structure nothing but a symbolic representation of mathematician and external world!

The definition of the infinite integers and rationals is a straightforward procedure. Infinite primes also allow generalization of the notion of ordinary number by allowing infinite-dimensional space of real units which are however non-equivalent in p-adic sense. This means that space-time points are infinitely structured. The fact that this structure completely invisible at the level of real physics suggests that it represents the space-time correlate for mathematical cognition.

3.2.1 The first level of hierarchy

In the following the concept of infinite prime is developed gradually by stepwise procedure rather than giving directly the basic definitions. The hope is that the development of the concept in the same manner as it actually occurred would make it easier to understand it.

Step 1

One could try to define infinite primes P by starting from the basic idea in the proof of Euclid for the existence of infinite number of primes. Take the product of all finite primes and add 1 to get a new prime:

$$\begin{aligned} P &= 1+X \ , \\ X &= \prod_p p \ . \end{aligned} \tag{3.2.1}$$

If P were divisible by finite prime then $P-X=1$ would be divisible by finite prime and one would encounter contradiction. One could of course worry about the possible existence of infinite primes smaller than P and possibly dividing P. The numbers $N=P-k$, $k>1$, are certainly not primes since k can be taken as a factor. The number $P'=P-2=-1+X$ could however be prime. P is certainly not divisible by $P-2$. It seems that one cannot express P and $P-2$ as product of infinite integer and finite integer. Neither it seems possible to express these numbers as products of more general numbers of form $\prod_{p\in U} p+q$, where U is infinite subset of finite primes and q is finite integer.

Step 2

P and $P-2$ are not the only possible candidates for infinite primes. Numbers of form

$$\begin{aligned} P(\pm,n) &= \pm 1+nX \ , \\ k(p) &= 0,1,..... \ , \\ n &= \prod_p p^{k(p)} \ , \\ X &= \prod_p p \ , \end{aligned} \tag{3.2.2}$$

where $k(p)\neq 0$ holds true only in finite set of primes, are characterized by a integer n, and are also good prime candidates. The ratio of these primes to the prime candidate P is given by integer n. In general, the ratio of two prime candidates $P(m)$ and $P(n)$ is rational number m/n telling which of the prime candidates is larger. This number provides ordering of the prime candidates $P(n)$. The reason why these numbers are good canditates for infinite primes is the same as above. No finite prime p with $k(p)\neq 0$ appearing in the product can divide these numbers since, by the same arguments as appearing in Euclid's theorem, it would divide also 1. On the other hand it seems difficult to invent any decomposition of these numbers containing infinite numbers. Already at this stage one can notice the structural analogy with the construction of multiboson states in quantum field theory: the numbers $k(p)$ correspond to the occupation numbers of bosonic states of quantum field theory in one-dimensional box, which suggests that the basic structure of QFT might have number theoretic interpretation in some very general sense. It turns out that this analogy generalizes.

Step 3

All $P(n)$ satisfy $P(n)\geq P(1)$. One can however also the possibility that $P(1)$ is not the smallest infinite prime and consider even more general candidates for infinite primes, which are smaller than $P(1)$. The trick is to drop from the infinite product of primes $X=\prod_p p$ some primes away by dividing it by integer $s=\prod_{p_i} p_i$, multiply this number by an integer n not divisible by any prime dividing s and to add to/subtract from the resulting number nX/s natural number ms such that m expressible as a product of powers of only those primes which appear in s to get

$$\begin{array}{l} P(\pm,m,n,s) = n\frac{X}{s} \pm ms \ , \\ m = \prod_{p|s} p^{k(p)} \ , \\ n = \prod_{p|\frac{X}{s}} p^{k(p)}, \ \ k(p) \geq 0 \ . \end{array} \tag{3.2.3}$$

Here $x|y$ means 'x divides y'. To see that no prime p can divide this prime candidate it is enough to calculate $P(\pm,m,n,s)$ modulo p: depending on whether p divides s or not, the prime divides only the second term in the sum and the result is nonzero and finite (although its precise value is not known). The ratio of these prime candidates to $P(+,1,1,1)$ is given by the rational number n/s: the ratio does not depend on the value of the integer m. One can however order the prime candidates with given values of n and s using the difference of two prime candidates as ordering criterion. Therefore these primes can be ordered.

One could ask whether also more general numbers of the form $n\frac{X}{s} \pm m$ are primes. In this case one cannot prove the indivisibility of the prime candidate by p not appearing in m. Furthermore, for $s \ mod \ 2 = 0$ and $m \ mod \ 2 \neq 0$, the resulting prime candidate would be even integer so that it looks improbable that one could obtain primes in more general case either.

Step 4

An even more general series of candidates for infinite primes is obtained by using the following ansatz which in principle is contained in the original ansatz allowing infinite values of n

$$\begin{array}{l} P(\pm,m,n,s|r) = nY^r \pm ms \ , \\ Y = \frac{X}{s} \ , \\ m = \prod_{p|s} p^{k(p)} \ , \\ n = \prod_{p|\frac{X}{s}} p^{k(p)}, \ \ k(p) \geq 0 \ . \end{array} \tag{3.2.4}$$

The proof that this number is not divisible by any finite prime is identical to that used in the previous case. It is not however clear whether the ansatz for given r is not divisible by infinite primes belonging to the lower level. A good example in $r = 2$ case is provided by the following unsuccessful ansatz

$$\begin{array}{l} N = (n_1Y + m_1s)(n_2Y + m_2s) = \frac{n_1n_2X^2}{s^2} - m_1m_2s^2 \ , \\ Y = \frac{X}{s} \ , \\ n_1m_2 - n_2m_1 = 0 \ . \end{array}$$

Note that the condition states that n_1/m_1 and $-n_2/m_2$ correspond to the same rational number or equivalently that (n_1, m_1) and (n_2, m_2) are linearly dependent as vectors. This encourages the guess that all other $r = 2$ prime candidates with finite values of n and m at least, are primes. For higher values of r one can deduce analogous conditions guaranteing that the ansatz does not reduce to a product of infinite primes having smaller value of r. In fact, the conditions for primality state that the polynomial $P(n,m,r)(Y) = nY^r + m$ with integer valued coefficients ($n > 0$) defined by the prime candidate is irreducible in the field of integers, which means that it does not reduce to a product of lower order polynomials of same type.

Step 5

A further generalization of this ansatz is obtained by allowing infinite values for m, which leads to the following ansatz:

$$\begin{array}{l} P(\pm,m,n,s|r_1,r_2) = nY^{r_1} \pm ms \ , \\ m = P_{r_2}(Y)Y + m_0 \ , \\ Y = \frac{X}{s} \ , \\ m_0 = \prod_{p|s} p^{k(p)} \ , \\ n = \prod_{p|Y} p^{k(p)}, \ \ k(p) \geq 0 \ . \end{array} \tag{3.2.5}$$

Here the polynomial $P_{r_2}(Y)$ has order r_2 is divisible by the primes belonging to the complement of s so that only the finite part m_0 of m is relevant for the divisibility by finite primes. Note that the part proportional to s can be infinite as compared to the part proportional to Y^{r_1}: in this case one must however be careful with the signs to get the sign of the infinite prime correctly. By using same arguments as earlier one finds that these prime candidates are not divisible by finite primes. One must also require that the ansatz is not divisible by lower order infinite primes of the same type. These conditions are equivalent to the conditions guaranteing the polynomial primeness for polynomials of form $P(Y) = nY^{r_1} \pm (P_{r_2}(Y)Y + m_0)s$ having integer-valued coefficients. The construction of these polynomials can be performed recursively by starting from the first order polynomials representing first level infinite primes: Y can be regarded as formal variable and one can forget that it is actually infinite number.

By finite-dimensional analogy, the infinite value of m means infinite occupation numbers for the modes represented by integer s in some sense. For finite values of m one can always write m as a product of powers of $p_i|s$. Introducing explicitly infinite powers of p_i is not in accordance with the idea that all exponents appearing in the formulas are finite and that the only infinite variables are X and possibly S (formulas are symmetric with respect to S and X/S). The proposed representation of m circumvents this difficulty in an elegant manner and allows to say that m is expressible as a product of infinite powers of p_i despite the fact that it is not possible to derive the infinite values of the exponents of p_i.

Summarizing, an infinite series of candidates for infinite primes has been found. The prime candidates $P(\pm, m, n, s)$ labeled by rational numbers n/s and integers m plus the primes $P(\pm, m, n, s|r_1, r_2)$ constructed as r_1:th or r_2:th order polynomials of $Y = X/s$: the latter ansatz reduces to the less general ansatz of infinite values of n are allowed.

One can ask whether the $p \ mod \ 4 = 3$ condition guaranteing that the square root of -1 does not exist as a p-adic number, is satisfied for $P(\pm, m, n, s)$. $P(\pm, 1, 1, 1) \ mod \ 4$ is either 3 or 1. The value of $P(\pm, m, n, s) \ mod \ 4$ for odd s on n only and is same for all states containing even/odd number of $p \ mod \ = 3$ excitations. For even s the value of $P(\pm, m, n, s) \ mod \ 4$ depends on m only and is same for all states containing even/odd number of $p \ mod \ = 3$ excitations. This condition resembles G-parity condition of Super Virasoro algebras. Note that either $P(+, m, n, s)$ or $P(-, m, n, s)$ but not both are physically interesting infinite primes ($2m \ mod \ 4 = 2$ for odd m) in the sense of allowing complex Hilbert space. Also the additional conditions satisfied by the states involving higher powers of X/s resemble to Virasoro conditions. An open problem is whether the analogy with the construction of the many-particle states in super-symmetric theory might be a hint about more deeper relationship with the representation of Super Virasoro algebras and related algebras.

It is not clear whether even more general prime candidates exist. An attractive hypothesis is that one could write explicit formulas for all infinite primes so that generalized theory of primes would reduce to the theory of finite primes.

3.2.2 Infinite primes form a hierarchy

By generalizing using general construction recipe, one can introduce the second level prime candidates as primes not divisible by any finite prime p or infinite prime candidate of type $P(\pm, m, n, s)$ (or more general prime at the first level: in the following we assume for simplicity that these are the only infinite primes at the first level). The general form of these prime candidates is exactly the same as at the first level. Particle-analogy makes it easy to express the construction receipe. In present case 'vacuum primes' at the lowest level are of the form

$$\begin{array}{l} \frac{X_1}{S} \pm S \ , \\ X_1 = X \prod_{P(\pm,m,n,s)} P(\pm, m, n, s) \ , \\ S = s \prod_{P_i} P_i \ , \\ s = \prod_{p_i} p_i \ . \end{array} \tag{3.2.6}$$

S is product or ordinary primes p and infinite primes $P_i(\pm, m, n, s)$. Primes correspond to physical states created by multiplying X_1/S (S) by integers not divisible by primes appearing S (X_1/S). The integer valued functions $k(p)$ and $K(p)$ of prime argument give the occupation numbers associated with X/s and s type 'bosons' respectively. The non-negative integer-valued function $K(P) = K(\pm, m, n, s)$ gives the occupation numbers associated with the infinite primes associated with X_1/S and S type 'bosons'. More general primes can be constructed by mimicking the previous procedure.

One can classify these primes by the value of the integer $K_{tot} = \sum_{P|X/S} K(P)$: for a given value of K_{tot} the ratio of these prime candidates is clearly finite and given by a rational number. At given level the ratio P_1/P_2 of two primes is given by the expression

$$\frac{P_1(\pm,m_1,n_1,s_1K_1,S_1}{P_2(\pm,m_2,n_2,s_2,K,S_2)} = \frac{n_1s_2}{n_2s_1}\prod_{\pm,m,n,s}(\frac{n}{s})^{K_1^+(\pm,n,m,s)-K_2^+(\pm,n,m,s)} \quad . \tag{3.2.7}$$

Here K_i^+ denotes the restriction of $K_i(P)$ to the set of primes dividing X/S. This ratio must be smaller than 1 if it is to appear as the first order term $P_1P_2 \rightarrow P_1/P_2$ in the canonical identification and again it seems that it is not possible to get all rationals for a fixed value of P_2 unless one allows infinite values of N expressed neatly using the more general ansatz involving higher power of S.

3.2.3 Construction of infinite primes as a repeated quantization of a super-symmetric arithmetic quantum field theory

The procedure for constructing infinite primes is very much reminiscent of the second quantization of an super-symetric arithmetic quantum field theory in which single particle fermion and boson states are labeled by primes. In particular, there is nothing especially frightening in the particle representation of infinite primes: theoretical physicists actally use these kind of representations quite routinely.

1. The binary-valued function telling whether a given prime divides s can be interpreted as a fermion number associated with the fermion mode labeled by p. Therefore infinite prime is characterized by bosonic and fermionic occupation numbers as functions of the prime labeling various modes and situation is super-symmetric. X can be interpreted as the counterpart of Dirac sea in which every negative energy state state is occupied and $X/s \pm s$ corresponds to the state containing fermions understood as holes of Dirac sea associated with the modes labeled by primes dividing s.

2. The multiplication of the 'vacuum' X/s with $n = \prod_{p|X/s} p^{k(p)}$ creates $k(p)$ 'p-bosons' in mode of type X/s and multiplication of the 'vacuum' s with $m = \prod_{p|s} p^{k(p)}$ creates $k(p)$ 'p-bosons'. in mode of type s (mode occupied by fermion). The vacuum states in which bosonic creation operators act, are tensor products of two vacuums with tensor product represented as sum

$$|vac(\pm)\rangle = |vac(\frac{X}{s})\rangle \otimes |vac(\pm s)\rangle \leftrightarrow \frac{X}{s} \pm s \tag{3.2.8}$$

obtained by shifting the prime powers dividing s from the vacuum $|vac(X)\rangle = X$ to the vacuum ± 1. One can also interpret various vacuums as many fermion states. Prime property follows directly from the fact that any prime of the previous level divides either the first or second factor in the decomposition $NX/S \pm MS$.

3. This picture applies at each level of infinity. At a given level of hierarchy primes P correspond to all the Fock state basis of all possible many-particle states of second quantized super-symmetric theory. At the next level these many-particle states are regarded as single particle states and further second quantization is performed so that the primes become analogous to the momentum labels characterizing various single-particle states at the new level of hierarchy.

4. There are two nonequivalent quantizations for each value of S due to the presence of $\pm$ sign factor. Two primes differing only by sign factor are like G-parity $+1$ and -1 states in the sense that these primes satisfy $P\ mod\ 4 = 3$ and $P\ mod\ 4 = 1$ respectively. The requirement that -1 does not have p-adic square root so that Hilbert space is complex, fixes G-parity to say $+1$. This observation suggests that there exists a close analogy with the theory of Super Virasoro algebras so that quantum TGD might have interpretation as number theory in infinite context. An alternative interpretation for the $\pm$ degeneracy is as counterpart for the possibility to choose the fermionic vacuum to be a state in which either all positive or all negative energy fermion states are occupied.

5. One can also generalize the construction to include polynomials of $Y = X/S$ to get infinite hierarchy of primes labeled by the two integers r_1 and r_2 associated with the polynomials in question. An entire hierarchy of vacuums labeled by r_1 is obtained. A possible interpretation of these primes is as counterparts for the bound states of quantum field theory. The coefficient for the power $(X/s)^{r_1}$ appearing in the highest term of the general ansatz, codes the occupation numbers associated with vacuum $(X/s)^{r_1}$. All the remaining terms are proportional to s and combine to form, in general infinite, integer m characterizing various infinite occupation numbers for the subsystem characterized by s. The additional conditions guaranteing prime number property are equivalent with the primality conditions for polynomials with integer valued coefficients and resemble Super Virasoro conditions. For $r_2 > 0$ bosonic occupation numbers associated with the modes with fermion number one are infinite and one cannot write explicit formula for the boson number.

6. One could argue that the analogy with super-symmetry is not complete. The modes of Super Virasoro algebra are labeled by natural number whereas now modes are labeled by prime. This need not be a problem since one can label primes using natural number n. Also 8-valued spin index associated with fermionic and bosonic single particle states in TGD world is lacking (space-time is surface in 8-dimensional space). This index labels the spin states of 8-dimensional spinor with fixed chirality. One could perhaps get also spin index by considering infinite octonionic primes, which correspond to vectors of 8-dimensional integer lattice such that the length squared of the lattice vector is ordinary prime:

$$\sum_{k=1,\ldots,8} n_k^2 = \text{ prime } .$$

   Thus one cannot exclude the possibility that TGD based physics might provide representation for octonions extended to include infinitely large octonions. The notion of prime octonion is well defined in the set of integer octonions and it is easy to show that the Euclidian norm squared for a prime octonion is prime. If this result generalizes then the construction of generalized prime octonions would generalize the construction of finite prime octonions. It would be interesting to know whether the results of finite-dimensional case might generalize to the infinite-dimensional context. One cannot exclude the possibility that prime octonions are in one-one correspondence with physical states in quantum TGD.

These observations suggest a close relationship between quantum TGD and the theory of infinite primes in some sense: even more, entire number theory and mathematics might be reducible to quantum physics understood properly or equivalently, physics might provide the representation of basic mathematics. Of course, already the uniqueness of the basic mathematical structure of quantum TGD points to this direction. Against this background the fact that 8-dimensionality of the imbedding space allows introduction of octonion structure (also p-adic algebraic extensions) acquires new meaning. Same is also suggested by the fact that the algebraic extensions of p-adic numbers allowing square root of real p-adic number are 4- and 8-dimensional.

What is especially interesting is that the core of number theory would be concentrated in finite primes since infinite primes are obtained by straightforward procedure providing explicit formulas for them. Repeated quantization provides also a model of abstraction process understood as construction of hierarchy of natural number valued functions about functions about ...... At the first level infinite primes are characterized by the integer valued function $k(p)$ giving occupation numbers plus subsystem-complement division (division to thinker and external world!). At the next level prime is characterized in a similar manner. One should also notice that infinite prime at given level is characterized by a pair $(R = MN, S)$ of integers at previous level. Equivalently, infinite prime at given level is characterized by fermionic and bosonic occupation numbers as functions in the set of primes at previous level.

3.2.4 Construction in the case of an arbitrary commutative number field

The basic construction recipe for infinite primes is simple and generalizes even to the case of algebraic extensions of rationals. Let $K = Q(\theta)$ be an algebraic number field (see the Appendix of [E1] for the basic definitions). In the general case the notion of prime must be replaced by the concept of irreducible defined as an algebraic integer with the property that all its decompositions to a product of two integers are such that second integer is always a unit (integer having unit algebraic norm, see Appendix of [E1]).

Assume that the irreducibles of $K = Q(\theta)$ are known. Define two irreducibles to be equivalent if they are related by a multiplication with a unit of K. Take one representative from each equivalence class of units. Define the irreducible to be positive if its first non-vanishing component in an ordered basis for the algebraic extension provided by the real unit and powers of θ, is positive. Form the counterpart of Fock vacuum as the product X of these representative irreducibles of K.

The unique factorization domain (UFD) property (see Appendix of [E1]) of infinite primes does not require the ring O_K of algebraic integers of K to be UFD although this property might be forced somehow. What is needed is to find the primes of K; to construct X as the product of all irreducibles of K but not counting units which are integers of K with unit norm; and to apply second quantization to get primes which are first order monomials. X is in general a product of powers of primes. Generating infinite primes at the first level correspond to generalized rationals for K having similar representation in terms of powers of primes as ordinary rational numbers using ordinary primes.

3.2.5 Mapping of infinite primes to polynomials and geometric objects

The mapping of the generating infinite primes to first order monomials labeled by their rational zeros is extremely simple at the first level of the hierarchy:

$$P_{\pm}(m,n,s) = \frac{mX}{s} \pm ns \rightarrow x_{\pm} \pm \frac{m}{sn} \ . \tag{3.2.9}$$

Note that a monomial having zero as its root is not obtained. This mapping induces the mapping of all infinite primes to polynomials.

The simplest infinite primes are constructed using ordinary primes and second quantization of an arithmetic number theory corresponds in one-one manner to rationals. Indeed, the integer $s = \prod_i p_i^{k_i}$ defining the numbers k_i of bosons in modes k_i, where fermion number is one, and the integer r defining the numbers of bosons in modes where fermion number is zero, are co-prime. Moreover, the generating infinite primes can be written as $(n/s)X \pm ms$ corresponding to the two vacua $V = X \pm 1$ and the roots of corresponding monomials are positive *resp.* negative rationals.

More complex infinite primes correspond sums of powers of infinite primes with rational coefficients such that the corresponding polynomial has rational coefficients and roots which are not rational but belong to some algebraic extension of rationals. These infinite primes correspond simply to products of infinite primes associated with some algebraic extension of rationals. Obviously the construction of higher infinite primes gives rise to a hierarchy of higher algebraic extensions.

It is possible to continue the process indefinitely by constructing the Dirac vacuum at the n:th level as a product of primes of previous levels and applying the same procedure. At the second level Dirac vacuum $V = X \pm 1$ involves X which is the product of all primes at previous levels and in the polynomial correspondence X thus correspond to a new independent variable. At the n:th level one would have polynomials $P(q_1|q_2|...)$ of q_1 with coefficients which are rational functions of q_2 with coefficients which are.... The hierarchy of infinite primes would be thus mapped to the functional hierarchy in which polynomial coefficients depend on parameters depending on ....

At the second level one representation of infinite primes would be as algebraic curve resulting as a locus of $P(q_1|q_2) = 0$: this certainly makes sense if q_1 and q_2 commute. At higher levels the locus is a higher-dimensional surface.

3.2.6 How to order infinite primes?

One can order the infinite primes, integers and rationals. The ordering principle is simple: one can decompose infinite integers to two parts: the 'large' and the 'small' part such that the ratio of the small part with the large part vanishes. If the ratio of the large parts of two infinite integers is different from one or their sign is different, ordering is obvious. If the ratio of the large parts equals to one, one can perform same comparison for the small parts. This procedure can be continued indefinitely.

In case of infinite primes ordering procedure goes like follows. At given level the ratios are rational numbers. There exists infinite number of primes with ratio 1 at given level, namely the primes with same values of N and same S with MS infinitesimal as compared to NX/S. One can order these primes using either the relative sign or the ratio of $(M_1S_1)/(M_2S_2)$ of the small parts to decide which of the two is larger. If also this ratio equals to one, one can repeat the process for the small parts of M_iS_i. In principle one can repeat this process so many times that one can decide which of the two primes is larger. Same of course applies to infinite integers and also to infinite rationals build from primes with infinitesimal MS. If NS is not infinitesimal it is not obvious whether this procedure works. If $N_iX_i/M_iS_i = x_i$ is finite for both numbers (this need not be the case in general) then the ratio $\frac{M_1S_1}{M_2S_2}\frac{(1+x_2)}{(1+x_1)}$ provides the needed criterion. In case that this ratio equals one, one can consider use the ratio of the small parts multiplied by $\frac{(1+x_2)}{(1+x_1)}$ of M_iS_i as ordering criterion. Again the procedure can be repeated if needed.

3.2.7 What is the cardinality of infinite primes at given level?

The basic problem is to decide whether Nature allows also integers S , $R = MN$ represented as infinite product of primes or not. Infinite products correspond to subsystems of infinite size (S) and infinite total occupation number (R) in QFT analogy.

1. One could argue that S should be a finite product of integers since it corresponds to the requirement of finite size for a physically acceptable subsystem. One could apply similar argument to R. In this case the set of primes at given level has the cardinality of integers ($alef_0$) and the cardinality of all infinite primes is that of integers. If also infinite integers R are assumed to involve only finite products of infinite primes the set of infinite integers is same as that for natural numbers.

2. NMP is well defined in p-adic context also for infinite subsystems and this suggests that one should allow also infinite number of factors for both S and $R = MN$. Super symmetric analogy suggests the same: one can quite well consider the possibility that the total fermion number of the universe is infinite. It seems however natural to assume that the occupation numbers $K(P)$ associated with various primes P in the representations $R = \prod_P P^{K(P)}$ are finite but nonzero for infinite number of primes P. This requirement applied to the modes associated with S would require the integer m to be explicitly expressible in powers of $P_i|S$ ($P_{r_2} = 0$) whereas all values of r_1 are possible. If infinite number of prime factors is allowed in the definition of S, then the application of diagonal argument of Cantor shows that the number of infinite primes is larger than $alef_0$ already at the first level. The cardinality of the first level is $2^{alef_0}2^{alef_0} == 2^{alef_0}$. The first factor is the cardinality of reals and comes from the fact that the sets S form the set of all possible subsets of primes, or equivalently the cardinality of all possible binary valued functions in the set of primes. The second factor comes from the fact that integers $R = NM$ (possibly infinite) correspond to all natural number-valued functions in the set of primes: if only finite powers $k(p)$ are allowed then one can map the space of these functions to the space of binary valued functions bijectively and the cardinality must be 2^{alef_0}. The general formula for the cardinality at given level is obvious: for instance, at the second level the cardinality is the cardinality of all possible subsets of reals. More generally, the cardinality for a given level is the cardinality for the subset of possible subsets of primes at the previous level.

3.2.8 How to generalize the concepts of infinite integer, rational and real?

The allowance of infinite primes forces to generalize also the concepts concepts of integer, rational and real number. It is not obvious how this could be achieved. The following arguments lead to a possible generalization which seems practical (yes!) and elegant.

Infinite integers form infinite-dimensional vector space with integer coefficients

The first guess is that infinite integers N could be defined as products of the powers of finite and infinite primes.

$$N = \prod_k p_k^{n_k} = nM \ , \ n_k \geq 0 \ , \tag{3.2.10}$$

where n is finite integer and M is infinite integer containing only powers of infinite primes in its product expansion.

It is not however not clear whether the sums of infinite integers really allow similar decomposition. Even in the case that this decomposition exists, there seems to be no way of deriving it. This would suggest that one should regard sums

$$\sum_i n_i M_i$$

of infinite integers as infinite-dimensional linear space spanned by M_i so that the set of infinite integers would be analogous to an infinite-dimensional algebraic extension of say p-adic numbers such that each coordinate axes in the extension corresponds to single infinite integer of form $N = mM$. Thus the most general infinite integer N would have the form

$$N = m_0 + \sum m_i M_i \ . \quad (3.2.11)$$

This representation of infinite integers indeed looks promising from the point of view of practical calculations. The representation looks also attractive physically. One can interpret the set of integers N as a linear space with integer coefficients m_0 and m_i:

$$N = m_0|1\rangle + \sum m_i|M_i\rangle \ . \quad (3.2.12)$$

$|M_i\rangle$ can be interpreted as a state basis representing many-particle states formed from bosons labeled by infinite primes p_k and $|1\rangle$ represents Fock vacuum. Therefore this representation is analogous to a quantum superposition of bosonic Fock states with integer, rather than complex valued, superposition coefficients. If one interprets M_i as orthogonal state basis and interprets m_i as p-adic integers, one can define inner product as

$$\langle N_a, N_b\rangle = m_0(a)m_0(b) + \sum_i m_i(a)m_i(b) \ . \quad (3.2.13)$$

This expression is well defined p-adic number if the sum contains only enumerable number of terms and is always bounded by p-adic ultrametricity. It converges if the p-adic norm of of m_i approaches to zero when M_i increases.

Generalized rationals

Generalized rationals could be defined as ratios $R = M/N$ of the generalized integers. This works nicely when M and N are expressible as products of powers of finite or infinite primes but for more general integers the definition does not look attractive. This suggests that one should restrict the generalized rationals to be numbers having the expansion as a product of positive and negative primes, finite or infinite:

$$N = \prod_k p_k^{n_k} = \frac{n_1 M_1}{nM} \ . \quad (3.2.14)$$

Generalized reals form infinite-dimensional real vector space

One could consider the possibility of defining generalized reals as limiting values of the generalized rationals. A more practical definition of the generalized reals is based on the generalization of the pinary expansion of ordinary real number given by

$$\begin{aligned} x &= \sum_{n\geq n_0} x_n p^{-n} \ , \\ x_n &\in \{0,..,p-1\} \ . \end{aligned} \quad (3.2.15)$$

It is natural to try to generalize this expansion somehow. The natural requirement is that sums and products of the generalized reals and canonical identification map from the generalized reals to generalized p-adcs are readily calculable. Only in this manner the representation can have practical value.

These requirements suggest the following generalization

$$\begin{aligned} X &= x_0 + \sum_N x_N p^{-N} \ , \\ N &= \sum_i m_i M_i \ , \end{aligned} \tag{3.2.16}$$

where x_0 and x_N are ordinary reals. Note that N runs over infinite integers which has *vanishing finite part*. Note that generalized reals can be regarded as infinite-dimensional linear space such that each infinite integer N corresponds to one coordinate axis of this space. One could interpret generalized real as a superposition of bosonic Fock states formed from single single boson state labeled by prime p such that occupation number is either 0 or infinite integer N with a vanishing finite part:

$$X = x_0|0\rangle + \sum_N x_N|N> \ . \tag{3.2.17}$$

The natural inner product is

$$\langle X,Y\rangle = x_0y_0 + \sum_N x_N y_N \ . \tag{3.2.18}$$

The inner product is well defined if the number of N:s in the sum is enumerable and x_N approaches zero sufficiently rapidly when N increases. Perhaps the most natural interpretation of the inner product is as R_p valued inner product.

The sum of two generalized reals can be readily calculated by using only sum for reals:

$$X+Y = x_0 + y_0 + \sum_N (x_N + y_N)p^{-N} \ , \tag{3.2.19}$$

The product XY is expressible in the form

$$XY = x_0y_0 + x_0Y + Xy_0 + \sum_{N_1,N_2} x_{N_1}y_{N_2}p^{-N_1-N_2} \ , \tag{3.2.20}$$

If one assumes that infinite integers form infinite-dimensional vector space in the manner proposed, there are no problems and one can calculate the sums N_1+N_2 by summing component wise manner the coefficients appearing in the sums defining N_1 and N_2 in terms of infinite integers M_i allowing expression as a product of infinite integers.

Canonical identification map from ordinary reals to p-adics

$$x = \sum_k x_k p^{-k} \quad \rightarrow \quad x_p = \sum_k x_k p^k \ ,$$

generalizes to the form

$$x = x_0 + \sum_N x_N p^{-N} \quad \rightarrow \quad (x_0)_p + \sum_N (x_N)_p p^N \ , \tag{3.2.21}$$

so that all the basic requirements making the concept of generalized real calculationally useful are satisfied.

There are several interesting questions related to generalized reals.

1. Are the extensions of reals defined by various values of p-adic primes mathematically equivalent or not? One can map generalized reals associated with various choices of the base p to each other in one-one manner using the mapping

$$X = x_0 + \sum_N x_N p_1^{-N} \rightarrow x_0 + \sum_N x_N p_2^{-N} . \tag{3.2.22}$$

   The ordinary real norms of *finite* (this is important!) generalized reals are identical since the representations associated with different values of base p differ from each other only infinitesimally. This would suggest that the extensions are physically equivalent. It these extensions are not mathematically equivalent then p-adic primes could have a deep role in the definition of the generalized reals.

2. One can generalize previous formulas for the generalized reals by replacing the coefficients x_0 and x_i by complex numbers, quaternions or octonions so as to get generalized complex numbers, quaternions and octonions. Also inner product generalizes in an obvious manner. The 8-dimensionality of the imbedding space provokes the question whether it might be possible to regard the infinite-dimensional configuration space of 3-surfaces, or rather, its tangent space, as a Hilbert space realization of the generalized octonions. This kind of identification could perhaps reduce TGD based physics to generalized number theory.

3.2.9 Comparison with the approach of Cantor

The main difference between the approach of Cantor and the proposed approach is that Cantor uses only the basic arithmetic concepts such as sum and multiplication and the concept of successor defining ordering of both finite and infinite ordinals. Cantor's approach is also purely set theoretic. The problems of purely set theoretic approach are related to the question what the statement 'Set is Many allowing to regard itself as One' really means and to the fact that there is no obvious connection with physics. The proposed approach is based on the introduction of the concept of prime as a basic concept whereas ordering is based on the use of ratios: using these one can recursively define ordering and get precise quantitative information based on finite reals. Together with canonical identification the concept of infinite primes becomes completely physical in the sense that all probabilities are always finite real numbers. The 'Set is Many allowing to regard itself as One' is defined as quantum physicist would define it: many particle states become single particle states in the second quantization describing the counterpart for the construction of the set of subsets of a given set. One could also say that integer as such corresponds to set as 'One' and its decomposition to a product of primes corresponds to the set as 'Many'. The concept of prime, the ultimate 'One', has as its physical counterpart the concept of elementary particle understood in very general sense. The new element is the physical interpretation: the sum of two numbers whose ratio is zero correspond to completely physical finite-subsystem-infinite complement division and the iterated construction of the set of subsets of a set at given level is basically p-adic evolution understood in the most general possible sense and realized as a repeated second quantization. What is attractive is that this repeated second quantization can be regarded also as a model of abstraction process and actually the process of abstraction itself.

The possibility to interpret the construction of infinite primes either as a repeated bosonic quantization involving subsystem-complement division or as a repeated super-symmetric quantization could have some deep meaning. A possible interpretation consistent with these two pictures is based on the hypothesis that fermions provide a reflective level of consciousness in the sense that the 2^N element Fock basis of many-fermion states formed from N single-fermion states can be regarded as a set of all possible statements about N basic statements. Statements about whether a given element of set X belongs to some subset S of X are certainly the fundamental statements from the point of view of mathematics. Hence one could argue that many-fermion states provide cognitive representation for the subsets of some set. Single fermion states represent the points of the set and many-fermion states represent possible subsets.

3.3 Generalizing the notion of infinite prime to the non-commutative context

The notion of prime and more generally, that of irreducible, makes sense also in more general number fields and even algebras. The considerations of [E2] suggests that the notion of infinite prime should be generalized to the case of complex numbers, quaternions, and octonions as well as to their hyper counterparts which seem to be physically the most interesting ones [E2]. Also the hierarchy of infinite primes should generalize as well as the representation of infinite primes as polynomials and as space-time surfaces. The proposed number theoretic realization of the dynamics defined by the absolute minimization of Kähler action can be realized if it is possible to assign hyper-octonion analytic functions to infinite hyper-octonionic primes [E2].

3.3.1 General view about the construction of generalized infinite primes

The consideration of basic objections against quaternionic and octonionic infinite primes allows to identify the basic philosophical ideas serving as guidelines for the construction of infinite primes.

Infinite primes should be commutative and associative

The basic objections against (hyper-)quaternionic and (hyper-)octonionic infinite primes relate to the non-commutativity and non-associativity.

1. In the case of quaternionic infinite primes non-commutativity, and in the case of octonionic infinite primes also non-associativity, might be expected to cause difficulties in the definition of X. Fortunately, the fact that all conjugates of a given finite prime appear in the product defining X, implies that the contribution from each irreducible with a given norm p is real and X is real. Therefore the multiplication and division of X with quaternionic or octonionic primes is a well-defined procedure, and generating infinite primes are well-defined apart from the degeneracy due to non-commutativity and non-associativity of the finite number of lower level primes. Also the products of infinite primes are well defined, since by the reality of X it is possible to tell how the products AB and BA differ. Of course, also infinite primes representing physical states containing infinite numbers of fermions and bosons are possible and infinite primes of this kind must be analogous to generators of a free algebra for which AB and BA are not related in any manner.

2. The sums of products of monomials of generating infinite primes define higher level infinite primes and also here non-commutativity and associativity cause potential difficulties. The assignment of a monomial to a quaternionic or octonionic infinite prime is not unique since the rational obtained by dividing the finite part mr with the integer n associated with infinite part can be defined either as $(1/n) \times mr$ or $mr \times (1/n)$ and the resulting non-commuting rationals are different.

If the polynomial associated with infinite prime has real-rational coefficients these difficulties do not appear. This would imply universality in the sense that the polynomials as such would not contain information about the number field in question. This number theoretic universality is highly attractive also physically.

The reduction of the roots of polynomials to complex roots encourages the idea about the analogy with quantum measurement theory. Although it is possible to define more general infinite primes, it seems that the primes having representation as space-time surface are reducible to those represented by polynomials with real-rational coefficients. This would mean that the number field field would not be seen at all in the characterization of the polynomial. The roots of the polynomial would be in general complex and effective 2-dimensionality would prevail in this sense. Complex planes of quaternions and octonions space define maximal commutative sub-fields of them. In the case of hyper-quaternions and hyper-octonions hyper-complex planes take the role of maximal sub-algebra which is closed and at the same time commutative. Interestingly, the hyper-octonionic solution ansatz involves a local fixing of a hyper-complex algebra at each point of $HO = M^8$ physically equivalent with the fixing the space of longitudinal polarizations.

At space-time level this should correspond to effective 2-dimensionality in the sense that quantum states and space-time surfaces are coded by the data associated with 2-dimensional partonic surfaces at the intersections of 3-D and 7-D light-like causal determinants. The tangent spaces of these surfaces should be dual to the local hyper-complex longitudinal polarization planes. The induced selection of the transversal polarization plane at each space-time point could be also seen as the number theoretical analog for the selection of a rest frame and of quantization axis for spin.

Commutativity requirement for infinite primes allows real-rationals or possibly algebraic extensions of them as the coefficients of the polynomials formed from hyper-octonionic infinite primes. If only infinite primes with complex rational coefficients are allowed and only the vacuum state $V_{\pm} = X \pm 1$ involving product over all primes of the number field, would reveal the number field. One could thus construct the generating infinite primes using the notion of hyper-octonionic prime for any algebraic extension of rationals.

Do hyper-octonionic infinite primes correspond to space-time surfaces?

The general philosophy behind the construction of infinite primes involves at least the following ideas.

1. Quantum TGD should result as an algebraic continuation of rational number based physics to various number fields. Similar continuation principle should hold true also for infinite primes. This means that the formal expressions for infinite primes should be essentially same as those associated with the infinite primes associated with the field or rational numbers or complex rationals. As far as space-time representation in terms of polynomials is considered, this means that the polynomials involved should have real coefficients. An analogous situation should prevail at the higher levels of the hierarchy.

2. Hyper-octonionic primes are favored physically and if they have representation as polynomials or more general rational functions of hyper-octonion with real-rational coefficients, it is possible to assign to each prime a 4-parameter foliation of $M^4 \times CP_2$ hyper-quaternionic space-time surfaces by the construction of [E2]. Also the dual of the foliation defines a foliation and canonically imbedded M^4 and CP_2 provide a basic example of dual 4-surfaces. The foliations are parameterized by functions $HO = M^8 \to S^6$ fixing the preferred octonionic imaginary unit. A possible identification is in terms of vacuum degeneracy. The fixing of the imaginary unit means fixing of complex plane of octonions and the physical interpretation is as a local fixing of longitudinal polarization directions having interpretation as gauge degrees of freedom.

The decomposition of rational infinite primes to hyper-octonionic could have a physical meaning

The requirement that hyper-octonionic infinite primes correspond at the highest level to polynomials with rational coefficients would mean an effective reducibility to rational infinite primes.

The reduction to rational infinite primes does not mean trivialization of the theory. One can decompose infinite rational primes to a product of hyper-octonionic primes just as one can decompose them to a product of primes in algebraic extensions of rational numbers and this decomposition might have a physical interpretation as a decomposition of a particle to its composites if one accepts the idea that the hierarchy of algebraic extensions corresponds to a hierarchy of increasing measurement resolutions. The reduction to a rational infinite prime implies that hyper-octonionic primes and their conjugates appear in a pairwise manner in the products involved. Hence the net values of the transversal parts of infinite hyper-octonic 8-momenta vanish and one could speak about the vanishing of transversal M^8 momenta in M^8 context. In H context this brings in mind the vanishing of transversal M^4 momenta for hadron and vanishing of color quantum numbers.

Commutativity and associativity for infinite primes does not imply commutativity and associativity for corresponding polynomials

The commutativity of infinite primes is not enough to eliminate completely the effects due to non-commutativity and non-associativity in case of corresponding polynomials. For the hyper-octonionic infinite primes at higher levels of hierarchy non-associativity causes delicate effects since the grouping of infinite primes affects the polynomial associated with the infinite prime and thus space-time surface associated with the infinite prime. Only for arguments $h_1, ..h_n$ restricted to a 2-dimensional subspace H_2 of M^8 the effects due to non-commutativity and non-associativity are completely absent and this conforms nicely with the notion of effective 2-dimensionality meaning that the physical on-associativity and non-commutativity are trivial and correspond to gauge degrees of freedom.

The unique solution to the problems is to assign to infinite hyper-octonionic primes polynomials for which all arguments h_i are identical $h_n = ... = h_1 = h$. A more general solution would be based on the assumption that the arguments of the polynomial are related by hyper-octonion real-analytic rational function. This option also allows to assign to hyper-octonionic infinite primes 4-D surfaces in a natural manner if hyper-octonion real-analyticity gives rise to a foliation of M^8 by quaternionic 4-surfaces. In this framework the proposed mapping of infinite primes to space-time surfaces could be seen as being natural because hyper-octonionic primes are associated with a maximal algebraic completion.

The interpretation of two vacuum primes in terms of positive and negative energy Fock states

In the rational case the positivity of primes means that $V_\pm = X \pm 1$ correspond to two non-equivalent Fock vacua. For hyper-octonionic primes the two vacua correspond to the to different signs of energy related by time reflection since the units with $n_0 < 0$ correspond to time reflection combined with Lorentz boost. The real part of a hyper-octonionic generating prime can be made non-vanishing by an application of a suitable boost represented by unit.

In TGD the time-orientation of the space-time sheet can be also negative and this means that energies can be either positive or negative [D3, D5]. The interpretation of the two vacua is as vacua associated with space-time sheets of negative and positive time orientation. The possibility that the sign of inertial energy is negative has profound implications and defines one of the most important differences between TGD and competing theories.

Physically it would be desirable that also more complex infinite primes having interpretation as representations of bound states could be interpreted as composites of states of unique positive and negative energy generating primes. If the positive and negative energy infinite primes correspond to states with fermion numbers, one must assume that the polynomials of the generating infinite primes are superpositions of products of monomials of degree n_+ and n_- with respect to the generating infinite primes $P_\pm(m, n, s)$ such that $n = n_+ - n_-$ is constant.

The vacua $X \pm 1$ can be interpreted as rational infinite primes, which are however not constructible from rational vacuum $X = \prod_p p$ by a finite number of steps since each rational prime p appears with some power $N(p)$ counting the number of positive primes with norm

$$N(\pi) = h_0^2 - \sum h_i^2 = p \ .$$

Thus one has

$$X = \prod_{\pi > 0} \pi = \prod_p p^{N(p)} \ .$$

Numbers with components in real algebraic extensions of rationals would pop-up dynamically, when one factorizes polynomials which are irreducible in the field of rationals.

If algebraic extensions of rationals are allowed as a fundamental number field, $N(\pi)$ must be replaced with

$$N(\pi) = N_K(h_0^2 - \sum_i h_i^2) = p \ .$$

Only one representative of positive primes related by a multiplication with real Dirichlet units representable as fractal scalings can be included (note that the number of Dirichlet units is always infinite for the real extensions of rationals). This gives a finite number of primes for given p. This option is however not attractive physically since it is in conflict with the idea that algebraic extensions pop up dynamically from the representations of the polynomial as space-time surface.

3.3.2 Quaternionic and octonionic primes and their hyper counterparts

The loss of commutativity and associativity implies that the definitions of (hyper-)quaternionic and (hyper-)octonionic primes are not completely straightforward.

Basic facts about quaternions and octonions

Both quaternions and octonions allow both Euclidian norm and the Minkowskian norm defined as a trace of the linear operator defined by the multiplication with octonion. Minkowskian norm has the metric signature of $H = M^4 \times CP_2$ or $M^4_+ \times CP_2$ so that H can be regarded locally as an octonionic space. Both norms are a multiplicative and the notions of both quaternionic and octonionic prime are well defined despite non-associativity. Quaternionic and octonionic primes have length squared equal to rational prime.

In the case of quaternions different basis of imaginary units I, J, K are related by 3-dimensional rotation group and different quaternionic basis span a 3-dimensional sphere. There is 2-sphere of complex structures since imaginary unit can be any unit vector of imaginary 3-space.

A basis for octonionic imaginary units J, K, L, M, N, O, P can be chosen in many manners and fourteen-dimensional subgroup G_2 of the group $SO(7)$ of rotations of imaginary units is the group labeling the octonionic structures related by octonionic automorphisms to each other. It deserves to be mentioned that G_2 is unique among the simple Lie-groups in that the ratio of the square roots of lengths for long and short roots of G_2 Lie-algebra are in ratio $3:1$ [33]. For other Lie-groups this ratio is either 2:1 or all roots have same length. The set of equivalence classes of the octonion structures is $SO(7)/G_2 = S^7$. In the case of quaternions there is only one equivalence class.

The group of automorphisms for octonions with a fixed imaginary part is $SU(3)$. The coset space $S^6 = G_2/SU(3)$ labels possible complex structures of the octonion space specified by a selection of a preferred imaginary unit. $SU(3)/U(2) = CP_2$ could be thought of as the space of octonionic structures giving rise to a given quaternionic structure with complex structure fixed. This can be seen as follows. The units $1, I$ are $SU(3)$ singlets whereas J, J_1, J_2 and K, K_1, K_2 form $SU(3)$ triplet and antitriplet. Under $U(2)$ J and K transform like objects having vanishing $SU(3)$ isospin and suffer only a $U(1)$ phase transformation determined by multiplication with complex unit I and are mixed with each other in orthogonal mixture. Thus $1, I, J, K$ is transformed to itself under $U(2)$.

Quaternionic and octonionic primes

Quaternionic primes with $p \ mod \ 4 = 1$ can correspond to (n_1, n_2) with n_1 even and n_2 odd or vice versa. For $p \ mod \ 4 = 3$ (n_1, n_2, n_3) with n_i odd is the minimal option. In this case there is however large number of primes having only two components: in particular, Gaussian primes with $p \ mod \ 4 = 1$ define also quaternionic primes. Purely real Gaussian primes with $p \ mod \ 4 = 3$ with norm $z\overline{z}$ equal to p^2 are not quaternionic primes, and are replaced with 3-component quaternionic primes allowing norm equal to p. Similar conclusions hold true for octonionic primes.

The reality condition for polynomials associated with Gaussian infinite primes requires that the products of generating prime and its conjugate are present so that the outcome is a real polynomial of second order.

Hyper primes

The notion of prime generalizes to hyper-quaternionic and octonionic case. The factorization $n_0^2 - n_3^2 = (n_0 + n_3)(n_0 - n_3)$ implies that any hyper-quaternionic and -octonionic primes can be represented as $(n_0, n_3, 0, ...) = (n_3 + 1, n_3, 0, ...)$, $n_3 = (p-1)/2$ for $p > 2$. $p = 2$ is exceptional: a representation with minimal number of components is given by $(2, 1, 1, 0, ...)$. Notice that the interpretation of hyper-quaternionic primes (or integers) as four-momenta implies that it is not possible to find rest system for them: only a system where energy is minimum is possible.

The notion of "irreducible" (see Appendix of [E1]) is defined as the equivalence class of primes related by a multiplication with a unit and is more fundamental than that of prime. All Lorentz boosts of a hyper prime combine to form an irreducible. Note that the units cannot correspond to real particles in corresponding arithmetic quantum field theory.

If the situation for $p > 2$ is effectively 2-dimensional in the sense that it is always possible to transform the hyper prime to a 2-component form by multiplying it by a suitable unit representing

Lorentz boost, the theory for time-like hyper primes effectively reduces to the 2-dimensional hyper-complex case when irreducibles are chosen to belong to H_2. The physical counterpart for the choice of H_2 would be the choice of the plane of longitudinal polarizations, or equivalently, of quantization axis for spin. This hypothesis is physically highly attractive since it would imply number theoretic universality and conform with the effective 2-dimensionality. Of course, the hyper-octonionic primes related by $SO(7,1)$ boosts need not represent physically equivalent states.

Also the rigorous notion of hyper primeness seems to require effective 2-dimensionality. If effective 2-dimensionality holds true, hyper integers have a decomposition to a product of hyper primes multiplied by a suitable unit. The representation is obtained by Lorentz boosting the hyper integer first to a 2-component form and then decomposing it to a product of hyper-complex primes.

The situation becomes certainly more complex if also space-like primes with negative norm squared $n_0^2 - n_1^2 - ... = -p$ are allowed. Gaussian primes with $p \ mod4 \ = 1$ are representable as space-like primes of form $(0, n_1, n_2, 0)$: $n_1^2 + n_2^2 = p$. Space-like primes with $p \ mod \ 4 = 3$ have at least 3 non-vanishing components which are odd integers.

3.3.3 Hyper-octonionic infinite primes

The infinite-primes associated with hyper-octonions are the most natural ones physically because of the underlying Lorentz invariance and the possibility to interpret them as 8-momenta with mass squared equal to prime. M^8 is consistent with the metric signature of the tangent space of H, and the four additional momentum components bring strongly in mind the tangent space counterpart of CP_2 contribution to the mass squared. Also the interpretation of quaternionic part of finite hyper-octonionic primes in terms of electro-weak and color quantum numbers could be considered since the total number of them is $2 + 2 = 4$.

Construction recipe at the lowest level of hierarchy assuming reduction to rational infinite primes

The condition that allowed hyper-octonionic infinite primes correspond to decompositions of rational infinite primes to products of their hyper-octonionic counterparts is the simplest manner to define them and generalizes the decomposition of rational infinite primes to products of primes in algebraic extensions of rationals.

This allows primes in algebraic extensions of rationals containing $\sqrt{-1}$ only if one interprets the commuting unit of hyper-octonionic integers as imaginary unit associated with the algebraic extensions of rationals. Composites of infinite primes in complexification of octonions would be in question. The reality of the coefficients of the polynomials assignable to infinite primes would also mean that the M^8 coordinates of M^8 stay real.

The physical interpretation for the reduction to rational infinite primes would be in terms of number theoretic analog of color confinement meaning decomposition of particles to their composites becoming visible in an improved algebraic resolution. Also the interpretation in terms of non-commutative geometry in transversal degrees of freedom meaning that only longitudinal momenta corresponding to non-vanishing of only hyper-complex part of hyper-octonionic 8-momentum. Indeed, the commutation relations $xy = qyz$, $q = exp(i\pi/n)$ for quantum plane would allow the vanishing of x and y identified now as components of transversal momentum.

More general construction recipe at the lowest level of hierarchy

The following argument represents the construction recipe for the first level hyper-octonionic primes without the assumption about the reduction to rational infinite primes.

1. Infinite prime property requires that X must be defined by taking one representative from each equivalence class representing irreducible and forming the product of their conjugates. The representative hyper-octonionic primes can be taken to be time-like positive energy primes. The conjugates of each irreducible appear in X so for a given norm p the net result is real for each rational prime p.

   The number of conjugates depends on the number of non-vanishing components of the the prime with norm p in the minimal representation having minimal energy. Several primes with a given norm p not related by a multiplication with unit or by automorphism are in principle possible. The degeneracy is determined by the number of elements of a subgroup of Galois group acting non-trivially on the prime. Galois group is generated by the permutations of 7 imaginary units and 7 conjugations of units consistent with the octonionic product. X is proportional to $p^{N(p)}$ where $N(p)$ in principle depends on p.

2. If the conjectured effective 2-dimensionality holds true, the situation reduces effectively to hyper-complex case and X is product of the squares of all primes multiplied by a power of 2. In the case of ordinary infinite primes there are two different vacuum primes $X \pm 1$. This is the case also now. Since the sign of the time-like component part corresponds to the sign of energy, the sign degeneracy $X \pm 1$ for the vacua could relate to the degeneracy corresponding to positive and negative energy space-time sheets. An alternative interpretation is in terms of fermion-antifermion degeneracy.

3. The product X of all hyper-octonionic irreducibles can be regarded as the counterpart of Dirac vacuum in a rather concrete sense. Moreover, in the hyper-quaternionic and octonionic case the norm of X is analogous to the Dirac determinant of a fermionic field theory with prime valued mass spectrum and integer valued momentum components. The inclusion of only irreducible eliminates from the infinite product defining Dirac determinant product over various Lorentz boosts of $p^k\gamma_k - m$.

4. An interesting question is what happens when the finite part of an infinite prime is multiplied by light like integer k. The obvious guess is that k describes the presence of a massless particle. If the resulting infinite integer is multiplied with conjugates $k_{c,i}$ of k an integer of form $\prod_i k_{c,i}mX/n$ having formally zero norm results. It would thus seem that there is a kind of gauge invariance in the sense that infinite primes for which both finite and infinite part are multiplied with the same light-like primes, are divisors of zero and correspond to gauge degrees of freedom.

5. More complex infinite hyper-octonionic primes can be always decomposed to products of generating infinite primes which correspond to polynomials with zeros in algebraic extensions of rationals so that the resulting polynomial has real-rational coefficients but has no rational zeros. An interpretation as bound states is suggestive and the replacement of the zero of corresponding polynomial with non-rational number is analogous to the change of particle rest mass in bound state formation. The sign of energy is well defined for each factor of this kind.

6. Hyper-octonionic infinite primes correspond to real-rational polynomials if all conjugates of given hyper-octonionic prime occur in the definition of generating infinite primes. The reality requirement satisfied in this manner would exclude the presence of light-like factors in the finite part of the infinite prime. Physically the presence of these factors would seem to be desirable (at least in the finite part of the infinite prime) since they could be interpreted physically as representations of massless particles. The reality condition can be also satisfied for a product of conjugates of infinite primes. In this case the constant part of the resulting infinite primes vanishes.

Zeta function and infinite primes

Fermionic Zeta function is expressible as a product of fermionic partition functions $Z_{F,p} = 1 + p^{-z}$ and could be seen as an image of X under algebraic homomorphism mapping prime p to $Z_{F,p}$ defining an analog of prime in the commutative function algebra of complex numbers. For hyper-octonionic infinite primes the contribution of each p to the norm of X is same finite power of p since only single representative from each Lorentz equivalence class is included, and there is one-one correspondence with ordinary primes so that an appropriate power of ordinary ζ_F might be regarded as a representation of X also in the case of hyper-octonionic primes.

Infinite primes suggest a generalization of the notion of ζ function. Real-rational infinite prime $X \pm 1$ would correspond to $\zeta_F \pm 1$. General infinite prime is mapped to a generalized zeta function by dividing ζ_F with the product of partition functions $Z_{F,p}$ corresponding to fermions kicked out from sea. The same product multiplies '1'. The powers p^n present in either factor correspond to the presence of n bosons in mode p and to a factor $Z_{p,B}^n$ in corresponding summand of the generalized Zeta. In the case of hyper-octonionic infinite primes some power of Z_F multiplied by p-dependent powers $Z_{F,p}^{n(p)}$ of fermionic partition functions with $n(p) \to 0$ for $p \to \infty$ should replace the image of X. If effective 2-dimensionality holds true $n(p) = 2$ holds true for $p > 2$.

For zeros of ζ_F which are same as those of Riemann ζ the image of infinite part of infinite prime vanishes and only the finite part is represented faithfully. Whether this might have some physical meaning is an interesting question.

3.3.4 Mapping of the hyper-octonionic infinite primes to polynomials

Infinite primes can be mapped to polynomial primes which in turn have geometric representation as algebraic surfaces. This inspires the idea that physics could be reduced to algebraic number theory and algebraic geometry [26, 26, 24] in some general sense. In the following consideration is restricted to hyper-octonionic primes which are the most interesting ones on basis of the considerations of [E2].

Mapping of infinite primes to polynomials at the first level of the hierarchy

The mapping of the generating infinite primes to first order monomials labeled by their rational zeros is extremely simple at the first level of the hierarchy:

$$P_{\pm}(m,n,s) = \frac{mX}{s} \pm ns \to h \pm \frac{m}{sn} \quad .$$

Note that a monomial having zero as its root is not obtained. This mapping induces the mapping of all infinite primes to polynomials.

The simplest infinite primes are constructed using ordinary primes and second quantization of an arithmetic number theory corresponds in one-one manner to rationals. Indeed, the integer $s = \prod_i p_i^{k_i}$ defining the numbers k_i of bosons in modes k_i, where fermion number is one, and the integer r defining the numbers of bosons in modes where fermion number is zero, are co-prime. Moreover, the generating infinite primes can be written as $(n/s)X \pm ms$ corresponding to the two vacua $V = X \pm 1$ and the roots of corresponding monomials are positive *resp.* negative rationals.

More complex infinite primes correspond sums of powers of infinite primes with rational coefficients such that the corresponding polynomial has real coefficients and roots which are not rational but belong to some algebraic extension of rationals. These infinite primes correspond simply to products of infinite primes associated with some algebraic extension of rationals. Obviously the construction of higher infinite primes gives rise to a hierarchy of higher algebraic extensions.

The representation of higher level infinite primes as polynomials

It is possible to continue the process indefinitely by constructing the Dirac vacuum at the n:th level as a product of primes of previous levels and applying the same procedure. At the second level Dirac vacuum $V = X \pm 1$ involves X which is the product of all primes at previous levels and in the polynomial correspondence X thus correspond to a new independent variable. At the n:th level one has polynomials $P(h_1|h_2|...)$ of h_1 with coefficients which are real-rational functions of h_2 with coefficients which are.... The hierarchy of infinite primes is thus mapped to the functional hierarchy in which polynomial coefficients depend on parameters depending on ....

The so called Slaving Hierarchy appearing in Haken's theory of self-organization has similar form: the non-dynamical coupling parameters of the system depend on slowly varying external parameters which in turn depend on... The lowest level of the hierarchy corresponding to the ordinary rationals takes the role of the highest boss in the hierarchy of infinite primes.

For higher level infinite primes the effects of non-commutativity and non-associativity cannot be avoided except when the arguments are restricted to the same hyper-complex sub-space of M^8 defining the polarization plane. The non-associativity implies that the grouping of the arguments in the polynomial matters and affects the space-time surface. It is not clear whether non-associativity and non-commutative can be really allowed for infinite primes.

A very attractive manner to avoid effects of non-associativity is to assume that all infinite primes are reducible to rational infinite primes and that representations in terms of infinite primes associated with various extensions of rationals (algebraic extensions of rationals and of non-commutative and non-associative completions of rationals) emerge from the decompositions of rational primes to these primes.

3.3.5 Mapping of infinite primes to space-time surfaces

At the lowest level of hierarchy the mapping of hyper-octonionic infinite primes to 4-surfaces is a special case of assigning to a hyper-octonion analytic function a foliation of imbedding space by 4-surfaces. At higher levels of hierarchy the mapping of infinite primes to space-time surfaces requires a generalization of this procedure and the constraints from non-commutativity and non-associativity dictate the generalization completely.

Associativity as the basic constraint

On basis of the general vision about how hyper-octonion analytic maps of M^8 to itself correspond to four-surfaces in $M^4 \times CP_2$ and perhaps also absolute minima of Kähler action, it is clear that the hyper-octonionic polynomials defined by the infinite primes at the first level of hierarchy indeed define a foliations of $M^4 \times CP_2$ by four-dimensional surfaces with an additional degeneracy corresponding to the possibility to choose freely the map $f : HO \rightarrow S^6$ characterizing the choice of preferred imaginary octonionic unit, or equivalently the plane defined by time-like polarizations. There is also a degeneracy related to the choice of the origin of M^8 coordinates and due to the $SO(7,1)$ invariance acting at the level of $M^8 = HO$.

The basic objection is that the polynomials representing infinite are ill defined at the higher levels of hierarchy due to the problems caused by non-associativity even in case that one restricts the consideration to rational functions with real coefficients. The only resolution of this objection is that the arguments h_i are functionally independent so that one can express h_i, $i > 1$ as hyper-octonion real-analytic function of h_1. Rational functions look especially natural and one can consider also the identification $h_n = h_{n-1} = ... = h_1$.

This assumption reduces the representation to one-dimensional case and if hyper-octonion real-analytic functions define foliations of imbedding space by quaternionic space-time surfaces, one obtains a hierarchy of increasingly complex space-time surfaces. An open question is whether the hierarchy of infinite primes indeed corresponds to a hierarchy of space-time sheets.

The requirement that the theory allows p-adicization is not only a challenge but also a heavy constraint. If everything is rational at the basic level in the proposed sense, there are indeed good hopes for the p-adicization at space-time level. This optimistic view is also encouraged by the recent formulation of quantum TGD as almost topological conformal field theory [C1].

The ordering of the arguments of the polynomials characterizes the thoughts about thoughts hierarchy as a hierarchy in which algebraic complexity increases and, as already noticed, also the Slaving Hierarchy. h_n corresponds to the highest level of the hierarchy and h_1 to its lowest level. Topological condensate indeed gives rise to this kind of hierarchy very naturally. This hierarchy is not lost even in the reduction of variables to single hyper-octonionic variable.

The identification allows a generalization of the basic philosophy of algebraic geometry. The rational functions associated with infinite primes have natural ordering with respect to their degree and dimension of algebraic extension of rationals associated with the roots of these polynomials. This makes sense for both functions of n complex arguments and single hyper-octonionic argument. Hence the space-time surfaces can be ordered in a natural manner with respect to their algebraic complexity. One could hope that this kind of ordering might be of decisive help in the physical interpretation of the predictions of the theory.

The most elegant theory results if all infinite primes are assumed to reduce to rational infinite primes and that the decomposition to primes in algebraic completions of rationals and to quaternionic, octonionic, hyper-octonionic infinite primes and their variants in the complexification of quaternions and octonions reflects to or is at least analogous to the possibility to decompose a particle into its more elementary constituents. One might hope that number theoretic analog of color confinement translates to a deep physical principle.

Interaction between infinite primes fixes the scaling of the polynomials associated with infinite primes

The assignment of a polynomial with an infinite prime is unique only up to an over-all scaling and the following argument suggests that the only physically acceptable scaling corresponds to the normalization of the constant term, call it c, of the polynomial to $c = 1$.

In algebraic geometry the zeros of polynomials as their representations has the property that the product of polynomials corresponds to a union of disjoint surfaces and there is no interaction between the surfaces. For infinite integers represented in terms of hyper-quaternionic surfaces this is not the case. This raises the question whether this state of affairs makes possible a realistic number theoretical description of interactions. This description could the counterpart for the description based on the absolute minima of Kähler action which are not simply disjoint unions of absolute minima associated with two 3-surfaces. It would also be analog for the description of the interaction between different space-time sheets in terms of polynomials defined by higher level infinite primes.

This interaction should be consistent with the idea that the interaction of the systems described by infinite primes is weak in some space-time regions. This is certainly the case if the polynomial approaches constant equal to one. To see what happens consider the product of polynomials associated with two infinite primes. The expectation is that in the regions where second hyper-octonion analytic polynomials P_1 approaches to a constant value, which must be real by real-analyticity, the product of infinite primes defines a 4-surface which resembles the surfaces associated with P_2.

The product of hyper-octonion analytic functions $g_1 = a_1 + b_1\overline{h}$ and $g_2 = a_2 + b_2\overline{h}$ is $a_1a_2 + b_1b_2\overline{h} \cdot \overline{h} + (a_1b_2 - a_2b_1)\overline{h}$. If b_1 approaches to zero, the product behaves as $a_1a_2 + a_1b_2\overline{h}$, so that a_1 should approach to $a_1 = 1$ in order that interaction would be negligible.

The observation would suggest that the mapping of infinite primes to polynomials must involve a scaling taking care that the constant term appearing in the polynomial equals to one. This kind of scaling is of course possible. It would however mean that infinite primes with polynomials for which constant term vanishes are not allowed. This would mean that products of conjugates of infinite primes for which finite part is proportional to a light-like integer are not allowed since in this case the constant term vanishes. This is true if one assumes that hyper-octonionic infinite primes reduce to rational infinite primes.

3.4 How to interpret the infinite hierarchy of infinite primes?

From the foregoing it should be clear that infinite primes might play key role in quantum physics. One can even consider the possibility that physics reduces to a generalized number theory, and that infinite primes are crucial for understanding mathematically consciousness and cognition. Of course, one must leave open the question whether infinite primes really provide really the mathematics of consciousness or whether they are only a beautiful but esoteric mathematical construct. In this spirit the following subsections give only different points of view to the problem with no attempt to a coherent overall view.

3.4.1 Infinite primes and hierarchy of super-symmetric arithmetic quantum field theories

Infinite primes are a generalization of the notion of prime. They turn out to provide number theoretic correlates of both free, interacting and bound states of a super-symmetric arithmetic quantum field theory. The mapping of infinite primes to polynomials in turn allows to assign to infinite prime space-time surface as a geometric correlate. Hence infinite primes serve as a bridge between classical and quantum and realize quantum classical correspondence stating that quantum states have classical counterparts, and has served as a basic heuristic guideline of TGD. More precisely, the natural hypothesis is that infinite primes code for the ground states of super-canonical representations (for instance, ordinary particles correspond to states of this kind).

Generating infinite primes as counterparts of Fock states of a super-symmetric arithmetic quantum field theory

The basic construction recipe for infinite primes is simple and generalizes to the quaternionic case.

1. Form the product of all primes and call it X:

$$X = \prod_p p \ .$$

2. Form the vacuum states

$$V_{\pm} = X \pm 1 \ .$$

3. From these vacua construct all *generating* infinite primes by the following process. Kick out from the Dirac sea some negative energy fermions: they correspond to a product s of first powers of primes: $V \rightarrow X/s \pm s$ (s is thus square-free integer). This state represents a state with some fermions represented as holes in Dirac sea but no bosons. Add bosons by multiplying by integer r, which decomposes into parts as $r = mn$: m corresponding to bosons in X/s is product of powers of primes dividing X/s and n corresponds to bosons in s and is product of powers of primes dividing s. This step can be described as $X/s \pm s \rightarrow mX/s \pm ns$.

Generating infinite primes are thus in one-one correspondence with the Fock states of a super-symmetric arithmetic quantum field theory and can be written as

$$P_{\pm}(m,n,s) = \frac{mX}{s} \pm ns \ ,$$

where X is product of all primes at previous level. s is square free integer. m and n have no common factors, and neither m and s nor n and X/s have common factors.

The physical analog of the process is the creation of Fock states of a super-symmetric arithmetic quantum field theory. The factorization of s to a product of first powers of primes corresponds to many-fermion state and the decomposition of m and n to products of powers of prime correspond to bosonic Fock states since p^k corresponds to k-particle state in arithmetic quantum field theory.

More complex infinite primes as counterparts of bound states

Generating infinite primes are not all that are possible. One can construct also polynomials of the generating primes and under certain conditions these polynomials are non-divisible by both finite primes and infinite primes already constructed.

The physical counterpart of n:th order irreducible polynomial is as a bound state of n particles whereas infinite integers constructed as products of infinite primes correspond to non-bound but interacting states. This process can be repeated at the higher levels by defining the vacuum state to be the product of all primes at previous levels and repeating the process. A repeated second quantization of a super-symmetric arithmetic quantum field theory is in question.

The fact that more general infinite primes can be constructed as polynomials of the generating infinite primes, suggest strongly that infinite primes can be mapped to ordinary polynomials by replacing the argument X in $V_{\pm} = X \pm 1$ with variable h. This indeed turns out to be the case. This correspondence allows to deduce that more general infinite primes correspond to irreducible polynomials of generating infinite primes not allowing decomposition to a product of generating infinite primes.

The infinite primes represented by irreducible polynomials correspond to quantum states obtained by mapping the superposition of the products of the generating infinite primes to a superposition of the corresponding Fock states. If complex rationals are the coefficient field for infinite integers, this gives rise to states in a complex Hilbert space and irreducibility corresponds to a superposition of states with varying particle number and the presence of entanglement. For instance, the superpositions of several products of type $\prod_{i=1,..,n} P_i$ of n generating infinite primes are possible and in general give rise to irreducible infinite primes decomposing into a product of infinite primes in algebraic extension of rationals.

3.4.2 Prime Hilbert spaces and infinite primes

There is a result of quantum information science providing an additional reason why for p-adic physics. Suppose that one has N-dimensional Hilbert space which allows $N+1$ unbiased basis. This means that the moduli squared for the inner product of any two states belonging to different basis equals to $1/N$. If one knows all transition amplitudes from a given state to all states of all $N+1$ mutually unbiased basis, one can fully reconstruct the state. For $N = p^n$ dimensional $N+1$ unbiased basis can be found and the article of Durt[57] gives an explicit construction of these basis by applying the properties of finite fields. Thus state spaces with p^n elements - which indeed emerge naturally in p-adic framework - would be optimal for quantum tomography. For instance, the discretization of one-dimensional line with length of p^n units would give rise to p^n-dimensional Hilbert space of wave functions.

The observation motivates the introduction of prime Hilbert space as as a Hilbert space possessing dimension which is prime and it would seem that this kind of number theoretical structure for the category of Hilbert spaces is natural from the point of view of quantum information theory. One might ask whether the tensor product of mutually unbiased bases in the general case could be constructed as a tensor product for the bases for prime power factors. This can be done but since the bases cannot have common elements the number of unbiased basis obtained in this manner is equal to $M+1$, where M is the smallest prime power factor of N. It is not known whether additional unbiased bases exists.

Hierarchy of prime Hilbert spaces characterized by infinite primes

The notion of prime Hilbert space provides also a new interpretation for infinite primes, which are in 1-1 correspondence with the states of a supersymmetric arithmetic QFT. The earlier interpretation was that the hierarchy of infinite primes corresponds to a hierarchy of quantum states. Infinite primes could also label a hierarchy of infinite-D prime Hilbert spaces with product and sum for infinite primes representing unfaithfully tensor product and direct sum.

1. At the lowest level of hierarchy one could interpret infinite primes as homomorphisms of Hilbert spaces to generalized integers (tensor product and direct sum mapped to product and sum) obtained as direct sum of infinite-D Hilbert space and finite-D Hilbert space. (In)finite-D Hilbert space is (in)finite tensor product of prime power factors. The map of N-dimensional Hilbert space to the set of N-orthogonal states resulting in state function reduction maps it to N-element set and integer N. Hence one can interpret the homomorphism as giving rise to a kind of shadow on the wall of Plato's cave projecting (shadow quite literally!) the Hilbert space to generalized integer representing the shadow. In category theoretical setting one could perhaps see generalize integers as shadows of the hierarchy of Hilbert spaces.

2. The interpretation as a decomposition of the universe to a subsystem plus environment does not seem to work since in this case one would have tensor product. Perhaps the decomposition could be to degrees of freedom to those which are above and below measurement resolution. One could of course consider decomposition to a tensor product of bosonic and fermionic state spaces.

3. The construction of the Hilbert spaces would reduce to that of infinite primes. The analog of the fermionic sea would be infinite-D Hilbert space which is tensor product of all prime Hilbert spaces H_p with given prime factor appearing only once in the tensor product. One can "add n bosons" to this state by replacing of any tensor factor H_p with its n+1:th tensor power. One can "add fermions" to this state by deleting some prime factors H_p from the tensor product and adding their tensor product as a finite-direct summand. One can also "add n bosons" to this factor.

4. At the next level of hierarchy one would form infinite tensor product of all infinite-dimensional prime Hilbert spaces obtained in this manner and repeat the construction. This can be continued ad infinitum and the construction corresponds to abstraction hierarchy or a hierarchy of statements about statements or a hierarchy of n:th order logics. Or a hierarchy of space-time sheets of many-sheeted space-time. Or a hierarchy of particles in which certain many-particle states at the previous level of hierarchy become particles at the new level (bosons and fermions). There are many interpretations.

5. Note that at the lowest level this construction can be applies also to Riemann Zeta function. ζ would represent fermionic vacuum and the addition of fermions would correspond to a removal of a product of corresponding factors ζ_p from ζ and addition of them to the resulting truncated ζ function. The addition of bosons would correspond to multiplication by a power of appropriate ζ_p. The analog of ζ function at the next level of hierarchy would be product of all these modified ζ functions and might well fail to exist since the product might typically converge to either zero or infinity.

Hilbert spaces assignable to infinite integers and rationals make also sense

1. Also infinite integers make sense since one can form tensor products and direct sums of infinite primes and of corresponding Hilbert spaces. Also infinite rationals exist and this raises the question what kind of state spaces inverses of infinite integers mean.

2. Zero energy ontology suggests that infinite integers correspond to positive energy states and their inverses to negative energy states. Zero energy states would be always infinite rationals with real norm which equals to real unit.

3. The existence of these units would give for a given real number an infinite rich number theoretic anatomy so that single space-time point might be able to represent quantum states of the entire universe in its anatomy (number theoretical Brahman=Atman). Also the world of classical worlds (light-like 3-surfaces of the imbedding space) might be imbeddable to this anatomy so that basically one would have just space-time surfaces in 8-D space and configuration space would have representation in terms of space-time based on generalized notion of number. Note that infinitesimals around a given number would be replaced with infinite number of number-theoretically non-equivalent real units multiplying it.

Should one generalize the notion of von Neumann algebra?

Especially interesting are the implications of the notion of prime Hilbert space concerning the notion of von Neumann algebra -in particular the notion of hyper-finite factors of type II_1 playing a key role in TGD framework. Does the prime decomposition bring in additional structure? Hyper-finite factors of type II_1 are canonically represented as infinite tensor power of 2×2 matrix algebra having a representation as infinite-dimensional fermionic Fock oscillator algebra and allowing a natural interpretation in terms of spinors for the world of classical worlds having a representation as infinite-dimensional fermionic Fock space.

Infinite primes would correspond to something different: a tensor product of all $p\times p$ matrix algebras from which some factors are deleted and added back as direct summands. Besides this some factors are replaced with their tensor powers. Should one refine the notion of von Neumann algebra so that one can distinguish between these algebras as physically non-equivalent? Is the full algebra tensor product of this kind of generalized hyper-finite factor and hyper-finite factor of type II_1 corresponding to the vibrational degrees of freedom of 3-surface and fermionic degrees of freedom? Could p-adic length scale hypothesis - stating that the physically favored primes are near powers of 2 - relate somehow to the naturality of the inclusions of generalized von Neumann algebras to HFF of type II_1?

3.4.3 Do infinite hyper-octonionic primes represent quantum numbers associated with Fock states?

Hyper-octonionic primes involve so much structure that one can seriously consider the possibility that they could code quantum numbers of elementary particles which in accordance with quantum-classical correspondence would be coded to the shape of space-time surfaces.

Hyper-octonionic infinite primes as representations for quantum numbers of Fock states?

Configuration space spinor fields assign infinite number of quantum states to a given 3-surface as components of configuration space spinor. This suggests that there cannot be one-to-one correspondence between Fock states and space-time surfaces except in the approximation that one replaces configuration space spinor field with single 'quantum average space-time'. This forces to consider critically the identification of the hyper-octonionic primes as quantum numbers.

Perhaps a more realistic identification of infinite primes is as coding for the quantum numbers for the ground states of the representations of super-canonical and Kac-Moody algebras. This identification would be in an agreement with the view that space-time surfaces represent only the classical aspects of physics but not quantum fluctuations. Arithmetic quantum field theory should represent only the sector of ground states of quantum TGD.

It is interesting to check whether hyper-octonionic infinite primes could allow a realistic coding for the quantum numbers of ground states of super Kac-Moody representations.

1. If it is assumed that each prime in the finite part of X corresponds to a fermion, the requirement that the Fock state possesses a well-defined fermion number poses constraints on the structure of the polynomial associated with the infinite prime. A product of generating infinite primes in algebraic extension of real-rationals interpreted as representing states for which rest mass is changed by bound state interactions, would however resolve these constraints. Also superpositions of products are allowed but in this case net fermion numbers associated with various monomials must be same.

2. Hyper-octonionic infinite prime could be interpreted as coding for the relationship between particle four-momentum represented by the hyper-quaternionic part of infinite prime and the quantum numbers associated with CP_2 degrees of freedom represented by the quaternionic part of the infinite prime. Electro-weak isospin and hyper charge and corresponding color quantum numbers indeed give rise to four quantum numbers.

   Mass squared formula for infinite primes, and more generally, infinite integers would be the basic string mass formula. For bound states the mass squared values would be primes in algebraic extension of rationals.

3. Space-like hyper-octonionic primes do not seem to be natural in the case of hyper-octonionic option. Octonionic option would allow them but in this case the interpretation in terms of momenta is lost. This not so plausible option would allow as a special case Gaussian and Eisenstein primes discussed in [E8]. Eisenstein primes correspond to algebraic extension involving $\sqrt{3}$. These primes correspond to time-like primes obtained by multiplying the prime with a suitable unit. The degeneracies of these primes due to units defined by complex phases are 4 and 8. One can ask whether these degeneracies might relate to the spin states of imbedding space spinors.

4. If the proposed interpretation is taken at face value, the question about distinction between quarks and leptons at the level of infinite primes, arises. Somehow the two different chiralities for induced imbedding space spinor fields should have space-time correlates. If the primes $p\ mod\ 4 = 1$ and $p\ mod\ 4 = 3$ correspond to leptons and quarks or vice versa it would be possible to assign to each generating infinite prime lepton or quark number. Bosons could be regarded as fermion-antifermion bound states and bosonic surfaces would correspond to the composites of two infinite primes with either $p\ mod\ 4 = 1$ or $p\ mod\ 4 = 3$ or superposition of this kind of monomials.

5. Since only polynomials with real coefficients are possible, kind of number theoretic analog of color confinement occurs, and requires that at least two generating infinite primes with the hyper-octonionic zero of the corresponding monomial with components belonging to an algebraic extension of real rationals appears in the state. This confinement has counterpart at the level of super-canonical conformal weights which are complex and expressible in terms of zeros of Riemann Zeta: only states with real net conformal weight are possible.

6. One can imagine several interpretations for the two vacua $V_{\pm} = X \pm 1$.
   i) The most plausible interpretation for these vacua is in terms of matter and antimatter and thus as representations for states having opposite fermion number. In number theoretic bound states represented by higher degree polynomials both matter and antimatter particles can occur.
   ii) A less plausible interpretation is as positive and negative energy vacua associated with the space-time sheets of opposite time orientation predicted by TGD. The fact that negative energy particles do not seem to appear in elementary particle reactions inspires the hypothesis that negative energies are associated with higher level infinite primes and correspond to the infinite primes defining the denominators of the rational functions appearing in the definitions of higher level infinite primes. Phase conjugate photons would be a basic example of negative energy particles.
   iii) Also the interpretation in terms of the vacua of associated Ramond and NS type super canonical algebras can be considered.

There are also other degrees of freedom besides Super Kac Moody degrees fo freedom.

1. Zero modes are an essential part of TGD and would correspond to the degrees of freedom associated with the maps $HO \rightarrow S^6$ and their generalization to the higher levels of the hierarchy.

Physical interpretation would be as a imbedding space dependent selection of longitudinal degrees of freedom in turn fixing at space-time level the spin quantization axis and the transversal degrees of freedom associated with polarizations of massless particles.

2. There is no obvious relation between super-canonical conformal weights and infinite primes. Perhaps the reason is that these quantum numbers are associated with configuration space spinor fields.

Family replication phenomenon and commutative sub-manifolds of space-time surface

The idea that complex Abelian sub-manifolds of space-time sheets are in preferred role by their commutativity in hyper-octonionic sense, is consistent with the topological explanation of family replication phenomenon [F1] by interpreting different particle families as particles with corresponding 3-surface having boundary with genus $g = 0, 1, 2, ...$

The representations $p = f(q)$ of the algebraic surfaces with real-analytic f, when restricted to complex numbers, define 2-dimensional Riemann surfaces in 4-dimensional complex space. These surfaces are characterized by genus so that genus emerges in very natural manner from the theory.

If the boundary component has same genus as the genus defined by hyper-quaternionicity, then the notion of elementary particle vacuum functional makes sense, and p-adic mass calculations [F2, F3, F4, F5] which rely crucially on the notion of genus, remain unchanged. natural possibility is that the 2-surface where hyper-quaternions are commutative in fact corresponds to a boundary component of 3-surface. The 2-dimensional intersections of 3-D light-like causal determinants X^3_l and 7-D light-like causal determinants defined by boundaries of future and past light-cones of M^4 are natural candidates for partonic 2-surfaces. If this picture is correct, one can also answer the troublesome question 'What is the two-dimensional sub-manifold of 3-dimensional boundary of space-time surface to which one assigns elementary particle vacuum functional?'. This question is of high relevance since the conformal equivalence class of boundary component depends on how the boundary component is identified.

3.4.4 The physical interpretation of infinite integers at the first level of the hierarchy

The idea that primes are for the number theory what elementary particles are for physics, suggests that the decomposition of an infinite integer to a product of infinite primes corresponds to the decomposition of a physical system to elementary systems allowing no further decomposition.

Higher degree polynomial primes as bound states

The sums for the products of infinite primes defining irreducible polynomials define infinite primes describing many particle states and the interpretation as composites of space-time surfaces associated with simpler 'effective' generating infinite primes belonging to the extension of quaternions is natural and leads to a dynamical generation of algebraic symmetries. A natural interpretation is as topological composites formed from space-time surfaces describing bound states. Each root of the polynomial equation defining a branch of the space-time surface would correspond to a particle present in the composite. Indeed, n:th order irreducible polynomial factors to product of monomials $x - l$, $l \notin K$. If the polynomial differs only slightly from a product of prime polynomials, it is natural to interpret the slight change of the roots as a slight change of the composite states induced by the mutual interaction.

Infinite integers as interacting many particle states

The space-time surfaces representing infinite integers could represent many-particle states. The space-time surface associated with the integer is in general not a union of the space-time surfaces associated with the primes composing the integer. This means that classical description of interactions emerges automatically. The description of classical states in terms of infinite integers is completely analogous to the description of many particle states as finite integers in arithmetic quantum field theory.

The finite primes which correspond to particles of an arithmetic quantum field theory present in Fock state, correspond to the space-time sheets of finite size serving as the building blocks of the space-time sheet characterized by infinite prime. Real topology is the space-time topology in the regions, where matter resides whereas 'mind stuff' corresponds to the regions obeying p-adic topology. This is in accordance with the fact that the physics based on real numbers is so successful. The success of p-adic physics could be understood as resulting from the fact that it describes the physics of the mind like regions mimicking the physics of the real matter-like regions.

3.4.5 What is the interpretation of the higher level infinite primes?

Interesting questions are related to the higher level infinite primes obtained by taking X to be a product of all lower level primes and repeating the construction.

Infinite hierarchy of infinite primes

Infinite hierarchy of infinite primes codes for a hierarchy of Fock states such that many-particle Fock states of a given level serve as elementary particles at next level. The unavoidable conclusion is that higher levels represent totally new physics not described by the standard quantization procedures. In particular, the assignment of fermion/boson property to arbitrarily large system would be in some sense exact. Topologically these higher level particles could correspond to space-time sheets containing many-particle states and behaving as higher level elementary particles.

This view suggests that the generating quantum numbers are present already at the lowest level and somehow coded by the hyper-octonionic primes taking the role of momentum quantum number they have in arithmetic quantum field theories. The task is to understand whether and how hyper-octonionic primes can code for quantum numbers predicted by quantum TGD.

The quantum numbers coding higher level states are collections of quantum numbers of lower level states. At geometric level the replacement of the coefficients of polynomials with rational functions is the equivalent of replacing single particle states with new single particle states consisting of many-particle states.

Rationals of the previous level appear at given level

What is remarkable is that the rationals formed from the integers of $n-1$:th level label the simplest primes of n:th level. The numerator and denominator of the rational number correspond to a pair of integers representing physical states at previous level, which suggests that the new states are higher level physical states representing information about pairs of physical states at the previous level. The most natural guess is that the states of the pair correspond to the initial and final states of a quantum jump. In this manner the infinite hierarchy give rise to physical states representing increasingly abstract information about dynamics. The fact that I am a physical system ponder physics problems could be seen as a direct evidence for the existence for these higher levels of physical existence.

At the next level physical states represent information about pairs of quantum jumps which in TGD inspired theory of consciousness correspond to memories about primary conscious experiences determined by quantum jumps. They clearly represent experiences about experiences. At n:th level quantum jump represent n-fold abstraction giving conscious information about experiences about.....about experiences.

TGD allows space-time sheets with both positive and negative time orientation and the sign of classical energy correlates with the orientation of the space-time sheet. This leads to a radical revision of the energy concept and clarifies the relationship between gravitational and inertial energy. The interpretation of the numerator and denominator of the infinite rational in terms of positive and negative energy space-time sheets looks natural. Of course, one must be ready to consider the possibility that "energy" might be replaced by some other conserved quantity. This interpretation would also explain why negative energy particles appear only at higher organization level of matter and are not detected in accelerators. Indeed, the basic TGD applications relate to quantum biology, consciousness [K1], and free energy [G2].

The interpretation of particle reactions as quantum jumps between zero energy states is implied by this vision, and this interpretation is consistent with crossing symmetry. Zero energy states can be seen also as representations of quantum jumps with positive and negative energy components of the state identifiable as counterparts of initial and final states. One could say that all states of the entire Universe, even at classical space-time level, represent reflective level of existence, being always about something. Only in the approximation that positive and negative energy components of the state do not interact the western view about objective reality with conserved energy makes sense.

3.4.6 Infinite primes and the structure of many-sheeted space-time

The mapping of infinite primes to space-time surfaces codes the structure of infinite prime to the structure of space-time surface in rather non-implicit manner, and the question arises about the concrete correspondence between the structure of infinite prime and topological structure of the space-time surface.

A possible interpretation for the lowest level infinite primes

The concrete prediction of the general vision is that the hierarchy of infinite primes should correspond to the hierarchy of space-time sheets. The challenge is to find space-time counterparts for infinite primes at the lowest level of hierarchy.

One could hope that the Fock space structure of infinite prime would have a more concrete correspondence with the structure of the many-sheeted space-time. One might that the space-time sheets labeled by primes p would directly correspond to the primes appearing in the definition of infinite prime. This expectation seems to be too simplistic.

1. What seems to be a safe guess is that the simplest infinite primes at the lowest level of the hierarchy should correspond to elementary particles. If inverses of infinite primes correspond to negative energy space-time sheets, this would explain why negative energy particles are not encountered in elementary particle physics.

2. More complex infinite primes at the lowest level of the hierarchy could be interpreted in terms of structures formed by connecting these structures by join along boundaries bonds to get space-time correlates of bound states. Even simplest infinite primes must correspond to bound state structures if the condition that the corresponding polynomial has real-rational coefficients is taken seriously.

Infinite primes at lowest level of hierarchy correspond to several finite primes rather than single finite prime. The number of finite primes is however finite. This conforms with the idea that this level indeed corresponds to space-time sheets associated with elementary particles.

1. A possible interpretation for multi-p property is in terms of multi-p p-adic fractality prevailing in the interior of space-time surface. The effective p-adic topology of these space-time sheets would depend on length scale. In the longest scale the topology would correspond to p_n, in some shorter length scale there would be smaller structures with $p_{n-1} < p_n$-adic topology, and so on... . A good metaphor would be a wave containing ripples, which in turn would contain still smaller ripples. The multi-p p-adic fractality would be assigned with the 4-D space-time sheets associated with elementary particles.

2. Effective 2-dimensionality would suggest that p-adic topologies could be assigned with the 2-dimensional partonic surfaces or corresponding 3-D light-like causal determinants. Thus infinite prime would characterize at the lowest level space-time sheet and corresponding partonic 2-surfaces. This interpretation is consistent with the fact that modified Dirac operator assigns to its generalized eigen modes p-adic prime p characterizing the p-adic topology of corresponding p-adic parton obeying same algebraic equations.

How to interpret higher level infinite primes?

A possible interpretation for higher level infinite primes is in terms of q-adicity assignable to the function spaces defined by the rational functions assignable to them. The role of finite prime p would be taken by the rational function defined by the infinite prime. This interpretation makes sense both when one assigns to infinite primes functions of rational arguments $q_1, ...q_n$ or when one identifies these arguments. This function space is q-adic for some rational number q. At the lowest level the infinite prime indeed defines naturally an ordinary rational number.

At higher levels of the hierarchy one can assign to infinite prime an infinite rational number of previous level. By continuing the assignments of lower level rationals to the infinite primes appearing in this infinite rational one ends up with an assignment of a unique rational number with a given infinite prime. This rational serves as a good candidate for a rational defining the q-adicity. The question is whether this q-adicity can be assign with space-time topology or some function space topology.

1. The modified Dirac operator associated with a partonic 2-surface assignable to the largest space-time sheet of topological condensation hierarchy would naturally assign q to its eigen modes. It is however not clear whether one can assign to partonic 2-surface characterized by algebraic equations unique q-adic space-time sheet. The problem is that q-adic numbers do not form number field so that the algebraic equations defining the partonic 2-surface need not make sense.

2. The q-adic function spaces might have a natural interpretation in terms of the fields assignable to the space-time sheet by replacing complex argument with quaternionic one. One possible interpretation is that primes appearing in the lowest level infinite prime correspond to partonic 2-surfaces and infinite prime itself defines q-adic topology for a functions space assignable to the space-time sheet. The q-adic topology associated with the function space associated with a space-time sheet containing topologically condensed space-time sheets would be characterized by the infinite prime and corresponding polynomial determined by the infinite primes associated with the topologically condensed space-time sheets that it contains. Note that the modified Dirac operator would assign to partonic 2-surfaces at all levels of hierarchy a p-adic prime.

3. Quantum criticality suggests strongly that configuration space of 3-surfaces effectively reduces to discrete spin glass energy landscape corresponding to the maxima of Kähler function. Spin glass property suggests strongly that this space obeys ultrametric topology. Therefore a natural conjecture is that the q-adic topology can be assigned with this space.

3.4.7 How infinite integers could correspond to p-adic effective topologies?

Besides the hierarchy of space-time sheets, TGD predicts, or at least suggests, several hierarchies such as the hierarchy of infinite primes, hierarchy of Jones inclusions [C6], dark matter hierarchy characterized by increasing values of $\hbar$ [F9, J6], the hierarchy of extensions of given p-adic number field, and the hierarchy of selves and quantum jumps with increasing duration with respect to geometric time. There are good reasons to expect that these hierarchies are closely related. Number theoretical considerations allow to develop more quantitative vision about the relationship between the hierarchy of infinite primes and p-adic length scale hierarchy.

How to define the notion of elementary particle?

p-Adic length scale hierarchy forces to reconsider carefully also the notion of elementary particle. p-Adic mass calculations led to the idea that particle can be characterized uniquely by single p-adic prime characterizing its mass squared [F3, F4, F5]. It however turned out that the situation is probably not so simple.

The work with modelling dark matter suggests that particle could be characterized by a collection of p-adic primes to which one can assign weak, color, em, gravitational interactions, and possibly also other interactions [F6, F8, F9]. It would also seem that only the space-time sheets containing common primes in this collection can interact. This leads to the notions of relative and partial darkness. An entire hierarchy of weak and color physics such that weak bosons and gluons of given physics are characterized by a given p-adic prime p and also the fermions of this physics contain space-time sheet characterized by same p-adic prime, say M_{89} as in case of weak interactions. In this picture the decay widths of weak bosons do not pose limitations on the number of light particles if weak interactions for them are characterized by p-adic prime $p \neq M_{89}$. Same applies to color interactions.

The p-adic prime characterizing the mass of the particle would perhaps correspond to the largest p-adic prime associated with the particle. Graviton which corresponds to infinitely long ranged interactions, could correspond to the same p-adic prime or collection of them common to all particles. This might apply also to photons. Infinite range might mean that the join along boundaries bonds mediating these interactions can be arbitrarily long but their transversal sizes are characterized by the p-adic length scale in question.

The natural question is what this collection of p-adic primes characterizing particle means? The hint about the correct answer comes from the number theoretical vision, which suggests that at fundamental level the branching of boundary components to two or more components, completely analogous to the branching of line in Feynman diagram, defines vertices [C2].

1. If space-time sheets correspond holographically to multi-p p-adic topology such that largest p determines the mass scale, the description of particle reactions in terms of branchings indeed makes sense. This picture allows also to understand the existence of different scaled up copies of QCD and weak physics. Multi-p p-adicity could number theoretically correspond to q-adic topology for $q = m/n$ a rational number consistent with p-adic topologies associated with prime factors of m and n ($1/p$-adic topology is homeomorphic with p-adic topology).

2. One could also imagine that different p-adic primes in the collection correspond to different space-time sheets condensed at a larger space-time sheet or boundary components of a given space-time sheet. If the boundary topologies for gauge bosons are completely mixed, as the model of hadrons forces to conclude, this picture is consistent with the topological explanation of the family replication phenomenon and the fact that only charged weak currents involve mixing of quark families. The problem is how to understand the existence of different copies of say QCD. The second difficult question is why the branching leads always to an emission of gauge boson characterized by a particular p-adic prime, say M_{89}, if this p-adic prime does not somehow characterize also the particle itself.

What effective p-adic topology really means?

The need to characterize elementary particle p-adically leads to the question what p-adic effective topology really means. p-Adic mass calculations leave actually a lot of room concerning the answer to this question.

1. At the fundamental level this problem seems to be well understood now. By the almost topological QFT property of quantum real and p-adic variants of light-like partonic 3-surfaces can satisfy same algebraic equations. Modified Dirac operator assigns well-defined p-adic prime p to its eigenmodes with non-vanishing eigenvalues. Zero modes are an exception.

2. The naivest option would be that each space-time sheet corresponds to single p-adic prime. This view is not favored by the view that each particle corresponds to a collection of p-adic primes each characterizing one particular interaction that the particle in question participates. A more natural possibility is that the boundary components of space-time sheet, and more generally, light-like 3-surfaces serving as causal determinants, correspond to different p-adic primes.

3. This implies that a given space-time sheet to several p-adic primes. Indeed, a power series in powers of given integer n gives rise to a well-defined power series with respect to all prime factors of n and effective multi-p-adicity could emerge at the level of field equations in this manner in the interior of space-time sheets. One could say that space-time sheet corresponds to several p-adic primes through its effective p-adic topology in a hologram like manner. This option is the most natural as far as physical interpretation is considered. It is also supported by the number theoretical considerations predicting the value of gravitational coupling constant.

An attractive hypothesis is that only space-time sheets characterized by integers n_i having common prime factors can be connected by join along boundaries bonds and can interact by particle exchanges and that each prime p in the decomposition corresponds to a particular interaction mediated by an elementary boson characterized by this prime.

Do infinite primes code for effective q-adic space-time topologies?

As found, one can assign to a given infinite prime a rational number. The most obvious question concerns the possible space-time interpretation of this rational number. Also the question arises about the possible relation with the integers characterizing space-time sheets having interpretation in terms of multi-p-adicity. On can assign to any rational number $q = m/n$ so called q-adic topology. This topology is not consistent with number field property like p-adic topologies. Hence the rational number q assignable to infinite prime could correspond to an effective q-adic topology.

If this interpretation is correct, arithmetic fermion and boson numbers could be coded into effective q-adic topology of the space-time sheets characterizing the non-determinism of Kähler action in the relevant length scale range. For instance, the power series of $q > 1$ in positive powers with integer coefficients in the range $[0, q)$ define q-adically converging series, which also converges with respect to the prime factors of m and can be regarded as a p-adic power series. The power series of q in negative powers define in similar converging series with respect to the prime factors of n.

I have proposed earlier that the integers defining infinite rationals and thus also the integers m and n characterizing finite rational could correspond at space-time level to particles with positive *resp.* negative time orientation with positive *resp.* negative energies. Phase conjugate laser beams would represent one example of negative energy states. With this interpretation super-symmetry exchanging the roles of m and n and thus the role of fermionic and bosonic lower level primes would correspond to a time reversal.

1. The first interpretation is that there is single q-adic space-time sheet and that positive and negative energy states correspond to primes associated with m and n respectively. Positive (negative) energy space-time sheets would thus correspond to p-adicity ($1/p$-adicity) for the field modes describing the states.

2. Second interpretation is that particle (in extremely general sense that entire universe can be regarded as a particle) corresponds to a pair of positive and negative energy space-time sheets labeled by m and n characterizing the p-adic topologies consistent with $m-$ and n-adicities. This looks natural since Universe has necessary vanishing net quantum numbers. Unless one allows the non-uniqueness due to $m/n = mr/nr$, positive and negative energy space-time sheets can be connected only by $\#$ contacts so that positive and negative energy space-time sheets cannot interact via the formation of $\#_B$ contacts and would be therefore dark matter with respect to each other. Antiparticles would also have different mass scales. If the rate for the creation of $\#$ contacts and their CP conjugates are slightly different, say due to the presence of electric components of gauge fields, matter antimatter asymmetry could be generated primordially.

These interpretations generalize to higher levels of the hierarchy. There is a homomorphism from infinite rationals to finite rationals. One can assign to a product of infinite primes the product of the corresponding rationals at the lower level and to a sum of products of infinite primes the sum of the corresponding rationals at the lower level and continue the process until one ends up with a finite rational. Same applies to infinite rationals. The resulting rational $q = m/n$ is finite and defines q-adic effective topology, which is consistent with all the effective p-adic topologies corresponding to the primes appearing in factorizations of m and n. This homomorphism is of course not 1-1.

q would associate with the particle q-adic topology consistent with a collection of p-adic topologies corresponding to the prime factors of m and n and characterizing the interactions that the particle can participate directly. In a very precise sense particles would represent both infinite and finite numbers.

Under what conditions boundary components can be connected by $\#_B$ contact?

Assume that particles are characterized by a p-adic prime determining it mass scale plus p-adic primes characterizing the gauge bosons to which they couple and assume that $\#_B$ contacts mediate gauge interactions. Assume that these primes label the boundary components of the space-time sheet representing the particle or more general light-like 3-surfaces. The question is what kind of space-time sheets can be connected by $\#_B$ contacts.

The first working hypothesis that comes in mind is that the p-adic primes associated with the two boundary components connected by $\#_B$ contact must be identical. If the notion of multi-p p-adicity is accepted, space-time sheets are characterized by integers and the largest prime dividing the integer might characterize the mass of the particle. This makes sense if the p-adic temperature $T = 1/n$ associated with small primes is small enough. In this case a common prime factor p for the integers characterizing the two space-time sheets could be enough for the possibility of $\#_B$ contact and this contact would be characterized by this prime. If no common prime factors exist, only $\#$ contacts could connect the space-time sheets. This option conforms with the number theoretical vision. This option would predict that the transition to large $\hbar$ phase occurs simultaneously for all interactions.

What about the integer characterizing graviton?

If one accepts the hypothesis that graviton couples to both visible and dark matter, graviton should be characterized by an integer dividing the integers characterizing all particles. This leaves two options.

Option I: gravitational constant characterizes graviton number theoretically

The argument leading to an expression for gravitational constant in terms of CP_2 length scale led to the proposal that the product of primes $p \leq 23$ are common to all particles and one interpretation was in terms of multi-fractality. If so, graviton would be characterized by a product of some or all primes $p \leq 23$ and would thus correspond to a very small p-adic length scale. This might be also the case for photon although it would seem that photon cannot couple to dark matter always. $p = 23$ might characterize the transversal size of the massless extremal associated with the space-time sheet of graviton.

Option II: gravitons are characterized by Mersenne prime M_{127}

The arguments related to the model of coupling constant evolution [C4] lead to the proposal that graviton coupling strength behaves as L_p^2 as a function of the p-adic length scale and that effective renormalization group invariance of tge gravitational coupling strength is due to the fact that gravitational interactions are carried by $\#_B$ contacts which correspond to Mersenne prime M_{127}. This would mean that each elementary particle contains partonic 2-surface labeled by M_{127}. This is possible if the p-adic temperature associated with M_{127} is $T = 1/n$, $n > 1$, for all particles lighter than electron so that p-adic thermodynamics does not contribute appreciably to the mass squared of the particle.

Option III: graviton behaves as a unit with respect to multiplication

One can also argue that if the largest prime assignable to a particle characterizes the size of the particle space-time sheet it does not make sense to assign any finite prime to a massless particle like graviton. Perhaps graviton corresponds to simplest possible infinite prime $P = X \pm 1$, X the product of all primes.

As found, one can assign to any infinite prime, integer, and rational a rational number $q = m/n$ to which one can assign a q-adic topology as effective space-time topology and as a special case effective p-adic topologies corresponding to prime factors of m and n.

In the case of $P = X \pm 1$ the rational number would be equal to ± 1. Graviton could thus correspond to $p = 1$-adic effective topology. The "prime" $p = 1$ indeed appears as a factor of any integer so that graviton would couple to any particle. Formally the 1-adic norm of any number would be 1 or 0 which would suggest that a discrete topology is in question.

The following observations help in attempts to interpret this.

1. CP_2 type extremals having interpretation as gravitational instantons are non-deterministic in the sense that M^4 projection is random light-like curve. This condition implies Virasoro conditions which suggests interpretation in terms topological quantum theory limit of gravitation involving vanishing four-momenta but non-vanishing color charges. This theory would represent gravitation at the ultimate CP_2 length scale limit without the effects of topological condensation. In longer length scales a hierarchy of effective theories of gravitation corresponds to the coupling of space-time sheets by join along boundaries bonds would emerge and could give rise to "strong gravities" with strong gravitational constant proportional to L_p^2. It is quite possible that the M-theory based vision about duality between gravitation and gauge interactions applies to electro-weak interactions and in these "strong gravities".

2. p-Adic length scale hypothesis $p \simeq 2^k$, k integer, implies that $L_k \propto \sqrt{k}$ corresponds to the size scale of causal horizon associated with $\#$ contact. For $p = 1$ k would be zero and the causal horizon would contract to a point which would leave only generalized Feynman diagrams consisting of CP_2 type vacuum extremals moving along random light-like orbits and obeying Virasoro conditions so that interpretation as a kind of topological gravity suggests itself.

3. $p = 1$ effective topology could make marginally sense for vacuum extremals with vanishing Kähler form and carrying only gravitational charges. The induced Kähler form vanishes identically by the mere assumption that X^4, be it continuous or discontinuous, belongs to $M^4 \times Y^2$, Y^2 a Lagrange sub-manifold of CP_2.

Why topological graviton, or whatever the particle represented by CP_2 type vacuum extremals should be called, should correspond to the weakest possible notion of continuity? The most plausible answer is that discrete topology is *consistent* with any other topology, in particular with any p-adic topology. This would express the fact that CP_2 type extremals can couple to any p-adic prime. The vacuum property of CP_2 type extremals implies that the splitting off of CP_2 type extremal leaves the physical state invariant and means effectively multiplying integer by $p = 1$.

It seems that Option I suggested by the deduction of the value of gravitational constant looks more plausible as far as the interpretation of gravitation is considered. This does not however mean that CP_2 type vacuum extremals carrying color quantum numbers could not describe gravitational interactions in CP_2 length scale.

3.4.8 An alternative interpretation for the hierarchy of functions defined by infinite primes

Suppose that infinite primes code for the ground states of super-conformal representations. Supersymmetry suggests that the corresponding polynomials or their zeros could code for the moduli space associated with these states. At the limit of algebraic closure of rationals the vanishing of the polynomial would code for a complex codimension one surface of C^n at n:th level of hierarchy.

The recent progress in the understanding of S-matrix [C2] relies on the idea that the data needed to construct S-matrix is provided by the intersection of real and p-adic parton 2-surface obeying same algebraic equations. Quantum TGD is almost topological QFT since only the light-likeness of orbits of partonic 2-surfaces brings in the notion of metric. This leads to the idea that the braiding S-matrices of topological quantum field theories generalize to give a realistic S-matrix in TGD framework. The number theoretical braids at partonic 2-surface for which the strands of the braid project to the same point of the geodesic sphere S^2 of CP_2 play a key role in this approach. Braids are thus characterized by complex numbers labeling the points of S^2.

In this framework the natural idea would be that that the n, in general complex, algebraic numbers, code for the positions of braids and that vanishing of the polynomial gives correlation between the positions of braids so that the position of n^{th} level braid is fixed almost uniquely once the positions of lower level braids are known. One must however admit that this kind of correlation does not look too convincing and that the interpretation involves ad hoc elements such as the selection of the geodesic sphere. It must be however added that infinite primes could allow several mutually consistent interpretations and that this interpretation or some interpretation analogous to it might make sense.

3.5 Does the notion of infinite-P p-adicity make sense?

In this section speculations related to infinite-P p-adicity are represented in the form of shy questions in order to not irritate too much the possible reader. The basic open question causing tension is whether infinite primes relate only to the physics of cognition or whether they might allow to say something non-trivial about the physics of matter too.

The obvious question is whether the notion of p-adic number field makes sense makes sense for infinite primes and whether it might have some physical relevance. One can certainly introduce power series in powers of any infinite prime P and the coefficients can be taken to belong to any ordinary number field. In the representation by polynomials P-Adic power series correspond to Laurent series in powers of corresponding polynomial and are completely finite.

For straightforward generalization of the norm all powers of infinite-P prime have vanishing norm. The infinite-p p-adic norm of infinite-p p-adic integer would be given by its finite part so that in this sense positive powers of P would represent infinitesimals. For Laurent series this would mean that the lowest term would give the whole approximation in the real topology. For finite-primes one could however replace the norm as a power of p by a power of some other number. This would allow to have a finite norm also for P-adic primes. Since the simplest P-adic primes at the lowest level of hierarchy define naturally a rational one might consider the possibility of defining the norm of P as the inverse of this rational.

3.5.1 Does infinite-P p-adicity reduce to q-adicity?

Any non-vanishing p-adic number is expressible as a product of power of p multiplied by a p-adic unit which can be infinite as a normal integer and has pinary expansion in powers of p:

$$x = p^n(x_0 + \sum_{k>0} x_k p^k) \ , \ x_k \in \{0,..,p-1\} \ , \ x_0 > 0 \ . \tag{3.5.1}$$

The p-adic norm of x is given by $N_p(x) = p^{-n}$. Each unit has p-adic inverse which for finite integers is always infinite as an ordinary integer.

To define infinite-P p-adic numbers one must generalize the pinary expansion to a infinite-P p-adic expansion of an infinite rational. In particular, one must identify what the statement 'infinite integer modulo P' means when P is infinite prime, and what are the infinite integers N satisfying the condition $N < P$. Also one must be able to construct the p-adic inverse of any infinite prime. The correspondence of infinite primes with polynomials allows to construct infinite-P p-adics in a straightforward manner.

Consider first the infinite integers at the lowest level.

1. Infinite-P p-adics at the first level of hierarhcy correspond to Laurent series like expansions using an irreducible polynomial P of degree n representing infinite prime. The coefficients of the series are numbers in the coefficient fields. Modulo p operation is replaced with modulo polynomial P operation giving a unique result and one can calculate the coefficients of the expansion in powers of P by the same algorithm as in the case of the ordinary p-adic numbers. In the case of n-variables the coefficients of Taylor series are naturally rational functions of at most $n-1$ variables. For infinite primes this means rationals formed from lower level infinite-primes.

2. Infinite-P p-adic units correspond to expansions of this type having non-vanishing zeroth order term. Polynomials take the role of finite integers. The inverse of a infinite integer in P-adic number field is obtained by developing the polynomial counterpart of $1/N$ in the following manner. Express N in the form $N = N_0(1 + x_1P + ..)$, where N_0 is polynomial with degree at most equal to $n-1$. The factor $1/(1 + x_1P + ...)$ can be developed in geometric series so that only the calculation of $1/N_0$ remains. Calculate first the inverse $\hat{N}_0^{-1}$ of N_0 as an element of the 'finite field' defined by the polynomials modulo P: a polynomial having degree at most equal to $n-1$ results. Express $1/N_0$ as

$$\frac{1}{N_0} = \hat{N}_0^{-1}(1 + y_1P + ...)$$

and calculate the coefficients in the expansion iteratively using the condition $N \times (1/N) = 1$ by applying polynomial modulo arithmetics. Generalizing this, one can develop any rational function to power series with respect to polynomial prime P. The expansion with respect to a polynomial prime can in turn be translated to an expansion with respect to infinite prime and also mapped to a superposition of Fock states.

3. What about the norm of infinite-P p-adic integers? Ultra-metricity suggest a straightforward generalization of the usual p-adic norm. The direct generalization of the finite-p p-adic norm would mean the identification of infinite-P p-adic norm as P^{-n}, where n corresponds to the lowest order term in the polymomial expansion. Thus the norm would be infinite for $n < 0$, equal to one for $n = 0$ and vanish for $n > 0$. Any polynomial integer N would have vanishing norm with respect to those infinite-P p-adics for which P divides N. Essentially discrete topology would result.

This seems too trivial to be interesting. One can however replace P^{-n} with a^{-n}, where a is any finite number a without losing the multiplicativity and ultra-metricity properties of the norm. The function space associated with the polynomial defined by P serves as a guideline also now. This space is naturally q-adic for some rational number q. At the lowest level the infinite prime defines naturally an ordinary rational number as the zero of the polynomial as is clear from the definition of the polynomial. At higher levels of the hierarchy the rational number is rational function of lower level infinite primes and by continuing the assignments of lower level rational functions to the infinite primes one ends up with an assignment of a unique rational number with a given infinite prime serving as an excellent candidate for a rational defining the q-adicity.

3.5.2 q-Adic topology determined by infinite prime as a local topology of the configuration space

Since infinite primes correspond to polynomials, infinite-P p-adic topology, which by previous considerations would be actually q-adic topology, is a natural candidate for a topology in function spaces, in particular in the configuration space of 3-surfaces.

This view conforms also with the idea of algebraic holography. The sub-spaces of configuration space can be modelled in terms of function spaces of rational functions, their algebraic extensions, and their P-adic completions. The mapping of the elements of these spaces to infinite rationals would make possible the correspondence between configuration space and number theoretic anatomy of point of the imbedding space.

The q-adic norm for these function spaces is in turn consistent with the ultra-metricity for the space of maxima of Kähler functions conjectured to be all that is needed to construct S-matrix. Ultra-metricity conforms nicely with the expected four-dimensional spin glass degeneracy due to the enormous vacuum degeneracy meaning that maxima of Kähler function define the analog of spin glass free energy landscape. That only maxima of Kähler function would be needed would mean that radiative corrections to the configuration space integral would vanish as quantum criticality indeed requires. This TGD can be regarded as an analog of for an integrable quantum theory. Quantum criticality is absolutely essential for guaranteing that S-matrix and U-matric elements are algebraic numbers which in turn guarantees number theoretic universality of quantum TGD.

3.5.3 The interpretation of the discrete topology determined by infinite prime

Also $p = 1$-adic topology makes formally sense and corresponds to a discrete topology in which all rationals have unit norm. It results also results if one naively generalizes p-adic topology to infinite-p p-adic topology by defining the norm of infinite prime at the lowest level of hierarchy as $|P|_P = 1/P = 0$. In this topology the distance between two points is either 1 or 0 and this topology is the roughest possible topology one can imagine.

It must be however noticed that if one maps infinite-P p-adics to real by the formal generalization of the canonical identification then one obtains real topology naturally if coefficients of powers of P are taken to be reals. This would mean that infinite-P p-adic topology would be equivalent with real topology.

Consider now the possible interpretations.

1. At the level of function spaces infinite-p p-adic topology in the naive sense has a completely natural interpretation and states that the replacement of the Taylor series with its lowest term.

2. The formal possibility of $p = 1$-adic topology at space-time level suggests a possible interpretation for the mysterious infinite degeneracy caused by the presence of the absolute minima of the Kähler function: one can add to any absolute minimum a vacuum extremal, which behaves completely randomly except for the constraints forcing the surface to be a vacuum extremal. This non-determinism is much more general than the non-determinism involving a discrete sequence of bifurcations (I have used the term association sequence about this kind of sequences). This suggests that one must replace the concept of 3-surface with a more general one, allowing also continuous association sequences consisting of a continuous family of space-like 3-surfaces with infinitesimally small time like separations. These continuous association sequences would be analogous to vacuum bubbles of the quantum field theories.

One can even consider the possibility that vacuum extremals are non-differentiable and even discontinuous obeying only effective $p = 1$-adic topology. Also modified Dirac operator vanishes identically in this case. Since vacuum surfaces are in question, $p = 1$ regions cannot correspond to material sheets carrying energy and also the identification as cognitive space-time sheets is questionable. Since

$p = 1$, the smallest possible prime in generalized sense, it must represent the lowest possible level of evolution, primordial chaos. Quantum classical correspondence suggests that $p = 1$ level is indeed present at the space-time level and might realized by the mysterious vacuum extremals.

3.6 Infinite primes and mathematical consciousness

The mathematics of infinity relates naturally with the mystery of consciousness and religious and mystic experience. In particular, mathematical cognition might have as a space-time correlate the infinitely structured space-time points implied by the introduction of infinite-dimensional space of real units defined by infinite (hyper-)octonionic rationals having unit norm in the real sense. I hope that the reader takes this section as a noble attempt to get a glimpse about unknown rather than final conclusions.

3.6.1 Infinite primes, cognition and intentionality

Somehow it is obvious that infinite primes must have some very deep role to play in quantum TGD and TGD inspired theory of consciousness. What this role precisely is has remained an enigma although I have considered several detailed interpretations, one of them above.

In the following an interpretation allowing to unify the views about fermionic Fock states as a representation of Boolean cognition and p-adic space-time sheets as correlates of cognition is discussed. Very briefly, real and p-adic partonic 3-surfaces serve as space-time correlates for the bosonic super algebra generators, and pairs of real partonic 3-surfaces and their algebraically continued p-adic variants as space-time correlates for the fermionic super generators. Intentions/actions are represented by p-adic/real bosonic partons and cognitions by pairs of real partons and their p-adic variants and the geometric form of Fermi statistics guarantees the stability of cognitions against intentional action. It must be emphasized that this interpretation is not identical with the one discussed above since it introduces different identification of the space-time correlates of infinite primes.

Infinite primes very briefly

Infinite primes have a decomposition to infinite and finite parts allowing an interpretation as a many-particle state of a super-symmetric arithmetic quantum field theory for which fermions and bosons are labeled by primes. There is actually an infinite hierarchy for which infinite primes of a given level define the building blocks of the infinite primes of the next level. One can map infinite primes to polynomials and these polynomials in turn could define space-time surfaces or at least light-like partonic 3-surfaces appearing as solutions of Chern-Simons action so that the classical dynamics would not pose too strong constraints.

The simplest infinite primes at the lowest level are of form $m_B X/s_F + n_B s_F$, $X = \prod_i p_i$ (product of all finite primes). The simplest interpretation is that X represents Dirac sea with all states filled and $X/s_F + s_F$ represents a state obtained by creating holes in the Dirac sea. m_B, n_B, and s_F are defined as $m_B = \prod_i p_i^{m_i}$, $n_B = \prod_i q_i^{n_i}$, and $s_F = \prod_i q_i$, m_B and n_B have no common prime factors. The integers m_B and n_B characterize the occupation numbers of bosons in modes labeled by p_i and q_i and $s_F = \prod_i q_i$ characterizes the non-vanishing occupation numbers of fermions.

The simplest infinite primes at all levels of the hierarchy have this form. The notion of infinite prime generalizes to hyper-quaternionic and even hyper-octonionic context and one can consider the possibility that the quaternionic components represent some quantum numbers at least in the sense that one can map these quantum numbers to the quaternionic primes.

The obvious question is whether configuration space degrees of freedom and configuration space spinor (Fock state) of the quantum state could somehow correspond to the bosonic and fermionic parts of the hyper-quaternionic generalization of the infinite prime. That hyper-quaternionic (or possibly hyper-octonionic) primes would define as such the quantum numbers of fermionic super generators does not make sense. It is however possible to have a map from the quantum numbers labeling super-generators to the finite primes. One must also remember that the infinite primes considered are only the simplest ones at the given level of the hierarchy and that the number of levels is infinite.

Precise space-time correlates of cognition and intention

The best manner to end up with the proposal about how p-adic cognitive representations relate bosonic representations of intentions and actions and to fermionic cognitive representations is through the following arguments.

1. In TGD inspired theory of consciousness Boolean cognition is assigned with fermionic states. Cognition is also assigned with p-adic space-time sheets. Hence quantum classical correspondence suggets that the decomposition of the space-time into p-adic and real space-time sheets should relate to the decomposition of the infinite prime to bosonic and fermionic parts in turn relating to the above mention decomposition of physical states to bosonic and fermionic parts.

   If infinite prime defines an association of real and p-adic space-time sheets this association could serve as a space-time correlate for the Fock state defined by configuration space spinor for given 3-surface. Also spinor field as a map from real partonic 3-surface would have as a space-time correlate a cognitive representation mapping real partonic 3-surfaces to p-adic 3-surfaces obtained by algebraic continuation.

2. Consider first the concrete interpretation of integers m_B and n_B. The most natural guess is that the primes dividing $m_B = \prod_i p^{m_i}$ characterize the effective p-adicities possible for the real 3-surface. m_i could define the numbers of disjoint partonic 3-surfaces with effective p_i-adic topology and associated with with the same real space-time sheet. These boundary conditions would force the corresponding real 4-surface to have all these effective p-adicities implying multi-p-adic fractality so that particle and wave pictures about multi-p-adic fractality would be mutually consistent. It seems natural to assume that also the integer n_i appearing in $m_B = \prod_i q_i^{n_i}$ code for the number of real partonic 3-surfaces with effective q_i-adic topology.

3. Fermionic statistics allows only single genuinely q_i-adic 3-surface possibly forming a pair with its real counterpart from which it is obtained by algebraic continuation. Pairing would conform with the fact that n_F appears both in the finite and infinite parts of the infinite prime (something absolutely essential concerning the consistency of interpretation!).

   The interpretation could be as follows.

   i) Cognitive representations must be stable against intentional action and fermionic statistics guarantees this. At space-time level this means that fermionic generators correspond to pairs of real effectively q_i-adic 3-surface and its algebraically continued q_i-adic counterpart. The quantum jump in which q_i-adic 3-surface is transformed to a real 3-surface is impossible since one would obtain two identical real 3-surfaces lying on top of each other, something very singular and not allowed by geometric exclusion principle for surfaces. The pairs of boson and fermion surfaces would thus form cognitive representations stable against intentional action.

   ii) Physical states are created by products of super algebra generators Bosonic generators can have both real or p-adic partonic 3-surfaces as space-time correlates depending on whether they correspond to intention or action. More precisely, m_B and n_B code for collections of real and p-adic partonic 3-surfaces. What remains to be interpreted is why m_B and n_B cannot have common prime factors (this is possible if one allows also infinite integers obtained as products of finite integer and infinite primes).

   iii) Fermionic generators to the pairs of a real partonic 3-surface and its p-adic counterpart obtained by algebraic continuation and the pictorial interpretation is as fermion hole pair. Unrestricted quantum super-position of Boolean statements requires that many-fermion state is accompanied by a corresponding many-antifermion state. This is achieved very naturally if real and corresponding p-adic fermion have opposite fermion numbers so that the kicking of negative energy fermion from Dirac sea could be interpreted as creation of real-p-adic fermion pairs from vacuum.

   If p-adic space-time sheets obey same algebraic expressions as real sheets (rational functions with algebraic coefficients), the Chern-Simons Noether charges associated with real partons defined as integrals can be assigned also with the corresponding p-adic partons if they are rational or algebraic numbers. This would allow to circumvent the problems related to the p-adic integration. Therefore one can consider also the possibility that p-adic partons carry

Noether charges opposite to those of corresponding real partons sheet and that pairs of real and p-adic fermions can be created from vacuum. This makes sense also for the classical charges associated with Kähler action in space-time interior if the real space-time sheet obeying multi-p p-adic effective topology has algebraic representation allowing interpretation also as p-adic surface for all primes involved.

iv) This picture makes sense if the partonic 3-surfaces containing a state created by a product of super algebra generators are unstable against decay to this kind of 3-surfaces so that one could regard partonic 3-surfaces as a space-time representations for a configuration space spinor field.

4. Are alternative interpretations possible? For instance, could $q = m_B/m_B$ code for the effective q-adic topology assignable to the space-time sheet. That q-adic numbers form a ring but not a number field casts however doubts on this interpretation as does also the general physical picture.

Number theoretical universality of S-matrix

The discreteness of the intersection of the real space-time sheet and its p-adic variant obtained by algebraic continuation would be a completely universal phenomenon associated with all fermionic states. This suggests that also real-to-real S-matrix elements involve instead of an integral a sum with the arguments of an n-point function running over all possible combinations of the points in the intersection. S-matrix elements would have a universal form which does not depend on the number field at all and the algebraic continuation of the real S-matrix to its p-adic counterpart would trivialize. Note that also fermionic statistics favors strongly discretization unless one allows Dirac delta functions.

3.6.2 The generalization of the notion of ordinary number field

The notion of infinite rationals leads also to the generalization of the notion of a finite number. The obvious generalization would be based on the allowance of infinitesimals. Much more interesting approach is however based on the observation that one obtains infinite number of real units by taking two infinite primes with a finite rational valued ratio q and by dividing this ratio by ordinary rational number q. As a real number the resulting number differs in no manner from ordinary unit of real numbers but in p-adic sense the points are not equivalent. This construction generalizes also to quaternionic and octonionic case.

Space-time points would become structured since infinite rationals normed to unity define naturally a gigantically infinite-dimensional free algebra generated by the units serving in well-define sense as Mother of All Algebras. The units of the algebra multiplying ordinary rational numbers (and also other elements) of various number fields are invisible at the level of real physics so that the interpretation as the space-time correlate of mathematical cognition realizing the idea of monad is natural. Universe would be an algebraic hologram with single point being able to represent the state of the Universe in its structure. Infinite rationals would allow the realization of the Platonia of all imaginable mathematical constructs at the level of space-time.

The generalized units for quaternions and octonions

In the case of real and complex rationals the group of generalized units generated by primes *resp.* infinite Gaussian primes is commutative. In the case of unit quaternions and hyper-quaternions group becomes non-commutative and in case of unit hyper-octonions the group is replaced by a kind non-associative generalization of group.

For infinite primes for which only finite number of bosonic and fermionic modes are excited it is possible to tell how the products AB and BA of two infinite primes explicitly since the finite hyper-octonionic primes can be assumed to multiply the infinite integer X from say left.

Situation changes if infinite number of bosonic excitations are present since one would be forced to move finite H- or O-primes past a infinite number of primes in the product AB. Hence one must simply assume that the group G generated by infinite units with infinitely many bosonic excitations is a free group. Free group interpretation means that non-associativity is safely localized inside infinite primes and reduced to the non-associativity of ordinary hyper-octonions. Needless to say free group is the best one can hope of achieving since free group allows maximal number of factor groups.

The free group G can be extended into a free algebra A by simply allowing superpositions of units with coefficients which are real-rationals or possibly complex rationals. Again free algebra fulfils the dreams as system with a maximal representative power. The analogy with quantum states defined as functions in the group is highly intriguing and unit normalization would correspond to the ordinary normalization of Schrödinger amplitudes. Obviously this would mean that single point is able to mimic quantum physics in its structure. Could state function reduction and preparation be represented at the level of space-time surfaces so that initial and final 3-surfaces would represent pure states containing only bound state entanglement represented algebraically, and could the infinite rationals generating the group of quaternionic units (no sums over them) represent pure states?

The free algebra structure of A together with the absolutely gigantic infinite-dimensionality of the endless hierarchy of infinite rational units suggests that the resulting free algebra structure is universal in the sense that any algebra defined with coefficients in the field of rationals can be imbedded to the resulting algebra or represented as a factor algebra obtained by the sequence $A \rightarrow 1_1 = A/I_1 \rightarrow A_1/I_2 ...$ where the ideal I_k is defined by $k : th$ relation in A_{k-1}.

Physically the embedding would mean that some field quantities defined in the algebra are restricted to the subalgebra. The representation of algebra B as an iterated factor algebra would mean that some field quantities defined in the algebra are constant inside the ideals I_k of A defined by the relations. For instance, the induced spinor field at space-time surface would have same value for all points of A which differ by an element of the ideal. At the configuration space level, the configuration space spinor field would be constant inside an ideal associated with the algebra of A-valued functions at space-time surfaces.

The units can be interpreted as defining an extension of rationals in C, H, or O. Galois group is defined as automorphisms of the extension mapping the original number field to itself and obviously the transformations $x \rightarrow gxg^{-1}$, where g belongs to the extended number field act as automorphisms. One can regard also the extension by real units as the extended number field and in this case the automorphisms contain also the automorphisms induced by the multiplication of each infinite prime Π_i by a real unit U_i: $\Pi_i \rightarrow \hat{\Pi}_i = U_i \Pi_i$.

The free algebra generated by generalized units and mathematical cognition

One of the deepest questions in theory of consciousness concerns about the space-time correlates of mathematical cognition. Mathematician can imagine endlessly different mathematical structures. Platonist would say that in some sense these structures exist. The claim classical physical worlds correspond to certain 4-surfaces in $M^4_+ \times CP_2$ would leave out all these beautiful mathematical structures unless they have some other realization than the physical one.

The free algebra A generated by the generalized multiplicative units of rationals allows to understand how Platonia is realized at the space-time level. A has no correlate at the level of real physics since the generalized units correspond to real numbers equal to one. This holds true also in quaternionic and octonionic cases since one can require that the units have net quaternionic and octonionic phases equal to one. By its gigantic size A and free algebra character might be able represent all possible algebras in the proposed manner. Also non-associative algebras can be represented.

Algebraic equations are the basic structural building blocks of mathematical thinking. Consider as a simple example the equation $AB = C$. The equations are much more than tautologies since they contain the information at the left hand side about the variables of the algebraic operation giving the outcome on the right hand side. For instance, in the case of multiplication $AB = C$ the information about the factors is present although it is completely lost when the product is evaluated. These equations pop up into our consciousness in some mysterious manner and the question is what are the space-time correlates of these experiences suggested to exist by quantum-classical correspondence.

The algebra of units is an excellent candidate for the sought for correlate of mathematical cognition. I must admit that that it did not occur to me that Leibniz might have been right about his monads! The idealization is however in complete accordance with the idea about the Universe as an algebraic hologram taken to its extreme. One can say that each point represents an equation. The left hand side of the equation corresponds to the element of the free algebra defined by octonionic units. Consider as an example product of powers of $X/\Pi(Q_q)$ representing infinite quaternionic rationals. Equality sign corresponds to the evaluation of this expression by interpreting it as a real quaternionic rational number: real physics does the evaluation automatically. The information about the primes appearing as factors of the result is not however lost at cognitive level. Note that the analogs of quantum states represented by superpositions of the unit elements of the algebra A can be interpreted as equations defining them.

When two points are cobordant?

Topological quantum field theories have led to a dramatic success in the understanding of 3- and 4-dimensional topologies and cobordisms of these manifolds (two n-manifolds are cobordant if there exists an $n+1$-manifold having them as boundaries). In his thought-provoking and highly inspiring article Pierre Cartier [45] poses a question which at first sounds absurd. What might be the the counterpart of cobordism for points? The question is indeed absurd unless the points have some structure.

If one takes seriously the idea that each point of space-time sheet corresponds to a unit defined by an infinite rational, the obvious question is under what conditions there is a continuous line connecting these points with continuity being defined in some generalized sense. In real sense the line is continuous always but in p-adic sense only if all p-adic norms of the two units are identical. Since the p-adic norm of the unit of $Y(n/m) = X/\Pi(n/m)$ is that of $q = n/m$, the norm of two infinite rational numbers is same only if they correspond to the same ordinary rational number.

Suppose that one has

$$\begin{aligned} Y_I &= \frac{\prod_i Y(q_{1i}^I)}{\prod_i Y(q_{2i}^I)} \ , \quad Y_F = \frac{\prod_i Y(q_{1i}^F)}{\prod_i Y(q_{2i}^F)} \ , \\ q_{ki}^I &= \frac{n_{k_i}^I}{m_{k_i}^I} \ , \qquad q_{ki}^F = \frac{n_{k_i}^F}{m_{k_i}^F} \ , \end{aligned} \tag{3.6.1}$$

Here $m^{\cdot}_{\cdot}$ representing arithmetic many-fermion state is a square free integer and $n^{\cdot}_{\cdot}$ representing arithmetic many-boson state is an integer having no common factors with $m^{\cdot}_{\cdot}$.

The two units have same p-adic norm in all p-adic number fields if the rational numbers associated with Y_I and Y_F are same:

$$\frac{\prod_i q_{1i}^I}{\prod_i q_{2i}^I} = \frac{\prod_i q_{1i}^F}{\prod_i q_{2i}^F} \ . \tag{3.6.2}$$

The logarithm of this condition gives a conservation law of energy encountered in arithmetic quantum field theories, where the energy of state labeled by the prime p is $E_p = log(p)$:

$$\begin{aligned} E^I &= \sum_i log(n_{1i}^I) - \sum_i log(n_{2i}^I) - \sum_i log(m_{1i}^I) + \sum_i log(m_{2i}^I) = \\ &= \sum_i log(n_{1i}^F) - \sum_i log(n_{2i}^F) - \sum_i log(m_{1i}^F) + \sum_i log(m_{2i}^F) = E^F \ . \end{aligned} \tag{3.6.3}$$

There are both positive and negative energy particles present in the system. The possibility of negative energies is indeed one of the basic predictions of quantum TGD distinguishing it from standard physics. As one might have expected, Y^I and Y^F represent the initial and final states of a particle reaction and the line connecting the two points represents time evolution giving rise to the particle reaction. In principle one can even localize various steps of the reaction along the line and different lines give different sequences of reaction steps but same overall reaction. This symmetry is highly analogous to the conformal invariance implying that integral in complex plane depends only on the end points of the curve.

Whether the entire four-surface should correspond to the same value of topological energy or whether E can be discontinuous at elementary particle horizons separating space-time sheets and represented by light-like 3-surfaces around wormhole contacts remains an open question. Discontinuity through elementary particle horizons would make possible the arithmetic analogs of poles and cuts of analytic functions since the limiting values of Y from different sides of the horizon are different. Note that the construction generalizes to the quaternionic and octonionic case.

TGD inspired analog for d-algebras

Maxim Kontsevich has done deep work with quantizations interpreted as a deformation of algebraic structures and there are deep connections with this work and braid group [58]. In particular, the Grothendienck-Teichmueller algebra believed to act as automorphisms for the deformation structures acts as automorphisms of the braid group at the limit of infinite number of strands. I must admit that my miserable skills in algebra does not allow to go to the horrendous technicalities but occasionally I have the feeling that I have understood some general ideas related to this work. In his article "Operads and Motives in Deformation Quantization" Kontsevich introduces the notions of operad and d-algebras over operad. Without going to technicalities one can very roughly say that d-algebra is essentially d-dimensional algebraic structure, and that the basic conjecture of Deligne generalized and proved by Kontsevich states in its generalized form that $d+1$-algebras have a natural action in all d-algebras.

In the proposed extension of various rationals a notion resembling that of universal d-algebra to some degree but not equivalent with it emerges naturally. The basic idea is simple.

1. Points correspond to the elements of the assumed to be universal algebra A which in this sense deserves the attribute $d = 0$ algebra. By its universality A should be able to represent any algebra and in this sense it cannot correspond $d = 0$-algebra of Kontsevich defined as a complex, that is a direct sum of vector spaces V_n and possessing d operation $V_n \rightarrow V_{n+1}$, satisfying $d^2 = 0$. Each point of a manifold represents one particular element of 0-algebra and one could loosely say that multiplication of points represents algebraic multiplication. This algebra has various subalgebras, in particular those corresponding to reals, complex numbers and quaternions. One can say that sub-algebra is non-associative, non-commutative, etc.. if its real evaluation has this property.

2. Lines correspond to evolutions for the elements of A which are continuous with respect to real (trivially) and all p-adic number fields. The latter condition is nontrivial and allows to interpret evolution as an evolution conserving number theoretical analog of total energy. Universal 1-group would consist of curves along which one has the analog of group valued field (group being the group of generalized units) having values in the universal 0-group G. The action of the 1-group in 0-group would simply map the element of 0-group at the first end of the curve its value at the second end. Curves define a monoid in an obvious manner. The interpretation as a map to A allows pointwise multiplication of these mappings which generalizes to all values of d.

   One could also consider the generalization of local gauge field so that there would be gauge potential defined in the algebra of units having values on A. This potential would define holonomy group acting on 0-algebra and mapping the element at the first end of the curve to its gauge transformed variant at the second end. In this case also closed curves would define non-trivial elements of the holonomy group. In fact, practically everything is possible since probably any algebra can be represented in the algebra generated by units.

3. Two-dimensional structures correspond to dynamical evolutions of one-dimensional structures. The simplest situation corresponds to 2-cubes with the lines corresponding to the initial and final values of the second coordinate representing initial and final states. One can also consider the possibility that the two-surface is topologically non-trivial containing handles and perhaps even holes. One interpret this cognitive evolutions represents 1-dimensional flow so that the initial points travel to final points. Obviously there is symmetry breaking involved since the second coordinate is in the role of time and this defines kind of time orientation for the surface.

4. The generalization to 4- and higher dimensional cases is obvious. One just uses d-manifolds with edges and uses their time evolution to define $d+1$-manifolds with edges. Universal 3-algebra is especially interesting from the point of view of braid groups and in this case the maps between initial and final elements of 2-algebra could be interpreted as braid operations if the paths of the elements along 3-surface are entangled. For instance field lines of Kähler gauge potential or of magnetic field could define this kind of braiding.

5. The d-evolutions define a monoid since one can glue two d-evolutions together if the outcome of the first evolution equals to the initial state of the second evolution. $d+1$-algebra also acts naturally in d-algebra in the sense that the time evolution $f(A \rightarrow B)$ assigns to the d-algebra

valued initial state A a d-algebra valued final state and one can define the multiplication as $f(A \rightarrow B)C = B$ for $A = C$, otherwise the action gives zero. If time evolutions correspond to standard cubes one gets more interesting structure in this manner since the cubes differing by time translation can be identified and the product is always non-vanishing.

6. It should be possible to define generalizations of homotopy groups to what might be called "cognitive" homotopy groups. Effectively the target manifold would be replaced by the tensor product of an ordinary manifold and some algebraic structure represented in A. All kinds of "cognitive" homotopy groups would result when the image is cognitively non-contractible. Also homology groups could be defined by generalizing singular complex consisting of cubes with cubes having the hierarchical decomposition into time evolutions of time evolutions of... in some sub-algebraic structure of A. If one restricts time evolutions to sub-algebraic structures one obtains all kinds of homologies. For instance, associativity reduces 3-evolutions to paths in rational $SU(3)$ and since $SU(3)$ just like any Lie group has non-trivial 3-homology, one obtains nontrivial "cognitive" homology for 3-surfaces with non-trivial 3-homology.

The following heuristic arguments are inspired by the proposed vision about algebraic cognition and the conjecture that Grothendienck-Teichmueller group acts as automorphisms of Feynman diagrammatics relating equivalent quantum field theories to each other.

1. The operations of $d + 1$-algebra realized as time evolution of d-algebra elements suggests an interpretation as cognitive counterparts for sequences of algebraic manipulations in d-algebra which themselves become elements of d+1 algebra. At the level of paths of points the sequences of algebraic operations correspond to transitions in which the number of infinite primes defining an infinite rational can change in discrete steps but is subject to the topological energy conservation guaranteing the p-adic continuity of the process for all primes. Different paths connecting a and b represent different but equivalent manipulations sequences.

   For instance, at $d = 2$ level one has a pile of these processes and this in principle makes it possible an abstraction to algebraic rules involved with the process by a pile of examples. Higher values of d in turn make possible further abstractions bringing in additional parameters to the system. All kinds of algebraic processes can be represented in this manner. For instance, multiplication table can be represented as paths assigning to an the initial state product of elements a and b represented as infinite rationals and to the final state their product ab represented as single infinite rational. Representation is of course always approximate unless the algebra is finite. All kinds abstract rules such as various commutative diagrams, division of algebra by ideal by choosing one representative from each equivalence class of A/I as end point of the path, etc... can be represented in this manner.

2. There is also second manner to represent algebraic rules. Entanglement is a purely algebraic notion and it is possible to entangle the many-particle states formed as products of infinite rationals representing inputs of an algebraic operation A with the outcomes of A represented in the same manner such that the entanglement is consistent with the rule.

3. There is nice analogy between Feynman diagrams and sequences of algebraic manipulations. Multiplication ab corresponds to a map $A \otimes A \rightarrow A$ is analogous to a fusion of elementary particles since the product indeed conserves the number theoretical energy. Co-algebra operations are time reversals of algebra operations in this evolution. Co-multiplication Δ assigns to $a \in A$ an element in $A \otimes A$ via algebra homomorphism and corresponds to a decay of initial state particle to two final state particles. It defines co-multiplication assign to $a \otimes b \in A \otimes A$ an element of $A \otimes A \rightarrow A \otimes A \otimes A$ and corresponds to a scattering of elementary particles with the emission of a third particle. Hence a sequence of algebraic manipulations is like a Feynman diagram involving both multiplications and co-multiplications and thus containing also loops. When particle creation and annihilation are absent, particle number is conserved and the process represents algebra endomorphism $A \rightarrow A$. Otherwise a more general operation is in question. This analogy inspires the question whether particle reactions could serve as a blood and flesh representation for $d = 4$ algebras.

4. The dimension $d = 4$ is maximal dimension of single space-time evolution representing an algebraic operation (unless one allows the possibility that space-time and imbedding space dimensions are come as multiples of four and 8). Higher dimensions can be effectively achieved only if several space-time sheets are used defining $4n$-dimensional configuration space. This could reflect some deep fact about algebras in general and also relate to the fact that 3- and 4-dimensional manifolds are the most interesting ones topologically.

3.6.3 Algebraic Brahman=Atman identity

The proposed view about cognition and intentionality emerges from the notion of infinite primes, which was actually the first genuinely new mathematical idea inspired by TGD inspired consciousness theorizing. Infinite primes, integers, and rationals have a precise number theoretic anatomy. For instance, the simplest infinite primes correspond to the numbers $P_{\pm} = X \pm 1$, where $X = \prod_k p_k$ is the product of all finite primes. Indeed, $P_{\pm}\ mod\ p = 1$ holds true for all finite primes. The construction of infinite primes at the first level of the hierarchy is structurally analogous to the quantization of super-symmetric arithmetic quantum field theory with finite primes playing the role of momenta associated with fermions and bosons. Also the counterparts of bound states emerge. This process can be iterated: at the second level the product of infinite primes constructed at the first level replaces X and so on.

The structural similarity with repeatedly second quantized quantum field theory strongly suggests that physics might in some sense reduce to a number theory for infinite rationals M/N and that second quantization could be followed by further quantizations. As a matter fact, the hierarchy of space-time sheets could realize this endless second quantization geometrically and have also a direct connection with the hierarchy of logics labeled by their order. This could have rather breathtaking implications.

1. One is forced to ask whether this hierarchy corresponds to a hierarchy of realities for which level below corresponds in a literal sense infinitesimals and the level next above to infinity.

2. Second implication is that there is an infinite number of infinite rationals behaving like real units ($M/N \equiv 1$ in real sense) so that space-time points could have infinitely rich number theoretical anatomy not detectable at the level of real physics. Infinite integers would correspond to positive energy many particle states and their inverses (infinitesimals with number theoretic structure) to negative energy many particle states and $M/N \equiv 1$ would be a counterpart for zero energy ontology to which oneness and emptiness are assigned in mysticism.

3. Single space-time point, which is usually regarded as the most primitive and completely irreducible structure of mathematics, would take the role of Platonia of mathematical ideas being able to represent in its number theoretical structure even the quantum state of entire Universe. Algebraic Brahman=Atman identity and algebraic holography would be realized in a rather literal sense.

Number theoretic anatomy of space-time point

This number theoretical anatomy should relate to mathematical consciousness in some manner. For instance, one can ask whether it makes sense to speak about quantum jumps changing the number theoretical anatomy of space-time points and whether these quantum jumps give rise to mathematical ideas. In fact, the identifications of Platonia as spinor fields in WCW on one hand and as the set number theoretical anatomies of point of imbedding space force the conclusion that configuration space spinor fields (recall also the identification as correlates for logical mind) can be realized in terms of the space for number theoretic anatomies of imbedding space points. Therefore quantum jumps would be correspond to changes in the anatomy of the space-time points. Or more precisely, to the changes of the configuration space spinor fields regarded as wave functions in the set of imbedding space points which are equivalent in real sense. Imbedding space would be experiencing genuine number theoretical evolution. The whole physics would reduce to the anatomy of numbers. All mathematical notions which are more than mere human inventions would be imbeddable to the Platonia realized as the number theoretical anatomies of single imbedding space point.

To realize this picture would require that both configuration space and configuration space spinor fields are mappable to the number theoretic anatomies of space-time point. The possibility to map infinite primes to polynomials and vice versa gives support for the possibility to map configuration space or at least the space of maxima of Kähler function defining the counterpart of spin glass energy landscape to the number theoretic anatomy of imbedding space point.

Function spaces provide a natural model for the subspaces of the world of classical worlds. The spaces of rational functions, their extensions, and q-adic completions, provide natural candidates for these function spaces, so that a mapping to real units defined by infinite rationals, their extensions, and q-adic completions emerge naturally. In the same manner Fock states can be mapped to infinite primes and one can see the polynomial-infinite prime correspondence also as an articulation of fermion-boson super-symmetry.

The commutativity requirement for infinite primes implies that infinite primes at n:th level can define rational functions of n complex variables. This relates naturally to the effective 2-dimensionality of TGD in the sense that configuration space geometry involves only data about 2-dimensional partonic surfaces at boundaries of $\delta M^4_\pm \times CP_2$. Allowing non-commutativity one would also obtain 4-D surfaces but algebraic continuation would mean that 2-D data is enough.

Could algebraic Brahman Atman identity represent a physical law?

Just for fun and to test these ideas it is interesting to find whether additional constraints coming from zero energy ontology and finite measurement resolution [C2] might give allow to realize algebraic Brahman Atman identity as a physical law dictating the number theoretic anatomy of some space-time points from the structure of quantum state of Universe.

The identification of quantum corrections as insertion of zero energy states in time scale below measurement resolution to positive or negative energy part of zero energy state and the identification of number theoretic braid as a space-time correlate for the finite measurement resolution give considerable additional constraints.

1. The fundamental representation space consists of wave functions in the Cartesian power U^8 of space U of real units associated with any point of H. That there are 8 real units rather than one is somewhat disturbing: this point will be discussed below. Real units are ratios of infinite integers having interpretation as positive and negative energy states of a super-symmetric arithmetic QFT at some level of hierarchy of second quantizations. Real units have vanishing net quantum numbers so that only zero energy states defining the basis for configuration space spinor fields should be mapped to them. In the general case quantum superpositions of these basis states should be mapped to the quantum superpositions of real units. The first guess is that real units represent a basis for configuration space spinor fields constructed by applying bosonic and fermionic generators of appropriate super Kac-Moody type algebra to the vacuum state.

2. What can one say about this map bringing in mind Gödel numbering? Each pair of bosonic and corresponding fermionic generator at the lowest level must be mapped to its own finite prime. If this map is specified, the map is fixed at the higher levels of the hierarchy. There exists an infinite number of this kind of correspondences. To achieve some uniqueness, one should have some natural ordering which one might hope to reflect real physics. The irreps of the (non-simple) Lie group involved can be ordered almost uniquely. For simple group this ordering would be with respect to the sum $N = N_F + N_{F,c}$ of the numbers N_F *resp.* $N_{F,c}$ of the fundamental representation *resp.* its conjugate appearing in the minimal tensor product giving the irrep. The generalization to non-simple case should use the sum of the integers N_i for different factors for factor groups. Groups themselves could be ordered by some criterion, say dimension. The states of a given representation could be mapped to subsequent finite primes in an order respecting some natural ordering of the states by the values of quantum numbers from negative to positive (say spin for $SU(2)$ and color isospin and hypercharge for $SU(3)$). This would require the ordering of the Cartesian factors of non-simple group, ordering of quantum numbers for each simple group, and ordering of values of each quantum number from positive to negative.

   The presence of conformal weights brings in an additional complication. One cannot use conformal as a primary orderer since the number of $SO(3) \times SU(3)$ irreps in the super-canonical sector is infinite. The requirement that the probabilities predicted by p-adic thermodynamics are rational numbers or equivalently that there is a length scale cutoff, implies a cutoff in conformal weight. The vision about M-matrix forces to conclude that different values of the total conformal weight n for the quantum state correspond to summands in a direct sum of HFFs. If so, the introduction of the conformal weight would mean for a given summand only the assignment n conformal weights to a given Lie-algebra generator. For each representation of the Lie group one would have n copies ordered with respect to the value of n and mapped to primes in this order.

3. Cognitive representations associated with the points in a subset, call it P, of the discrete intersection of p-adic and real space-time sheets, defining number theoretic braids, would be in question. Large number of partonic surfaces can be involved and only few of them need to contribute to P in the measurement resolution used. The fixing of P means measurement of N positions of H and each point carries fermion or anti-fermion numbers. A more general situation corresponds to plane wave type state obtained as superposition of these states. The condition of rationality or at least algebraicity means that discrete variants of plane waves are in question.

4. By the finiteness of the measurement resolution configuration space spinor field decomposes into a product of two parts or in more general case, to their superposition. The part Ψ_+, which is above measurement resolution, is representable using the information contained by P, coded by the product of second quantized induced spinor field at points of P, and provided by physical experiments. Configuration space "orbital" degrees of freedom should not contribute since these points are fixed in H.

5. The second part of the configuration space spinor field, call it Ψ_-, corresponds to the information below the measurement resolution and assignable with the complement of P and mappable to the structure of real units associated with the points of P. This part has vanishing net quantum numbers and is a superposition over the elements of the basis of CH spinor fields and mapped to a quantum superposition of real units. The representation of Ψ_- as a Schrödinger amplitude in the space of real units could be highly unique. Algebraic holography principle would state that the information below measurement resolution is mapped to a Schrödinger amplitude in space of real units associated with the points of P.

6. This would be also a representation for perceiver-external world duality. The correlation function in which P appears would code for the information appearing in M-matrix representing the laws of physics as seen by conscious entity about external world as an outsider. The quantum superposition of real units would represent the purely subjective information about the rest of the universe. Hence number theoretic Brahman=Atman would correspond also to the original Brahman=Atman. Note that one must perceive external world in order to have the representation of the rest of the Universe.

There is an objection against this picture. One obtains an 8-plet of arithmetic zero energy states rather than one state only. What this strange 8-fold way could mean?

1. The crucial observation is that hyper-finite factor of type II_1 (HFF) creates states for which center of mass degrees of freedom of 3-surface in H are fixed. One should somehow generalize the operators creating local HFF states to fields in H, and an octonionic generalization of conformal field suggests itself. I have indeed proposed a quantum octonionic generalization of HFF extending to an HFF valued field Ψ in 8-D quantum octonionic space with the property that maximal quantum commutative sub-space corresponds to hyper-octonions [C6]. This construction raises $X^4 \subset M^8$ and by number theoretic compactification also $X^4 \subset H$ in a unique position since non-associativity of hyper-octonions does not allow to identify the algebra of HFF valued fields in M^8 with HFF itself.

2. The value of Ψ in the space of quantum octonions restricted to a maximal commutative subspace can be expressed in terms of 8 HFF valued coefficients of hyper-octonion units. By the hyper-octonionic generalization of conformal invariance all these 8 coefficients must represent zero energy HFF states. The restriction of Ψ to a given point of P would give a state, which has 8 HFF valued components and Brahman=Atman identity would map these components to U^8

associated with P. One might perhaps say that 8 zero energy states are needed in order to code the information about the H positions of points P.

One-element field realized in terms of real units with number theoretic anatomy

One-element field [59] looks rather self-contradictory notion since 1 and 0 should be represented by same element. The real units expressible as ratios of infinite rationals could however provide a well-defined realization of this notion.

1. The condition that same element represents the neutral element of both sum and product gives strong constraint on one-element field. Consider an algebra formed by reals with sum and product defined in the following manner. Sum, call it $\oplus$, corresponds to the ordinary product $x \times y$ for reals whereas product, call it $\otimes$, is identified as the non-commutative product $x \otimes y = x^y$. $x = 1$ represents both the neutral element (0) of $\oplus$ and the unit of $\otimes$. The sub-algebras generated by 1 and multiple powers $P_n(x) = P_{n-1}(x) \otimes x = x \otimes ... \otimes x$ form commutative sub-algebras of this algebra. When one restricts the consideration to $x = 1$ one obtains one-element field as sub-field which is however trivial since $\oplus$ and $\otimes$ are identical operations in this subset.

2. One can get over this difficulty by keeping the operations $\oplus$ and $\otimes$, by assuming one-element property only with respect to the real and various p-adic norms, and by replacing ordinary real unit 1 with the algebra of real units formed from infinite primes by requiring that the real and various p-adic norms of the resulting numbers are equal to one. As far as real and various p-adic norms are considered, one has commutative one-element field. When number theoretic anatomy is taken into account, the algebra contains infinite number of elements and is non-commutative with respect to the product since the number theoretic anatomies of x^y and y^x are different.

3.6.4 Leaving the world of finite reals and ending up to the ancient Greece

If strong number theoretic vision is accepted, all physical predictions of quantum TGD would be numbers in finite algebraic extensions of rationals. Just the numbers which ancient Greeks were able to construct by the technical means at use! This seems rather paradoxical but conforms also with the hypothesis that the dicrete algebraic intersections of real and p-adic 2-surfaces provide the fundamental cognitive representations.

The proposed construction for infinite primes gives a precise division of infinite primes to classes: the ratios of primes in given class span a *subset of rational numbers*. These classes give much more refined classification of infinities than infinite ordinals or alephs. They would correspond to separate phases in the evolution of consciousness identified as a sequence of quantum jumps defining sequence of primes $\rightarrow p_1 \rightarrow p_2$...... Infinite primes could mean a transition from space-time level to the level of function spaces. Configuration space is example of a space which can be parameterized by a space of functions locally.

The minimal assumption is that infinite primes reflect their presence only in the possibility to multiply the coordinates of imbedding space points by real units formed as ratios of infinite integers. The correspondence between polynomials and infinite primes gives hopes of mapping at least the reduced configuration space consisting of the the maxima of Kähler function to the anatomy of space-time point. Also configuration space spinors and perhaps also the the modes of configuration space spinor fields would allow this kind of map.

One can consider also the possibility that infinite integers and rationals give rise to a hierarchy of imbedding spaces such that given level represents infinitesimals from the point of view of higher levels in hierarchy. Even 'simultaneous' time evolutions of conscious experiences at different aleph levels with completely different time scales (to put it mildly) are possible since the time values around which the contents of conscious experience are possibly located, are determined by the quantum jump: also multi-snapshots containing snapshots also from different aleph levels are possible. Un-integrated conscious experiences with all values of p could be contained in given quantum jump: this would give rise to a hierarchy of conscious beings: the habitants above given level could be called Gods with full reason: those above us would probably call us just 'epsilons' if ready to admit that we exist at all except in non-rigorous formulations of elementary calculus!

Quantum entanglement between subsystems belonging to different aleph levels of infinity would make possible experiences containing information about this finite world and about the higher level worlds, too. Perhaps our brightest mathematical thoughts (at least) could correspond to cognitive space-time sheets of infinite duration glued to cognitive space-time sheets with even more infinite duration whereas the contents of sensory experiences would be located around finite values of geometric time.

3.6.5 Infinite primes and mystic world view

The proposed interpretation deserves some additional comments from the point of consciousness theory.

1. An open problem is whether the finite integer S appearing in the infinite prime is product of only finite or possibly even infinite number of lower level primes at a given level of hierarchy. The proposed physical identification of S indeed allows S to be a product of infinitely many primes. One can allow also M and N appearing in the infinite and infinite part to be contain infinite number of factors. In this manner one obtains a hierarchy of infinite primes expressible in the form

$$\begin{aligned} &P = nY^{r_1} + mS \ \ , \ \ r = 1, 2, ... \\ &m = m_0 + P_{r_2}(Y) \ \ , \\ &Y = \frac{X}{S} \ \ , \\ &S = \prod_i P_i \ \ . \end{aligned}$$

   Note that this ansatz is in principle of the same general form as the original ansatz $P = nY + mS$. These primes correspond in physical analogy to states containing infinite number of particles.

   If one poses no restrictions on S this implies that that the cardinality for the set of infinite primes at first level would be $c = 2^{alef_0}$ ($alef_0$ is the cardinality of natural numbers). This is the cardinality for *all* subsets of natural numbers equal to the cardinality of reals. At the next level one obtains the cardinality 2^c for *all* subsets of reals, etc....

   If S were always a product of *finite number of primes* and $k(p)$ would differ from zero for finite number of primes only, the cardinality of infinite primes would be $alef_0$ at each level. One could pose the condition that mS is infinitesimal as compared to nX/S. This would guarantee that the ratio of two infinite primes at the same level would be well defined and equal to n_1S_2/n_2S_1. On the other hand, the requirement that all rationals are obtained as ratios of infinite primes requires that no restrictions are posed on $k(p)$: in this case the cardinality coming from possible choices of $r = ms$ is the cardinality of reals at first level.

   The possibility of primes for which also S is finite would mean that the algebra determined by the infinite primes must be generalized. For the primes representing states containing infinite number of bosons and/or fermions it would be be possible to tell how P_1P_2 and P_2P_1 differ and these primes would behave like elements of free algebra. As already found, this kind of free algebra would provide single space-time point with enormous algebraic representative power and analog of Brahman=Atman identity would result.

2. There is no physical subsystem-complement decomposition for the infinite primes of form $X \pm 1$ since fermionic degrees of freedom are not excited at all. Mystic could interpret it as a state of consciousness in which all separations vanish and there is no observer-observed distinction anymore. A state of pure awareness would be in question if bosonic and fermionic excitations represent the contents of consciousness! Since fermionic many particle states identifiable as Boolean statements about basic statements are identified as representation for reflective level of consciousness, $S = 1$ means that the reflective level of consciousness is absent: enlightment as the end of thoughts according to mystics.

   The mystic experiences of oneness ($S = 1!$), of emptiness (the subset of primes defined by S is empty!) and of the absence of all separations (there is no subsystem-complement separation and hence no division between observer and observed) could be related to quantum jumps to this kind of sectors of the configuration space. In super-symmetric interpretation $S = 1$ means that state contains no fermions.

3. There is entire hierarchy of selves corresponding to the hierarchy of infinite primes and the relationship between selves at different levels of the hierarchy is like the relationship between God and human being. Infinite primes at the lowest level would presumably represent elementary particles. This implies a hierarchy for moments of consciousness and it would be un-natural to exclude the existence of higher level 'beings' (one might call them Angels, Gods, etc...).

3.6.6 Infinite primes and evolution

The original argument leading to the notion of infinite primes was simple. Generalized unitarity implies evolution as a gradual increase of the p-adic prime labeling the configuration space sector D_p to which the localization associated with quantum jump occurs. Infinite p-adic primes are forced by the requirement that p-adic prime increases in a statistical sense and that the number of quantum jumps already occurred is infinite (assuming finite number of these quantum jumps and therefore the first quantum jump, one encounters the problem of deciding what was the first configuration space spinor field).

Quantum classical correspondence requires that p-adic evolution of the space-time surface with respect to geometric time repeats in some sense the p-adic evolution by quantum jumps implied by the generalized unitarity [E6]. Infinite p-adic primes are in a well defined sense composites of the primes belonging to lower level of infinity and at the bottom of this de-compositional hierarchy are finite primes. This decomposition corresponds to the decomposition of the space-time surface into p-adic regions which in TGD inspired theory of consciousness correspond to selves. Therefore the increase of the composite primes at lower level of infinity induces the increase of the infinite p-adic prime. p-Adic prime can increase in two manners.

1. One can introduce the concept of the p-adic sub-evolution: the evolution of infinite prime P is induced by the sub-evolution of infinite primes belonging to a lower level of infinity being induced by .... being induced by the evolution at the level of finite primes. For instance, the increase of the cell size means increase of the p-adic prime characterizing it: neurons are indeed very large and complicated cells whereas bacteria are small. Sub-evolution occurs both in subjective and geometric sense.

   i) For a given value of geometric time the p-adic prime of a given space-time sheet gradually increases in the evolution by quantum jumps: our geometric past evolves also!

   ii) The p-adic prime characterizing space-time sheet also increases as the geometric time associated with the space-time sheet increases (say during morphogenesis).

   The notion of sub-evolution is in accordance with the "Ontogeny recapitulates phylogeny" principle (ORP): the evolution of organism, now the entire Universe, contains the evolutions of the more primitive organisms as sub-evolutions.

2. Infinite prime increases also when entirely new finite primes emerge in the decomposition of an infinite prime to finite primes. This means that entirely new space-time sheets representing new structures emerge in quantum jumps. The creation of space-time sheets in quantum jumps could correspond to this process. By quantum classical correspondence this process corresponds at the space-time level to phase transitions giving rise to new material space-time sheets with more and more refined effective p-adic effective topology.

3.7 Local zeta functions, Galois groups, and infinite primes

The recent view about TGD leads to some conjectures about Riemann Zeta.

1. Non-trivial zeros should be algebraic numbers.

2. The building blocks in the product decomposition of ζ should be algebraic numbers for non-trivial zeros of zeta.

3. The values of zeta for their combinations with positive imaginary part with positive integer coefficients should be algebraic numbers.

These conjectures are motivated by the findings that Riemann Zeta seems to be associated with critical systems and by the fact that non-trivial zeros of zeta are analogous to complex conformal weights. The necessity to make such a strong conjectures, in particular conjecture c), is an unsatisfactory feature of the theory and one could ask how to modify this picture. Also a clear physical interpretation of Riemann zeta is lacking.

3.7.1 Local zeta functions and Weil conjectures

Riemann Zeta is not the only zeta [26, 25]. There is entire zoo of zeta functions and the natural question is whether some other zeta sharing the basic properties of Riemann zeta having zeros at critical line could be more appropriate in TGD framework.

The so called local zeta functions analogous to the factors $\zeta_p(s) = 1/(1-p^{-s})$ of Riemann Zeta can be used to code algebraic data about say numbers about solutions of algebraic equations reduced to finite fields. The local zeta functions appearing in Weil's conjectures [27] associated with finite fields $G(p,k)$ and thus to single prime. The extensions $G(p,nk)$ of this finite field are considered. These local zeta functions code the number for the points of algebraic variety for given value of n. Weil's conjectures also state that if X is a mod p reduction of non-singular complex projective variety then the degree for the polynomial multiplying the product $\zeta(s) \times \zeta(s-1)$ equals to Betti number. Betti number is 2 times genus in 2-D case.

It has been proven that the zetas of Weil are associated with single prime p, they satisfy functional equation, their zeros are at critical lines, and rather remarkably, they are rational functions of p^{-s}. For instance, for elliptic curves zeros are at critical line [27].

The general form for the local zeta is $\zeta(s) = exp(G(s))$, where $G = \sum g_n p^{-ns}$, $g_n = N_n/n$, codes for the numbers N_n of points of algebraic variety for n^{th} extension of finite field F with nk elements assuming that F has $k = p^r$ elements. This transformation resembles the relationship $Z = exp(F)$ between partition function and free energy $Z = exp(F)$ in thermodynamics.

The exponential form is motivated by the possibility to factorize the zeta function into a product of zeta functions. Note also that in the situation when N_n approaches constant N_∞, the division of N_n by n gives essentially $1/(1-N_\infty p^{-s})$ and one obtains the factor of Riemann Zeta at a shifted argument $s - log_p(N_\infty)$. The local zeta associated with Riemann Zeta corresponds to $N_n = 1$.

3.7.2 Local zeta functions and TGD

The local zetas are associated with single prime p, they satisfy functional equation, their zeros lie at the critical lines, and they are rational functions of p^{-s}. These features are highly desirable from the TGD point of view.

Why local zeta functions are natural in TGD framework?

In TGD framework modified Dirac equation assigns to a partonic 2-surface a p-adic prime p and inverse of the zeta defines local conformal weight. The intersection of the real and corresponding p-adic parton 2-surface is the set containing the points that one is interested in. Hence local zeta sharing the basic properties of Riemann zeta is highly desirable and natural. In particular, if the local zeta is a rational function then the inverse images of rational points of the geodesic sphere are algebraic numbers. Of course, one might consider a stronger constraint that the inverse image is rational. Note that one must still require that p^{-s} as well as s are algebraic numbers for the zeros of the local zeta (conditions 1) and 2) listed in the beginning) if one wants the number theoretical universality.

Since the modified Dirac operator assigns to a given partonic 2-surface a p-adic prime p, one can ask whether the inverse $\zeta_p^{-1}(z)$ of some kind of local zeta directly coding data about partonic 2-surface could define the generalized eigenvalues of the modified Dirac operator and radial super-canonical conformal weights so that the conjectures about Riemann Zeta would not be needed at all.

The eigenvalues of the modified Dirac operator would in a holographic manner code for information about partonic 2-surface. This kind of algebraic geometric data are absolutely relevant for TGD since U-matrix and probably also S-matrix must be formulated in terms of the data related to the intersection of real and partonic 2-surfaces (number theoretic braids) obeying same algebraic equations and consisting of algebraic points in the appropriate algebraic extension of p-adic numbers. Note that the hierarchy of algebraic extensions of p-adic number fields would give rise to a hierarchy of zetas so that the algebraic extension used would directly reflect itself in the eigenvalue spectrum of the modified Dirac operator and super-canonical conformal weights. This is highly desirable but not achieved if one uses Riemann Zeta.

One must of course leave open the possibility that for real-real transitions the inverse of the zeta defined as a product of the local zetas (very much analogous to Riemann Zeta) defines the conformal weights. This kind of picture would conform with the idea about real physics as a kind of adele formed from p-adic physics.

Finite field hierarchy is not natural in TGD context

That local zeta functions are assigned with a hierarchy of finite field extensions do not look natural in TGD context. The reason is that these extensions are regarded as abstract extensions of $G(p,k)$ as opposed to a large number of algebraic extensions isomorphic with finite fields as abstract number fields and induced from the extensions of p-adic number fields. Sub-field property is clearly highly relevant in TGD framework just as the sub-manifold property is crucial for geometrizing also other interactions than gravitation in TGD framework.

The $O(p^n)$ hierarchy for the p-adic cutoffs would naturally replace the hierarchy of finite fields. This hierarchy is quite different from the hierarchy of finite fields since one expects that the number of solutions becomes constant at the limit of large n and also at the limit of large p so that powers in the function G coding for the numbers of solutions of algebraic equations as function of n should not increase but approach constant N_∞. The possibility to factorize $exp(G)$ to a product $exp(G_0)exp(G_\infty)$ would mean a reduction to a product of a rational function and factor(s) $\zeta_p(s) = 1/(1-p^{-s_1})$ associated with Riemann Zeta with argument s shifted to $s_1 = s - log_p(N_\infty)$.

What data local zetas could code?

The next question is what data the local zeta functions could code.

1. It is not at clear whether it is useful to code global data such as the numbers of points of partonic 2-surface modulo p^n. The notion of number theoretic braid occurring in the proposed approach to S-matrix suggests that the zeta at an algebraic point z of the geodesic sphere S^2 of CP_2 or of light-cone boundary should code purely local data such as the numbers N_n of points which project to z as function of p-adic cutoff p^n. In the generic case this number would be finite for non-vacuum extremals with 2-D S^2 projection. The n^{th} coefficient $g_n = N_n/n$ of the function G_p would code the number N_n of these points in the approximation $O(p^{n+1}) = 0$ for the algebraic equations defining the p-adic counterpart of the partonic 2-surface.

2. In a region of partonic 2-surface where the numbers N_n of these points remain constant, $\zeta(s)$ would have constant functional form and therefore the information in this discrete set of algebraic points would allow to deduce deduce information about the numbers N_n. Both the algebraic points and generalized eigenvalues would carry the algebraic information.

3. A rather fascinating self referentiality would result: the generalized eigen values of the modified Dirac operator expressible in terms of inverse of zeta would code data for a sequence of approximations for the p-adic variant of the partonic 2-surface. This would be natural since second quantized induced spinor fields are correlates for logical thought in TGD inspired theory of consciousness. Even more, the data would be given at points $\zeta(s)$, s a rational value of a super-canonical conformal weight or a value of generalized eigenvalue of modified Dirac operator (which is essentially function $s = \zeta_p^{-1}(z)$ at geodesic sphere of CP_2 or of light-cone boundary).

3.7.3 Galois groups, Jones inclusions, and infinite primes

Langlands program [21, 22] is an attempt to unify mathematics using the idea that all zeta functions and corresponding theta functions could emerge as automorphic functions giving rise to finite-dimensional representations for Galois groups (Galois group is defined as a group of automorphisms of the extension of field F leaving invariant the elements of F). The basic example corresponds to rationals and their extensions. Finite fields $G(p,k)$ and their extensions $G(p,nk)$ represents another example. The largest extension of rationals corresponds to algebraic numbers (algebraically closed set). Although this non-Abelian group is huge and does not exist in the usual sense of the word its finite-dimensional representations in groups $GL(n, Z)$ make sense.

For instance, Edward Witten is working with the idea that geometric variant of Langlands duality could correspond to the dualities discovered in string model framework and be understood in terms of topological version of four-dimensional $N = 4$ super-symmetric YM theory [36]. In particular, Witten assigns surface operators to the 2-D surfaces of 4-D space-time. This brings unavoidably in mind partonic 2-surfaces and TGD as $N = 4$ super-conformal almost topological QFT.

This observation stimulates some ideas about the role of zeta functions in TGD if one takes the vision about physics as a generalized number theory seriously.

Galois groups, Jones inclusions, and quantum measurement theory

The Galois representations appearing in Langlands program could have a concrete physical/cognitive meaning.

1. The Galois groups associated with the extensions of rationals have a natural action on partonic 2-surfaces represented by algebraic equations. Their action would reduce to permutations of roots of the polynomial equations defining the points with a fixed projection to the above mentioned geodesic sphere S^2 of CP_2 or δM^4_+. This makes possible to define modes of induced spinor fields transforming under representations of Galois groups. Galois groups would also have a natural action on configuration space-spinor fields. One can also speak about configuration space spinors invariant under Galois group.

2. Galois groups could be assigned to Jones inclusions having an interpretation in terms of a finite measurement resolution in the sense that the discrete group defining the inclusion leaves invariant the operators generating excitations which are not detectable.

3. The physical interpretation of the finite resolution represented by Galois group would be based on the analogy with particle physics. The field extension K/F implies that the primes (more precisely, prime ideals) of F decompose into products of primes (prime ideals) of K. Physically this corresponds to the decomposition of particle into more elementary constituents, say hadrons into quarks in the improved resolution implied by the extension $F \to K$. The interpretation in terms of cognitive resolution would be that the primes associated with the higher extensions of rationals are not cognizable: in other words, the observed states are singlets under corresponding Galois groups: one has algebraic/cognitive counterpart of color confinement.

4. For instance, the system labeled by an ordinary p-adic prime could decompose to a system which is a composite of Gaussian primes. Interestingly, the biologically highly interesting p-adic length scale range 10 nm-5 μm contains as many as four Gaussian Mersennes ($M_k = (1+i)^k - 1$, $k = 151, 157, 163, 167$), which suggests that the emergence of living matter means an improved cognitive resolution.

Galois groups and infinite primes

In particular, the notion of infinite prime suggests a manner to realize the modular functions as representations of Galois groups. Infinite primes might also provide a new perspective to the concrete realization of Langlands program.

1. The discrete Galois groups associated with various extensions of rationals and involved with modular functions which are in one-one correspondence with zeta functions via Mellin transform defined as $\sum x_n n^{-s} \to \sum x_n z^n$ [28]. Various Galois groups would have a natural action in the space of infinite primes having interpretation as Fock states and more general bound states of an arithmetic quantum field theory.

2. The number theoretic anatomy of space-time points due to the possibility to define infinite number of number theoretically non-equivalent real units using infinite rationals [17] allows the imbedding space points themselves to code holographically various things. Galois groups would have a natural action in the space of real units and thus on the number theoretical anatomy of a point of imbedding space.

3. Since the repeated second quantization of the super-symmetric arithmetic quantum field theory defined by infinite primes gives rise to a huge space of quantum states, the conjecture that the number theoretic anatomy of imbedding space point allows to represent configuration space (the world of classical worlds associated with the light-cone of a given point of H) and configuration space spinor fields emerges naturally [17].

4. Since Galois groups G are associated with inclusions of number fields to their extensions, this inclusion could correspond at quantum level to a generalized Jones inclusion $\mathcal{N} \subset \mathcal{M}$ such that G acts as automorphisms of $\mathcal{M}$ and leaves invariant the elements of $\mathcal{N}$. This might be possible if one allows the replacement of complex numbers as coefficient fields of hyper-finite factors of type II_1 with various algebraic extensions of rationals. Quantum measurement theory with a finite measurement resolution defined by Jones inclusion $\mathcal{N} \subset \mathcal{M}$ [16] could thus have also a purely number theoretic meaning provided it is possible to define a non-trivial action of various Galois groups on configuration space spinor fields via the imbedding of the configuration space spinors to the space of infinite integers and rationals (analogous to the imbedding of space-time surface to imbedding space).

This picture allows to develop rather fascinating ideas about mathematical structures and their relationship to physical world. For instance, the functional form of a map between two sets the points of the domain and target rather than only its value could be coded in a holographic manner by using the number theoretic anatomy of the points. Modular functions giving rise to generalized zeta functions would emerge in especially natural manner in this framework. Configuration space spinor fields would allow a physical realization of the holographic representations of various maps as quantum states.

3.8 Remarks about correspondence between infinite primes, space-time surfaces, and configuration space spinor fields

The correspondence of CH points with infinite primes and thus with real units can be understood if one assume that the points of CH correspond to infinite rationals via their mapping to hyper-octonion real-analytic rational functions conjectured to define foliations of M^8 to hyper-quaternionic 4-surfaces inducing corresponding foliations of H. The correspondence of CH spinors with the real units identified as infinite rationals with varying number theoretical anatomies is not so obvious. It is good to approach the problem by making questions.

1. How the points of CH and CH spinors at given point of CH correspond to various real units? Configuration space Hamiltonians and their super-counterparts characterize modes of configuration space spinor fields rather than only spinors. Does this mean that only ground states of super-conformal representations, which are expected to correspond elementary particles, correspond to configuration space spinors and are coded by infinite primes?

2. How do CH spinor fields (as opposed to CH spinors) correspond to infinite rationals? Configuration space spinor fields are generated by elements of super-conformal algebra from ground states. Should one code the matrix elements of the operators between ground states and creating zero energy states in terms of time-like entanglement between ground states represented by real units and assigned to the preferred points of H characterizing the tips of future and past light-cones and having also interpretation as arguments of n-point functions?

The argument to be represented is in a nutshell following.

1. CH itself and CH spinors are by super-symmetry characterized by ground states of super-conformal representations and can be mapped to infinite rationals defining real units U_k multiplying the eight preferred H coordinates h^k whereas configuration space spinor fields correspond to discrete analogs of Schrödinger amplitudes in the space whose points have U_k as coordinates. The 8-units correspond to ground states for an 8-fold tensor power of a fundamental super-conformal representation or to a product of representations of this kind.

2. General states are coded by quantum entangled states defined as entangled states of positive and negative energy ground states with entanglement coefficients defined by the product of operators creating positive and negative energy states represented by the units. Normal ordering prescription makes the mapping unique.

3. The condition that various symmetries have number theoretical correlates leads to rather detailed view about the map of ground states to real units.

4. It seems that quantal generalization of the fundamental associativity and commutativity conditions might be needed.

Before continuing it is perhaps good to represent the most obvious objection against the idea. The correspondence between CH and CH spinors with infinite rationals and their discreteness means that also CH (world of classical worlds) and space of CH spinors should be discrete. First this looks non-sensible but is indeed what one obtains if space-time surfaces correspond to light-like 3-surfaces expressible in terms of algebraic equations involving rational functions with rational coefficients.

3.8.1 How CH and CH spinor fields correspond to infinite rationals?

The basic question is how CH and CH spinor fields on quantum fluctuating degrees of freedom (degrees of freedom for which configuration space metric is non-vanishing) correspond to infinite rationals.

Associativity and commutativity or only their quantum variants?

Associativity and commutativity conditions are absolutely essential notions in quantum TGD and also in the mapping of infinite primes to the space-time sheets. Associativity, guaranteed by hyper-octonion real-analyticity and implying rational infinite primes, seems to be necessary in order to obtain well-defined representations but might be too strong a condition.

Associativity implies hyper-quaternionicity and commutativity requirement in turn leads to rational infinite primes. Since one can decompose rational primes to hyper-quaternionic and even hyper-octonionic primes, one might hope that this could allow to represent states which consist of colored constituents. This representations has however the flavor of a formal trick and the considerations related to concrete representations of infinite primes suggest that the rationality of infinite primes might be a too restrictive condition.

A more radical possibility is that physical states are only quantum associative. This means that they are obtained as quantum superpositions in the space of real units over all possible associations performed for a given product of hyper-octonion primes (for instance, $|A(BC)\rangle + |(AB)C\rangle$). These states would be associative in quantum sense but would not reduce to hyper-quaternionic primes. Also the notion of quantum commutativity makes sense. The fact that mesons are quantum superpositions of quark-antiquark pairs which each corresponds to different pair of hyper-quaternionic primes and are thus not representable classically, suggests that one can require only quantum associativity and quantum commutativity.

How this idea relates to the representation of space-time surfaces in terms of rational functions of hyper-octonionic variable obtained as an image of rational infinite prime? If one replaces the coefficients of the polynomial which complex or more complex rational, hyper-octonion real analyticity is lost and one must consider some manner to map associative quantum state defined as superposition of various associations to single hyper-quaternionic prime.

1. The first approach is based on the assumption that only infinite integers reduce to infinite rational integers in the sense that the corresponding rational function has rational coefficients. This would allow partons as colored partons represented as non-associative constituents of infinite integers and there would be no problems with space-time correlates. It is however not clear whether this kind infinite integers are possible.

2. In the case of non-commutative group one can speak about commutator group and define Abelian group as coset group of these. Could it be that one can speak about associator algebra and define associative algebra by identifying additive associators $A(BC) - (AB)C$ with zero or multiplicative associators $(A(BC))((AB)C)^{-1}$ with unit. Hyper-octonionic primes would be mapped to

something represented by matrices. A good guess for the representation is in terms of 8-D analog of Pauli spin matrices.

Basic assumptions

The following assumptions serve as constraints when one tries to guess the map of quantum states to infinite primes.

1. Free many-particle states correspond to infinite integers and bound states to infinite primes mappable to irreducible polynomials. The numerator/denominator of the infinite rational should correspond to positive/negative energy states of which zero energy states consist of. At higher levels the mapping should be induced from that for the lowest level. Bosonic (fermionic) elementary particles in ground states should correspond to bosonic (fermionic primes). Phase conjugation as a generalization of that for laser beams) would correspond to the replacement of infinite integer with its inverse.

2. Concerning charge conjugation one can imagine several options but the detailed study of the realization of color symmetry leaves only one option. For this option the two singlets $1 \pm ie_7$ and triplet and antitriplet correspond to leptons and quarks with spin and electro-weak spin represented by the moduli space associated with the hyper-octonionic structures. One must leave open the interpretation of the change of the sign of the small part of the infinite prime, which looks excellent candidate for some discrete symmetry (parity perhaps?).

3. Discrete super-canonical and Super Kac-Moody algebras with bosonic and fermionic generators label the states. One should map the ground states of these representations to infinite primes and thus to real units in a natural manner. The requirement that standard model symmetries reduce to number theory serves as a powerful constraint and will be analyzed in detail later.

4. The excited states of various super-conformal representations can be mapped to quantum superpositions of many particle states formed from infinite primes. The operators creating the positive and negative energy parts are unique combinations of the operators of algebra if normal ordering prescription is applied. The matrix elements of these operators between ground states can be calculated. The entangled state formed from ground states with entanglement coefficients represented by these matrix elements gives the representation of the general state. Note that the real units would be associated with different points of H identifiable as arguments of n-point function in S-matrix elements.

How to map ground states of super-conformal representations to infinite primes?

Under the assumptions just stated the problem reduces to that of guessing the detailed form of the map of the ground states of super-conformal representations to primes at the first level of the hierarchy. The mapping of infinite primes to rational functions could provide a clue about how to achieve a natural one-to-one correspondence.

1. The decomposition of the irreducible polynomials in the algebraic extension of rationals gives interpretation in terms of many-particle states labeled by primes in the extension. This brings in Galois groups and their representations. This seems to be something new to present day physics. Note that color group plays the role of Galois group for octonions regarded as extension of reals.

2. Partonic two-surfaces should correspond to infinite primes but in such a manner that an infinite number of infinite primes are mapped to the same partonic 2-surface since given 3-surface should be able to to carry an arbitrary state of super-canonical and super Kac-Moody representation. This is the case since each light-like 3-surface traversing a given partonic 2-surface corresponds to an infinite prime in turn assumed to code for a foliation of hyper-quaternionic or co-hyper-quaternionic surfaces via corresponding rational function of hyper-octonionic variable. Light-like 3-surfaces and corresponding 4-D space-time sheets would thus code for the ground states of super-conformal representations. Quantum classical correspondence would apply to ground states but not to the excited states of super-conformal representations.

3. One should also understand how light-like partonic 3-surfaces are mapped to the number theoretic anatomies of a point of imbedding space. The natural choice for this point would be the preferred point of H defining the tip of the light-cone and the origin of complex coordinates of CP_2 transforming linearly under $U(2) \subset SU(3)$. This choice should be coded as a zero/pole of infinite rational with unit real norm coding for the zero energy states. Zeros would correspond to the positive energy state and poles to the negative energy state.

The treatment of zero modes

There are also zero modes which are absolutely crucial for quantum measurement theory. They entangle with quantum fluctuating degrees of freedom in quantum measurement situation and thus map quantum numbers to positions of pointers. The interior degrees of freedom of space-time interior must correspond to zero modes and they represent space-time correlates for quantum states realized at light-like partonic 3-surfaces.

As long as states associated with zero modes are represented by operators (such as CH Hamiltonians), the same description applies to them as to the representation of excited states of super-conformal representations. The absence of metric in zero modes means that there is no integration measure. The problems are avoided if one assumes that wave functions in zero modes have a discrete locus as suggested already earlier.

According to the argument represented in [C1], the quantum fluctuating configuration space degrees of freedom are by definition super-symmetrizable since configuration space gamma matrices correspond to the super counterparts of Hamiltonians in the case of super-canonical algebra. Super-symmetrizability condition means that the Poisson brackets of bosonic Hamiltonians reduce to 1-dimensional integrals over "stringy" curves of partonic 2-surface [C1]. This happens for the sub-algebra of super-canonical algebra having vanishing S^2 spin and color charges.

This would mean that zero modes include also the charged Hamiltonians of the super-canonical algebra. This brings in mind induced representations for which one has coset space structure with entire super-canonical group divided by the group generated by neutral super-canonical algebra. The necessary discretization zero modes of freedom suggests a reduction of the representations of isometry groups of H and CH to those for discrete subgroups of isometry groups which indeed appear naturally in Jones inclusions.

One must take this suggestion with some grain of salt. The coset construction for Kac-Moody representations allows to consider the possibility of extending the representations to charged Hamiltonians in such a manner that "stringy" commutators are preserved. The generation of Virasoro and Kac-Moody central extension parameters might be seen as the price paid for the stringy commutation relations.

Configuration space spinor fields as discrete Schrödinger amplitudes in the space of number theoretic anatomies?

It would seem that the analog of a complex Schrödinger amplitude in the space of number-theoretic anatomies of a given imbedding space point represented by single point of H and represented as 8-tuples of real units could naturally represent the dependence of CH spinors understood as ground states of super-conformal representations obtained as an 8-fold tensor power of a fundamental representation or product of representations perhaps differing somehow. The open question is why eight of them are needed. The excited states of super-conformal representations would be represented as time entangled states with entanglement between real units associated with the preferred points characterizing the tips future and past directed light-cones.

This picture conforms with the simple idea that infinite primes label the points in the fibers of the spinor field bundle having CH_h, h a preferred point of H characterizing the preferred origin of hyper-octonion structure, as a base space and that physical states correspond to discrete analogs of Schrödinger amplitude in this kind of bundles and product bundles formed from them. These 8-tuples define a number theoretical analog of $U(1)^8$ group in terms of which all number theoretical symmetries are represented.

3.8.2 Can one understand fundamental symmetries number theoretically?

One should understand symmetries number theoretically.

1. The basic idea is that color $SU(3) \subset G_2$ acts as automorphisms of hyper-octonion structure with a preferred imaginary unit and preferred point with respect to which hyper-octonionic power series are developed. $SO(7,1)$ would act as symmetries in the moduli space of hyper-octonion structures. Associativity implies symmetry breaking so that only hyper-quaternionic structures are considered and $SO(3,1) \times SO(4)$ acts as symmetries of the moduli space for these structures.

2. Color group is the analog of Galois group for the extension of reals to octonions and has a natural action on the decompositions of rational infinite primes to hyper-octonionic infinite primes. Color confinement is implied by hyper-quaternionicity of primes implied by associativity necessary to assign space-time surfaces to the infinite rationals. If one assumes only quantum associativity, one should have a generalization of the condition guaranteing color confinement. A possible more general condition is that infinite integers give rise to rational polynomials whereas infinite primes can be non-associative and non-commutative if they appear as constituents of N-particle state. This would predict that free quarks are not possible.

3. Electro-weak symmetries and Lorentz group act in the moduli space of hyper-octonionic structures and their actions deform space-time in H picture. CP_2 parameterizes the moduli space of hyper-quaternionic structures induced from a given hyper-octonionic structure with preferred imaginary unit.

4. Four-momenta correspond to translational degrees of freedom associated with the preferred points of M^4 coded by the infinite rational (tip of the light-cone). Color quantum numbers in cm degrees of freedom can be assigned to the CP_2 projection of the preferred point of H. As a matter fact, the definition of hyper-octonionic structure involves the choice of origin of M^8 giving rise to the preferred point of H.

Automorphisms and the symmetries of moduli space of hyper structures as basic symmetries

Consider now in more detail various symmetries.

1. G_2 acts as automorphisms on octonionic imaginary units and $SU(3)$ respects the choice of preferred imaginary unit. Associativity requires a reduction to hyper-quaternionic primes and implies color confinement. For hyper-quaternionic primes the automorphisms restrict to $SO(3)$ which has right/left action of fermionic hyper-quaternionic primes and adjoint action on bosonic hyper-quaternionc primes. The choice of hyper-quaternionic structure is global as opposed to the local choice of hyper-quaternionic tangent space of space-time surface assigning to a point of $HQ \subset HO$ a point of CP_2. $U(2) \subset SU(3)$ leaves invariant given hyper-quaternionic structure which are thus parameterized by CP_2. Color partial waves can be interpreted as partial waves in this moduli space.

2. The choice of global hyper-octonionic coordinate is dictated only modulo a transformation of $SO(1,7)$ acting as isometries of hyper-octonionic norm and as transformations in moduli space of hyper-octonion structures $SO(7)$ acting leaves invariant the choice of real unit. $SO(1,3) \times SO(4)$ acts in the moduli space of global hyper-quaternionic structures identified as sub-structures of hyper-octonionic structure. The choice of global HO structures involves also choice of origin implying preferred point of H. The M^4 projection of this point corresponds to the tip of light-cone. Since the integers representing physical states must be hyper-quaternionic by associativity conditions, the symmetry breaking ("number theoretic compactification") to $SO(1,3) \times SO(4)$ occurs very naturally. This group acts as spinor rotations in H picture and as isometries in M^8 picture.

3. $SO(1,7)$ allows 3 different 8-dimensional representations (8_v, 8_s, and $\overline{8}_s$). All these representations must decompose under $SU(3)$ as $1+1+3+\overline{3}$ as little exercise with $SO(8)$ triality demonstrates. Under $SO(6) \cong SU(4)$ the decompositions are $1+1+6$ and $4+\overline{4}$ for 8_v and 8_s and its conjugate. Both hyper-octonion spinors and gamma matrices are identified as hyper-octonion units rather than as matrices. It would be natural to assign to bosonic M^8 primes 8_v and to fermionic M^8 primes 8_s and $\overline{8}_s$. One can distinguish between $8_v, 8_s$ and $\overline{8}_s$ for hyper-octonionic units only if one considers the full $SO(1,3) \times SO(4)$ action in the moduli space of hyper-octonionic structures.

Physical interpretation of the decomposition of rational primes to various hyper-primes

Consider now the physical interpretation for the decomposition of rational primes to hyper-complex, hyper-quaternionic, and hyper-octonionic primes. Here one must keep doors open by allowing also the notion of quantum commutativity and quantum associativity so that infinite hyper-octonionic primes would not in general have these properties whereas their images to gamma matrices would define primes of an associative algebra so that a unique space-time representation in terms of hyper-octonionic polynomial would result. Abelianization would produce a generalization of hyper-complex algebra with 7 commuting imaginary units satisfying $e_i^2 = 1$. I have considered earlier also the possibility that hyper-analytic functions of this kind of variable could define space-time surfaces. At this stage one cannot distinguish between this and hyper-octonion real-analytic option.

1. The net quantum numbers of physical states must vanish in zero energy ontology. This is implied by the reduction of infinite rationals to infinite rationals associated with rationals but one must consider also more general options. The vanishing of net quantum numbers could be achieved in many manners. In the most general case the quantum numbers of positive and negative energy state represented by integers in the numerator and denominator of the infinite rational would compensate. If one requires only associativity for infinite primes (or integers) then positive (negative) energy state can correspond to hyper-quaternionic integer and one ends up with H picture and breaking of M^8 symmetries to those of H.

2. Commutativity condition implies a restriction to hyper-complex numbers. The only restriction would be due to fermion number conservation. Bosonic rational primes could be decomposed to fermionic and antifermionic hyper-quaternionic/octonionic primes such that the net fermion number vanishes. Fermionic primes could correspond to neutrinos and antineutrinos.

3. Giving up commutativity condition but requiring that the primes are associative gives hyper-quaternionic primes and color confinement. One obtains two states which possess non-vanishing and opposite color hypercharges equal to $\pm 2/3$. Thus only the interpretation as lepton, antilepton, quark and antiquark with no color isospin is possible. Spin, weak spin, and color would not be manifest since it would correspond to degree of freedom in the moduli space of hyper-quaternionic structures.

4. Hyper-quaternionic primes can be decomposed to hyper-octonionic primes. In the fermionic sector the three quark states consisting of hyper-octonion units would give color singlets as linear combination of hyper-octonion real unit and the preferred imaginary unit. A state analogous to baryon would result. Is this representation just a formal trick or does it have a real physical content must be left open. In TGD framework, color quantum numbers correspond to color partial waves in CP_2 labeling the moduli space of hyper-quaternionic structures associated with a given hyper-octonionic structure. One might hope that the decomposition provides a formal representation of information about these partial waves.

5. Giving up also associativity for single hyper-octonionic prime and requiring only quantum associativity and requiring that only infinite integers reduces to rational infinite integers leads to the most general framework allowing to describe entangled many particle states formed from elementary particles with quantum numbers of quark and lepton and basic gauge bosons. Gauge bosons and would correspond to locally entangled fermion antifermion pairs (as predicted by TGD) represented as locally entangled real units.

Electro-weak and color symmetries

The crucial test for this picture is whether color and electro-weak symmetries can be understood number theoretically.

Electro-weak group acts as transformations in the hyper-quaternionic moduli space inducing left or right actions of fermions which cannot interpreted as $U(2) \subset SU(3)$ automorphisms realized via adjoint action. For bosons one adjoint action results. Therefore color singlet states can possess non-vanishing electro-weak quantum numbers as also spin. For bosonic hyper-quaternionic primes one obtains singlet and triplet and for fermionic primes two doublets. The interpretation in terms of electro-weak gauge bosons and electro-weak doublets seems natural. Spin degrees of freedom are not manifestly visible but correspond to the moduli space resulting by $SL(2,C)$ action on hyper-quaternionic units.

Some more detailed comments about color symmetries are in order.

1. Color group SU(3) corresponds to subgroup of G_2 which acts as a Galois group for the extension of reals to octonions. $SU(3)$ leaves invariant real unit and a preferred octonionic imaginary unit. As noticed 8_v, 8_s and $\overline{8}_s$ decompose in a similar manner under $SU(3)$ and only the action of $SL(2,C)\times SO(4)$ modifying hyper-octonionic structure can distinguish between them.

2. Color group would act as a symmetry group on the composites of hyper-octonionic primes and color confinement in spinorial degrees of freedom would follow automatically from (complex) rationality (and even hyper-quaternionicity) of infinite integers necessitated by associativity. This does not however imply color singlet property in color rotational degrees of freedom in imbedding space. The value of color hypercharge (em charge) assignable to the spinors is the only signature of whether lepton or quark is in question.

3.9 A little crazy speculation about knots and infinite primes

D-dimensional knots correspond to the isotopy equivalence classes of the imbeddings of spheres S^d to S^{d+2}. One can consider also the isotopy equivalence classes of more general manifolds $M^d \subset M^{d+2}$. Knots [54] are very algebraic objects. The product (or sum, I prefer to talk about product) of knots is defined in terms of connected sum. Connected sum quite generally defines a commutative and associative product, and one can decompose any knot into prime knots.

Knots can be mapped to Jones polynomials $J(K)$ (for instance - there are many other polynomials and there are very general mathematical results about them [54]) and the product of knots is mapped to a product of corresponding polynomials. The polynomials assignable to prime knots should be prime in a well-defined sense, and one can indeed define the notion of primeness for polynomials $J(K)$: prime polynomial does not factor to a product of polynomials of lower degree in the extension of rationals considered.

This raises the idea that one could define the notion of zeta function for knots. It would be simply the product of factors $1/(1-J(K)^{-s})$ where K runs over prime knots. The new (to me) but very natural element in the definition would be that ordinary prime is replaced with a polynomial prime. This observation led to the idea that the hierarchy of infinite primes could correspond to the hierarchy of knots in various dimensions and this in turn stimulated quite fascinating speculations.

3.9.1 Do knots correspond to the hierarchy of infinite primes?

A very natural question is whether one could define the counterpart of zeta function for infinite primes. The idea of replacing primes with prime polynomials would resolve the problem since infinite primes can be mapped to polynomials. For some reason this idea however had not occurred to me earlier.

The correspondence of both knots and infinite primes with polynomials inspires the question whether $d=1$-dimensional prime knots might be in correspondence (not necessarily 1-1) with infinite primes. Rational or Gaussian rational infinite primes would be naturally selected: these are also selected by physical considerations as representatives of physical states although quaternionic and octonionic variants of infinite primes can be considered.

If so, knots could correspond to the subset of states of a super-symmetric arithmetic quantum field theory with bosonic single particle states and fermionic states labeled by quaternionic primes.

1. The free Fock states of this QFT are mapped to first order polynomials and irreducible polynomials of higher degree have interpretation as bound states so that the non-decomposability to a product in a given extension of rationals would correspond physically to the non-decomposability into many-particle state. What is fascinating that apparently free arithmetic QFT allows huge number of bound states.

2. Infinite primes form an infinite hierarchy, which corresponds to an infinite hierarchy of second quantizations for infinite primes meaning that n-particle states of the previous level define single particle states of the next level. At space-time level this hierarchy corresponds to a hierarchy

defined by space-time sheets of the topological condensate: space-time sheet containing a galaxy can behave like an elementary particle at the next level of hierarchy.

3. Could this hierarchy have some counterpart for knots?In one realization as polynomials, the polynomials corresponding to infinite prime hierarchy have increasing number of variables. Hence the first thing that comes into my uneducated mind is as the hierarchy defined by the increasing dimension d of knot. All knots of dimension d would in some sense serve as building bricks for prime knots of dimension $d+1$ or possibly $d+2$ (the latter option turns out to be the more plausible one). A canonical construction recipe for knots of higher dimensions should exist.

4. One could also wonder whether the replacement of spherical topologies for d-dimensional knot and $d+2$-dimensional imbedding space with more general topologies could correspond to algebraic extensions at various levels of the hierarchy bringing into the game more general infinite primes. The units of these extensions would correspond to knots which involve in an essential manner the global topology (say knotted non-contractible circles in 3-torus). Since the knots defining the product would in general have topology different from spherical topology the product of knots should be replaced with its category theoretical generalization making higher-dimensional knots a groupoid in which spherical knots would act diagonally leaving the topology of knot invariant. The assignment of d-knots with the notion of n-category, n-groupoid, etc.. by putting d=n is a highly suggestive idea. This is indeed natural since are an outcome of a repeated abstraction process: statements about statements about .....

5. The lowest ($d=1, D=3$) level would be the fundamental one and the rest would be somewhat boring repeated second quantization;-). This is why the dimension $D=3$ (number theoretic braids at light-like 3-surfaces!) would be fundamental for physics.

3.9.2 Further speculations

Some further speculations about the proposed structure of all structures are natural.

1. The possibility that algebraic extensions of infinite primes could allow to describe the refinements related to the varying topologies of knot and imbedding space would mean a deep connection between number theory, manifold topology, sub-manifold topology, and n-category theory.

2. Category theory appears already now in fundamental role in the construction of the generalization of M-matrix unifying the notions of density matrix and S-matrix. Generalization of category to n-category theory and various n-structures would have very direct correspondence with the physics of TGD Universe if one assumes that repeated second quantization makes sense and corresponds to the hierarchical structure of many-sheeted space-time where even galaxy corresponds to elementary fermion or boson at some level of hierarchy.

   This however requires that the unions of light-like 3-surfaces and of their sub-manifolds at different levels of topological condensate are able to represent higher-dimensional manifolds physically albeit not in the standard geometric sense since imbedding space dimension is just 8. This might be possible.

3. As far as physics is considered, the disjoint union of sub-manifolds of dimensions d_1 and d_2 behaves like a d_1+d_2-dimensional Cartesian product of the corresponding manifolds. This is of course used in standard manner in wave mechanics (the configuration space of n-particle system is identified as E^{3n}/S_n with division coming from statistics).

4. If the surfaces have intersection points, one has a union of Cartesian product with punctures (intersection points) and of lower-dimensional manifold corresponding to the intersection points.

5. Note also that by posing symmetries on classical fields one can effectively obtain from a given n-manifold manifolds (and orbifolds) with quotient topologies.

The megalomanic conjecture is that this kind of physical representation of d-knots and their imbedding spaces is possible using many-sheeted space-time. Perhaps even the entire magnificent mathematics of n-manifolds and their sub-manifolds might have a physical representation in terms of sub-manifolds of 8-D $M^4 \times CP_2$ with dimension not higher than space-time dimension $d=4$.

3.9.3 The idea survives the most obvious killer test

All this looks nice and the question is how to give a death blow to all this reckless speculation. Torus knots are an excellent candidate for performing this unpleasant task but the hypothesis survives!

1. Torus knots [56] are labeled by a pair integers (m, n), which are relatively prime. These are prime knots. Torus knots for which one has $m/n = r/s$ are isotopic so that any torus knot is isotopic with a knot for which m and n have no common prime power factors.

2. The simplest infinite primes correspond to free Fock states of the supersymmetric arithmetic QFT and are labeled by pairs (m, n) of integers such that m and n do not have any common prime factors. Thus torus knots would correspond to free Fock states! Note that the prime power $p^{k(p)}$ appearing in m corresponds to $k(p)$-boson state with boson "momentum" p and the corresponding power in n corresponds to fermion state plus $k(p) - 1$ bosons.

3. A further property of torus knots is that (m, n) and (n, m) are isotopic: this would correspond at the level of infinite primes to the symmetry $mX + n \rightarrow nX + m$, X product of all finite primes. Thus infinite primes are in $2 \rightarrow 1$ correspondence with torus knots and the hypothesis survives also this murder attempt. Probably the assignment of orientation to the knot makes the correspondence 1-1 correspondence.

3.9.4 How to realize the representation of the braid hierarchy in many-sheeted space-time?

One can consider a concrete construction of higher-dimensional knots and braids in terms of the many-sheeted space-time concept.

1. The basic observation is that ordinary knots can be constructed as closed braids so that everything reduces to the construction of braids. In particular, any torus knot labeled by (m,n) can be made from a braid with m strands: the braid word in question is $(\sigma_1....\sigma_{m-1})^n$ or by $(m, n) = (n, m)$ equivalence from n strands. The construction of infinite primes suggests that also the notion of d-braid makes sense as a collection of d-braids in $d+2$-space, which move and and define $d+1$-braid in $d+3$ space (the additional dimension being defined by time coordinate).

2. The notion of topological condensate should allow a concrete construction of the pairs of d- and $d + 2$-dimensional manifolds. The 2-D character of the fundamental objects (partons) might indeed make this possible. Also the notion of length scale cutoff fundamental for the notion of topological condensate is a crucial element of the proposed construction.

3. Infinite primes have also interpretation as physical states and the representation in terms of knots would mean a realization of quantum classical correspondence.

The concrete construction would proceed as follows.

1. Consider first the lowest non-trivial level in the hierarchy. One has a collection of 3-D light-like 3-surfaces X^3_i representing ordinary braids. The challenge is to assign to them a 5-D imbedding space in a natural manner. Where do the additional two dimensions come from? The obvious answer is that the new dimensions correspond to the partonic 2-surface X^2 assignable to the $3-D$ lightlike surface X^3 at which these surfaces have suffered topological condensation. The geometric picture is that X^3_i grow like plants from ground defined by X^2 at 7-dimensional $\delta M^4_+ \times CP_2$.

2. The degrees of freedom of X^2 should be combined with the degrees of freedom of X^3_i to form a 5-dimensional space X^5. The natural idea is that one first forms the Cartesian products $X^5_i = X^3_i \times X^2$ and then the desired 5-manifold X^5 as their union by posing suitable additional conditions. Braiding means a translational motion of X^3_i inside X^2 defining braid as the orbit in X^5. It can happen that X^3_i and X^3_j intersect in this process. At these points of the union one must obviously pose some additional conditions. Same applies to intersection of more than two X^3_i.

Finite (p-adic) length scale resolution suggests that all points of the union at which an intersection between two or more light-like 3-surfaces occurs must be regarded as identical. In general the intersections would occur in a 2-d region of X^2 so that the gluing would take place along 5-D regions of X_i^5 and there are therefore good hopes that the resulting 5-D space is indeed a manifold. The imbedding of the surfaces X_i^3 to X^5 would define the braiding.

3. At the next level one would consider the 5-d structures obtained in this manner and allow them to topologically condense at larger 2-D partonic surfaces in the similar manner. The outcome would be a hierarchy consisting of $2n+1$-knots in $2n+3$ spaces. A similar construction applied to partonic surfaces gives a hierarchy of $2n$-knots in $2n+2$-spaces.

4. The notion of length scale cutoff is an essential element of the many-sheeted space-time concept. In the recent context it suggests that d-knots represented as space-time sheets topologically condensed at the larger space-time sheet representing $d+2$-dimensional imbedding space could be also regarded effectively point-like objects (0-knots) and that their d-knottiness and internal topology could be characterized in terms of additional quantum numbers. If so then d-knots could be also regarded as ordinary colored braids and the construction at higher levels would indeed be very much analogous to that for infinite primes.

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Part II

TGD AND P-ADIC NUMBERS

Chapter 4

p-Adic Numbers and Generalization of Number Concept

4.1 Introduction

There have been a lot of early speculations about the role of the p-adic numbers in Physics [16, 17, 18]. In [9] one can find a review of the work done. In general the work is related to the quantum theory and based on the assumption that the quantum mechanical state space is an ordinary complex Hilbert space. This is not absolutely necessary since p-adic unitarity and probability concepts make sense [16]. What is however essential is some kind of correspondence between the p-adic and real numbers since the predictions of, say p-adic quantum mechanics, should be expressed in terms of the real numbers.

One can imagine two kinds of correspondences between reals and p-adics.

1. The correspondence defined by rational numbers regarded as common to real and p-adic number fields and their extensions applies at the level of geometry. The generalization of the number concept obtained by gluing all number fields together along common rational numbers generalizes also to the level of manifolds and Hilbert spaces.

2. Another correspondence is based on the canonical identification and can be used to map p-adic probabilities to their real counterparts. Also the predictions of p-adic thermodynamics for mass squared values of elementary particles can be mapped to the p-adic numbers using the correspondence. Canonical identification does not however work at space-time level since it does not respect field equations nor even differentiability although it is continuous.

3. A compromize between canonical correspondence and identification via common rationals is achieved by using a modification of canonical identification $I_{R_p \rightarrow R}$ defined as $I_1(r/s) = I(r)/I(s)$.

The formulation of the p-adic physics requires the construction of the p-adic differential and integral calculus. Also the p-adic counterparts of Hilbert space, group theory, and Fourier analysis are needed as also the generalization of manifold concept, Riemann geometry, sub-manifold geometry, and even configuration space geometry. These generalizations are discussed in this and subsequent chapter.

4.1.1 Canonical identification

The notion of canonical identification dominated p-adic TGD almost for a decade. Canonical identification is a canonical correspondence between the p-adic numbers and nonnegative real numbers defined by the "pinary" expansion of real number: positive real number $x = \sum x_n p^n$ ($x = 0, 1, .., p-1$, p prime) is mapped to p-adic number $\sum x_n p^{-n}$. This canonical correspondence allows to induce p-adic topology to the real axis. p-Adically differentiable functions define typically fractal like real functions via the canonical identification so that p-adic numbers provide analytic tool for producing fractals. p-Adic numbers allow algebraic extensions of arbitrary dimension and the concept of complex analyticity generalizes to p-adic analyticity.

The concepts of the p-adic probability and unitarity make sense and one can associate with the p-adic probabilities unique real probabilities using the canonical correspondence and this predicts novel physical effects. The successful p-adic description of the particle massivation relies heavily on the canonical correspondence.

4.1.2 Identification via common rationals

Besides canonical identification there is also a second natural correspondence between reals and p-adics. This correspondence is induced via common rationals in the sense that one can regard p-adics and reals as different completions of rationals and given rational number can be identified as an element or reals or of any p-adic number field.

For instance, if S-matrix is complex rational matrix or belongs to finite-dimensional extension or rationals, one can regard it as either real or p-adic S-matrix. The assumption that the so called CKM matrix describing quark mixings is complex rational, fixes with some empirical inputs the CKM matrix essentially uniquely. Second example: if it is assumed that fundamental state space has complex rationals as a coefficient field, it becomes sensible to define tensor factors of Hilbert spaces belonging to different number fields because entanglement is possible with complex rational coefficients. One could also see the basic physics as essentially rational and real and p-adic physics as different algebraic continuations of it. Also much more general vision encouraged by TGD inspired theory of consciousness and p-adic physics as physics of cognition and intentionality is possible.

On can generalize the concepts of the definite integral, Hilbert space, Riemannian manifold, and Lie group to the p-adic context in a relatively straightforward manner and the correspondence via common rationals makes it possible to carry out these generalizations as an algebraic continuation with clear interpretation about what is involved. The generalization of the number concept generalizes these structures so that real and various p-adic variants of the structure can be seen as various facets of the generalized structure.

4.1.3 Hybrid of canonical identification and identification via common rationals

A compromize between canonical correspondence and identification via common rationals is achieved by using a modification of canonical identification $I_{R_p \rightarrow R}$ defined as $I_1(r/s) = I(r)/I(s)$. If the conditions $r \ll p$ and $s \ll p$ hold true, the map respects algebraic operations and also unitarity and various symmetries. It seems that this option must be used to relate p-adic transition amplitudes to real ones and vice versa [F5]. In particular, real and p-adic coupling constants are related by this map. Also some problems related to p-adic mass calculations find a nice resolution when I_1 is used.

This variant of canonical identification is not equivalent with the original one using the infinite expansion of q in powers of p since canonical identification does not commute with product and division. The variant is however unique in the recent context when r and s in $q = r/s$ have no common factors. For integers $n < p$ it reduces to direct correspondence. R_{p1} and R_{p_2} are glued together along common rationals by an the composite map $I_{R \rightarrow R_{p_2}} I_{R_{p_1} \rightarrow R}$.

4.1.4 Topics of the chapter

The topics of the chapter are the following:

1. p-Adic numbers, their extensions (also those involving transcendentals) are described. The existence of a square root of an ordinary p-adic number is necessary in many applications of the p-adic numbers (p-adic group theory, p-adic unitarity, Riemannian geometry) and its existence implies a unique algebraic extension, which is 4-dimensional for $p > 2$ and 8-dimensional for $p = 2$. Contrary to the first expectations, all possible algebraic extensions are possible and one cannot interpret the algebraic dimension of the algebraic extension as a physical dimension.

2. The concepts of the p-adic differentiability and analyticity are discussed. The notion of p-adic fractal is introduced the properties of the fractals defined by p-adically differentiable functions are discussed.

3. Various approaches to the problem of defining p-adic valued definite integral are discussed. The only reasonable generalizations rely on algebraic continuation and correspondence via common rationals. p-Adic field equations do not necessitate p-adic definite integral but algebraic continuation allows to assign to a given real space-time sheets a p-adic space-time sheets if the definition of space-time sheet involves algebraic relations between imbedding space coordinates. There are also hopes that one can algebraically continue the value of Kähler action to p-adic context if finite-dimensional extensions are allowed.

4. Symmetries are discussed from p-adic point of view starting from the identification via common rationals. Also possible p-adic generalizations of Fourier analysis are considered. Besides a number theoretical approach, group theoretical approach providing a direct generalization of the ordinary Fourier analysis based on the utilization of exponent functions existing in algebraic extensions containing some root of e and its powers up to e^{p-1} is discussed. Also the generalization of Fourier analysis based on the Pythagorean phases is considered.

4.2 p-Adic numbers

4.2.1 Basic properties of p-adic numbers

p-Adic numbers (p is prime: 2,3,5,...) can be regarded as a completion of the rational numbers using a norm, which is different from the ordinary norm of real numbers [8]. p-Adic numbers are representable as power expansion of the prime number p of form:

$$x = \sum_{k\geq k_0} x(k)p^k ,\ x(k) = 0, ..., p-1 \ . \tag{4.2.1}$$

The norm of a p-adic number is given by

$$|x| = p^{-k_0(x)} \ . \tag{4.2.2}$$

Here $k_0(x)$ is the lowest power in the expansion of the p-adic number. The norm differs drastically from the norm of the ordinary real numbers since it depends on the lowest pinary digit of the p-adic number only. Arbitrarily high powers in the expansion are possible since the norm of the p-adic number is finite also for numbers, which are infinite with respect to the ordinary norm. A convenient representation for p-adic numbers is in the form

$$x = p^{k_0}\varepsilon(x) \ , \tag{4.2.3}$$

where $\varepsilon(x) = k + ...$ with $0 < k < p$, is p-adic number with unit norm and analogous to the phase factor $exp(i\phi)$ of a complex number.

The distance function $d(x, y) = |x - y|_p$ defined by the p-adic norm possesses a very general property called ultra-metricity:

$$d(x,z) \leq max\{d(x,y), d(y,z)\} \ . \tag{4.2.4}$$

The properties of the distance function make it possible to decompose R_p into a union of disjoint sets using the criterion that x and y belong to same class if the distance between x and y satisfies the condition

$$d(x,y) \leq D \ . \tag{4.2.5}$$

This division of the metric space into classes has following properties:

1. Distances between the members of two different classes X and Y do not depend on the choice of points x and y inside classes. One can therefore speak about distance function between classes.
2. Distances of points x and y inside single class are smaller than distances between different classes.
3. Classes form a hierarchical tree.

Notice that the concept of the ultra-metricity emerged in physics from the models for spin glasses and is believed to have also applications in biology [10]. The emergence of p-adic topology as the topology of the effective space-time would make ultra-metricity property basic feature of physics.

4.2.2 p-Adic ultrametricity and divergence cancellation

p-Adic ultrametricity implies that the p-adic norm for a sum of p-adic numbers cannot be larger than the maximum of the p-adic norm for the summands. In p-adic QFT this has an overall important consequence: p-adic loops sums over the discrete labels characterizing p-adic planewaves are bounded from above. This means an automatic cancellation of the ultraviolet divergences. The finite volume of the p-adic space-time region in turn implies the cancellation of the infrared divergences and the convergence of the p-adic loops sums to a well defined limit.

It must be emphasized that the finiteness of the terms appearing in the loop sums is not trivially true in the coordinate-space formulation of the perturbation theory and it will be found that finiteness, or equivalently, the p-adic pseudo-constancy of the coordinate space propagators, might necessitate the natural p-adic cutoff provided by the CP_2 radius below which the assumption about the effective quantum average space-time representable locally as a map $M^4_+ \rightarrow CP_2$ fails. One must however emphasize that the formulation of the theory is not yet so detailed that one could draw any strong conclusions in this respect.

4.2.3 Extensions of p-adic numbers

Algebraic democracy suggests that all possible real algebraic extensions of the p-adic numbers are possible. This conclusion is also suggested by various physical requirements, say the fact that the eigenvalues of a Hamiltonian representable as a rational or p-adic $N \times N$-matrix, being roots of N:th order polynomial equation, in general belong to an algebraic extension of rationals or p-adics. The dimension of the algebraic extension cannot be interpreted as physical dimension. Algebraic extensions are characteristic for cognitive physics and provide a new manner to code information. A possible interpretation for the algebraic dimension is as a dimension for a cognitive representation of space and would explain how it is possible to mathematically imagine spaces with all possible dimensions although physical space-time dimension is four (TGD as a number theory vision suggest that also space-time dimensions which are multiples of four are possible). The idea of algebraic hologram and other ideas related to the physical interpretation of the algebraic extensions of p-adics are discussed in the chapter "TGD as a generalized number theory".

It seems however that algebraic democracy must be extended to include also transcendentals in the sense that finite-dimensional extensions involving also transcendental numbers are possible: for instance, Neper number e defines a p-dimensional extension. It has become clear that these extensions fundamental for understanding how p-adic physics as physics of cognition is able to mimick real physics. The evolution of mathematical cognition can be seen as a process in which p-adic space-time sheets involving increasing value of p-adic prime p and increasing dimension of algebraic extension appear in quantum jumps.

Recipe for constructing algebraic extensions

Real numbers allow only complex numbers as an algebraic extension. For p-adic numbers algebraic extensions of arbitrary dimension are possible
[8]. The simplest manner to construct (n+1)-dimensional extensions is to consider irreducible polynomials $P_n(t)$ in R_p assumed to have rational coefficients: irreduciblity means that the polynomial does not possess roots in R_p so that one cannot decompose it into a product of lower order R_p valued polynomials. This condition is equivalent with the condition with irreducibility in the finite field $G(p,1)$, that is modulo p in R_p.

Denoting one of the roots of $P_n(t)$ by θ and defining $\theta^0 = 1$ the general form of the extension is given by

$$Z = \sum_{k=0,..,n-1} x_k\theta^k \quad . \tag{4.2.6}$$

Since θ is root of the polynomial in R_p it follows that θ^n is expressible as a sum of lower powers of θ so that these numbers indeed form an n-dimensional linear space with respect to the p-adic topology.

Especially simple odd-dimensional extensions are cyclic extensions obtained by considering the roots of the polynomial

$$\begin{aligned} P_n(t) &= t^n + \epsilon d \ , \\ \epsilon &= \pm 1 \ . \end{aligned} \tag{4.2.7}$$

For $n = 2m+1$ and $(n = 2m, \epsilon = +1)$ the irreducibility of $P_n(t)$ is guaranteed if d does not possess n:th root in R_p. For $(n = 2m, \epsilon = -1)$ one must assume that $d^{1/2}$ does not exist p-adically. In this case θ is one of the roots of the equation

$$t^n = \pm d \ , \tag{4.2.8}$$

where d is a p-adic integer with a finite number of pinary digits. It is possible although not necessary to identify the roots as complex numbers. There exists n complex roots of d and θ can be chosen to be one of the real or complex roots satisfying the condition $\theta^n = \pm d$. The roots can be written in the general form

$$\begin{aligned} \theta &= d^{1/n}exp(i\phi(m)), \ m = 0,1,....,n-1 \ , \\ \phi(m) &= \frac{m2\pi}{n} \ \text{or} \ \frac{m\pi}{n} \ . \end{aligned} \tag{4.2.9}$$

Here $d^{1/n}$ denotes the real root of the equation $\theta^n = d$. Each of the phase factors $\phi(m)$ gives rise to an algebraically equivalent extension: only the representation is different. Physically these extensions need not be equivalent since the identification of the algebraically extended p-adic numbers with the complex numbers plays a fundamental role in the applications. The cases $\theta^n = \pm d$ are physically and mathematically quite different.

p-Adic valued norm for numbers in algebraic extension

The p-adic valued norm of an algebraically extended p-adic number x can be defined as some power of the ordinary p-adic norm of the determinant of the linear map $x :^e R_p^n \rightarrow^e R_p^n$ defined by the multiplication with x: $y \rightarrow xy$

$$N(x) = det(x)^{\alpha} \ , \ \alpha > 0 \ . \tag{4.2.10}$$

Real valued norm can be defined as the p-adic norm of $N(x)$. The requirement that the norm is homogenous function of degree one in the components of the algebraically extended 2-adic number (like also the standard norm of R^n) implies the condition $\alpha = 1/n$, where n is the dimension of the algebraic extension.

The canonical correspondence between the points of R_+ and R_p generalizes in obvious manner: the point $\sum_k x_k\theta^k$ of algebraic extension is identified as the point $(x_R^0, x_R^1, ..., x_R^k, ..,)$ of R^n using the pinary expansions of the components of p-adic number. The p-adic linear structure of the algebraic extension induces a linear structure in R_+^n and p-adic multiplication induces a multiplication for the vectors of R_+^n.

Algebraic extension allowing square root of ordinary p-adic numbers

The existence of a square root of an ordinary p-adic number is a common theme in various applications of the p-adic numbers and for long time I erratically believed that only this extension is involved with p-adic physics. Despite this square root allowing extension is of central importance and deserves a more detailed discussion.

1. The p-adic generalization of the representation theory of the ordinary groups and Super Kac Moody and Super Virasoro algebras exists provided an extension of the p-adic numbers allowing square roots of the 'real' p-adic numbers is used. The reason is that the matrix elements of the raising and lowering operators in Lie-algebras as well as of oscillator operators typically involve square roots. The existence of square root might play a key role in various p-adic considerations.

2. The existence of a square root of a real p-adic number is also a necessary ingredient in the definition of the p-adic unitarity and probability concepts since the solution of the requirement that $p_{mn} = S_{mn}\bar{S}_{mn}$ is ordinary p-adic number leads to expressions involving square roots.

3. p-Adic length scales hypothesis states that the p-adic length scale is proportional to the square root of p-adic prime.

4. Simple metric geometry of polygons involves square roots basically via the theorem of Pythagoras. p-Adic Riemannian geometry necessitates the existence of square root since the definition of the infinitesimal length ds involves square root. Note however that p-adic Riemannian geometry can be formulated as a mere differential geometry without any reference to global concepts like lengths, areas, or volumes.

The original belief that square root allowing extensions of p-adic numbers are exceptional seems to be wrong in light of TGD as a generalized number theory vision. All algebraic extensions of p-adic numbers a possible and the interpretation of algebraic dimension of the extension as a physical dimension is not the correct thing to do. Rather, the possibility of arbitrarily high algebraic dimension reflects the ability of mathematical cognition to imagine higher-dimensional spaces. Square root allowing extension of the p-adic numbers is the simplest one imaginable, and it is fascinating that it indeed is the dimension of space-time surface for $p > 2$ and dimension of imbedding space for $p = 2$. Thus the square root allowing extensions deserve to be discussed.

The results can be summarized as follows.

1. In $p > 2$ case the general form of extension is

$$Z = (x + \theta y) + \sqrt{p}(u + \theta v) \ , \tag{4.2.11}$$

where the condition $\theta^2 = x$ for some p-adic number x not allowing square root as a p-adic number. For $p \ mod \ 4 = 3$ θ can be taken to be imaginary unit. This extension is natural for p-adication of space-time surface so that space-time can be regarded as a number field locally. Imbedding space can be regarded as a cartesian product of two 4-dimensional extensions locally.

2. In $p = 2$ case 8-dimensional extension is needed to define square roots. The extension is defined by adding $\theta_1 = \sqrt{-1} \equiv i$, $\theta_2 = \sqrt{2}$, $\theta_3 = \sqrt{3}$ and the products of these so that the extension can be written in the form

$$Z = x_0 + \sum_k x_k\theta_k + \sum_{k<l} x_{kl}\theta_{kl} + x_{123}\theta_1\theta_2\theta_3 \ . \tag{4.2.12}$$

Clearly, $p = 2$ case is exceptional as far as the construction of the conformal field theory limit is considered since the structure of the representations of Virasoro algebra and groups in general changes drastically in $p = 2$ case. The result suggest that in $p = 2$ limit space-time surface and H are in same relation as real numbers and complex numbers: space-time surfaces defined as the absolute minima of 2-adiced Kähler action are perhaps identifiable as surfaces for which the imaginary part of 2-adically analytic function in H vanishes.

The physically interesting feature of p-adic group representations is that if one doesn't use $\sqrt{p}$ in the extension the number of allowed spins for representations of $SU(2)$ is finite: only spins $j < p$ are allowed. In $p = 3$ case just the spins $j \leq 2$ are possible. If 4-dimensional extension is used for $p = 2$ rather than 8-dimensional then one gets the same restriction for allowed spins.

Finite-dimensional extensions involving transcendentals

The transcendentals e and π appear repeatedly in the basic formulas of calculus and physics. Also logarithms are unavoidable. The idea that rational numbers are common for all number fields suggests that the p-adic variant of logarithm function should be well-defined and be equivalent with the real logarithm in the subset of rationals. This boils down to the requirement that the logarithms $log(p)$, p prime exist for all primes.

The requirement that cognition has as its space-time correlates p-adic space-time sheets corresponding to finite-dimensional extensions of p-adic numbers implies that the extensions involving transcendentals must be finite-dimensional. This requirement discussed in the chapter "Riemann Hypothesis and Physics" looks extremely strong.

The intuitive expectation is that the extension containing e, π, and logarithms $log(p)$ of primes is finite-dimensional for any prime p. Log(p) is contained in the extension if $\pi/log(p)$ is rational number for any prime p. π is contained in the extension of $\pi/log(log(....(log(\pi)...))$ is rational number for sum finite-fold logarithmic iterate of π. The detailed argument is discussed in the chapter "Riemann Hypothesis and Physics" and here only a rough sketch is given.

1. The extension containing e is finite-dimensional. The reason is that e^x exists as a p-adic series for $|x|_p < 1$. Thus only the powers $e, e^2, ..., e^{p-1}$ need to be introduced and this gives to a p-dimensional extension.

2. One might think that π can be defined in the extension containing $\sqrt{-1}$ ($\sqrt{-1}$ is an ordinary p-adic number for $p \ mod \ 4 = 1$) by using the identity $log(-1) = \sqrt{-1}\pi$ and by writing $log(-1) = log[(p-1)/(1-p)] = 1/2 log[(p-1)^2] - log(1-p)$ and by applying power series of logarithm $log(1+y)$ converging for $|y|_p < 1$. Unfortunately, the constraint $exp(i\pi) = -1$ is not satisfied for this identification of π. Thus the only hope is that e/π is rational number or an analogous statement holds true for some higher logarithmic iterate of π.

3. The logarithms $log(q)$, $q \neq p$, can be defined by writing

$$log(q) = log[q^{d(p,q)}]/d(p,q) \ ,$$

   where $d(p,q)$ is an integer such that $q^{d(p,q)} \ mod \ p = 1$. The difficult part is thus the identification of $log(p)$ for R_p. This logarithm exists if $log(p)/\pi$ is a rational number. This number theoretical conjecture is unproven and implies that $log(x)/\pi$ is rational number for any rational number x. The conjecture follows from the requirement that Riemann Zeta is a universal function existing in the field of real numbers and in various p-adic number fields and is algebraically continuable from its representation in the set of rationals. This is achieved if the values of the functions p^{iy} appearing as building blocks of Riemann Zeta $\zeta(x+iy)$ are algebraic numbers when y is a rational number. A stronger condition is that y is rational number for the zeros $z = 1/2 + iy$ of Riemann zeta so that also zeros would be universal.

4.2.4 p-Adic Numbers and Finite Fields

Finite fields (Galois fields) consists of finite number of elements and allow sum, multiplication and division. A convenient representation for the elements of a finite field is as the roots of the polynomial equation $t^{p^m} - t = 0 \ mod \ p$, where p is prime, m an arbitrary integer and t is element of a field of characteristic p ($pt = 0$ for each t). The number of elements in a finite field is p^m, that is power of prime number and the multiplicative group of a finite field is group of order $p^m - 1$. $G(p,1)$ is just cyclic group Z_p with respect to addition and $G(p,m)$ is in rough sense m:th Cartesian power of $G(p,1)$.

The elements of the finite field $G(p,1)$ can be identified as the p-adic numbers $0, ..., p-1$ with p-adic arithmetics replaced with modulo p arithmetics. The finite fields $G(p,m)$ can be obtained from m-dimensional algebraic extensions of the p-adic numbers by replacing the p-adic arithmetics with the modulo p arithmetics. In TGD context only the finite fields $G(p>2,2)$, $p\ mod\ 4=3$ and $G(p=2,4)$ appear naturally. For $p>2$, $p\ mod\ 4=3$ one has: $x+iy+\sqrt{p}(u+iv)\rightarrow x_0+iy_0\in G(p,2)$.

An interesting observation is that the unitary representations of the p-adic scalings $x\rightarrow p^kx$ $k\in Z$ lead naturally to finite field structures. These representations reduce to representations of a finite cyclic group Z_m if $x\rightarrow p^mx$ acts trivially on representation functions for some value of m, $m=1,2,...$ Representation functions, or equivalently the scaling momenta $k=0,1,...,m-1$ labelling them, have a structure of cyclic group. If $m\neq p$ is prime the scaling momenta form finite field $G(m,1)=Z_m$ with respect to the summation and multiplication modulo m. Also the p-adic counterparts of the ordinary plane waves carrying p-adic momenta $k=0,1...,p-1$ can be given the structure of Finite Field $G(p,1)$: one can also define complexified plane waves as square roots of the real p-adic plane waves to obtain Finite Field $G(p,2)$.

4.3 What is the correspondence between p-adic and real numbers?

There must be some kind of correspondence between reals and p-adic numbers. This correspondence can depend on context. In p-adic mass calculations one must map p-adic mass squared values to real numbers in a continuous manner and canonical identification is a natural guess. Presumably also p-adic probabilities should be mapped to their real counterparts. One can wonder whether p-adic valued S-matrix has any physical meaning or whether one should assume that the elements of S-matrix are algebraic numbers allowing interpretation as real or p-adic numbers: this would pose extremely strong constraints on S-matrix. If one wants to introduce p-adic physics at space-time level one must be able to relate p-adic and real space-time regions to each other and the identification along common rational points of real and various p-adic variants of the imbedding space suggests itself here.

4.3.1 Generalization of the number concept

The recent view about the unification of real and p-adic physics is based on the generalization of number concept obtained by fusing together real and p-adic number fields along common rationals.

Rational numbers as numbers common to all number fields

The unification of real physics of material work and p-adic physics of cognition and intentionality leads to the generalization of the notion of number field. Reals and various p-adic number fields are glued along their common rationals (and common algebraic numbers too) to form a fractal book like structure. Allowing all possible finite-dimensional extensions of p-adic numbers brings additional pages to this "Big Book".

This generalization leads to a generalization of the notion of manifold as a collection of a real manifold and its p-adic variants glued together along common rationals. The precise formulation involves of course several technical problems. For instance, should one glue along common algebraic numbers and Should one glue along common transcendentals such as e^p? Are algebraic extensions of p-adic number fields glued together along the algebraics too?

At space-time level the book like structure corresponds to the decomposition of space-time surface to real and p-adic space-time sheets. This has deep implications for the view about cognition. For instance, two points infinitesimally near p-adically are infinitely distant in real sense so that cognition becomes a cosmic phenomenon.

How large p-adic space-time sheets can be?

Space-time region having finite size in the real sense can have arbitrarily large size in p-adic sense and vice versa. This raises a rather thought provoking questions. Could the p-adic space-time sheets have cosmological or even infinite size with respect to the real metric but have be p-adically finite? How large space-time surface is responsible for the p-adic representation of my body? Could the large or even infinite size of the cognitive space-time sheets explain why creatures of a finite physical size can invent the notion of infinity and construct cosmological theories? Could it be that pinary cutoff $O(p^n)$

defining the resolution of a p-adic cognitive representation would define the size of the space-time region needed to realize the cognitive representation?

In fact, the mere requirement that the neighborhood of a point of the p-adic space-time sheet contains points, which are p-adically infinitesimally near to it can mean that points infinitely distant from this point in the real sense are involved. A good example is provided by an integer valued point $x = n < p$ and the point $y = x + p^m$, $m > 0$: the p-adic distance of these points is p^{-m} whereas at the limit $m \to \infty$ the real distance goes as p^m and becomes infinite for infinitesimally near points. The points $n + y$, $y = \sum_{k>0} x_k p^k$, $0 < n < p$, form a p-adically continuous set around $x = n$. In the real topology this point set is discrete set with a minimum distance $\Delta x = p$ between neighboring points whereas in the p-adic topology every point has arbitrary nearby points. There are also rationals, which are arbitrarily near to each other both p-adically and in the real sense. Consider points $x = m/n$, m and n not divisible by p, and $y = (m/n) \times (1 + p^k r)/(1 + p^k s)$, $s = r + 1$ such that neither r or s is divisible by p and $k >> 1$ and $r >> p$. The p-adic and real distances are $|x - y|_p = p^{-k}$ and $|x - y| \simeq (m/n)/(r + 1)$ respectively. By choosing k and r large enough the points can be made arbitrarily close to each other both in the real and p-adic senses.

The idea about astrophysical size of the p-adic cognitive space-time sheets providing representation of body and brain is consistent with TGD inspired theory of consciousness, which forces to take very seriously the idea that even human consciousness involves astrophysical length scales.

Generalizing complex analysis by replacing complex numbers by generalized numbers

One general idea which results as an outcome of the generalized notion of number is the idea of a universal function continuable from a function mapping rationals to rationals or to a finite extension of rationals to a function in any number field. This algebraic continuation is analogous to the analytical continuation of a real analytic function to the complex plane. Rational functions with rational coefficients are obviously functions satisfying this constraint. Algebraic functions with rational coefficients satisfy this requirement if appropriate finite-dimensional algebraic extensions of p-adic numbers are allowed. Exponent function is such a function. Logarithm is also such a function provided that the above mentioned number theoretic conjecture holds true.

For instance, residy calculus might be generalized so that the value of an integral along the real axis could be calculated by continuing it instead of the complex plane to any number field via its values in the subset of rational numbers forming the rim of the book like structure having number fields as its pages. If the poles of the continued function in the finitely extended number field allow interpretation as real numbers it might be possible to generalize the residy formula. One can also imagine of extending residy calculus to any algebraic extension. An interesting situation arises when the poles correspond to extended p-adic rationals common to different pages of the "great book". Could this mean that the integral could be calculated at any page having the pole common. In particular, could a p-adic residy integral be calculated in the ordinary complex plane by utilizing the fact that in this case numerical approach makes sense.

4.3.2 Canonical identification

There exists a natural continuous map $Id : R_p \to R_+$ from p-adic numbers to non-negative real numbers given by the "pinary" expansion of the real number for $x \in R$ and $y \in R_p$ this correspondence reads

$$\begin{aligned} y &= \sum_{k>N} y_k p^k \to x = \sum_{k<N} y_k p^{-k} \quad , \\ y_k &\in \{0, 1, .., p-1\} \quad . \end{aligned} \tag{4.3.1}$$

This map is continuous as one easily finds out. There is however a little difficulty associated with the definition of the inverse map since the pinary expansion like also desimal expansion is not unique ($1 = 0.999...$) for the real numbers x, which allow pinary expansion with finite number of pinary digits

$$
\begin{aligned}
x &= \sum_{k=N_0}^{N} x_k p^{-k} \ , \\
x &= \sum_{k=N_0}^{N-1} x_k p^{-k} + (x_N - 1)p^{-N} + (p-1)p^{-N-1} \sum_{k=0,..} p^{-k} \ .
\end{aligned}
\tag{4.3.2}
$$

The p-adic images associated with these expansions are different

$$
\begin{aligned}
y_1 &= \sum_{k=N_0}^{N} x_k p^{k} \ , \\
y_2 &= \sum_{k=N_0}^{N-1} x_k p^{k} + (x_N - 1)p^{N} + (p-1)p^{N+1} \sum_{k=0,..} p^{k} \\
&= y_1 + (x_N - 1)p^N - p^{N+1} \ ,
\end{aligned}
\tag{4.3.3}
$$

so that the inverse map is either two-valued for p-adic numbers having expansion with finite number of pinary digits or single valued and discontinuous and nonsurjective if one makes pinary expansion unique by choosing the one with finite number of pinary digits. The finite number of pinary digits expansion is a natural choice since in the numerical work one always must use a pinary cutoff on the real axis.

Canonical identification is continuous map of non-negative reals to p-adics

The topology induced by the canonical identification map in the set of positive real numbers differs from the ordinary topology. The difference is easily understood by interpreting the p-adic norm as a norm in the set of the real numbers. The norm is constant in each interval $[p^k, p^{k+1})$ (see Fig. A-6) and is equal to the usual real norm at the points $x = p^k$: the usual linear norm is replaced with a piecewise constant norm. This means that p-adic topology is coarser than the usual real topology and the higher the value of p is, the coarser the resulting topology is above a given length scale. This hierarchical ordering of the p-adic topologies will be a central feature as far as the proposed applications of the p-adic numbers are considered.

Ordinary continuity implies p-adic continuity since the norm induced from the p-adic topology is rougher than the ordinary norm. This allows two alternative interpretations. Either p-adic image of a physical systems provides a good representation of the system above some pinary cutoff or the physical system can be genuinely p-adic below certain length scale L_p and become in good approximation real, when a length scale resolution L_p is used in its description. The first interpretation is correct if canonical identification is interpreted as a cognitive map. p-Adic continuity implies ordinary continuity from right as is clear already from the properties of the p-adic norm (the graph of the norm is indeed continuous from right). This feature is one clear signature of the p-adic topology.

If one considers seriously the application of canonical identification to basic quantum TGD one cannot avoid the question about the p-adic counterparts of the negative real numbers. It has turned out that there is no satisfactory manner to circumvent the fact that canonical images of p-adic numbers are naturally non-negative. The correct conclusion is that canonical interpretation applies only in p-adic thermodynamics, where it is used only in the direction $R_p \rightarrow R$ and real images are naturally non-negative numbers.

The notion of p-adic linearity

The linear structure of the p-adic numbers induces a corresponding structure in the set of the non-negative real numbers and p-adic linearity in general differs from the ordinary concept of linearity. For example, p-adic sum is equal to real sum only provided the summands have no common pinary digits. Furthermore, the condition $x +_p y < max\{x, y\}$ holds in general for the p-adic sum of the

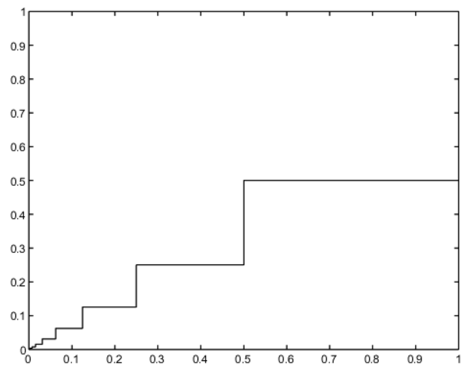

Figure 4.1: The real norm induced by canonical identification from 2-adic norm.

real numbers. p-Adic multiplication is equivalent with the ordinary multiplication only provided that either of the members of the product is power of p. Moreover one has $x \times_p y < x \times y$ in general. An interesting possibility is that p-adic linearity might replace the ordinary linearity in some strongly nonlinear systems so these systems would look simple in the p-adic topology.

Does canonical identification define a generalized norm?

Canonical correspondence is quite essential in TGD:eish applications. Canonical identification makes it possible to define a p-adic valued definite integral. Canonical identification is in a key role in the successful predictions of the elementary particle masses. Canonical identification makes also possible to understand the connection between p-adic and real probabilities. These and many other successfull applications suggests that canonical identification is involved with some deeper mathematical structure. The following inequalities hold true:

$$\begin{aligned} (x+y)_R &\leq x_R + y_R \ , \\ |x|_p |y|_R \leq (xy)_R &\leq x_R y_R \ , \end{aligned} \tag{4.3.4}$$

where $|x|_p$ denotes p-adic norm. These inequalities can be generalized to the case of $(R_p)^n$ (a linear vector space over the p-adic numbers).

$$\begin{aligned} (x+y)_R &\leq x_R + y_R \ , \\ |\lambda|_p |y|_R \leq (\lambda y)_R &\leq \lambda_R y_R \ , \end{aligned} \tag{4.3.5}$$

where the norm of the vector $x \in T_p^n$ is defined in some manner. The case of Euclidian space suggests the definition

$$(x_R)^2 = (\sum_n x_n^2)_R \ . \tag{4.3.6}$$

These inequalities resemble those satisfied by the vector norm. The only difference is the failure of linearity in the sense that the norm of a scaled vector is not obtained by scaling the norm of the original vector. Ordinary situation prevails only if the scaling corresponds to a power of p.

These observations suggests that the concept of a normed space or Banach space might have a generalization and physically the generalization might apply to the description of some nonlinear systems. The nonlinearity would be concentrated in the nonlinear behavior of the norm under scaling.

4.3.3 The interpretation of canonical identification

During the development of p-adic TGD two seemingly mutually inconsistent competing identifications of reals and p-adics have caused a lot of painful tension. Canonical identification provides one possible identification map respecting continuity whereas the identification of rationals as points common to p-adics and reals respects algebra of rationals. The resolution of the tension came from the realization that canonical identification naturally maps the predictions of p-adic probability theory and thermodynamics to real numbers. Canonical identification also maps p-adic cognitive representations to symbolic ones in the real world world or vice versa. The identification by common rationals is in turn the correspondence implied by the generalized notion of number and natural in the construction of quantum TGD proper.

Canonical identification maps the predictions of the p-adic probability calculus and statistical physics to real numbers

p-Adic mass calculations based on p-adic thermodynamics were the first and rather successful application of the p-adic physics (see the four chapters in [TGDpad]. The essential element of the approach was the replacement of the Boltzmann weight $e^{-E/T}$ with its p-adic generalization p^{L_0/T_p}, where L_0 is the Virasoro generator corresponding to scaling and representing essentially mass squared operator instead of energy. T_p is inverse integer valued p-adic temperature. The predicted mass squared averages were mapped to real numbers by canonical identification.

One could also construct a real variant of this approach by considering instead of the ordinary Boltzman weights the weights p^{-L_0/T_p}. The quantization of temperature to $T_p = log(p)/n$ would be a completely ad hoc assumption. In the case of real thermodynamics all particles are predicted to be light whereas in case of p-adic thermodynamics particle is light only if the ratio for the degeneracy of the lowest massive state to the degeneracy of the ground state is integer. Immense number of particles disappear from the spectrum of light particles by this criterion. For light particles the predictions are same as of p-adic thermodynamics in the lowest non-trivial order but in the next order deviations are possible.

The success of the p-adic mass calculations led to the idea that canonical identification generalizes also to the space-time level and appears even in the formulation of fundamental quantum TGD. However, when real space-time surfaces (absolute minima of Kähler action) are mapped by I^{-1} to their p-adic counterparts, one encounters several problems. The inverse of the canonical identification is two-valued; canonical identification map is not defined for negative real numbers; canonical identification is not manifestly General Coordinate Invariant concept; the direct canonical image of the space-time surface is not p-adically differentiable. What is needed is smooth surface perhaps satisfying the p-adic counterparts of the field equations associated with the absolute minimization of the Kähler action.

Already the problems with the general covariance definitely exclude canonical identification and its variants at space-time level, and that the generalization of the number concept provides the correct approach. Even such a simple fact that canonical images are always non-negative suggests that the applications must be such that this restriction is naturally satisfied. Canonical identification can indeed be used to map the predictions of the p-adic valued statistical physics to real numbers. For instance, p-adic probabilities and the p-adic entropy can be mapped to real numbers by canonical identification. The general idea is that a faithful enough cognitive representation of the real physics can by the number theoretical constraints involved make predictions, which would be extremely difficult to deduce from real physics.

Canonical identification as cognitive map mapping real external world to p-adic internal world or vice versa

It is interesting to look what canonical identification does assuming that rationals are common to p-adics and reals. Canonical identication maps the rationals $q = m/n$, n not divisible by p in the range $[1, \infty]$ to the range $[0, 1]$ and vice versa. One can say that real axis is defined 'inside' $[0, 1]$ and 'outside' $[1, \infty]$ and canonical identification maps these regions to each other in a p-adically continuous manner. This suggests that canonical identification and its generalizations could provide basic building blocks for cognitive maps mapping external world to a cognitive representation inside brain. Symbolic representations of thoughts in real world would in turn involve canonical identification in the reverse sense.

The physical counterpart of the pinary cutoff is very natural. The larger the pinary cutoff p^n is, the larger the real counterpart of the p-adic image via the correspondence by common rationals is. What is small p-adically is large in real sense at the level of integers. The better the resolution of the cognitive map is, the larger the p-adic space-time sheet giving rise to the representation is. For the p-adic primes associated with elementary particles already the pinary cutoff $O(p^3) = 0$ requires macroscopic and even astrophysical length scales. The idea that our consciousness might involve astrophysical length scales via p-adic cognitive representations, is in accordance with the views forced by TGD inspired theory of consciousness but using considerations based on quite different premises [H8].

4.3.4 Variants of canonical identification

One can also imagine variants of canonical identification.

The variant of canonical identification commuting with division of integers

The basic problems of canonical identification is that it does not respect unitarity. For this reason it is not well suited for relating p-adic and real scattering amplitudes. The problem of the correspondence via direct rationals is that it does not respect continuity.

A compromize between algebra and topology is achieved by using a modification of canonical identification $I_{R_p \rightarrow R}$ defined as $I_1(r/s) = I(r)/I(s)$. If the conditions $r \ll p$ and $s \ll p$ hold true, the map respects algebraic operations and also unitarity and various symmetries. It seems that this option must be used to relate p-adic transition amplitudes to real ones and vice versa [F5]. In particular, real and p-adic coupling constants are related by this map. Also some problems related to p-adic mass calculations find a nice resolution when I_1 is used.

This variant of canonical identification is not equivalent with the original one using the infinite expansion of q in powers of p since canonical identification does not commute with product and division. The variant is however unique in the recent context when r and s in $q = r/s$ have no common factors. For integers $n < p$ it reduces to direct correspondence.

Generalized numbers would be regarded in this picture as a generalized manifold obtained by gluing different number fields together along rationals. Instead of a direct identification of real and p-adic rationals, the p-adic rationals in R_p are mapped to real rationals (or vice versa) using a variant of the canonical identification $I_{R \rightarrow R_p}$ in which the expansion of rational number $q = r/s = \sum r_n p^n / \sum s_n p^n$ is replaced with the rational number $q_1 = r_1/s_1 = \sum r_n p^{-n} / \sum s_n p^{-n}$ interpreted as a p-adic number:

$$q = \frac{r}{s} = \frac{\sum_n r_n p^n}{\sum_m s_n p^n} \rightarrow q_1 = \frac{\sum_n r_n p^{-n}}{\sum_m s_n p^{-n}} \quad . \tag{4.3.7}$$

R_{p1} and R_{p_2} are glued together along common rationals by an the composite map $I_{R \rightarrow R_{p_2}} I_{R_{p_1} \rightarrow R}$.

This variant of canonical identification seems to be excellent candidate for mapping the predictions of p-adic mass calculations to real numbers and also for relating p-adic and real scattering amplitudes to each other [F5].

Phase preserving canonical identification

Before the emergence of new view about p-adic physics, the above listed problems forced to consider a modification of the canonical identification map and several options have been considered. The requirement of General Coordinate Invariance finally led to what seemed to be a unique solution to these problems. One must define canonical identification in preferred imbedding space coordinates: if preferred coordinates are not unique, the transformations between the preferred coordinates systems must commute with the modified canonical identification. Although this mapping is not relevant for the definition of fundamental theory, it might make sense if taken as a map defining cognitive representations at the level of Schröndinger amplitudes. In particular, the beautiful mathematical properties of this map and the direct connection with quantum measurement theory, suggest that one should not not keep mind open for possible applications of this map in some future theory of cognition.

The preferred coordinates are Minkowski coordinates $(m^0, m^3, m^1, m^2))$ and complex coordinates of CP_2 transforming linearly under certain Cartan sugroup $U(1) \times U(1)$ determined by the surface

Y^3: these coordinates are determined modulo rotations of subgroup $SO(2) \times U(1) \times U(1)$ of Cartan subgroup of $SO(3,1) \times SU(3)$ acting as multiplication by a phase factor in case of $m^1 + im^2$ and CP_2 complex coordinates. Lorentz boosts in Cartan subgroup of $SO(3,1)$ act as multiplication by hyperbolic 'phase factor' in case of the coordinate pair $(m^0, m^3) \equiv a(cosh(\eta), sinh(\eta))$. The mapping commutes with these transformations if the phase factors are mapped as such to their p-adic counterparts, that is without canonical identification. The mapping is only possible for rational complex phase factors: they correspond to Pythagorean triangles. The coordinate $a = \sqrt{(m^0)^2 - (m^3)^2}$ and moduli of the complex coordinates are mapped using canonical identification.

Since phase preserving canonical identification is discontinuous in phase degrees of freedom, the image of the space-time surface induced by the mapping of H is in the generic case discrete and does not form a subset of any p-adic 4-surface. One can however require that p-adic space-time surface is a smooth completion of a minimal pinary cutoff of the image fixed by the requirement that p-adic counterparts of the field equations guaranteing absolute minimization of the Kähler action are satisfied. The phenomenon of p-adic pseudo constants and nondeterminism of Kähler action give good hopes of achieving this. There is a direct connection with quantum measurement theory since the transformations of Cartan algebra commuting with the canonical identification map corresponds to a maximal set of commuting observables in the algebra of the isometry charges.

Although it seems that phase preserving canonical identification might not be useful at the level of imbdeding space, it can be applied to map real spinor fields to their p-adic counterparts. The natural requirement is that the modulus squared is mapped continuously in the cognitive map so that canonical identification is the natural possibility. The phases of eigenstate basis represent typically quantum numbers such as momentum components and spin. Therefore Pythagorean phases are a natural representation of the phase factors and must be mapped as such to their p-adic counterparts. Thus phase preserving canonical identification is natural for spinor fields and Schödinger amplitudes.

4.4 p-Adic differential and integral calculus

p-Adic differential calculus differs from its real counterpart in that piecewise constant functions depending on a finite number of pinary digits have vanishing derivative. This property implies p-adic nondeterminism, which has natural interpretation as making possible imagination if one identifies p-adic regions of space-time as cognitive regions of space-time.

One of the stumbling blocks in the attempts to construct p-adic physics have been the difficulties involved with the definition of the p-adic version of a definite integral. There are several alternative options as how to define p-adic definite integral and it is quite possible that there is simply not a single correct version since p-adic physics itself is a cognitive model.

1. The first definition of the p-adic integration is based on three ideas. The ordering for the limits of integration is defined using canonical correspondence. $x < y$ holds true if $x_R < y_R$ holds true. The integral functions can be defined for Taylor series expansion by defining indefinite integral as the inverse of the differentiation. If p-adic pseudoconstants are present in the integrand one must divide the integration range into pieces such that p-adic integration constant changes its value in the points where new piece begins.

2. Second definition is based on p-adic Fourier analysis based on the use of p-adic planewaves constructed in terms of Pythagorean phases. This definition is especially attractive in the definition of p-adic QFT limit and is discussed in detail later in the section 'p-Adic Fourier analysis'. In this case the integral is defined in the set of rationals and the ordering of the limits of integral is therefore not a problem.

3. For p-adic functions which are direct canonical images of real functions, p-adic integral can be defined also as a limit of Riemann sum and this in principle makes the numerical evaluation of p-adic integrals possible. As found in the chapter 'Mathematical Ideas', Riemann sum representation leads to an educated guess for an *exact formula for the definite integral* holding true for functions which are p-adic counterparts of real-continuous functions and for p-adically analytic functions. The formula provides a calculational recipe of p-adic integrals, which converges extremely rapidly in powers of p. Ultrametricity guarantees the absence of divergences in arbitrary dimensions provided that integrand is a bounded function. It however seems that this definition

of integral cannot hold true for the p-adically differentiable function whose real images are not continuous.

4.4.1 p-Adic differential calculus

The rules of the p-adic differential calculus are formally identical to those of the ordinary differential calculus and generalize in a trivial manner for the algebraic extensions.

The class of the functions having vanishing p-adic derivatives is larger than in the real case: any function depending on a finite number of positive pinary digits of p-adic number and of arbitrary number of negative pinary digits has a vanishing p-adic derivative. This becomes obvious, when one notices that the p-adic derivative must be calculated by comparing the values of the function at nearby points having the same p-adic norm (here is the crucial difference with respect to real case!). Hence, when the increment of the p-adic coordinate becomes sufficiently small, p-adic constant doesn't detect the variation of x since it depends on finite number of positive p-adic pinary digits only. p-Adic constants correspond to real functions, which are constant below some length scale $\Delta x = 2^{-n}$. As a consequence p-adic differential equations are non-deterministic: integration constants are arbitrary functions depending on a finite number of the positive p-adic pinary digits. This feature is central as far applications are considered and leads to the interpretation of p-adic physics as physics of cognition which involves imagination in essential manner. The classical non-determinism of the Kähler action, which is the key feature of quantum TGD, corresponds in a natural manner to the non-determinism of volition in macroscopic length scales.

p-analytic maps $g : R_p \rightarrow R_p$ satisfy the usual criterion of differentiability and are representable as power series

$$g(x) = \sum_k g_k x^k \quad . \tag{4.4.1}$$

Also negative powers are in principle allowed.

4.4.2 p-Adic fractals

p-Adically analytic functions induce maps $R_+ \rightarrow R_+$ via the canonical identification map. The simplest manner to get some grasp on their properties is to plot graphs of some simple functions (see Fig. 4.4.2 for the graph of p-adic x^2 and Fig. 4.4.2 for the graph of p-adic $1/x$). These functions have quite characteristic features resulting from the special properties of the p-adic topology. These features should be universal characteristics of cognitive representations and should allow to deduce the value of the p-adic prime p associated with a given cognitive system.

1. p-Analytic functions are continuous and differentiable from right: this peculiar asymmetry is a completely general signature of the p-adicity. As far as time dependence is considered, the interpretation of this property as a mathematical counterpart of the irreversibility looks attractive. This suggests that the transition from the reversible microscopic dynamics to irreversible macroscopic dynamics could correspond to the transition from the ordinary topology to an effective p-adic topology.

2. There are large discontinuities associated with the points $x = p^n$. This implies characteristic threshold phenomena. Consider a system whose output $f(n)$ is a function of input, which is integer n. For $n < p$ nothing peculiar happens but for $n = p$ the real counterpart of the output becomes very small for large values of p. In the bio-systems threshold phenomena are typical and p-adicity might be the key to their understanding. The discontinuities associated with the powers of $p = 2$ are indeed encountered in many physical situations. Auditory experience has the property that a given frequency ω_0 and its multiples $2^k\omega_0$, octaves, are experienced as the same frequency, this suggests that the auditory response function for a given frequency ω_0 is a 2-adicallly analytic function. Titius-Bode law states that the mutual distances of planets come in powers of 2, when suitable unit of distance is used. In turbulent systems period doubling spectrum has peaks at frequencies $\omega = 2^k\omega_0$.

3. A second signature of the p-adicity is "p-plicity" appearing in the graph of simple p-analytic functions. As an example, consider the graph of the p-adic x^2 demonstrating clearly the decomposition into p steps at each interval $[p^k, p^{k+1})$.

4. The graphs of the p-analytic functions are in general ordered fractals as the examples demonstrate. For example, power functions x^n are self-similar (the values of the function at some any interval (p^k, p^{k+1}) determines the function completely) and in general p-adic x^n with non-negative (negative) n is smaller (larger) than real x^n expect at points $x = p^n$ as the graphs of p-adic x^2 and $1/x$ show (see Fig. 4.4.2 and 4.4.2) These properties are easily understood from the properties of the p-adic multiplication. Therefore the first guess for the behavior of a p-adically analytic function is obtained by replacing x and the coefficients g_k with their p-adic norms: at points $x = p^n$ this approximation is exact if the coefficients of the power series are powers of p. This step function approximation is rather reasonable for simple functions such as x^n as the figures demonstrate. Since p-adically analytic function can be approximated with $f(x) \sim f(x_0) + b(x - x_0)^n$ or as $a(x - x_0)^n$ (allowing non-analyticity at x_0) around any point the fractal associated with p-adically analytic function has universal geometrical form in sufficiently small length scales.

p-Adic analyticity is well defined for the algebraic extensions of R_p, too. The figures 4.4.2 and 4.4.2 visualize the behavior of the real and imaginary parts of the 2-adic z^2 function as a function of the real x and y coordinates in the parallelpiped I^2,$I = [1+2^{-7}, 2-2^{-7}]$. An interesting possibility is that the order parameters describing various phases of some physical systems are p-adically differentiable functions. The p-analyticity would therefore provide a means for coding the information about ordered fractal structures.

The order parameter could be one coordinate component of a p-adically analytic map $R^n \rightarrow R^n$, $n = 3, 4$. This is analogous to the possibility to regard the solution of the Laplace equation in two dimensions as a real or imaginary part of an analytic function. A given region V of the order parameter space corresponds to a given phase and the volume of the ordinary space occupied by this phase corresponds to the inverse image $g^{-1}(V)$ of V. Very beautiful images are obtained if the order parameter is the the real or imaginary part of a p-analytic function $f(z)$. A good example is p-adic z^2 function in the parallelpiped $[a, b] \times [a, b]$, $a = 1 + 2^{-9}$, $b = 2 - 2^9$ of C-plane. The value range of the order parameter can be divided into, say, 16 intervals of the same length so that each interval corresponds to a unique color. The resulting fractals possess features, which probably generalize to higher-dimensional extensions.

1. The inverse image is an ordered fractal and possesses lattice/cell like structure, with the sizes of cells appearing in powers of p. Cells are however not identical in analogy with the differentiation of the biological cells.

2. p-Analyticity implies the existence of a local vector valued order parameter given by the p-analytic derivative of $g(z)$: the geometric structure of the phase portrait indeed exhibits the local orientation clearly.

A second representation of the fractals is obtained by dividing the value range of z into a finite number of intervals and associating different color to each interval. In a given resolution this representation makes obvious the presence of 0, 1- and 2-dimensional structures not obvious from the graph representation used in the figures of this book.

These observations suggests that p-analyticity might provide a means to code the information about ordered fractal structures in the spatial behavior of order parameters (such as enzyme concentrations in bio-systems). An elegant manner to achieve this is to use purely real algebraic extension for 3-space coordinates and for the order parameter: the image of the order parameter $\Phi = \phi_1 + \phi_2\theta + \phi_3\theta^2$ under the canonical identification is real and positive number automatically and might be regarded as concentration type quantity.

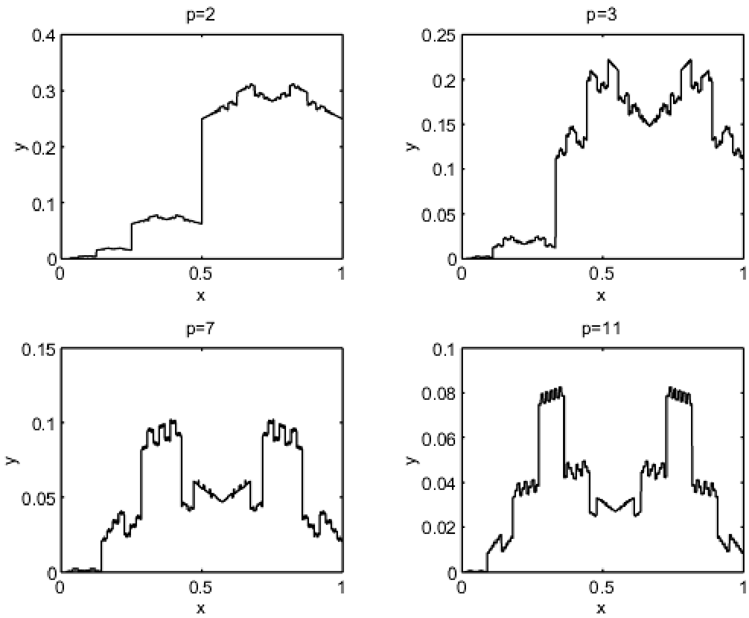

Figure 4.2: p-Adic x^2 function for some values of p

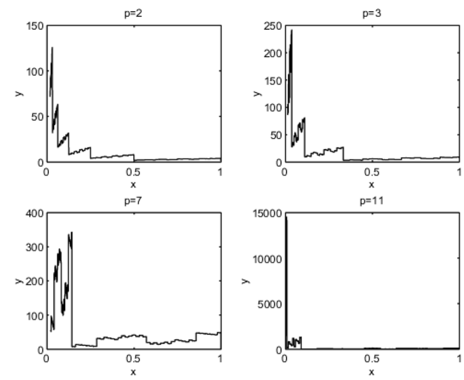

Figure 4.3: p-Adic $1/x$ function for some values of p

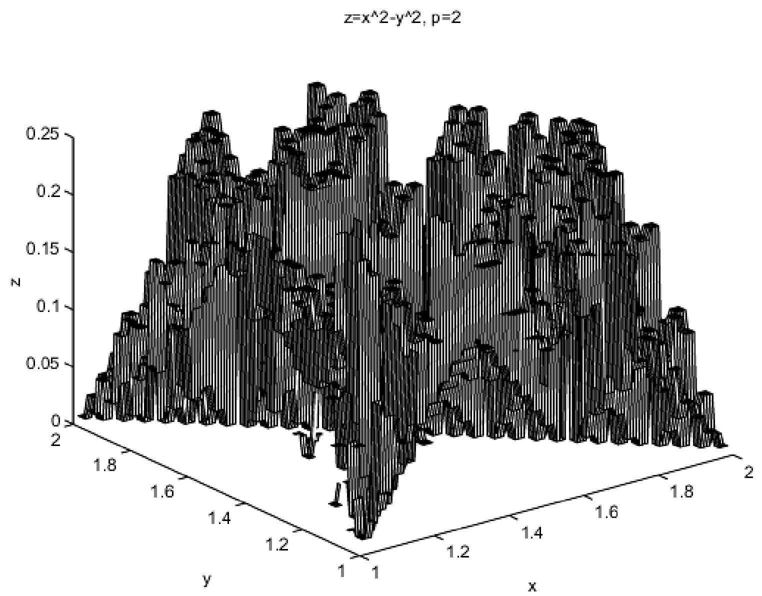

Figure 4.4: The graph of the real part of 2-adically analytic $z^2 =$ function.

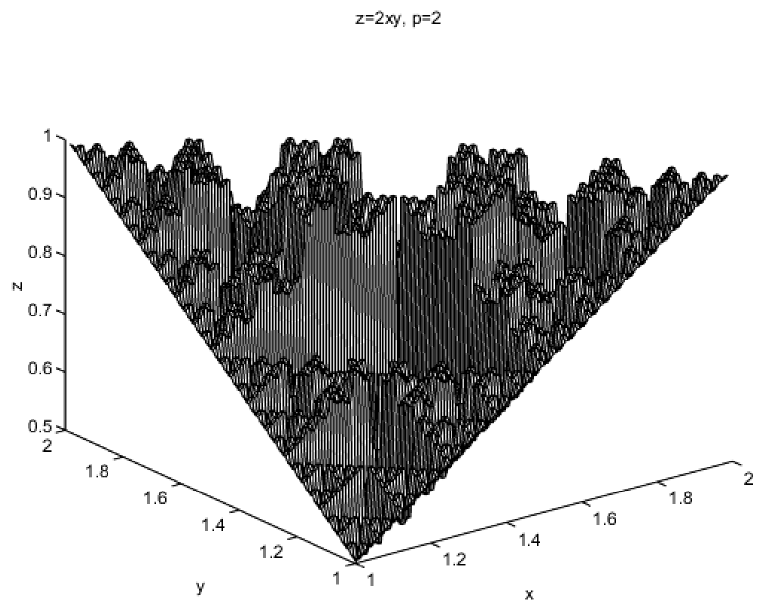

Figure 4.5: The graph of 2-adically analytic $Im(z^2) = 2xy$ function.

4.4.3 p-Adic integral calculus

The basic problems of the integration with p-adic values of integral are caused by the facts that p-adic numbers are not well-ordered and by the properties of p-adic norm. The general idea that p-adic physics can mimic real physics only at the algebraic level, leads to the idea that p-adic integration could be algebraized whereas numerical approaches analogous to Riemann sum are not possible. In the following three examples are discussed.

1. Definite integral can be defined using integral function and by defining integration limits via canonical identification: the drawback is the loss of general coordinate invariance. A more elegant general coordinate invariant approach is based on the identification of rationals as common to both reals and p-adics. This works for rational valued integration limits.

2. Residy calculus allows to realize integrals of analytic functions over closed curves of complex plane. The generalization of the residy calculus makes possible to realize conformal invariance at elementary particle horizons which are metrically 2-dimensional and allow conformal invariance and has also p-adic counterpart.

3. The perturbative series using Gaussian integration is the only to perform in practice infinite-dimensional functional integrals and being purely algebraic procedure, allows a straightforward p-adic generalization. This is the only option for p-adicizing configuration space integral.

Definition of the definite integral using integral function concept and canonical identification or identification by common rationals

The concept of the p-adic definite integral can be defined for functions $R_p \rightarrow C$ [9] using translationally invariant Haar measure for R_p. In present context one is however interested in definining a p-adic valued definite integral for functions $f : R_p \rightarrow R_p$: target and source spaces could of course be also some some algebraic extensions of the p-adic numbers.

What makes the definition nontrivial is that the ordinary definition as the limit of a Riemann sum doesn't seem to work: it seems that Riemann sum approaches to zero in the p-adic topology since, by ultra-metricity, the p-adic norm of a sum is never larger than the maximum p-adic norm for the summands. The second difficulty is related to the absence of a well-ordering for the p-adic numbers. The problems might be avoided by defining the integration essentially as the inverse of the differentiation and using the canonical correspondence to define ordering for the p-adic numbers. More generally, the concepts of the form, cohomology and homology are crucially based on the concept of the boundary. The concept of boundary reduces to the concept of an ordered interval and canonical identification makes it indeed possible to define this concept.

The definition of the p-adic integral functions defining integration as inverse of the differentiation is straightforward and one obtains just the generalization of the standard calculus. For instance, one has $\int z^n = \frac{z^{n+1}}{(n+1)} + C$ and integral of the Taylor series is obtained by generalizing this. One must however notice that the concept of integration constant generalizes: any function $R_p \rightarrow R_p$ depending on a finite number of pinary digits only, has a vanishing derivative.

Consider next the definite integral. The absence of the well ordering implies that the concept of the integration range (a, b) is not well defined as a purely p-adic concept. As already mentioned there are two solutions of the problem.

1. The identification of rational numbers as common to both reals and p-adics allows to order the integration limits when the end points of the integral are rational numbers. This is perhaps the most elegant solution of the problem since it is consistent with the restricted general coordinate invariance allowing rational function based coordinate changes. This approach works for rational functions with rational coefficients and more general functions if algebraic extension or extension containing transcendentals like e and logarithms of primes are allowed. The extension containing e, π, and $log(p)$ is finite-dimensional if e/π and $\pi/log(p)$ are rational numbers for all primes p. Essentially algebraic continuation of real integral to p-adic context is in question.

2. An alternative resolution of the problem is based on the canonical identification. Consider p-adic numbers a and b. It is natural to define a to be smaller than b if the canonical images of a and b satisfy $a_R < b_R$. One must notice that $a_R = b_R$ does not imply $a = b$, since the inverse of the

canonical identification map is two-valued for the real numbers having a finite number of pinary digits. For two p-adic numbers a, b with $a < b$, one can define the integration range (a, b) as the set of the p-adic numbers x satisfying $a \leq x \leq b$ or equivalently $a_R \leq x_R \leq b_R$. For a given value of x_R with a finite number of pinary digits, one has two values of x and x can be made unique by requiring it to have a finite number of pinary digits.

One can define definite integral $\int_a^b f(x)dx$ formally as

$$\int_a^b f(x)dx = F(b) - F(a) \ , \tag{4.4.2}$$

where $F(x)$ is integral function obtained by allowing only ordinary integration constants and $b_R > a_R$ holds true. One encounters however a problem, when $a_R = b_R$ and a and b are different. Problem is avoided if the integration limits are assumed to correspond to p-adic numbers with a finite number of pinary digits.

One could perhaps relate the possibility of the p-adic integration constants depending on finite number of pinary digits to the possibility to decompose integration range $[a_R, b_R]$ as $a = x_0 < x_1 < \ldots . x_n = b$ and to select in each subrange $[x_k, x_{k+1}]$ the inverse images of $x_k \leq x \leq x_{k+1}$, with x having finite number of pinary digits in two different manners. These different choices correspond to different integration paths and the value of the integral for different paths could correspond to the different choices of the p-adic integration constant in integral function. The difference between a given integration path and 'standard' path is simply the sum of differences $F(x_k) - F(y_k)$, $(x_k)_R = (y_k)_R$.

This definition has several nice features.

1. The definition generalizes in an obvious manner to the higher dimensional case. The standard connection between integral function and definite integral holds true and in the higher-dimensional case the integral of a total divergence reduces to integral over the boundaries of the integration volume. This property guarantees that p-adic action principle leads to same field equations as its real counterpart. It this in fact this property, which drops other alternatives from the consideration.

2. The basic results of the real integral calculus generalize as such to the p-adic case. For instance, integral is a linear operation and additive as a set function.

The ugly feature is the loss of the general coordinate invariance due to the fact that canonical identification does not commute with coordinate changes (except scalings by powers of p) and it seems that one cannot use canonical identification at the fundamental level to define definite integrals.

Definite integrals in p-adic complex plane using residy calculus

Residy calculus allows to calculate the integrals $\oint_C f(z)dz$ around complex curves as sums over poles of the function inside the curve:

$$\oint f(z)dz = i2\pi \sum_k Res(f(z_k)) \ , \tag{4.4.3}$$

where $Res(f(z_k))$ at pole $z = z_k$ is defined as $Res(f(z_k)) = \lim_{z\rightarrow z_k}(z - z_k)f(z)$. This definition applies in case of 2-dimensional $\sqrt{-1}$-containing algebraic extension of p-adic numbers ($p \ mod \ 4 = 3$) but its seems that this is not relevant for quantum TGD.

Quaternion conformal invariance corresponds to the conformal invariance associated with topologically 3-dimensional elementary particle horizons surrounding wormhole contacts which have Euclidian signature of induced metric. The induced metric is degenerate at the elementary particle horizon so that these surfaces are metrically two-dimensional. This implies a generalization of conformal invariance analogous to that at light cone cone boundary. In particular, a subfield of quaternions isomorphic with complex numbers is selected. One expects that residy calculus generalizes.

Elementary particle horizons are defined by a purely algebraic condition stating that the determinant of the induced metric vanishes, and thus the notion makes sense for p-adic space-time sheets too.

Also residy calculus should make sense for all algebraic extensions of p-adic numbers and the algebra of quaternion conformal invariance would generalize to the p-adic context too. Note however that the notion of p-adic quaternions does not make sense: the reason is that p-adic Euclidian length squared for a non-vanishing p-adic quaternion can vanish so that the inverse of quaternion is not well defined always. In the set of rational numbers this failure does not however occur and this might be enough for p-adicization to work.

Definite integrals using Gaussian perturbation theory

In quantum field theories functional integrals are defined by Gaussian perturbation theory. For real infinite-dimensional Gaussians the procedure has a rigorous mathematical basis deriving from measure theory. For the imaginary infinite-dimensional Gaussians defining the Feynman path integrals of quantum field theory the rigorous mathematical justification is lacking.

In TGD framework the integral over the configuration space of three surface can be reduced to a real Gaussian perturbation theory around the maxima of Kähler function. The integration is over quantum fluctuating degrees of freedom defining infinite-dimensional symmetric space for given values of zero modes. According to the more detailed arguments about how to construct p-adic counterpart of real configuration space physics described in the chapter "Construction of Quantum Theory", the following conjectures are trued.

1. The symmetric space property implies that there is only one maximum of Kähler function for given values of zero modes.

2. The generalization of Duistermaat-Heecke theorem holding true in finite-dimensional case suggests that by symmetric space property the integral of the exponent of Kähler gives just the exponent of Kähler function at the maximum and Gaussian determinant and metric determinant cancel each other.

3. The fact that free Gaussian field theory corresponds to a flat symmetric space inspires the hypothesis that S-matrix elements involving configuration space spinor fields in the representations of the isometry group reduce to those given by free field theory with propagator defined by the inverse of the configuration space covariant Kähler metric evaluated in the tangent space basis defined by the isometry currents at the maximum of Kähler function. This implies that there is no perturbation series which would spoil any hopes about proving the rationality. The reduction to a free field theory does not make quantum TGD non-interacting since interactions are described as topologically (as decays and fusions of 3-surfaces) rather than algbraically as non-linearities of local action.

4. If the exponent function is a rational function with rational coefficients in the sense that for the points of configuration space having finite number of rational valued coordinates (also zero modes), then the exponent $e^{K_{max}}$ is a rational number for rational values of zero modes. From the rationality of the exponent of the Kähler function follows the rational valuedness of the matrix elements of the metric. The undeniably very optimistic conclusion is that for rational values of the zero modes the S-matrix elements would be rational valued or have values if finite extension of rationals, so that they could be continued to the p-adic sectors of the configuration space. The S-matrix would have the same form in all number fields.

5. One could also interpret the outcome as an algebraic continuation of the rational quantum physics to real and p-adic physics. Configuration space-integrals can be thought of as being performed in the rational configuration space. Of course, one can define also ordinary integrals over R^n numerically using Riemann sums by considering the division of the integration region to very small n-cubes for which the sides have rational-number valued lengths and such that the value of the function is taken at rational valued point inside each cube.

The finite-dimensional real one-dimensional Gaussian $exp(-ax^2/2)$ provides a natural testing ground for this rather speculative picture. The integral of the Gaussian is $(2\pi)^{1/2}/\sqrt{a}$: in n-dimensional case where a is replaced by a quadratic form defined by a matrix A one obtains $(2\pi)^{n/2}/\sqrt{det(A)}$ in n-dimensional case. The integral of a function $exp(-ax^2+kx^n)x^k$ reduces to a perturbation series as sum of graphs containing single vertex containing k lines and arbitrary number of vertices containing

n lines and endowed with a factor k, and assigning with the lines the propagator factor $1/a$. For n-dimensional case the propagator factor would be inverse of the matrix A.

The result makes sense in the p-adic context if a and k are rational numbers. In the n-dimensional case matrix A and the coefficients defining the polynomial defining the interaction term must be rational numbers. The only problematic factor is the power of 2π, which seems to require algebraic extension containing π. Of course, one could define the normalization of the functional integral by dividing it by $(2\pi)^{n/2}$ to get rid of this fact. In the definition of S-matrix elements this normalization factor always disappears so that this problem has no physical significance.

In the case of free scalar quantum field theory n-point functions the perturbation theory are simply products of 2-point functions defined by the inverse of the infinite-dimensional Gaussian matrix. For plane wave basis for scalar field labelled by 4-momentum k the inverse of the Gaussian matrix reduces to the propagator ($i/(k^2+i\epsilon)$ for scalar field), which is rational function of the square of 4-momentum vector. In case of interacting quantum field the infinite summation over graphs spoils the hopes of obtaining end result which could be proven to be rational valued for rational values of incoming and outgoing four-momenta. The loop integrals are source of divergence problems and also number-theoretically problematic.

4.5 p-Adic symmetries and Fourier analysis

4.5.1 p-Adic symmetries and generalization of the notion of group

The most basic questions physicist can ask about the p-adic numbers are related to symmetries. It seems obvious that the concept of a Lie-group generalizes: nothing prevents from replacing the real or complex representation spaces associated with the definitions of the classical Lie-groups with the linear space associated with some algebraic extension of the p-adic numbers: the defining algebraic conditions, such as unitarity or orthogonality properties, make sense for the algebraically extended p-adic numbers, too.

For orthogonal groups one must replace the ordinary real inner product with the inner product $\sum_k X_k^2$ with a Cartesian power of a purely real extension of p-adic numbers. In the unitary case one must consider the complexification of a Cartesian power of a purely real extension with the inner product $\sum \bar{Z}_k Z_k$. Here $p \ mod \ 4 = 3$ is required. It should be emphasized however that the p-adic inner product differs from the ordinary one so that the action of, say, p-adic counterpart of a rotation group in R_p^3 induces in R^3 an action, which need not have much to do with ordinary rotations so that the generalization is physically highly nontrivial. Extensions of p-adic numbers also mean extreme richness of structure.

The exponentiation $t \rightarrow exp(tJ)$ of the Lie-algebra element J is a central element of Lie group theory and allows to coordinatize that elements of Lie group by mapping tangent space points the points representing group elements. Without algebraic extensions involving e or its roots one can exponentiate only the group parameters t satisfying $|t|_p < 1$. Thus the values of the exponentiation parameter which are too small/large in real/p-adic sense are not possible and one can say that the standard p-adic Lie algebra is a ball with radius $|t|_p = 1/p$.

The study of ordinary one-dimensional translations gives an idea about what it is involved. For finite values of the p-adic integer t the exponentiated group element corresponds in the case of translation group to a power of e so that the points reached by exponentiation cannot correspond to rational points. Since logarithm function exist as an inverse of p-adic exponent and since rationals correspond to infinite but periodic pinary expansions, rational points having the same p-adic norm can be reached by p-adic exponentials using t which is infinite as ordinary integer. This result is expected to generalize to the case of groups represented using rational-valued matrices.

One can define a hierarchy of p-adic Lie-groups by allowing extensions allowing e and even its roots such that the algebras have p-adic radii p^k. Hence the fact that the powers $e, ..., e^{p-1}$ define a finite-dimensional extensions of p-adic numbers seems to have a deep group theoretical meaning. One can define a hierarchy of increasingly refined extensions by taking the generator of extension to be $e^{1/n}$. For instance, in the case of translation group this makes possible p-adic variant of Fourier analysis by using discrete plane wave basis.

One can generalize also the notion of group by using the generalized notion of number. This means that one starts from the restriction of the group in question to a group acting in say rational and complex rational linear space and requires that real and p-adic groups have rational group transformations as common. By performing various completions one obtains a generalized group having the characteristic book like structure. In this kind of situation the relationship between various groups is clear and also the role of extensions of p-adic numbers can be understood. The notion of Lie-algebra generalizes also to form a book like structure. Coefficients of the pages of the Lie-algebra belong to various number fields and rational valued coefficients correspond to a part partially (because of the restriction $|t|_p < p^k$) common to all Lie-algebras.

$SO(2)$ as example

A simple example is provided by the generalization of the rotation group $SO(2)$. The rows of a rotation matrix are in general n orthonormalized vectors with the property that the components of these vectors have p-adic norm not larger than one. In case of $SO(2)$ this means the the matrix elements $a_{11} = a_{22} = a, a_{12} = -a_{21} = b$ satisfy the conditions

$$\begin{aligned} a^2 + b^2 &= 1 \ , \\ |a|_p &\leq 1 \ , \\ |b|_p &\leq 1 \ . \end{aligned} \tag{4.5.1}$$

One can formally solve a as $a = \sqrt{1-b^2}$ but the solution doesn't exists always. There are various possibilities to define the orthogonal group.

1. One possibility is to allow only those values of a for which square root exists as p-adic number. In case of orthogonal group this requires that both $b = sin(\Phi)$ and $a = cos(\Phi)$ exist as p-adic numbers. If one requires further that a and b make sense also as ordinary rational numbers, they define a Pythagorean triangle (orthogonal triangle with integer sides) and the group becomes discrete and cannot be regarded as a Lie-group. Pythagorean triangles emerge for rational counterpart of any Lie-group.

2. Other possibility is to allow an extension of the p-adic numbers allowing a square root of any ordinary p-adic number. The minimal extensions has dimension 4 (8) for $p > 2$ ($p = 2$). Therefore space-time dimension and imbedding space dimension emerge naturally as minimal dimensions for spaces, where p-adic $SO(2)$ acts 'stably'. The requirement that a and b are real is necessary unless one wants the complexification of $so(2)$ and gives constraints on the values of the group parameters and again Lie-group property is expected to be lost.

3. The Lie-group property is guaranteed if the allowed group elements are expressible as exponents of a Lie-algebra generator Q. $g(t) = exp(iQt)$. This exponents exists only provided the p-adic norm of t is smaller than one. If one uses square root allowing extension, one can require that t satisfies $|t| \leq p^{-n/2}$, $n > 0$ and one obtains a decreasing hierarchy of groups $G_1, G_2, ...$ For the physically interesting values of p (typically of order $p = 2^{127} - 1$) the real counterparts of the transformations of these groups are extremely near to the unit element of the group. These conclusions hold true for any group. An especially interesting example physically is the group of 'small' Lorentz transformations with $t = O(\sqrt{p})$. If the rest energy of the particle is of order $O(\sqrt{p})$: $E_0 = m = m_0\sqrt{p}$ (as it turns out) then the Lorentz boost with velocity $\beta = \beta_0\sqrt{p}$ gives particle with energy $E = m/\sqrt{1-\beta_0^2 p} = m(1 + \frac{\beta_0^2 p}{2} + ..)$ so that $O(p^{1/2})$ term in energy is Lorentz invariant. This suggests that non-relativistic regime corresponds to small Lorentz transformations whereas in genuinely relativistic regime one must include also the discrete group of 'large' Lorentz transformations with rational transformations matrices.

4. One can extend the group to contain products $G_1 G_2$, such that G_1 is a rational matrix belonging to the restriction of the Lie-group to rational matrices not obtainable from a unit matrix p-adically by exponentiation, and G_2 is a group element obtainable from unit element by exponentiation. For instance, rational CP_2 is obtained from the group of rational 3×3 unitary matrices as by dividing it by the $U(2)$ subgroup of rational unitary matrices.

Even the construction of the representations of the translation group raises nontrivial issues since the construction of p-adic Fourier analysis is by no means a nontrivial task. One can however define the concept of p-adic plane wave group theoretically and p-adic plane waves are orthogonal with respect to the inner product defined by the proposed p-adic integral.

The representations of 3-dimensional rotation group $SO(3)$ can be constructed as homogenous functions of Cartesian coordinates of E^3 and in this case the phase factors $exp(im\phi)$ typically appearing in the expressions of spherical harmonics do not pose any problems. The construction of p-adic spherical harmonics is possible if one assumes that allowed spherical angles (θ,ϕ) correspond to Pythagorean triangles.

A similar situation is encountered also in the case of CP_2 spherical harmonics in in fact, quite generally. This number theoretic quantization of angles could be perhaps interpreted as a kind of cognitive quantum effect consistent with the fact that only rationals can be visualized concretely and relate directly to the sensory experience. More generally, the possibility to realize only rationals numerically might reflect the facts that only rationals are common to reals and p-adics and that cognition is basically p-adic.

Fractal structure of the p-adic Poincare group

p-Adic Poincare group, just as any other p-adic Lie group, contains entire fractal hierarchy of subgroups with the same Lie-algebra. For instance, translations $m^k \rightarrow m^k + p^N a^k$, where a^k has p-adic norm not larger than one form subgroup for all values of N. The larger the value of N is, the smaller this subgroup is. Quite generally this implies orbits within orbits and representations within representations like structure so that p-adic symmetry concept contains hologram like aspect. This property of the p-adic symmetries conforms nicely with the interpretation of p-adic symmetries as cognitive representations of real symmetries since the symmetries can be realized in a p-adically finite spatiotemporal volume of the cognitive space-time sheet. Even more, this volume can be p-adically arbitrarily small. If one identifies both p-adics and reals as a completion of rationals, the corresponding real volumes are however strictly speaking infinite in absence of a pinary cutoff.

The hierarchy of subgroups implies that M^4_+ decomposes in a natural manner to 4-cubes with side $L_0 = N_p(L)L_p$, where $N_p(L) = p^{-N}$ denotes the p-adic norm of L such that these 4-cubes are invariant under the group of sufficiently small Poincare transformations. In real context these cubes define a hierarchy of exteriors of cubes with decreasing sizes. One can have full p-adic Poincare invariance in p-adically arbitrarily small volume. Only those Poincare transformations, which leave the minimal p-adic cube invariant are symmetries. Also this picture suggest that the p-adic space-time sheets providing cognitive representations about finite space-time regions by canonical identification can have very large size.

The construction of the p-adic Fourier analysis is a nontrivial problem. The usual exponent functions $f_P(x) = exp(iPx)$, providing a representation of the p-adic translations do not make sense as a Fourier basis: f_P is not a periodic function; f_P does not converge if the norm of Px is not smaller than one and the natural orthogonalization of the different momentum eigenstates does not seem to be possible using the proposed definition of the definite integral.

This state of affairs suggests that p-adic Fourier analysis involves number theory. It turns out that one can construct what might be called number theoretical plane waves and that p-adic momentum space has a natural fractal structure in this case. The basic idea is to reduce p-adic Fourier analysis to a Fourier analysis in a finite field $G(p,1)$ plus fractality in the sense that all p^m-scaled versions of the $G(p,1)$ plane waves are used. This means that p-adic plane waves in a given interval $[n, n+1)p^m$ are piecewise constant plane waves in a finite field $G(p,1)$. Number theoretical p-adic plane waves are pseudo constants so that the construction does not work for p-adically differentiable functions. The pseudo-constancy however turns out to be a highly desirable feature in the construction of the p-adic QFT limit of TGD based on the mapping of the real H-quantum fields to p-adic quantum fields using the canonical identification.

The unsatisfactory feature of this approach is that number theoretic p-adic plane waves do not behave in the desired manner under translations. It would be nice to have a p-adic generalization of the plane wave concept allowing a generalization of the standard Fourier analysis and a direct connection with the theory of the representations of the translation group. A natural idea is to to define exponential function as a solution of a p-adic differential equation representing the action of a translation generator and to introduce multiplicative pseudo constant making possible to define exponential function for all values of its argument. One can develop an argument suggesting that the plane waves obtained in this manner are indeed orthogonal.

Infinitesimal form of translational symmetry might be argued to be too strong requirement since p-adically infinitesimal translations typically correspond to real translations which are arbitrarily large: this is not consistent with the idea that cognitive representations with a finite spatial resolution are in question. This motivates a third approach to the p-adic Fourier analysis. The basic requirement is that discrete subgroup of translations commutes with the map of the real plane waves to their p-adic counterparts. This means that the products of the real phase factors are mapped to the products of the corresponding p-adic phase factors. This is possible if the phase factor is a rational complex number so that the phase angle corresponds to a Pythagorean triangle. The p-adic images of the real plane waves are defined for the momenta $k = nk_G$, $k_G = \phi_G/\Delta x$, where $\phi_G \in [0, 2\pi]$ is a Pythagorean phase angle and where the points $x_n = n\Delta x$ define a discretization of x-space, Δx being a rational number. These plane waves form a complete and orthogonalized set.

4.5.2 p-Adic Fourier analysis: number theoretical approach

Contrary to the original expectations, number theoretical Fourier analysis is probably not basic mathematical tools of p-adic QFT since it fails to provide irreducible representation for the translational symmetries. Despite this it deserves documentation.

Fourier analysis in a finite field $G(p,1)$

The p-adic numbers of unit norm modulo p reduce to a finite field $G(p,1)$ consisting of the integers $0, 1, ..., p-1$ with arithmetic operations defined by those of the ordinary integers taken modulo p. Since the elements $1, ..., p-1$ form a multiplicative group there must exists an element a of $G(p,1)$ (actually several) such that $a^{p-1} = 1$ holds true in $G(p,1)$. This kind of element is called primitive root. If n is a factor of $p-1$: $(p-1) = nm$, then also $a^m = 1$ holds true. This reflects the fact that Z_{p-1} decomposes into a product $Z_{m_1}^{n_1} Z_{m_2}^{n_2} ... Z_{m_s}^{n_s}$ of commuting factors Z_{m_i}, such that $m_i^{n_i}$ divides $p-1$.

A Fourier basis in $G(p,1)$ can be defined using p functions $f_k(n)$, $k = 0, .., p-1$. For $k = 0, 1, ..., p-2$ these functions are defined as

$$f_k(n) \quad = \quad a^{nk} \ , \ n = 0 \ , ..., p-1 \ , \tag{4.5.2}$$

and satisfy the periodicity property

$$f_k(0) = f_k(p-1) \ .$$

The problem is to identify the lacking p:th function. Since $f_k(n)$ transforms irreducibly under translations $n \rightarrow n + m$ it is natural to require that also the p:th function transforms in a similar manner and satisfies the periodicity property. This is achieved by defining

$$f_{p-1}(n) \quad = \quad (-1)^n \ . \tag{4.5.3}$$

The counterpart of the complex conjugation for f_k for $k \neq p-1$ is defined as $f_k \rightarrow f_{p-1-k}$. f_{p-1} is invariant under the conjugation. The inner product is defined as

$$\langle f_k, f_l \rangle = \sum_{n=0}^{p-2} f_{p-1-k}(n) f_l(n) = \delta(k,l)(p-1) \ . \tag{4.5.4}$$

The dual basis $\hat{f}_k$ clearly differs only by the normalization factor $1/(p-1)$ from the basis f_{p-k}. The counterpart of Fourier expansion for any real function in $G(p,1)$ can be obviously constructed using this function basis and Fourier components are obtained as the inner products of the dual Fourier basis with the function in question.

A natural interpretation for the integer k is as a p-adic momentum since in the translations $n \rightarrow n+m$ the plane wave with $k \neq p-1$ changes by a phase factor a^{km}. For $k = p-1$ it transforms by $(-1)^m$ so that also now an eigen state of finite field translations is in question.

p-Adic Fourier analysis based on p-adic plane waves

The basic idea is to reduce p-adic Fourier analysis to the Fourier analysis in $G(p,1)$ by using fractality.

1. Let the function $f(x)$ be such that the maximum p-adic norm of $f(x)$ is p^{-m}. One can uniquely decompose $f(x)$ to a sum of functions $f_n(x)$ such that $|f_n(x)|_p = p^n$ or vanishes in the entire range of definition for f:

$$\begin{aligned} f(x) &= \sum_{n\geq m} f_n(x) \ , \\ f_n(x) &= g_n(x)p^n \ , \\ |g_n(x)| &= 1 \ \text{for} \ g(x)\neq 0 \ . \end{aligned} \tag{4.5.3}$$

   The higher the value of n, the smaller the contribution of f_n. The expansion converges extremely rapidly for the physically interesting large values of p.

2. Assume that $f(x)$ is such that for each value of n one can find some resolution $p^{m(n)}$ below which $g_n(x)$ is constant in the sense that for all intervals $[r,r+1)p^{m(n)}$ (defined in terms of the canonical identification) the function $f_n(x)$ is constant. For p-adically differentiable functions this cannot be the case since they would be pseudo constants if this were true. In the physical situation CP_2 size provides a natural p-adic cutoff so that only a finite number of f_n:s are needed and the resolution in question corresponds to CP_2 length scale. Hence ordinary plane waves (possibly with a natural UV cutoff) should have an expansion in terms of the p-adic plane waves.

3. The assumption implies that in each interval $(r,r+1)p^{m(n)-1}$, g_n can be regarded as a function in $G(p,1)$ identified as the set $x=(r+sp)p^{m(n)-1}$, $s=0,1,...,p-1$. Hence one can Fourier expand $f_n(x)$ using $G(p,1)$ plane waves f^{ks}. In this manner one obtains a rapidly converging expansion using p-adic plane waves.

Periodicity properties of the number theoretic p-adic plane waves

The periodicity properties of the p-adic plane waves make it possible to associate a definite wavelength with a given p-adic plane wave. For the p-adic momenta k not dividing $p-1$, the wavelength corresponds to the entire range $(n,n+1)p^m$ and its real counterpart is

$$\lambda = p^{-m-1/2}l \ ,$$

where $l\sim 10^4\sqrt{\hbar G}$ is the fundamental p-adic length scale. If k divides $p-1=\prod_i m_i^{n_i}$, the period is m_i and the real wavelength is

$$\lambda(m_i) = m_i p^{-m-1-1/2}l \ .$$

One might wonder whether this selection of preferred wavelengths has some physical consequences. The first thing to notice is that p-adic plane waves do *not* replace ordinary plane waves in the construction of the p-adic QFT limit of TGD. Rather, ordinary plane waves are expanded using the p-adic plane waves so that the selection of the preferred wavelengths, if it occurs at all, must be a dynamical process. The average value of the prime divisors, and hence the number of different wavelengths for a given value of p, counted with the degeneracy of the divisor is given by [22]

$$\Omega(n) = ln(ln(n)) + 1.0346 \ ,$$

and is surprisingly small, or order 6 for numbers of order M_{127}! If one can apply probabilistic arguments or [22] to the numbers of form $p-1$, too then one must conclude that very few wavelengths are possible for general prime p! This in turn means that to each p there are associated only very few characteristic length scales, which are predictable. Furthermore, all the p^k-multiples of these scales are also possible if p-adic fractality holds true in macroscopic length scales.

Mersenne primes M_n can be considered as an illustrative example of the phenomenon. From [23] one finds that $M_{127}-1$ has 11 distinct prime factors and 3 and 7 occurs three and 2 times respectively.

The number of distinct length scales is $3 \cdot 2^{11} - 1 \sim 2^{12}$. $M_{107} - 1$ and $M_{89} - 1$ have 7 and 11 singly occurring factors so that the numbers of length scales are $2^7 - 1 = 127 = M_7$ and $2^{11} - 1$. Note that for hadrons (M_{107}) the number of possible wavelengths is especially small: does this have something to do with the collective behavior of color confined quarks and gluons? An interesting possibility is that this length scale generation mechanism works even macroscopically (for p-adic length scale hypothesis at macroscopic length scales see the third part of the book). One cannot exclude the possibility that long wavelength photons, gravitons and neutrinos might therefore provide a completely new mechanism for generating periodic structures with preferred sizes of period.

4.5.3 p-Adic Fourier analysis: group theoretical approach

The problem with the straightforward generalization of the Fourier analysis is that the standard Taylor expansion of the plane wave $exp(ikx)$ converges only provided x has p-adic norm smaller than one and that the p-adic exponential function does not have the periodicity properties of the ordinary exponential function guaranteing orthogonality of the functions of the Fourier basis. Besides this one must assume $p \ mod \ 4 = 3$ to guarantee that $\sqrt{-1}$ does not exist as ordinary p-adic number.

The approach based on algebraic extensions allowing trigonometry

In an attempt to construct Fourier analysis the safest approach is to start from the ordinary Fourier analysis at circle or that for a particle in a one-dimensional box. The function basis uses as the basic building blocks the functions $e^{in\phi}$ in the case of circle and functions $e^{in\pi x/L}$ in the case of a particle in a box of side L.

The view about rationals as common to both reals and p-adics, and the possibility of finite-dimensional extensions of p-adics generated by the roots $e^{i2\pi/p^k}$ suggest how to realize this idea.

1. Consider first the case of the circle. Fix some value of N and select a set of points $\phi_n = in2\pi/p^k$ at which the phases are defined meaning p^{k+1}-dimensional algebraic extension. That powers of p appear is consistent with p-adic fractality. If so spin 1/2 *resp.* spin 1 particles would be inherently 2-adic *resp.* 3-adic. The plane wave basis corresponds $exp(ik\phi_n)$, $k = 0, ..., N-1$. In the case of particle in the one-dimensional box such that L corresponds to a rational number, the box is decomposed into N intervals of length L/N.

2. One can assign to the phases a well defined angular momentum as integer $n = 0, ..., N-1$ whereas the momentum spectrum for a particle in a box are given by $n\pi/L$. It is possible to continue the phase factor to the neighborhood of each point by requiring that the differential equation

$$\frac{d}{dx} exp(ikx) = ikexp(ikx)$$

defining the exponential function is satisfied.

3. The inner product of the plane waves $f_{k_1})$ and f_{k_2} can be defined as the sum

$$\langle k_1 \rangle \equiv \sum_n \overline{f}_{k_1}(x_n) f_{k_2}(x_n) \ , \qquad (4.5.4)$$

and orthogonality and completeness differ by no means from those of ordinary Fourier analysis.

p-Adic Fourier analysis, Pythagorean phases, and Gaussian primes

An alternative approach is based on Pythagorean phases and discretization in x-space, which is very natural thing to do if p-adic field theory is taken as a cognitive model rather than 'real' physics. This is also natural because rational Minkowski space is in the algebraic approach the fundamental object and reals and p-adics emerge as its completions.

Rational phase factors are common to the complexified p-adics ($p\ mod\ 4 = 3$) and reals and this suggests that one should define p-adic plane waves so that their values are in the set of the Pythagorean phases. Pythagorean phases are in one-one correspondence with the phases of the squares of Gaussian integers N_G and thus generated as products of squares of Gaussian primes π_G, which are complex integers with modulus squared equal to prime $p\ mod\ 4 = 1$. Thus the set of phases $\phi(\pi_G)$ for the phases for π_G^2 form an algebraically infinite-dimensional linear space in the sense that the phases representable as superpositions

$$2\phi_G = \sum_{\pi_G} n_{\pi_G} 2\phi(\pi_G)$$

of these phases with integer coefficients belong to the set.

Consider now the definition of the plane wave basis based on Pythagorean phases and the identification of the p-adics and reals via common rationals.

1. Let $x_0 = q = m/n$ denote a value of x-coordinate and let k denote some value of momentum. If $exp(ikx_0)$ is a Pythagorean phase then also the multiples nk correspond to Pythagorean phases. k itself cannot be a rational number so that k is not defined as an ordinary p-adic number: this could be seen as a defect of the approach since one cannot speak of a well-defined momentum. Neither can k be a rational multiple of π so that Pythagorean phases have nothing to do with the phases defined by algebraic extensions containing the phase $exp(i\pi/n)$ already discussed.

   For a given value of $x_0 = q$ the momenta k for which $exp(ikq)$ is a Pythagorean phase are in one-one correspondence with Pythagorean phases. Moreover, Pythagorean phases result in the lattice defined by the multiples of the x_0. Thus a natural definition of the p-adic plane waves emerges predicting a maximal momentum spectrum with one-one correspondence with Pythagorean phases, and selecting a preferred lattice of points at the real axis. This definition is also in accordance with the idea that p-adic plane waves are related with a cognitive representation for real physics.

2. Pythagorean phases are in one-one correspondence with the phase factors associated with the squares of the Gaussian integers and generating phases correspond to the phases $\phi(\pi_G)$ associated with the squares of Gaussian primes π_G. The moduli squared for the Gaussian primes correspond to squares of rational primes $p\ mod\ 4 = 1$. Thus set of allowed momenta k_G for given spatial resolution m/n is the set

$$\{k_G(q)\} = \{\tfrac{2\phi_G/}{q} + \tfrac{2\pi n}{q} | n \in Z\} \ ,$$
$$\{\phi_G\} = \{\textstyle\sum_{\pi_G} n_{\pi_G}\phi(\pi_G)\} \ .$$

   When the spatial resolution $x_0 = q$ is replaced with $q_1 = r/s$, the spectrum is scaled by a rational factor q/q_1. The set of momenta is a dense subset of the real axis. There is no observable difference between the real momenta differing by a multiple of $2\pi/q$ and one must drop them from consideration. This conclusion is forced also by the fact that p-adically the momenta $k = nk_0$ do not exist, it is only the phase factors which exist.

3. It is easy to see that the p-adic plane waves with different momenta are orthogonal to each other as complex rational numbers:

$$\sum_n exp\left[in(k_G(1) - k_G(2))\right] = 0 \ .$$

4. Also completeness relations are satisfied in the sense that the condition

$$\sum_{k_G} exp\left[i(n_1 - n_2)k_G\right] = 0$$

   is satisfied for $n_1 \neq n_2$. This is due to the fact that all integer multiples of k_G define Pythagorean phases. This means that the Fourier series of a function with respect to Pythagorean phases makes sense and one can expand p-adic-valued functions of space-time coordinates as Fourier series using Pythagorean phases. In particle expansion of the the imbedding space coordinates as functions of p-adic space-time coordinates might be carried out in this manner.

5. One can criticise this approach for the fact that there is no unique continuation of the phase factors from the set of the rationals $x_n = nx_0$ to p-adic numbers neighborhoods of these points. Although eigen states of finite translations are in question one cannot regard the states as eigen states of infinitesimal translations since the momenta are not well defined as p-adic numbers. One could of course arbitrarily assign momentum eigenstate $e^{in\pi(x-x_k)}$ the point x_k to the eigenstate characterized by the dimensionless momentum n but the momentum spectrum associated with different Pythagorean phases would be same.

4.6 Generalization of Riemann geometry

In real context the coordinatization of manifold is regarded as a trivial problem. It took long time to realize that in p-adic context the proper treatment of coordinatization problem leads to deep insights about p-adic symmetries and about the origin og the p-adic length scales hypothesis. There are several approaches to the construction of the p-adic Riemann geometry. The most simple minded approach relies on a direct generalization of the real line element and to the proposed integral for p-adically analytic functions. A more refined approach relies on the general physical consistency conditions provided by quantum TGD and by the proposed definition of the Riemann integral.

4.6.1 p-Adic Riemannian geometry as a direct formal generalization of real Riemannian geometry

It is possible to generalize the concept of the (sub)manifold geometry to a p-adic (sub)manifold geometry and it seems that this definition of p-adic geometry indeed works at the level of the imbedding space. The formal definition of p-adic Riemannian geometry is based on p-adic line element

$$ds^2 = g_{kl}dx^k dx^l \quad .$$

The minimal requirement is that inner products of tangent space vectors exist. Lengths and angles are defined in the usual manner.

A stronger and somewhat questionable requirement is that also curve lengths, areas, volumes, etc.. exist. This requires the definition of the square root ds of the line element. In general case the existence of a square root forces an extension of the p-adic numbers allowing square roots of ordinary p-adic numbers. As found, the extension is 4-dimensional for $p > 2$ and 8-dimensional in $p = 2$ case. It must be emphasized that the algebraic dimensions do not have interpretation as physical dimensions. The extension in question must appear as a coefficient ring of the p-adic tangent space so that p-adic Riemann spaces must be locally Cartesian powers of $4-$ $(p > 2)$ or 8-dimensional $(p = 2)$ extension. Therefore the TGD:eish dimensions of the space-time and imbedding space emerge very naturally in the p-adic context. In order to avoid the appearance of an imaginary unit in $p \ mod \ 4 = 3$ case, one must multiply ds^2 with -1 if the square root of $(\frac{ds}{dt})^2$ is imaginary so that one has

$$s = \int ds = \int \sqrt{\epsilon g_{kl}\frac{dx^k}{dt}\frac{dx^l}{dt}}dt \quad ,$$

where ϵ is a sign factor. The p-adic length of a curve can be calculated if the integrand is integrable in the sense defined previously.

The definition of a pseudo-Riemannian metric poses problem: it seems that one should be able to make distinction between negative and positive p-adic numbers. A possible manner to make this distinction is to define p-adic numbers with unit norm to be positive or negative according to whether they are squares or not. This definition makes sense if -1 does not possess square root: this is true for $p \ mod \ 4 = 3$. This condition will be encountered in most applications of the p-adic numbers. At analytic level the definition generalizes in an obvious manner: what is required that the components of the metric are ordinary p-adic numbers. The p-adic counter part of the Minkowski metric can be defined as

$$ds_p^2 = (dm^0)^2 - ((dm^1)^2 + (dm^2)^2 + (dm^3)^2) \ . \tag{4.6.1}$$

The real image of this line element under canonical identification is non-negative but the metric allows to define the p-adic counterpart of M^4 lightcone as the surface $(m^0)^2 - ((m^1)^2 + (m^2)^2 + (m^3)^2) = 0$ and this surface can be regarded as a fractal counterpart of the ordinary light cone. Furthermore, this metric allows the p-adic counterpart of the Lorentz group as its group of symmetries.

The p-adic length of a curve can be finite also in the case when the real length diverges. This is the case for fractal curves contained in a finite volume of space: coast of Britain is the canonical example. The reason is that by p-adic ultra-metricity p-adic length is necessarily bounded. It is not clear whether the generalized p-adic Riemann sum has well defined limit for curves, which are general fractals. An interesting possibility is that one could define the length of a fractal curve ('coast line of Britain') using p-adic Riemannian geometry. A possible model of this curve is obtained by identifying the ordinary real plane with its p-adic counterpart via the canonical identification and modelling the fractal curve with p-adically analytic curve $x = x(t)$. The real counterpart of this curve is certainly a fractal and need not have a well defined real length. The p-adic length of this curve can be defined as the p-adic integral of $s_p = \int ds$ and its real counterpart s_R obtained by the canonical identification can be defined to be the real length of the curve.

p-Adic Riemann geometry has some special features resulting from ultra-metricity. For instance, the real counterpart for the p-adic length can be longer for a portion of a curve than for the entire curve! A good example is the p-adic length for the portion $(0 < x < 1, y > 0)$ of the unit circle $x^2 + y^2 = 1$, which can be written as

$$s(\phi) = arcsin(x) \ .$$

$arcsin(x = 1)$ is not well defined p-adically so that one must actually define the p-adic counterpart of $x_R = 1$ as $x = -p$. The length of a quadrant is $s(\pi/2) = arcsin(-p)$ so that the length of a half circle is $s(\pi) = 2arcsin(-p)$. In order $O(p)$ the length of a quadrant is $s(\pi/2) =\simeq -p \simeq (p-1)p$ whereas the length of a semicircle is $s(\pi) \simeq -2p \simeq (p-2)p$ so that the real counterpart $s_R(\pi) \simeq (p-2)/p$ for the p-adic length of a half circle is shorter than the length $s_R(\pi/2) \simeq (p-1)/p$ of a quadrant for sufficiently large values of p! For very large values of p the lengths are identical in excellent approximation. If one uses the length of a quadrant as a definition of p-adic $\pi/2$ one has "$\pi/2$" $= -arcsin(p)$ which gives for the real counterpart of p-adic "$\pi/2$"$_R$: ("$\pi/2$")$_R \simeq 1$ for large values of p.

4.6.2 Topological condensate as a generalized manifold

It seems that the concept of the p-adic Riemann manifold is not as such enough for the mathematization of the topological condensate concept. This manifold can be given locally p-adic topology but decomposes into regions with different values of the p-adic prime p. Also real regions are possible. These regions are glued together along their boundaries.

One can consider two possibilities for performing the identification map. Gluing together along common rationals at the boundaries defined by the rational topology is the first option, and certainly the fundamental one if one assumes that space-times are surfaces in a rational imbedding space which can be completed to either real or p-adic imbedding space. This kind of gluing operation is very natural for the solutions of the field equations obtained by a completion of rationals to various number fields in which the power series representing the solution of the field equations converge. This will be discussed in detail in the chapter "TGD as a generalized number theory".

The second option is the use of canonical identification map or some generalization of this map mapping real space-time regions to their p-adic counterparts. This gluing operation makes sense in case of cognitive representations and is not so fundamental. In this case p-adic space-time surfaces, possibly characterized by different value of prime p, are like different sheets of a chart having common overlap region. Although the p-adic regions can be disjoint they correspond to cognitive images of the real regions such that some overlap region is mapped to the both p-adic chart sheets. This common region defines the gluing of the p-adic surfaces together.

If one requires that the p-adic space-time surface is differentiable and even more, satisfies the p-adic counterparts of the field equations, one must loosen the cognitive mapping so that the image of the real space-time surface is discrete. Therefore one must weaken also the gluing conditions by introducing pinary cutoff.

4.6.3 p-Adic conformal geometry?

It would be nice to have a generalization of the ordinary conformal geometry to the p-adic context. A possibility worth of studying is that the induced Kähler form defining a Maxwell field on the space-time surface, could be the basic entity of the 4-dimensional conformal geometry rather than metric. If the existence of square root is required the dimension of this geometry is $D = 4$ of $D = 8$ depending on the value of p. In the following it is assumed that the extension used is the minimal extension allowing square root and $p \ mod \ 4 = 3$ condition holds so that the imaginary unit belongs to the generators of the extension.

In 2-dimensional case line element transforms by a conformal scale factor in p-analytic map $Z \rightarrow f(Z)$. In the four-dimensional case this requirement leads to a degenerate line element

$$\begin{aligned} ds^2 &= g(Z, Z_c, ...)dZdZ_c \ , \\ &= g(Z, Z_c, ..)(dx^2 + dy^2 + p(du^2 + dv^2) + 2\sqrt{p}(dxdu + dydv)) \ , \end{aligned} \tag{4.6.1}$$

where the conformal factor $g(Z, Z_c, ..)$ is invariant under the complex conjugation. The metric tensor associated with the line element does not possess an inverse. This is obvious from the fact that the line element depends on two coordinates Z, Z_c only so that the p-adic conformal metric is effectively 2-dimensional rather than 4-dimensional. It therefore seems that one must give up conformal covariance requirement for the line element.

In two-dimensional conformal geometry angles are the simplest conformal invariants and are expressible in terms of the inner product. In 4-dimensional case one can define invariants, which are analogous to angles. Let A and B be two vectors in the 4-dimensional quadratic extension allowing a square root. Denote A (B) and its various conjugates by A_i (B_i), $i = 1, 2, 3, 4$. Define phase like quantities $X_{ij} =$ "$exp(i2\Phi_{ij})$" between A and B by the following formulas

$$X_{ij} \equiv \frac{A_i A_j B_k B_l}{\sqrt{A_1 A_2 A_3 A_4}\sqrt{B_1 B_2 B_3 B_4}} \ . \tag{4.6.2}$$

where i, j, k, l is permutation of $1, 2, 3, 4$. Each quantity X_{ij} is invariant under one of the conjugations $_c, \hat{}$ or $\hat{}_c$ and X_{ij} has values in 2-dimensional subspace of the 4-dimensional extension. As in ordinary case the angles are invariant under conjugation and this means that only 3 angle like quantities exists: this is in accordance with the fact that 3-angles are needed to specify the orientation of the vector A with respect to the vector B.

One can define also more general invariants using four vectors A, B, C, D and permutations i, j, k, l and r, s, t, u of $1, 2, 3, 4$

$$\begin{aligned} U_{ijkl} &= \frac{X_{ijkl}}{X_{rstu}} \ , \\ X_{ijkl} &\equiv A_i B_j C_k D_l \ . \end{aligned} \tag{4.6.2}$$

The number of the functionally independent invariants is reduced if various conjugates of the invariants are not counted as different invariants. If 2 or 3 vectors are identical one obtains as a special case invariants associated with 3 and 2 vectors. If there are only two vectors the number of the functionally independent invariants is 6.

There exists quadratic conformal covariants associated with tensors of weight two. The general form of the covariant is given by

$$X = g^{ij:kl} A_{ij} B_{kl} \ . \tag{4.6.3}$$

The tensor $g^{ij:kl}$ has the property that in complex coordinates $Z, \bar{Z}, \hat{Z}, \hat{\bar{Z}}$ the only nonvanishing components of the tensor have $i \neq j \neq k \neq l$. This guarantees the multiplicative transformation property in the conformal transformations $Z \rightarrow W(Z)$:

$$X(W) = \frac{dW}{dZ}\frac{d\bar{W}}{d\bar{Z}}\frac{d\hat{W}}{d\hat{Z}}\frac{d\bar{\hat{W}}}{d\bar{\hat{Z}}}X(Z) \ . \tag{4.6.4}$$

The simplest example of tensor $g^{ij:kl}$ is permutation symbol and the instanton density of any gauge field defines a p-adic conformal covariant (the quantity is actually $Diff^4$ invariant).

4.7 Appendix: p-Adic square root function and square root allowing extension of p-adic numbers

The following arguments demonstrate that the extension allowing square roots of ordinary p-adic numbers is 4-dimensional for $p < 2$ and 8-dimensional for $p = 2$.

4.7.1 $p > 2$ resp. $p = 2$ corresponds to $D = 4$ resp. $D = 8$ dimensional extension

What is important is that only the square root of ordinary p-adic numbers is needed: the square root need not exist outside the real axis. It is indeed impossible to find a finite-dimensional extension allowing square root for all ordinary p-adic numbers numbers. For $p > 2$ the minimal dimension for algebraic extension allowing square roots near real axis is $D = 4$. For $p = 2$ the dimension of the extension is $D = 8$.

For $p > 2$ the form of the extension can be derived by the following arguments.

1. For $p > 2$ a p-adic number y in the range $(0, p-1)$ allows square root only provided there exists a p-adic number $x \in \{0, p-1\}$ satisfying the condition $y = x^2 \ mod \ p$. Let x_0 be the smallest integer, which does not possess a p-adic square root and add the square root θ of x_0 to the number field. The numbers in the extension are of the form $x + \theta y$. The extension allows square root for every $x \in \{0, p-1\}$ as is easy to see. p-adic numbers $mod\ p$ form a finite field $G(p,1)$ [8] so that any p-adic number y, which does not possess square root can be written in the form $y = x_0 u$, where u possesses square root. Since θ is by definition the square root of x_0 then also y possesses square root. The extension does not depend on the choice of x_0.

   The square root of -1 does not exist for $p\ mod\ 4 = 3$ [7] and $p = 2$ but the addition of θ gurantees its existence automatically. The existence of $\sqrt{-1}$ follows from the existence of $\sqrt{p-1}$ implied by the extension by θ. $\sqrt{(-1+p)-p}$ can be developed in power in powers of p and series converges since the p-adic norm of coefficients in Taylor series is not larger than 1. If $p - 1$ does not possess a square root, one can take θ to be equal to $\sqrt{-1}$.

2. The next step is to add the square root of p so that the extension becomes 4-dimensional and an arbitrary number in the extension can be written as

$$Z = (x + \theta y) + \sqrt{p}(u + \theta v) \ . \tag{4.7.1}$$

In $p = 2$ case 8-dimensional extension is needed to define square roots. The addition of $\sqrt{2}$ implies that one can restrict the consideration to the square roots of odd 2-adic numbers. One must be careful in defining square roots by the Taylor expansion of square root $\sqrt{x_0 + x_1}$ since n:th Taylor coefficient is proportional to 2^{-n} and possesses 2-adic norm 2^n. If x_0 possesses norm 1 then x_1 must possess norm smaller than 1/8 for the series to converge. By adding square roots $\theta_1 = \sqrt{-1}, \theta_2 = \sqrt{2}$ and $\theta_3 = \sqrt{3}$ and their products one obtains 8-dimensional extension.

The emergence of the dimensions $D = 4$ and $D = 8$ for the algebraic extensions allowing the square root of an ordinary p-adic number stimulates an obvious question: could one regard space-time as this kind of an algebraic extension for $p > 2$ and the imbedding space $H = M^4_+ \times CP_2$ as a similar 8-dimensional extension of the 2-adic numbers? Contrary to the first expectations, it seems that algebraic dimension cannot be regarded as a physical dimension, and that quaternions and octonions provide the correct framework for understanding space-time and imbedding space dimensions. One could perhaps say that algebraic dimensions are additional dimensions of the world of cognitive physics rather than those of the real physics and there presence could perhaps explain why we can imagine all possible dimensions mathematically.

By construction, any ordinary p-adic number in the extension allows square root. The square root for an arbitrary number sufficiently near to p-adic axis can be defined through Taylor series expansion of the square root function $\sqrt{Z}$ at a point of p-adic axis. The subsequent considerations show that the p-adic square root function does not allow analytic continuation to R^4 and the points of the extension allowing a square root consist of disjoint converge cubes forming a structure resembling future light cone in certain respects.

4.7.2 p-Adic square root function for $p > 2$

The study of the properties of the series representation of a square root function shows that the definition of the square root function is possible in certain region around the real p-adic axis. What is nice that this region can be regarded as the p-adic analog (not the only one) of the future light cone defined by the condition

$$N_p(Im(Z)) \quad < \quad N_p(t = Re(Z)) = p^k \quad , \tag{4.7.2}$$

where the real p-adic coordinate $t = Re(Z)$ is identified as a time coordinate and the imaginary part of the p-adic coordinate is identified as a spatial coordinate. The p-adic norm for the four-dimensional extension is analogous to ordinary Euclidian distance. p-Adic light cone consists of cylinders parallel to time axis having radius $N_p(t) = p^k$ and length $p^{k-1}(p-1)$. As a real space (recall the canonical correspondence) the cross section of the cylinder corresponds to a parallelpiped rather than ball.

The result can be understood heuristically as follows.

1. For the four-dimensional extension allowing square root ($p > 2$) one can construct square root at each point $x(k,s) = sp^k$ represented by ordinary p-adic number, $s = 1,...,p-1$, $k \in Z$. The task is to show that by using Taylor expansion one can define square root also in some neighbourhood of each of these points and find the form of this neighbourhood.

2. Using the general series expansion of the square root function one finds that the convergence region is p-adic ball defined by the condition

$$N_p(Z - sp^k) \quad \leq \quad R(k) \quad , \tag{4.7.3}$$

   and having radius $R(k) = p^d, d \in Z$ around the expansion point.

3. A purely p-adic feature is that the convergence spheres associated with two points are either disjoint or identical! In particular, the convergence sphere $B(y)$ associated with any point inside convergence sphere $B(x)$ is identical with $B(x)$: $B(y) = B(x)$. The result follows directly from the ultra-metricity of the p-adic norm. The result means that stepwise analytic continuation is not possible and one can construct square root function only in the union of p-adic convergence spheres associated with the points $x(k,s) = sp^k$ which correspond to ordinary p-adic numbers.

4. By the scaling properties of the square root function the convergence radius $R(x(k,s)) \equiv R(k)$ is related to $R(x(0,s)) \equiv R(0)$ by the scaling factor p^{-k}:

$$R(k) \quad = \quad p^{-k}R(0) \quad , \tag{4.7.4}$$

   so that the convergence sphere expands as a function of the p-adic time coordinate. The study of the convergence reduces to the study of the series at points $x = s = 1,...,k-1$ with a unit p-adic norm.

5. Two neighboring points $x = s$ and $x = s + 1$ cannot belong to the same convergence sphere: this would lead to a contradiction with the basic results of about square root function at integer points. Therefore the convergence radius satisfies the condition

$$R(0) < 1 . \quad (4.7.5)$$

The requirement that the convergence is achieved at all points of the real axis implies

$$\begin{aligned} R(0) &= \frac{1}{p} , \\ R(p^k s) &= \frac{1}{p^{k+1}} . \end{aligned} \quad (4.7.5)$$

If the convergence radius is indeed this, then the region, where the square root is defined, corresponds to a connected light cone like region defined by the condition $N_p(Im(Z)) = N_p(Re(Z))$ and $p > 2$-adic space time is the p-adic analog of the M^4 lightcone. If the convergence radius is smaller, the convergence region reduces to a union of disjoint p-adic spheres with increasing radii.

How the p-adic light cone differs from the ordinary light cone can be seen by studying the explicit form of the p-adic norm for $p > 2$ square root allowing extension $Z = x + iy + \sqrt{p}(u + iv)$

$$\begin{aligned} N_p(Z) &= (N_p(det(Z)))^{\frac{1}{4}} , \\ &= (N_p((x^2 + y^2)^2 + 2p^2((xv - yu)^2 + (xu - yv)^2) + p^4(u^2 + v^2)^2))^{\frac{1}{4}} , \end{aligned} \quad (4.7.4)$$

where $det(Z)$ is the determinant of the linear map defined by a multiplication with Z. The definition of the convergence sphere for $x = s$ reduces to

$$N_p(det(Z_3)) = N_p(y^4 + 2p^2y^2(u^2 + v^2) + p^4(u^2 + v^2)^2)) < 1 . \quad (4.7.5)$$

For physically interesting case $p \ mod \ 4 = 3$ the points (y, u, v) satisfying the conditions

$$\begin{aligned} N_p(y) &\leq \frac{1}{p} , \\ N_p(u) &\leq 1 , \\ N_p(v) &\leq 1 , \end{aligned} \quad (4.7.4)$$

belong to the sphere of convergence: it is essential that for all u and v satisfying the conditions one has also $N_p(u^2 + v^2) \leq 1$. By the canonical correspondence between p-adic and real numbers, the real counterpart of the sphere $r = t$ is now the parallelpiped $0 \leq y < 1, 0 \leq u < p, 0 \leq v < p$, which expands with an average velocity of light in discrete steps at times $t = p^k$.

4.7.3 Convergence radius for square root function

In the following it will be shown that the convergence radius of $\sqrt{t + Z}$ is indeed non-vanishing for $p > 2$. The expression for the Taylor series of $\sqrt{t + Z}$ reads as

$$\begin{aligned} \sqrt{t + Z} &= = \sqrt{x} \sum_n a_n , \\ a_n &= (-1)^n \frac{(2n - 3)!!}{2^n n!} x^n , \\ x &= \frac{Z}{t} . \end{aligned} \quad (4.7.3)$$

The necessary criterion for the convergence is that the terms of the power series approach to zero at the limit $n \rightarrow \infty$. The p-adic norm of the n:th term is for $p > 2$ given by

$$N_p(a_n) = N_p(\frac{(2n-3)!!}{n!})N_p(x^n) < N_p(x^n)N_p(\frac{1}{n!}) \ . \tag{4.7.4}$$

The dangerous term is clearly the $n!$ in the denominator. In the following it will be shown that the condition

$$U \equiv \frac{N_p(x^n)}{N_p(n!)} < 1 \ for \ \ N_p(x) < 1 \ , \tag{4.7.5}$$

holds true. The strategy is as follows:
a) The norm of x^n can be calculated trivially: $N_p(x^n) = p^{-Kn}, K \geq 1$.
b) $N_p(n!)$ is calculated and an upper bound for U is derived at the limit of large n.

p-Adic norm of $n!$ for $p > 2$

Lemma 1: Let $n = \sum_{i=0}^{k} n(i)p^i$, $0 \leq n(i) < p$ be the p-adic expansion of n. Then $N_p(n!)$ can be expressed in the form

$$\begin{aligned} N_p(n!) &= \prod_{i=1}^{k} N(i)^{n(i)} \ , \\ N(1) &= \frac{1}{p} \ , \\ N(i+1) &= N(i)^{p-1}p^{-i} \ . \end{aligned} \tag{4.7.4}$$

An explicit expression for $N(i)$ reads as

$$N(i) = p^{-\sum_{m=0}^{i} m(p-1)^{i-m}} \ . \tag{4.7.5}$$

Proof: $n!$ can be written as a product

$$\begin{aligned} N_p(n!) &= \prod_{i=1}^{k} X(i,n(i)) \ , \\ X(k,n(k)) &= N_p((n(k)p^k)!) \ , \\ X(k-1,n(k-1)) &= N_p(\prod_{i=1}^{n(k-1)p^{k-1}} (n(k)p^k + i)) = N_p((n(k-1)p^{k-1})!) \ , \\ X(k-2,n(k-2)) &= N_p(\prod_{i=1}^{n(k-2)p^{k-2}} (n(k)p^k + n(k-1)p^{k-1} + i) \ ,) \\ &= N_p((n(k-2)p^{k-2})!) \ , \\ X(k-i,n(k-i)) &= N_p((n(k-i)p^{k-i})!) \ . \end{aligned} \tag{4.7.1}$$

The factors $X(k,n(k))$ reduce in turn to the form

$$\begin{aligned} X(k,n(k)) &= \prod_{i=1}^{n(k)} Y(i,k) \ , \\ Y(i,k) &= \prod_{m=1}^{p^k} N_p(ip^k + m) \ . \end{aligned} \tag{4.7.1}$$

The factors $Y(i,k)$ in turn are indentical and one has

$$\begin{aligned} X(k,n(k)) &= X(k)^{n(k)} \ , \\ X(k) &= N_p(p^k!) \ . \end{aligned} \tag{4.7.1}$$

The recursion formula for the factors $X(k)$ can be derived by writing explicitely the expression of $N_p(p^k!)$ for a few lowest values of k:
1) $X(1) = N_p(p!) = p^{-1}$.
2) $X(2) = N_p(p^2!) = X(1)^{p-1}p^{-2}$ ($p^2!$ decomposes to $p-1$ products having same norm as $p!$ plus the last term equal to p^2.
i) $X(i) = X(i-1)^{p-1}p^{-i}$

Using the recursion formula repeatedly the explicit form of $X(i)$ can be derived easily. Combining the results one obtains for $N_p(n!)$ the expression

$$\begin{aligned} N_p(n!) &= p^{-\sum_{i=0}^{k} n(i)A(i)} \ , \\ A(i) &= \sum_{m=1}^{i} m(p-1)^{i-m} \ . \end{aligned} \tag{4.7.1}$$

The sum $A(i)$ appearing in the exponent as the coefficient of $n(i)$ can be calculated by using geometric series

$$\begin{aligned} A(i) &= (\frac{p-1}{p-2})^2(p-1)^{i-1}(1+\frac{i}{(p-1)^{i+1}} - \frac{(i+1)}{(p-1)^i}) \ , \\ &\leq (\frac{p-1}{p-2})^2(p-1)^{i-1} \ . \end{aligned} \tag{4.7.1}$$

Upper bound for $N_p(\frac{x^n}{n!})$ for $p > 2$

By using the expressions $n = \sum_i n(i)p^i$, $N_p(x^n) = p^{-Kn}$ and the expression of $N_pn!$ as well as the upper bound

$$A(i) \leq (\frac{p-1}{p-2})^2(p-1)^{i-1} \ . \tag{4.7.2}$$

For $A(i)$ one obtains the upper bound

$$N_p(\frac{x^n}{n!}) \leq p^{-\sum_{i=0}^{k} n(i)p^i(K-(\frac{(p-1)}{(p-2)})^2(\frac{(p-1)}{p})^{i-1})} \ . \tag{4.7.2}$$

It is clear that for $N_p(x) < 1$ that is $K \geq 1$ the upper bound goes to zero. For $p > 3$ exponents are negative for all values of i: for $p = 3$ some lowest exponents have wrong sign but this does not spoil the convergence. The convergence of the series is also obvious since the real valued series $\frac{1}{1-\sqrt{N_p(x)}}$ serves as a majorant.

4.7.4 $p = 2$ case

In $p = 2$ case the norm of a general term in the series of the square root function can be calculated easily using the previous result for the norm of $n!$:

$$N_p(a_n) = N_p(\frac{(2n-3)!!}{2^n n!})N_p(x^n) = 2^{-(K-1)n+\sum_{i=1}^{k} n(i)\frac{i(i+1)}{2^{i+1}}} \ . \tag{4.7.3}$$

At the limit $n \to \infty$ the sum term appearing in the exponent approaches zero and convergence condition gives $K > 1$, so that one has

$$N_p(Z) \equiv (N_p(det(Z)))^{\frac{1}{8}} \leq \frac{1}{4} \ . \tag{4.7.4}$$

The result does not imply disconnected set of convergence for square root function since the square root for half odd integers exists:

$$\sqrt{s+\frac{1}{2}} = \frac{\sqrt{2s+1}}{\sqrt{2}} \ , \tag{4.7.5}$$

so that one can develop square as a series in all half odd integer points of the p-adic axis (points which are ordinary p-adic numbers). As a consequence, the structure for the set of convergence is just the 8-dimensional counterpart of the p-adic light cone. Space-time has natural binary structure in the sense that each $N_p(t) = 2^k$ cylinder consists of two identical p-adic 8-balls (parallelpipeds as real spaces).

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Chapter 5

p-Adic Physics: Physical Ideas

5.1 Introduction

The basic implication of 'TGD as a generalized number theory' philosophy is that p-adic regions of the space-time surface result dynamically. Space-time surface is defined by the vanishing condition of a rational function of two quaternion-valued variables q_1 and p_1. This condition gives p_1 as a function of q_1. It can however happen that some components of the quaternion p_1 fail to be real numbers and become complex. It might be however possible to perform the completion of the rational space-time surface to a p-adic space-time surface and for some values of the p-adic prime the series defining the power series representing $p_1 = f(q_1)$ can converge to a number in some algebraic extension of the ordinary p-adic numbers. It is also quite possible that p-adic and real power roots $p_1 = f(q_1)$ converge simultaneously. Even more general rational-adic topologies in which norm is a power of a rational number are possible: rational-adic numbers do not however form a ring. p-Adic numbers are thus very closely related with quaternion-conformal invariance and criticality.

p-Adic topologies form an infinite hierarchy and p-adic physics leads to a vision about many-sheeted space-time as a hierarchical structure consisting of p-adic and real space-time sheetse of increasing size and increasing value of prime p. These surfaces are glued together using topological sum or join along boundaries bonds. Contrary to the original expectations, p-adic space-time regions represent 'mind-stuff' rather than 'matter' which is also present and represented by real and infinite-p p-adic regions. Thus p-adic provide 'cognitive representations' for matter like regions and this is why their physics provides a manner to understand real physics. If p-adic-to-real phase transitions are possible, one can understand why it is possible to assign p-adic prime even to real regions. In fact, the hypothesis that p-adic regions provide a cognitive model for real physics, poses very strong constraints on real physics.

There is a "holy trinity" of non-determinisms in TGD in the sense that there is the non-determinism associated with the quantum jumps, the classical non-determinism of the Kähler action and p-adic non-determinism. The non-determinism of quantum jumps can involve also a selection between various multifurcations for various absolute minima of the Kähler action in which case it represents a genuine volitional act. p-Adic non-determinism in turn corresponds to the non-determinism of pure imagination with no material consequences. Also real space-time sheets with finite time duration are also possible and they might represent what might be called 'sensory space-time sheets' as opposed to cognitive space-time sheets. Cognitive space-time sheets can be transformed to real ones in quantum jumps inducing change of control parameters of the polynomial defining space-time surface: if the change is such that the p-adic root is replaced with a real root, one can say that thought is transformed into action. The reverse of this process is the transformation of sensory input into cognition.

"Holy trinity" implies that it should be possible to determine the p-adic prime characterizing a given space-time region (or space-time sheet) by observing a large number of quantum time developments of this system. The characteristic p-adic fractality, that is the presence of time scales $T(p,k) = p^k T_p$, should become manifest in the statistical properties of the cognitive time developments which in should turn reflect the properties of the real physics since cognitive representations are in question. For instance, quantum jumps with especially large amplitude would tend to occur at time scales $T(p,k) = p^k T_p$. $T(p,k)$ could also provide series of characteristic correlation times. Needless to say, this prediction means definite departure from the non-determinism of ordinary quantum mechanics and only at the limit of infinite p the predictions should be identical. An interesting possibility is that $1/f$ noise [33] is a direct manifestation of the classical non-determinism: if this is the case, it should be possible to associate a definite value of p to $1/f$ noise. Also transformations of the p-adic cognitive space-time sheets to real space-time sheets of a finite time duration and vice versa might be involved with the $1/f$ noise so that $1/f$ noise would be a direct signature of cognitive consciousness.

The 'physical' building blocks of p-adic TGD, as opposed to the philosophical mathematical ones briefly summarized above, and in more detail in previous chapters, are spin glass analogy leading to the general picture about how finite-p p-adicity emerges from quantum TGD, the identification of elementary particles as CP_2 type extremals, and elementary particle black hole analogy. These building blocks have been present as stable pieces of theory from beginning whereas philosophical ideas and interpretations have undergone rather wild fluctuations during an almost last decade of p-adic TGD.

5.2 p-Adic numbers and spin glass analogy

Spin glass phase decomposes into regions in which the direction of the magnetization varies randomly with respect to spatial coordinates but remains constant in time. What makes spin glass special is that the boundary regions between regions of different magnetization do not give rise to large surface energies. Spin glass structure emerges in two manners in TGD framework.

1. Spin glass behavior at the level of real physics is encountered in TGD framework because of the classical non-determinism of the Kähler action. The classical non-determinism of CP_2 type extremals represents the manifestation of the spin glass analogy at the level of elementary particle physics. In macroscopic length scales real physics spin glass analogy makes possible 'real world engineering'.

2. Spin glass behavior at the level of cognition is encountered because of the p-adic non-determinism and makes possible what might be called imagination or 'cognitive engineering'. The point is that any piecewise constant function has a vanishing p-adic derivative. Therefore any function of the spatial coordinates depending on a finite number of the pinary digits is a pseudo constant. The discontinuities of this kind in the field variables do not lead to infinite surface energies in the p-adic context as they would in the real context.

Spin glass energy landscape is characterized by an ultra-metric distance function. The reduced configuration space CH_{red} consisting of the maxima of the Kähler function with respect to quantum fluctuating degrees of freedom and zero modes defines the TGD counter part of the spin glass energy landscape. This notion makes sense only in real context since p-adic space-time regions do not contribute to the Kähler function and all p-adic configurations are equally probable. The original vision was that if the ultra-metric distance function in CH_{red} is induced from a p-adic norm, a connection between p-adic physics and real physics also at the level of space-time might emerge somehow. It seems however that the ultra-metricity of CH_{red} need not directly relate to the p-adicity at the space-time level which can be understood if p-adic space-time regions give rise to cognitive representations of the real regions. Of course, it *might* be that the p-adic prime characterizing cognitive representation of a real region characterizes also the reduced configuration space associated with the region in question (one must of course assume that the reduced configuration space approximately decomposes into a Cartesian product of the reduced configuration spaces associated with real regions).

5.2.1 General view about how p-adicity emerges

In TGD classical theory is exact part of the quantum theory and in a well defined sense appears already at the level of the configuration space geometry: the definition of the configuration space Kähler metric [B1] associates a unique space-time surface to a given 3-surface. The vacuum functional of the theory (exponent of the Kähler function) is analogous to the exponent $exp(H/T_c)$ appearing in the definition of the partition function of a critical system so that the Universe described by TGD is quantum critical system. Critical system is characterized by the presence of two phases, which can be present in arbitrary large volumes. The TGD:eish counter part of this seems to be the presence of two kinds of

3-surfaces for which either Kähler electric or Kähler magnetic field energy dominates. These 3-surfaces have outer boundaries for purely topological reasons and these boundaries can be of a macroscopic size. Therefore it seems that 3-space should be regarded as what could be called topological condensate with a hierarchical, fractal like structure: there are 3-surfaces (with boundaries) condensed on 3-surfaces condensed on...... .

This leads to a radically new manner to see the world around us. The outer surfaces of the macroscopic bodies correspond to the boundaries of 3-surfaces in the condensate so that one can see the 3-topology in all its complexity just by opening one's eyes! A rather compelling evidence for the basic ideas of TGD if one is willing to give up the nebulous concept of "material object in topologically trivial 3-space" and to allow nontrivial 3-topology in macroscopic length scales. A second rather radical departure from the conventional picture of the 3-space is that TGD:eish 3-space is not connected but contains arbitrary many disjoint components. In fact the actual Universe should consist of infinitely many 3-surfaces condensed on each other.

In two-dimensional critical systems conformal transformations act as symmetries and conformal invariance implies the Universality of critical systems. This suggests that one should try to find the generalization of the conformal invariance to higher dimensional, in particular, 4-dimensional case. If finally turned out that quaternion-conformal invariance realizes quantum criticality four 4-surfaces imbedded to 8-dimensional space. As a by product an explanation for space-time and imbedding space dimensions results.

In this approach the p-adic regions of the space-time surface result dynamically. Space-time surface is defined by the vanishing condition of a polynomial of two quaternion-valued variables q and p. This condition gives p as a function of q. It can however occur that some components of p become complex numbers. They must be however real so that the solution fails to exist in the real sense. It might be however possible to perform the completion of the rational space-time surface to a p-adic space-time surface and for some values of the p-adic prime the series defining the power series representing $p = f(q)$ might converge to a number in some algebraic extension of the ordinary p-adic numbers. Even more general rational-adic topologies in which norm is power of a rational number are possible. p-Adic numbers would thus be very closely related with quaternion-conformal invariance and criticality.

p-Adic topologies form an infinite hierarchy and p-adic physics leads to a vision about many-sheeted space-time as a hierarchical structure consisting of p-adic 4-surfaces of increasing size and increasing value of prime p. These surfaces are glued together using topological sum operation. Contrary to the original expectations, this hierachy is the hierarchy for the regions of space-time representing 'mind-stuff' rather than 'matter' which is also present and represented by real and infinite-p p-adic regions. p-Adic provide 'cognitive representations' for matterlike regions and this is why their physics provides a manner to understand real physics.

5.2.2 p-Adic numbers and the analogy of TGD with spin-glass

The vacuum degeneracy of the Kähler action leads to precise spin glass analogy at the level of the configuration space geometry and the generalization of the energy landscape concept to the TGD:eish context leads to the hypothesis about how p-adicity is realized at the level of the configuration space. Also the concept of p-adic space-time surface emerges rather naturally.

Spin glass briefly

The basic characteristic of the spin glass phase [34] is that the direction of the magnetization varies spatially, being constant inside a given spatial region, but does not depend on time. In the real context this usually leads to large surface energies on the surfaces at which the magnetization direction changes. Regions with different direction of magnetization clearly correspond non-vacuum regions separated by almost vacuum regions. Amusingly, if 3-space is effectively p-adic and if magnetization direction is p-adic pseudo constant, no surface energies are generated so that p-adics might be useful even in the context of the ordinary spin glasses.

Spin glass phase allows a great number of different ground states minimizing the free energy. For the ordinary spin glass, the partition function is the average over a probability distribution of the coupling constants for the partition function with Hamiltonian depending on the coupling constants.

Free energy as a function of the coupling constants defines 'energy landscape' and the set of free energy minima can be endowed with an ultra-metric distance function using a standard construction [35].

Vacuum degeneracy of the Kähler action

The Kähler action defining configuration space geometry allows enormous vacuum degeneracy: any four-surface for which the induced Kähler form vanishes, is an extremal of the Kähler action. Induced Kähler form vanishes if the CP_2 projection of the space-time surface is Lagrange manifold of CP_2: these manifolds are at most two-dimensional and any canonical transformation of CP_2 creates a new Lagrange manifold. An explicit representation for Lagrange manifolds is obtained using some canonical coordinates P_i, Q_i for CP_2: by assuming

$$P_i = \partial_i f(Q_1, Q_2) \ ,$$

where f arbitrary function of its arguments. One obtains a 2-dimensional sub-manifold of CP_2 for which the induced Kähler form proportional to $dP_i \wedge dQ^i$ vanishes. The roles of P_i and Q_i can obviously be interchanged. A familiar example of Lagrange manifolds are $p_i = constant$ surfaces of the ordinary (p_i, q_i) phase space.

Since vacuum degeneracy is removed only by classical gravitational interaction there are good reasons to expect large ground state degeneracy, when system corresponds to a small deformation of a vacuum extremal. This degeneracy is very much analogous to the ground state degeneracy of spin glass.

Vacuum degeneracy of the Kähler action and physical spin glass analogy

Quite generally, the dynamical reason for the physical spin glass degeneracy is the fact that Kähler action has a huge vacuum degeneracy. Any 4-surface with CP_2 projection, which is a Legendre sub-manifold (generically two-dimensional), is vacuum extremal. This implies that space-time decomposes into non-vacuum regions characterized by non-vanishing Kähler magnetic and electric fields such that the (presumably thin) regions between the the non-vacuum regions are vacuum extremals. Therefore no surface energies are generated. Also the fact that various charges and momentum and energy can flow to larger space-time sheets via wormholes is an important factor making possible strong field gradients without introducing large surfaces energies. From a given absolute minimum of Kähler action one obtains a new one by adding arbitrary space-time surfaces which is vacuum exremal. Uniqueness of the absolute minima in the sense that real regions of space-time $X^4(X^3)$ are unique could be acieved by requiring that vacuum regions are p-adic and represent thus cognitive regions whereas real regions carry non-vanishing induced Kähler field.

The canonical invariance of the Kähler action for vacuum extremals allows a further understanding of the vacuum degeneracy. The presence of the classical gravitational interaction spoils the canonical group $Can(CP_2)$ as gauge symmetries of the action and transforms it to the isometry group of CH. As a consequence, the $U(1)$ gauge degeneracy is transformed to a spin glass type degeneracy and several, perhaps even infinite number of maxima of Kähler function for given values of the zero modes, become possible. Thus locally, the space maxima of Kähler function should look like a union of copies of the space of zero modes. Given sheet has naturally as its boundary the 3-surfaces for which two maxima of the Kähler function coalesce or are created from single maximum by a cusp catastrophe. In catastrophe regions there are several sheets and the value of the maximum Kähler function determines which give a measure for the importance of various sheets. The quantum jumps selecting one of these sheets can be regarded as phase transitions.

In TGD framework classical non-determinism forces to generalize the notion of the 3-surface by replacing it with a sequence of space like 3-surfaces having time like separations such that the sequence characterizes uniquely one branch of multifurcation. This characterization works when non-determinism has discrete nature. For CP_2 type extremals which are bosonic vacua, basic objects are essentially four-dimensional since M^4_+ projection of CP_2 type extremal is random light like curve. This effective four-dimensionality of the basic objects makes it possible to topologize Feynman diagrammatics of quantum field theories by replacing the lines of Feynman diagrams with CP_2 type extremals.

In TGD framework spin glass analogy holds true also in the time direction, which reflects the fact that the vacuum extremals are non-deterministic. For instance, by gluing vacuum extremals with a finite space-time extension (also in time direction!) to a non-vacuum extremal and deforming slightly, one obtains good candidates for the degenerate absolute minima. This non-determinism is expected to make the absolute minima of the Kähler action highly degenerate. The construction of S-matrix at the high energy limit suggests that since a localization selecting one degenerate maximum occurs, one must accept as a fact that each choice of the parameters corresponds to a particular S-matrix and one must average over these choices to get scattering rates. This averaging for scattering rates corresponds to the averaging over the thermodynamical partition functions for spin glass. A more general is that one allows final state wave functions to depend on the zero modes which affect S-matrix elements: in the limit that wave functions are completely localized, one ends up with the simpler scenario.

The real effective action is expected to be Einstein-Yang-Mills action for the induced gauge fields. This action does not possess any vacuum degeneracy. The space-time surfaces are certainly absolute minima of the Kähler action and EYM-action could take a dynamical role only in the sense that extremality with respect to classical part of EYM action selects one of the degenerate absolute minima of the Kähler action. On the other hand, the construction of S-matrix suggests that the choice of particular parameter values characterizing zero modes affects only the coupling constants and propagators of the effective Einstein-Yang-Mills theory, and that one must perform averaging over the predictions of these theories. Thus EYM action could at most fix a gauge.

p-Adic non-determinism and spin glass analogy

One must carefully distinguish between cognitive and physical spin-glass analogy. Cognitive spin-glass analogy is due to the p-adic non-determinism. p-Adic pseudo constants induce a non-determinism which essentially means that p-adic extrema depend on the p-adic pseudo constants which depend on a finite number of positive pinary digits of their arguments only. Thus p-adic extremals are glued from pieces for which the values of the integration constants are genuine constants. Obviously, an optimal cognitive representation is achieved if pseudo constants reduce to ordinary constants.

More precisely, any function

$$\begin{aligned} f(x) &= f(x_N) \ , \\ x_N &= \sum_{k\leq N} x_k p^k \ , \end{aligned} \tag{5.2.0}$$

which does not depend on the pinary digits x_n, $n > N$ has a vanishing p-adic derivative and is thus a pseudo constant. These functions are piecewise constant below some length scale, which in principle can be arbitrary small but finite. The result means that the constants appearing in the solutions the p-adic field equations are constants functions only below some length scale. For instance, for linear differential equations integration constants are arbitrary pseudo constants. In particular, the p-adic counterparts of the absolute minima (defined by the correspondence with infinite primes) are highly degenerate because of the presence of the pseudo constants. This in turn means a characteristic randomness of the spin glass also in the time direction since the surfaces at which the pseudo constants change their values do not give rise to infinite surface energy densities as they would do in the real context.

The basic character of cognition would be spin glass like nature making possible 'engineering' at the level of thoughts (planning) whereas classical non-determinism of the Kähler action would make possible 'engineering' at the level of the real world.

Localization in zero modes

The Kähler function defining configuration space metric possesses infinite number of zero modes which represent non-quantum-fluctuating degrees of freedom. The requirement that physics is local at the level of zero modes implies that each quantum jump involves a localization in zero modes. This localization could be complete or in a region whose size is determined by the p-adic length scale hypothesis.

Localization would mean an enormous calculational simplification: functional integral reduces into ordinary functional integral over the quantum-fluctuating degrees of freedom and there is no need to integrate over the zero modes. The complete or partial localization in zero modes would explain why the world of conscious experience looks classical. Perhaps the complete localization is however too much to wish for: it could however be that one must use wave functionals in the zero modes only in the case that one is interested in a comparison of the transition rates associated with different values of zero modes rather than in transition rates with the condition that a localization has occurred to definite values of zero modes.

The functional integral over the fiber degrees of freedom can be approximated by a Gaussian integrals around maxima. Classical non-determinism would suggest the possibility of several maxima in fiber degrees of freedom but the symmetric space property of the fiber suggests that there is only single maximum of Kähler function. The existence of single maximum gives good hopes that the configuration space integration reduces effectively to Gaussian integration of free field theory.

5.2.3 The notion of the reduced configuration space

Quantum jumps occur with highest probability to those values of zero modes which correspond to the maxima of the Kähler function and a simplified description of the situation is obtained by considering the reduced configuration space CH_{red} consisting of the maxima of Kähler function with respect to both zero modes and and quantum fluctuating degrees of freedom.

The hypothesis that the space CH_{red} is an enumerable set is a natural first guess. In macroscopic length scales, one might indeed hope that the generation of Kähler electric fields reducing the vacuum degeneracy could imply a discrete degeneracy for the maxima of the Kähler action.

In elementary particle length scales this hypothesis fails and it is good to analyze the situation in more detail since it gives some about how complex the situation can be. For the so called CP_2 type extremals the classical non-determinism gives rise to a functional continuum of degenerate maxima of the Kähler function. The degenerate maxima correspond to random zitterbewegung orbits for which the 'time parameter' u is an arbitrary function of CP_2 coordinates. In this case however zero modes characterizing light like random curve representing the zitterbewegung orbit behave exactly like conformal gauge degrees of freedom. The choice of the 'time parameter' u however affects S-matrix elements: dependence is very weak and only through the volumes of the propagator lines determined by the selection of u (Kähler action for CP_2 type extremal is proportional to its volume) occurring in quantum jump. Effectively the functional continuum is replaced with the real continuum of the volume of the propagator line varying from zero to the volume of CP_2.

A localization for the positions of the vertices of the Feynman diagrams defined by CP_2 type extremals cannot however be assumed. Neither can one assume that only single Feynman diagram is selected if one wants that a generalization of ordinary Feynman diagrammatics results. There are several alternative identifications.

1. The degrees represented by Feynman diagrams with varying positions of vertices represent fiber degrees of freedom so that there would be slight dependence of the Kähler function on the positions of the vertices. Certainly the Feynman diagrams with different topologies have different value of Kähler action and must correspond to fiber degrees of freedom. The reason is that vertex regions of the Feynman diagrams must involve deformations of CP_2 extremals since otherwise Feynman diagrams are singular as 4-manifolds. Note that the idea about localization in fiber degrees of freedom is not favored by this example.

2. The positions for the vertices of the Feynman diagram are excellent candidates for zero modes and localization is not possible now. The fact that these degrees of freedom correspond to center of mass degrees of freedom related to the isometries of the theory might distinguish between them and other zero modes. One can consider also a refinement for localization in the zero modes hypothesis: localization occurs only in length scale resolution defined by the p-adic length scale. In fact, the assumption that CP_2 type extremals have suffered topological condensation on space-time sheets with size of order p-adic length scale characterizing the elementary particle implies this.

Whether the notion of CH_{red} makes sense for the p-adic space-time regions is not at all obvious. For the proposed construction of the configuration space metric p-adic regions do not contribute to the Kähler function which is real-valued. Only in case that the p-adic contribution is rational number, it could be interpreted as a real valued contribution to the Kähler function. In case of CP_2 type extremals this is not the case although the exponent of the Kähler function for a full CP_2 type extremal is a rational number if the proposed model for the p-adic evolution of Kähler coupling strength is correct.

If it does not make sense to distinguish between the maxima of the Kähler function in the p-adic context, one cannot define CH_{red} on basis of this criterion. From the point of view of cognition this means maximal freedom of imagination.

An interesting question is whether one must count the cognitive degeneracy as a degeneracy of physical states. If localization occurs in each quantum jump with respect to both real and p-adic zero mode degeneracy, and if all cognitive options are equally probable, then the only conclusion seems to be that space-time surfaces for which the cognitive degeneracy is highest, represent the most probable final states. This would mean that the systems with the highest cognitive resources would be winners in the struggle for survival. An alternative manner to see the same thing is that systems with a high cognitive degeneracy are able to undergo a rich repertoire of p-adic-to-real phase transitions and thus to adapt with the environment.

Explicit definition of the ultra-metric distance function for energy landscape

The points of CH_{red} are completely analogous to the minima of the free energy and the precise analogy with spin glass suggests that CH_{red} must possess naturally an ultra-metric topology. One can quite generally construct an explicit ultra-metric distance function for the set of energy minima in a given energy landscape describing energy as a function of the coordinates of some configuration space using existing recipes [36]. The concept is useful when the energy landscape has fractal like structure. An attractive metaphor is to regard energy as a height function for a landscape with mountains.

The distance function between two energy minima should describe the difficulty of getting from a given minimum to another one. A concrete measure for this difficulty is obtained by considering all possible paths from x to y. The height for the highest point on this path, absolute maximum $h_{max}(\gamma)$ of the height function on this path gives the measure for the difficulty for reaching y along the path γ. There exists some easiest path from x to y. The difficulty to reach y from x can be defined as the height of the highest point associated with the easiest path and hence the minimum of $h_{max}(\gamma)$ in the set of all possible paths from x to y:

$$d(x,y) = Min(h_{max}(\gamma(x,y)) \ .$$

It is easy check that this distance function is ultra-metric:

$$d(x,z) \leq Max\{d(x,y), d(y,z)\} \ .$$

All what is needed is to notice that for any path $x \rightarrow z$ going through y highest point of the path is either the highest point associated with the path from $x \rightarrow y$ or $y \rightarrow z$: from this the inequality follows trivially since one can in principle find also easier paths.

Identification of the height function in the case of the reduced configuration space?

Obviously the negative for the maximum of Kähler function as function of zero modes is the counterpart of free energy. This function could well be many valued but this is an unessential complication. It is not clear whether K is negative definite (there are strong reasons to believe that this is the case). One can however consider any positive definite function of K as a height function defining an ultra-metric norm in the manner suggested. The requirement that p-adic norm results should fix the definition uniquely.

The exponential $exp(-K_{max})$ of the maximum of Kähler function as function of the zero modes, which is the inverse for the vacuum functional of the theory, is the first guess for the height function defining the ultra-metric norm (the wandering from 3-surface X^3 to Y^3 corresponds to quantum tunnelling physically.). The justification for this identification is that the integration over the fiber degrees of freedom gives Gaussian determinant cancelling the metric determinant and leaves on the exponent of Kähler function to the functional integral over zero modes. The intuitive expectation is that ultra-metric norm is p-adic for some p and that the space of zero modes decomposes into regions D_p In order to get a power of p as required by p-adicity, one can expand h as powers of p and identify p-adic norm as p^n for the highest pinary digit n with non-vanishing coefficient.

The height function can have a normalization factor and this factor could be chosen so that the ultra-metric norm is a power of p for CP_2 type extremals, which are certainly very important building blocks of absolute minimum space-time surfaces. The argument relating the gravitational coupling constant to the Kähler coupling strength and fixing the dependence of the Kähler coupling strength on the prime p, suggests that one must define the height function as

$$h_p = \frac{exp(-K(p))}{exp(-K(p=1))} \ ,$$

where the Kähler function at $p = 1$ is formally obtained by regarding the value of the Kähler coupling strength as a function in the set of all natural numbers.

Does the proposed height function h_p define p-adic topology?

The great question is whether one can obtain p-adic ultra-metricity in this manner. There is some evidence for this.

1. Criticality and spin glass analogy suggests that $exp(K)$ as a function of zero modes is fractal. If it is p-adic fractal then p-adic topology is expected to be a natural consequence: in this case the map of CH_{red} to its p-adic counterpart could make it possible to replaced CH_{red} with a smooth function.

2. CP_2 type extremals, the counterparts of black holes and a model of elementary particle in TGD, have finite negative Kähler action. One can glue CP_2 type extremals to any space-time surface to lower the Kähler action. 3-surfaces Z^3 on path from X^3 to Y^3 containing CP_2 extremals on $X^4(Z^3)$ are excellent candidates for 'mountains' in the landscape metaphor. The height of Z^3 is roughly described by the number of CP_2 type extremals glued on $X^4(Z^3)$.

3. The argument leading to a correct prediction of gravitational constant in terms of assuming that Kähler coupling strength α_K depends on zero modes only through the p-adic prime assumed to characterize a given region D_p of the configuration space for which the set of maxima of Kähler function as function of zero modes should obey has p-adic topology. The crucial input is the relationship

$$exp(K_p(CP_2))\frac{R^2}{G} = \frac{1}{p} \ ,$$

which is equivalent with $G = exp(K_p(CP_2)L_p^2$, where $L_p \simeq \sqrt{p}\times R$ is the p-adic length scale and $R \simeq 10^4\sqrt{G}$ is CP_2 size and the fundamental p-adic length scale. This formula is a dimensional estimate for gravitational coupling strength in terms of the p-adic length scale squared and the exponential of Kähler function for CP_2 type extremal describing graviton. The exponent gives the probability for the appearance of one virtual graviton in a given quantum state. The probability is very small since the exponent is negative for CP_2type extremal and gravitation is consequently a very weak interaction.

4. If one makes the identification

$$\frac{R^2}{G}(\sim 10^8) = exp(-K_{p=1}),$$

then the function

$$h_p = \frac{exp(-K_p)}{exp(-K_{p=1})}/$$

is the n:th power of p for a vacuum extremal to which n CP_2 type extremals are glued. This is just the p-adic norm p^n! If h_p were p^n-valued in the general case it would be a p-adic pseudo constant and rather tame as a fractal. Very probably, this is not true in the general case and the p-adic norm of the p-adic counterpart of h_p in the canonical identification

$$\begin{aligned} &N_p \equiv |Id(h_p)|_p \ , \\ &Id(\sum x_n p^n) = \sum_n x_n p^{-n} \ . \end{aligned}$$

depending on the most significant pinary digit of h_p only, is a good candidate for a p-adically ultra-metric height function having also a correct normalization. In any case, it seems that the number of virtual CP_2 type extremals (gravitons!) glued to an absolute minimum space-time surface $X^4(X^3)$ could define the height function. p-Adicity would emerge naturally and would have a direct physical meaning. Of course, this identification works for $n \geq 0$ only: the physical interpretation of the p-adic norm in $n < 0$ case is open.

A possible interpretation in terms of virtual graviton emission suggests the interpretation of the factor $\frac{R^2}{G} = exp(-K_{p=1})$ as a Gaussian determinant $\sqrt{det_G}$ associated with the integration over the zero modes around the maximum. The definition of Gaussian determinant in the real context is problematic and p-adicization plus adelic decomposition of the functional integral might provide a precise definition of $\sqrt{det_G}$. The divergence of the Gaussian determinant in the real context would lead to the vanishing of the gravitational constant. This picture is in accordance with the assumption that gravitational constant does not appear in quantum TGD as a fundamental constant and that the curvature scalar term in the low energy effective action essentially results from radiative corrections and hence derives from the logarithm of det_G.

5.3 p-Adic numbers and quantum criticality

TGD Universe is quantum critical in the sense that the value of Kähler coupling constant is completely analogous to critical temperature. Therefore the obvious question is how p-adicity might relate to quantum criticality.

5.3.1 Connection with quantum criticality

p-Adicization of the reduced configuration space relates in an interesting manner to quantum criticality. At quantum criticality the number of the absolute minima of Kähler action for a surface Y^3 belonging to light cone boundary measures the cognitive resources of this surface and of its diffeomorphs. N_d is assumed to behave as $N_d \sim exp(-K_{cr})$, where Kähler function is evaluated for the critical value α_{cr} of the Kähler coupling strength. α_{cr} is like Hagedorn temperature appearing in the thermodynamics of strings. Above α_{cr} the theory might not be mathematically well defined since (at least real) the sum over the configuration space integrals associated with the maxima of Kähler function would diverge exponentially at the limit when the value of Kähler function increases. In string thermodynamics this corresponds to the growth of number $g(E)$ of the states of given energy more rapidly than the inverse of the Boltzmann factor $exp(-E/T_H)$. Below α_{cr} the theory is certainly well defined but in TGD framework the cognitive resources of the Universe would not be maximal since vacuum functional would differ significantly from zero for very few space-time surfaces only. At quantum criticality the situation is optimal but it is not clear whether the real theory makes sense at quantum criticality: at least in string thermodynamics the partition function diverges also at Hagedorn temperature.

The cognitive resources of p-adic space-time sheet are measured by the entropy type quantity $log(N_d)/log(2)$ having lower bound $log(p)/log(2)$ bits for the 3-surfaces allowed by the vacuum functional. For instance, the maximal cognitive resources of electronic space-time sheet ($M_{127} = 2^{127} - 1$) would be 127 bits. In TGD one must allow even infinite primes and for these cognitive resources can be literally infinite.

5.3.2 Geometric description of the critical phenomena?

The idea that critical systems might have a geometric description is not new. There is a lot of evidence that simple, purely geometric lattice models based on the bond concept reproduce same critical exponents as the thermal models [20]. The probability for a bond to exist corresponds to temperature in these models. For example, in a bond percolation model it is possible to relate the critical exponents to various fractal dimensions. This provides a nice manner to reduce the problem of predicting critical temperature to that of predicting the critical probability for the bond. This problem is local and once the temperature dependence of the bond probability and critical bond probability are known one can calculate the critical temperature.

What is new that in TGD approach the concept of bond ceases to be a phenomenological concept related to the simple modelling of the critical systems. TGD predicts that the boundaries of 3-surfaces can have arbitrarily large sizes. Furthermore, the formation of the join along boundaries bonds connecting the boundaries of two disjoint 3-surfaces seems to provide the basic mechanism for the formation of macroscopic quantum systems with long range correlations. This means that phase transitions should basically correspond to changes in the connectedness of the boundary of the 3-space. The description of the super fluidity, super conductivity and Quantum Hall effect based on the join along boundaries bond concept is suggested in [D7, E9] and also other phase transitions might be describable in the same manner. In hadronic length scale join along boundaries bonds correspond to color flux tubes connecting valence quarks. In nuclear length scale the short range part of the nuclear force corresponds to the formation of join along boundaries bonds between nucleons.

p-Adic approach suggests a concrete description for the phase transition changing the connectedness of the 3-surface. Disjoint 3-surfaces are labelled by p-adic numbers, whose p-adic expansion does not contain powers p^n with $n > N$, where N is some finite integer: the larger the value of N the larger the degree of disjointness. This means that phase transitions (say evaporation or condensation) changing the connectedness of the 3-surface should correspond to transitions changing the value of N. In evaporation process N increases and in condensation process N decreases. Also catastrophic processes like the breaking of a solid object to pieces might correspond to increase in N. Typical self organization processes such as biological growth and healing might correspond to a gradual decrease of N.

Fractal like configurations with a discrete scale invariance are known to play important role in the description of the critical phenomena: they are the most probable configurations at the critical point. The idea that fractal corresponds to a fixed point of a discrete scaling transformation, is in accordance with the definition of the fractals as fixed points for a set of affine transformations acting on subsets of some metric space [21]. A natural candidate for the discrete scaling transformation is the transformation of the 4-surface induced by the multiplication of the p-adic argument Z of H-coordinate $h(Z)$ by a power of p: $Z \to p^n Z$. A tempting idea is that most probable 3-spaces indeed are invariant under these scalings. This even suggests that something, which might be called "Mandelbrot cosmology", might provide a description of the Universe in all length scales as a 4-dimensional analog of Mandelbrot set. The breaking of the discrete scaling invariance is bound to occur, when one considers finite subsystem instead of the whole Universe. p-Adic cutoff might provide an elegant description for the breaking of the exact scaling invariance: 3-surface in question depends on finite number of the pinary digits of Z only.

5.3.3 Initial value sensitivity and p-adic differentiability

Initial value sensitivity is one of the basic properties of the critical systems and implies unpredictability in practice. p-Adic differentiability seems to be related to this property in a very general manner. Consider a configuration of an initial value sensitive system, which can possess very high dimension. For definiteness, assume that the dynamics is described by some differential equations, which can be reduced to equations of first order for the configuration space coordinates X (we do not bother to write indices):

$$\frac{dX}{dt} = J(X) \ . \tag{5.3.1}$$

Space-time coordinate is a p-adic number one can assume that time coordinate is a p-adic number, too.

The purely p-adic feature of this differential equation follows from the fact that any function depending on a finite number of pinary digits of a p-adic number possesses a vanishing p-adic derivative! This implies that the integration constants are not just ordinary constants but functions of the p-adic number t depending on finite number of pinary digits of t! Obviously this implies classical non-determinism in long time scales! One can construct solutions of the differential equation in the form $X(t) = X_0(t) + X_1(t)$, where $X_0(t)$ depends on a finite number of pinary digits of the p-adic time t and equations reduce to

$$\frac{dX_1}{dt} = J(X_0 + X_1) \ . \tag{5.3.2}$$

Of course, one must be careful in defining what "finite number of pinary digits" means, when p-adic cutoff is actually present. The simplest integration constants depend on the p-adic norm of t (or on the lowest pinary digit of t) only.

The result is in accordance with the so called Slaving Principle [18]. One can think that the dynamics in long time scales (low pinary digits of p-adic number t) is given by the integration constants having arbitrary dependence on these pinary digits and the dynamics in short length scales is determined by the differential equations in the "background" given by these time dependent integration constants.

Initial value sensitivity implies effectively non-deterministic behavior and p-adic numbers perhaps provide a possibility to describe it properly. The properties of the Kähler function suggests that the classical non-determinism might be in fact actual. The point is that the classical space time surface associated with a given 3-surface need not be unique. This surface is determined as an absolute minimum of the so called Kähler action and Kähler action possesses enormous vacuum degeneracy [D1]: the most general vacuum extremal has 2- dimensional CP_2 projection, which is so called Lagrange manifold possessing a vanishing induced Kähler form. Symplectic transformations and $Diff(M^4)$ act as exact dynamical symmetries of the vacuum extremals and $Diff(M^4)$ contains p-adically analytic transformations of M^4 as subgroup. It might well happen that those absolute minima, which are obtainable as small deformations of the vacuum extremals inherit the characteristic degeneracy of the vacuum extremals.

The classical macroscopic non-determinism might be essential to the possibility of the quantum measurements. In TGD the state function reduction is described as 'jump between histories' that is two deterministic time developments [H1]. In quantum measurement microscopic and macroscopic system are strongly correlated and microscopic transition induces a phase transition like phenomenon in a macroscopic critical system. The general belief is that quantum effects become unimportant in macroscopic systems. The situation need not be this if macroscopic system is critical, or even non-deterministic.

In the TGD inspired theory of 'thinking systems', conscious thoughts correspond to quantum jumps selecting one of the possible time developments in the quantum superposition of several quantum average effective space-time times allowed by the non-determinism. p-Adic pseudo constants could provide a mathematical description for this non-determinism. These 'cognitive' quantum jumps are certainly involved with a realistic description of a quantum measurement modelling also the presence of the observer quantum mechanically.

In turns out that quantum non-determinism, classical non-determinism of Kähler action and p-adic non-determinism are very closely related in quantum TGD: one could even speak of a holy trinity of non-determinisms. Quantum non-determinism corresponds closely to the classical non-determinism of Kähler action: quantum jumps select between various branches of the branches of multifurcations of classical space-time surface. The p-adic counterparts of these branches are in turn obtained by varying pseudo constants in the solution of the p-adic Euler-Lagrange equations for the Kähler action: this requirement in fact makes it possible to assign unique p-adic prime to a given, sufficiently small space-time region.

5.3.4 There are very many p-adic critical orbits

An interesting connection between the p-adicity and initial value sensitive systems is related to the possibility to replace also the configuration space (possibly infinite dimensional) with an algebraic extension of the p-adic numbers. The underlying motivation is the need to get a proper mathematical description of the finite accuracy for the observables and p-adic cutoff provides this description.

This in turn suggests Universality in some aspects of the dynamical behavior. The dynamical equations $dX/dt = J(X)$ define a flow that is a diffeomorphism $X \to F(X,t)$ of configuration space. This flow contains as integration constants arbitrary functions of the p-adic time coordinate t depending on a finite number of pinary digits of t so that classical non-determinism is present. By p-adic conformal invariance this diffeomorphism ought to be p-adically analytic map that is representable as a power series of the algebraically extended p-adic numbers x and t.

The p-adic analyticity of the dynamic diffeomorphism gives strong constraints on the properties of the dynamic map. A particularly interesting map is in this respect Poincare map. One can ask several interesting questions. How does the Universal behavior of one- dimensional and 2-dimensional analytic iterated maps generalize to the p-adic case? What do attractors look like? What are the counterparts of Julia set and Mandelbrot set? What about routes to chaos? Could p-adic hypothesis provide deeper explanation for the fact that period doubling seems to be a rather general mechanism for the transition to turbulence. It might be possible to answer these questions since p-adic analyticity is very strong constraint on the behavior of the maps.

Already the study of the simplest p-adic complex maps reveal some surprises. The simplest map to study is the map $Z \rightarrow Z^n$ for any extension of p-adic numbers (dimension is arbitrary!). The repeller consists of the points p-adic norm equal to one. Due to the roughness of the p-adic topology, the real counterpart of the repeller is of same dimension as the configuration space itself so that the critical orbits form a set with a non-vanishing measure! For example, in the 2-dimensional case and for the 2-adic extension, the set of the critical orbits corresponds in the real plane to a square $(1/2, 1] \times (1/2, 1]$.

How do the small deformations of $Z \rightarrow Z^n$ of form $Z \rightarrow Z^n + \epsilon Z^m$ affect the set of the critical orbits? If the norm of the parameter ϵ is sufficiently small, the previous repeller belongs to the repeller also now. Also new points can appear in repeller. These considerations suggest that the repellers/attractors of the p-adically analytic maps have rather simple structure as compared to their real and complex counter parts. An interesting possibility is that in general case these sets are fractal like objects resembling the fractals associated with p-adic order parameters.

The fact that set of critical orbits is n-dimensional rather than $(n-1)$ or lower-dimensional in the p-adic case suggests an interesting physical interpretation in accordance with the general idea that p-adic topology corresponds to criticality. In ordinary situation these orbits are not very interesting because a small deformation spoils their criticality. In p-adic case the situation is different since the critical orbits are meta-stable and their are very many of them. In TGD one can even identify good candidates for the set of of these meta-stable critical orbits as small deformations of the vacuum extremals of the Kähler action. Needless to emphasize, this vacuum degeneracy is a phenomenon not encountered in the standard field theories.

5.4 p-Adic Slaving Principle and elementary particle mass scales

The understanding of the elementary particle mass scales is a fundamental problem in the unified field theories. The attempts to understand the generation of the mass scales dynamically have not been successful. The basic problem is the fine tuning difficulty: the predicted mass scale hierarchy is not stable under the small changes of the model parameters. A possible explanation for the failure is that the fundamental mass scales are really fundamental and therefore cannot depend on the details of the dynamical model.

Criticality is known to imply Universality and criticality indeed is the fundamental property of Kähler action. Therefore the derivation of the elementary particle length scale(s) should be based on a proper formulation of the criticality concept. p-Adic numbers indeed provide a promising tool in this respect and the following arguments show that it is possible not only to understand some general elementary particle length scale but leptonic, hadronic and intermediate gauge boson length scales plus a small number of shorter length scales in terms of primes near prime powers of two. The most important length scales correspond to Mersenne primes: there are only sixteen Mersenne primes below electron length scale and the remaining Mersenne primes correspond to super astronomical length scales.

What is nice that the p-adic hypothesis makes possible to express these length scales as square roots of Mersenne primes and possibly Fermat primes, that is prime numbers of type $p = 2^m \pm 1$. What is amusing is that Mersenne primes are closely related to the so called Perfect Numbers $n = 2^{m-1}(2^m - 1)$ representable not only as a product of their prime factors but also as a sum of their proper divisors. The ancient number mystics believed that this property makes these numbers very exceptional in the World Order!

5.4.1 p-Adic length scale hypothesis

p-Adic length scale hypothesis has served as a basic hypothesis of p-adic TGD for several years. This hypothesis states that the scales $L_p = \sqrt{p}l$, $l = 1.376 \cdot 10^4 \sqrt{G}$ are fundamental length scale at p-adic condensate level p. The original interpretation of the hypothesis was following:

1. Above the length scale L_p p-adicity sets on and effective course grained space-time topology is p-adic rather than ordinary real topology.

2. The length scale L_p serves as a p-adic length scale cutoff for the field theory description of particles. This means that space-time begins to look like Minkowski space so that quantum field theory $M^4 \to CP_2$ becomes a realistic approximation. Below this length scale string like objects and other particle like 3-surfaces dominate.

3. It is un-natural to assume that just single p-adic field would be chosen from the infinite number of possibilities. Rather, there is an infinite number of cutoff length scales. To each prime p there corresponds a cutoff length scale L_p above which p-adic quantum field theory $M^4 \to CP_2$ makes sense and one has a hierarchy of p-adic quantum field theories. These different p-adic field theories correspond to different hierarchically levels possibly present in the topological condensate. Hierarchical ordering $< p_1 < p_2 < ...$ means that only the surface $p_1 < p_2$ can condense on the surface p_2. The condensed surface can in practice be regarded as a point like particle at level p_2 described by the p-adic conformal field theory below length scale L_{p_2}.

The work with p-adic QFT has however demonstrated that the hypothesis a) and b) are probably wrong and the following interpretation is closer to the truth.

1. The length scale $L_p = \sqrt{p}l$ defines an *infrared* cutoff rather than ultraviolet cutoff for a p-adic quantum field theory formulated in terms of quarks and leptons and gauge bosons. For instance, for hadrons this length scale is of order hadron size and L_p defines UV cutoff for possibly existing field theory describing hadrons as basic objects. Above L_p real topology effectively replaces the p-adic one (real continuity implies p-adic continuity) and if length scale resolution L_p is used real physics is excellent approximation.

2. p-Adic QFT is free of UV divergences with any UV cutoff and there is no need to assume that p-adicity fails below some length scale. Rather, p-adicity is completely general property of the effective quantum average space-time defined by the Quantum TGD, which is based on the real number field. The concept of the effective space-time, or topological condensate, is in turn necessary for the formulation of field theory limit of TGD. The analogy of Quantum TGD with spin glass phase gives strong support for the p-adic topological condensate consisting of p-adic regions with different p glued together along their boundaries.

p-Adic topologies form a hierarchy of increasingly coarser topologies. The p-adic norm $N(x_p)$ defines a function of a real argument via the canonical identification of the nonnegative real numbers and p-adic numbers. The p-adic norm is same as ordinary real norm for $x = p^k$ and is constant at each interval $[p^k, p^{k+1})$. This means that

1. p-adic topologies are coarser than real topologies so that the functions, which are continuous in the p-adic topology need not be continuous in the real topology.

2. p-adic topologies are ordered: the larger the value of p, the coarser the topology in the long lenght scales. In short length scales the situation is just the opposite.

5.4.2 Slaving Principle and p-adic length scale hypothesis

Slaving Principle states that there exists a hierarchy of dynamics with increasing characteristic length (time) scales and the dynamical variables of a given length scale obey dynamics, where the dynamical variables of the longer length (time) scale serve as "masters" that is effectively as external parameters or integration constants. The dynamics of the "slave" corresponds to a rapid adaptation to the conditions posed by the "master".

p-Adic length scale hierarchy suggests a quantitive realization of this philosophy.

1. By the previous considerations there is an infinite hierarchy of length scales L_p such that the space-time surfaces below the length scale L_p look like Minkowski space and p-adic quantum field theory $M^4 \to CP_2$ makes sense below the length scale L_p. These length scales are associated with the different condensation levels present in the topological condensate and define the typical

size of the p-adic surface in absence of the collective quantum effects, which should correspond to the formation of the join along boundaries bonds between objects with size of order L_p. The reason why the typical size is just this is that the imbedding of the p-adic coordinate space into space H has strongest discontinuities in the real topology, when coordinate values correspond to powers of p so that a typical imbedding decomposes into separate pieces with size of order L_p. Of course, this kind of discontinuity is possible for all powers of p but is not observable in shorter length scales for the physically most interesting values of p due to the extreme smallness of the corresponding length scales.

2. The lowest level of the hierarchy corresponds to 2-adic dynamics and this field theory makes sense below the cutoff length scale $L_2 = \sqrt{2}l$ defining the typical size for a 2-adic surface. Solutions of the 2-adic field equations are non-deterministic due to the possibility of the integration constants depending on finite number of binary digits. The dependence on a finite number of positive bits of the real coordinates only means that they are genuine constants below some length scale $L_2(lower) < L_2$, which in principle depends on the state of the system.

3. 2-adic pseudo-constants are analogous to external parameters and should be determined by the dynamics associated with the longer length and time scales. The properties of the p-adic numbers suggest that these constants in turn are p-adically differentiable functions of their argument with some value of $p_1 > 2$ determined by the p_1-adic dynamics describing the interaction between $p = 2$ surface condensed on $p = p_1$ level and $p = p_1$ background surface. The p_1-adic integration constants associated with these functions are actual constants above the length scale $L_{p_1}(lower) \geq L_2(lower)$ but also these in principle depend on a finite number of pinary digits and their values are determined by the interaction of p_1 level with the next level in the condensation hierarchy.

4. At the next level p_1 one encounters p_1-adic dynamics and new p-adic integration constants. The net effect is that one obtains a hierarchy of p-adic numbers $2 < p_1 < p_2 < ...$ in correspondence with the length and time scales $L_2 < L_{p_1} < L_{p_2} < ...$: the higher the boss the larger the p. In TGD it is very tempting to interpret the various levels of the slaving hierarchy as the levels of the topological condensate so that the surfaces at level p are condensed on the surfaces of level $p_1 > p$ (see Fig. 5.4.2). Not all values of p need be present in the hierarchy and it might well happen that certain values of p are in an exceptional position physically.

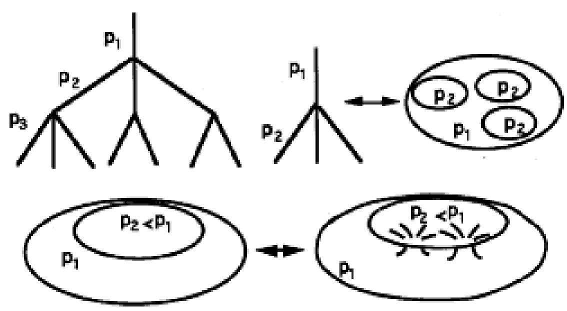

Figure 5.1: Two-dimensional visualization of topological condensate concept

5.4.3 Primes near powers of two and Slaving Hierarchy: Mersenne primes

All values of p are in principle present in the Slaving Hierarchy but the assumption that all values of p are equally important physically is not realistic. The point is that the number $N(n)$ of primes smaller than n behaves as $N(n) \sim n/ln(n)$ and there are just too many prime numbers. For example, for $n = 10^{38}$ there are about one prime number per 87 natural numbers!

A natural looking assumption is that a new physically important length scale emerges, when a fixed number of powers of 2 combine to form a new length scale. The reason is that a given interval $[2^k, 2^{k+1})$ forms an independent fractal unit (for the simplest fractals these intervals are related by a similarity, see figures in [E4] and it is therefore unnatural to cut this unit into pieces as would happen if p were far from a power of two. This breaking would indeed happen since p-adically differentiable functions have sharp gradients at points p^k. This non-breaking or "synergy" is reached provided the allowed primes are as close as possible to powers of 2: $p \simeq 2^m$. It should be noticed that this condition also guarantees that the frequency peaks associated with various powers of p in good approximation correspond to period doubling frequencies characteristic to fractal and chaotic systems.

The best approximation achievable corresponds to Fermat and Mersenne primes

$$p = 2^m \pm 1 \ . \tag{5.4.1}$$

It can be shown that for Fermat primes (+) the condition $m = 2^k$ must be satisfied and for Mersenne primes (-) m must be itself prime.

How abundant are the prime numbers of type $p = 2^m \pm 1$? The great surprise was that there are very few numbers of this kind!

1. The primes of type $2^m + 1$, Fermat primes, are very rare: only 5 numbers in the range $1 < n < 2^{2^{21}} \simeq 10^{10^6}$(!) [7] and there are good arguments suggesting that the number of the Fermat primes is finite! The known Fermat primes correspond to $m = 2^k$, with $k = 0, 1, 2, 3, 4$. The corresponding primes are $p = 3, 5, 17, 257, 65537$. Note that the lowest Fermat prime 3 is also a Mersenne prime. It will be later found that p-adic conformal invariance is in TGD possible for primes p satisfying the condition $p \ mod \ 4 = 3$ and this condition is not satisfied by Fermat primes $F > 3$.

2. The primes of form $2^m - 1$, Mersenne primes, are also there as follows from the requirement that m is prime. The list of allowed exponents of m consists of the following numbers:

$$2, 3, 5, 7, 13, 17, 19, 31, 61, 89, 107, 127, 521, ...$$

.

One can make two observations about these numbers:

1. $m = 127$ corresponds to the number 10^{38} fundamental to Physics. The square root of this number gives the ratio of the proton length scale to Planck length scale. This suggests the possibility that fundamental physical length scales are given by square roots of Mersenne and possibly Fermat primes using some length scale of order Planck scale as a unit.

2. $m = 61$ corresponds to the number of order 10^{19}: this in turn allows the possibility that fundamental physical length scales are linearly related to Fermat and Mersenne primes. This alternative however turns out to be not the correct one.

These observations lead to following scenario for the fundamental length scales:

1. The p-adic length scale L_p, below which p-adic quantum field theory approximation makes sense, is proportional to the square root of p and these length scales are p-adically the most interesting length scales:

$$\begin{aligned} L_p &= \sqrt{p} l \ , \\ l &\sim k \cdot 10^4 \sqrt{G} \ , \\ k &\simeq 1.376 \ . \end{aligned} \tag{5.4.0}$$

   Only quite recently the physical interpretation of the length scale l was found. Contrary to the original expectations, CP_2 is not of order Planck length but of order l. At this length scale Euclidian regions of space-time, in particular CP_2 type extremals representing elementary particles, become important. Above this length scale a field theory in Minkowski space is expected to be a good approximation to quantum physics.

2. Physically the most interesting length scales correspond to the p-adic cutoff length scales L_p associated with the Mersenne primes M_n.

3. The fact that l is of the same order of magnitude as the length scale at which the coupling constants of the standard model become approximately equal, is not probably an accident. Below l it is not anymore sensible to speak about the topological condensation of CP_2 type extremals since CP_2 type extremals themselves have size of order l. Hence the symmetry breaking effects caused by the topological condensation cannot be present in the string model type desciption applying below l.

The predictions are as follows:

1. $m = 127$ corresponds to electron Compton length.

2. $m = 107$ corresponds to proton Compton length L_P.

3. $m = 89$ corresponds to length scale of order $1/256$ times proton Compton length and is identifiable approximately as $L_W/2\sqrt{2}$, where L_W is intermediate boson length scale of about $L_P/100$.

4. $m = 61$ corresponds to length scale of the order of $10^{-6}L_P$ is not reachable by the present day accelerators.

5. $m = 521$ corresponds to a completely super-astronomical length scale of order 10^{27} light years!

It seems that the proposed scenario might have catched something essential in the problem of the elementary particle mass scales: it predicts correctly 3 fundamental length scales associated with leptons, hadrons and intermediate gauge bosons from number theory; there is extremely large gap in the length scale hierarchy after electron Compton length and new shorter length scales exist but unfortunately they are outside the reach of the present day experiments. The calculations of the third part of the book show that not only the mass scales can be understood but also particle masses can be predicted with errors below one per cent using the length scale hypothesis combined with the p-adic Super Virasoro invariance and p-adic thermodynamics.

5.4.4 Length scales defined by prime powers of two and Finite Fields

Above M_{127} there is an extremely large gap for Mersenne primes and this suggests that there must be also other physically important primes. Certainly all primes near powers of two define physically interesting length scales by 2-adic fractality but there are two many of them. The first thing, which comes into mind is to consider the set of primes near prime powers of two containing as special case Mersenne primes. The following argument is one of the many arguments in favor of these length scales developed during last years.

TGD Universe is critical at quantum level and criticality is related closely to the scaling invariance. This suggests that unitary irreducible representations of p-adic scalings $x \rightarrow p^m x$, $m \in Z$ should play central role in quantum theory. Unitarity requires that scalings are represented by a multiplication with phase factor and the reduction to a representation of a finite cyclic group Z_m requires that scalings $x \rightarrow p^m x$, m some integer, act trivially. In ordinary complex case the representations in question correspond to the phase factors $\Psi_k(x) = |x|^{(\frac{ik2\pi}{ln(p)})} = exp(iln(|x|)\frac{k2\pi}{ln(p)})$, $k \in Z$ and the reduction to a representation of Z_m is also possible but there is no good reason for restricting the consideration to discrete scalings.

1. The Schrödinger amplitudes in question are p-adic counterparts of the ordinary complex functions $\Psi_k(x) = exp(iln(|x|)k\frac{ik2\pi}{ln(p)})$, $k \in Z$. They have a unit p-adic norm, they are analogous to plane waves, they depend on p-adic norm only and satisfy the scaling invariance condition

$$\begin{aligned}
\Psi_k(p^m x|p \rightarrow p_1) &= \Psi_k(x|p \rightarrow p_1) \ , \\
\Psi_k(x|p \rightarrow p_1) &= \Psi_k(|x|_p|p \rightarrow p_1) \ , \\
|\Psi_k(x|p \rightarrow p_1)|_p &= 1 \ ,
\end{aligned} \tag{5.4.-1}$$

which guarantees that these functions are effectively functions on the set of the p-adic numbers with cutoff performed in m:th power.

2. The solution to the conditions is suggested by the analogy with the real case:

$$\begin{aligned}\Psi_k(x|p \rightarrow p_1) &= exp(i\frac{kn(x)2\pi}{m}) \ , \\ n(x) &= ln_p(N(x)) \in N \ , \end{aligned} \quad (5.4.-1)$$

where $n(x)$ is integer (the exponent of the lowest power of the p-adic number) and $k = 0, 1, ..., m-1$ is integer. The existence of the functions is however not obvious. It will be shortly found that the functions in question exist in $p > 2$-adic for all m relatively prime with respect to p but exist for all odd m and $m = 2$ in the 2-adic case.

3. If m is prime (!) the functions $K = \Psi_k$ form a finite field $G(m, 1) = Z_m$ with respect to the p-adic sum defined as the p-adic product of the Schrödinger amplitudes

$$K + L = \Psi_{k+l} = \Psi_k \Psi_l \ , \quad (5.4.0)$$

and multiplication defined as

$$KL = \Psi_{kl} \ . \quad (5.4.1)$$

Hence, if the proposed Schödinger amplitudes possessing definite scaling invariance properties are physically important, then the length scales defined by the prime powers of two must be physically special since Schrödinger amplitudes or equivalently, the p-adic scaling momenta k labeling them, have a natural finite field structure. By the Slaving Hierarchy Hypothesis, also the p-adic length scales near prime powers of two (and perhaps of prime $p > 2$, too) are therefore physically interesting. p-Adic scalings correspond to p-adic translations if p-adic coordinates correspond to exponentials of the ordinary linear coordinates so that translations are represented by scalings.

The generalized plane waves exist p-adically if nontrivial $N = p$:th root of the quantity $exp(i2\pi) = 1$ exists.

1. $N = 2$:th roots of 1 exist trivially for all values of p.

2. In 2-adic case the roots exist always for odd values of N and especially so for prime values of N: the trick is to write $1^{1/N} = -(-1)^{1/N} = -(1-2)^{1/N}$ and use the Taylor series

$$\begin{aligned}(1+x)^{1/N} &= \sum_n \frac{A_n}{n!} x^n \ , \\ A_n &= \prod_{k=0}^{n-1} (\frac{1}{N} - k)(-1)^n \ , \\ x &= -2 \ . \end{aligned} \quad (5.4.0)$$

to show the existence of one root different from the trivial root. In 2-adic case the powers of $x = 2$ converge to zero rapidly and compensate the powers of 2 coming from $n!$ in the denominator. The coefficients A_n possess 2-adic norm not larger than 1.

3. For $p > 2$ nontrivial $N = p$:th roots do not allow representation as plane waves for the simple reason that only the trivial p:th root of 1 exists p-adically. Roots of unity must have p-adic norm equal to one and by writing the condition modulo p one obtains a condition $a^N\ mod\ p = 1$ in $G(p,1)$. The roots of unity in $G(p,1)$ satisfy always $a^{p-1} = 1$ and the possible orders N are factors of $p-1$. In particular, prime roots with $p_1 > p-1$ are not possible. The number of prime factors is typically quite small. For instance, for primes of order $p = 2^{127}$ the number of prime roots is of order 6.

The conclusion is that for $p > 2$ only those finite fields $G(p_1,1)$ for which p_1 is factor of $p-1$ are realizable as representation of phase factors whereas for $p = 2$ all fields $G(p_1,1)$ allow this kind of representation. Therefore $p = 2$-adic numbers are clearly exceptional. In the p-adic case the functions $\Psi_p(x,|p \rightarrow p_1)$ give irreducible representations for the group of p-adic scalings $x \rightarrow p^m x$, $m \in Z$ and the integers k can be regarded as scaling momenta. This suggests that these functions should play the role of the ordinary momentum eigenstates in the quantum theory of fractal structures. The result motivates the hypothesis that prime powers of two and also of p define physically especially interesting p-adic length scales: this hypothesis will be of utmost importance in future applications of TGD.

The ordinary (number theoretic) p-adic plane waves associated with the translations can be constructed as functions $f_k(x) = a^{kx}$, $k = 0,...,n$, $a^n = 1$. For $p > 2$ these plane waves are periodic with period n, which is factor of $p-1$ so that wavelengths correspond to factors of $p-1$ and generate a finite number of physically favored length scales. The p-adic plane waves with the momenta $k = 0,...,p-2$ form finite field $G(p,1)$, when p-adic arithmetics is replaced with the modulo p arithmetics, that is to accuracy $O(p)$ (note that the definition of the arithmetic operations is *not* the same as in the previous case). The square roots of the p-adic plane waves are also well defined

The important property of the p-adic plane waves is that they are pseudo constants: this property played profound role in the earlier formulations of the p-adic QFT limit. It took a considerable time to discover that the counterparts of the ordinary real plane waves providing representations for translation group exists and satisfy the appropriate orthogonality relations. Therefore number theoretic plane waves do not play so essential role in p-adic QFT as was originally believed.

5.5 CP_2 type extremals

CP_2 type extremals are perhaps the most important vacuum extremals of the Kähler action. The reason is that they are vacuum extremals with a negative and finite Kähler action and hence favored by the absolute minimization of the Kähler action. On the other hand, maximization of Kähler function does not favor CP_2 type extremals because the virtual CP_2 type extremals are exponentially suppressed. CP_2 type extremals seem to play the same role as black holes possess in General Relativity. p-Adic thermodynamics, leading to excellent predictions for the masses of the elementary particles, predicts that elementary particles should possess p-adic entropy and Hawking-Bekenstein law for the entropy generalizes.

In GRT based cosmology black holes populate the most probable Universe, which is of course a problem: in TGD black holes are replaced by elementary particles. The second law of thermodynamics requires that the very early Universe should have a low entropy and hence that black holes should populate the recent day Universe: in TGD the very early cosmology is dominated by cosmic strings, which is a low entropy state. By the absolute minimization of the Kähler action, most cosmic strings however decay to elementary particles and produce p-adic entropy. To get a grasp of the orders of magnitude, it is good to notice that electron, which corresponds to $p = M_{127} = 2^{127}-1$, has entropy equal to 127 bits.

The basic observation is that the M^4_+ projection of the CP_2 type extremal corresponds to a light like random curve and the quantization of this motion leads to Virasoro algebra and Kac Moody algebra characterizing quantized transversal motion superposed with the cm motion. CP_2 type extremals allow covariantly constant right handed neutrino spinors as solutions of the Dirac equation for the induced spinors in the interior and this leads to $N = 1$ super symmetry and a generalization of the Virasoro invariance to Super Virasoro invariance.

The previous p-adic mass calculations were based on this picture but it turned out that the Super Virasoro invariance and related Kac Moody symmetries generalize to the level of the configuration space geometry and in an extended form provide the basic symmetries of the quantum TGD. Although the quantization of the zitterbewegung motion of the CP_2 type extremals is a phenomenological procedure only, and is not needed in the fundamental theory, it deserves to be described because of its key role in the development of quantum TGD. There were however some strange features involved: for instance, $N = 1$ super-symmetry generated by righthanded neutrino was exact only for minimal surfaces.

The realization that super-symmetry requires modified Dirac action led to the final breakthrough. CP_2 type extremals allow quaternion-conformal symmetries and the super-generators associated with quark and lepton numbers are non-vanishing despite the fact that vacuum extremals are in question. Even Super-Kac-Moody generators are non-vanishing. Even more, CP_2 type extremals cease to be vacua for Dirac action. Especially beautiful feature of CP_2 type extremals is that they can describe also massive states and zitterbewegung is the geometric correlate of massivation.

5.5.1 Zitterbewegung motion classically

The M^4_+ projection of a CP_2 type extremal is a random light like curve. Also Dirac equation, which gives also classically rise to a motion with light velocity and this motivates the term 'zitterbewebung'. Zitterbewegung occurs at the light of velocity and any given 3-velocity gives rise to the solution of light likeness condition if one fixes the time component of velocity to be

$$\frac{dm^0}{d\tau} = \sqrt{m_{ij}\frac{dm^i}{d\tau}\frac{dm^j}{d\tau}} \ . \tag{5.5.0}$$

The vanishing of CP_2 part of the second fundamental form requires that velocity and acceleration are orthogonal:

$$m_{kl}\frac{dm^k}{d\tau}\frac{d^2m^l}{d\tau^2} = 0 \ . \tag{5.5.1}$$

This condition is identically satisfied.

A very general solution to the conditions is provided by the equations

$$\frac{d^2m^k}{d\tau^2} = F^{kl}\frac{dm^l}{d\tau} \ , \tag{5.5.2}$$

describing the motion the of massless charged particle in external Maxwell field.

5.5.2 Basic properties of CP_2 type extremals

CP_2 type extremal has the following explicit representation

$$m^k = f^k(u(s^k)) \ , \quad m_{kl}\frac{df^k}{du}\frac{df^l}{du} = 0 \ . \tag{5.5.3}$$

The function $u(s^k)$ is an arbitrary function of CP_2 coordinates and serves effectively as a time parameter in CP_2 defining a slicing of CP_2 to time=constant sections. The functions f^k are arbitrary apart from the restriction coming from the light likeness. When one expands the functions f^k to Fourier series with respect to the parameter u, light likeness conditions reduce to classical Virasoro conditions $L_n = 0$.

It is possible to write the expression for m^k in a physically more transparent form by separating the center of mass motion and by introducing p-adic length scale L_p as a normalization factor.

$$\frac{m^k}{L_p} = m^k_0 + p^k_0 u + \sum_n \frac{1}{\sqrt{n}} a^k_n exp(i2\pi nu) + c.c. \ . \tag{5.5.4}$$

The first term corresponds to the center of mass term responsible for rectilinear motion along geodesic line and second term corresponds to the zitterbewegung motion. p^k serves as an effective classical momentum which can be normalized as $p_k p^k = \epsilon$, $\epsilon = \pm 1$ or $\epsilon = 0$. What has significance is whether

p^k is time like, light like, or space like. Conformal invariance corresponds to the freedom to replace u with a new 'time parameter' $f(u)$.

The physically most natural representation of u is as a function $f(U)$ of the fractional volume U for a 4-dimensional sub-manifold of CP_2 spanned by the 3-surfaces $X^3(U=0)$ and $X^3(U)$:

$$u = f(U) \ , \quad U = \frac{V(s^k)}{V(CP_2)} = \frac{S_K(u)}{S_K(CP_2)} \ . \tag{5.5.5}$$

The range of the values for U is bounded from above: $U \leq V_{max}/V(CP_2)$ and the value $U = 1$ is possible only if CP_2 type extremal begins and ends as a point. U represents also Kähler action using the value of the Kähler action for CP_2 as a unit.

The requirement that CP_2 type extremal extends over an infinite time and spatial scale implies the requirement

$$f(U_{max}) = \infty \ . \tag{5.5.6}$$

For $f(U_{max}) < \infty$ CP_2 type extremal can exist only in a finite temporal and spatial interval for finite values of 'momentum ' components p^k. This suggest a precise geometric distinction between real and virtual particles: virtual particles correspond to the functions $f(U_{max}) < \infty$ in contrast to the incoming and outgoing particles for which one has $f(U_{max}) = \infty$. This hypothesis, although it looks like an ad hoc assumption, is at least worth of studying.

The mere requirement that virtual CP_2 type extremal extends over a temporal or spatial distance of order $L > L_p$ implies that for $L < L_p$ the value of U is smaller than one. Kähler action, which is given by

$$S_K(X^4) = U \times S_K(CP_2) \ , \tag{5.5.7}$$

remains small for distances much smaller than L. For $f(U_{max}) = \infty$ this is even more true. This has an important implication: below a certain length scale the exponential of the Kähler action associated with the internal line of a Feynman diagram does not give rise to a suppression factor whereas above some characteristic length L and time scale there is an exponential suppression of the propagator by the factor $exp(-S_K(CP_2))$ practically hindering the propagation over distances larger than this length scale.

The presence of the exponential obviously introduces an effective infrared cutoff: this cutoff is prediction of the fundamental theory rather than ad hoc input as in quantum field theories. Of course, infrared cutoff results also from the condition $f(U_{max}) < \infty$. Physically the infrared cutoff results from the topological condensation of the CP_2 type extremals to larger space-time sheets. These could correspond to massless extremals (MEs). p-Adic length scale L_p is an excellent candidate for the cutoff length scale in the directions transversal to ME.

The suppression factor coming from the exponent of the Kähler action implies a distance dependent renormalization of the propagators. In the long length scale limit the suppression factor approaches to a constant value

$$exp\left[-\frac{V_{max}}{V(CP_2)} S_K(CP_2)\right] \ ,$$

and can be absorbed to the coupling constant so that the dependence on the maximal length of the internal lines can be interpreted as an effective coupling constant evolution. For instance, the smallness of the gravitational constant could be understood as follows. Since gravitons propagate over macroscopic distances, the virtual CP_2 type extremals develops a full Kähler action and there is huge suppression factor reducing the value of the gravitational coupling to its observed value: at short length scales the values of the gravitational coupling approaches to $G_{short} = L_p^2$ which means strong gravitation for momentum transfers $Q^2 > 1/L_p^2$. The values of V_{max} and thus those of the suppression factor can vary: only at the limit when CP_2 extremal has point like contact with the lines it joins together, one has $V_{max} = V(CP_2)$. If the boundary component characterizing elementary particle family belongs to CP_2 type extremal (it could be associated with a larger space-time sheet), CP_2 type extremal contains a hole: also this reduces the maximal volume of the CP_2 extremal.

5.5.3 Quantized zitterbewegung and Super Virasoro algebra

Calculating various Fourier components of right left hand side of the light likeness condition $m_{kl}p^kp^l = 0$ for $p^k = dm^k/du$ explicitly using the general expansion for m^k separating center of mass motion from zitterbewegung, one obtains classical Virasoro conditions

$$\begin{aligned} p_0^2 &= L_0 \ , \\ L_n|phys\rangle &= 0 \ , \ . \end{aligned} \tag{5.5.7}$$

where L_n are defined by by their classical expressions as bi-linears of the Fourier coefficients. Therefore interior degrees of freedom give Virasoro algebra and zitterbewegung is more or less equivalent with the classical string dynamics.

It is not however not obvious whether a quantization of this dynamics is needed. If quantization is needed (perhaps to formulate the unitarity conditions in zero modes properly), it corresponds to the construction of the bosonic wave functionals in zero modes defined by the zitterbewegung degrees of freedom. Quantization could be carried out in the same manner as in string models.

The simplest assumption motivated by the Euclidian metric of CP_2 type extremal is that the commutator of p^k and m^k is proportional to a delta function as in ordinary quantization. One can Fourier expand m^k and p_k in the form

$$\begin{aligned} m^k &= m_0^k + p_0^k s + \frac{1}{K}\sum \frac{1}{n} a_n^{k,\dagger} exp(inKs) + \sum \frac{1}{n} a_n^k exp(-inKs) \ , \\ p^k &= p_0^k + i\sum a_n^{k,\dagger} exp(inKs) - i\sum a_n^k exp(-inKs) \ . \end{aligned} \tag{5.5.7}$$

Here cm motion has been extracted and the formula is identical with the formula expressing the motion for a fixed point of string. The parameter K is Kac Moody central charge. Note that the exponents $exp(iKns)$ exist provided that Ks is p-adically of order $O(p)$ or, if algebraic extension by introducing $\sqrt{p}$ is allowed, of order $O(\sqrt{p})$.

The commutator of p_i and m^j is of the standard form if the oscillator operators obey Kac-Moody algebra

$$\begin{aligned} \left[p_{i,0}, m_0^j\right] &= m_i^{\ j} \ , \\ Comm(a_{i,m}^\dagger, a_n^j) &= Km\delta(m,n) m_i^{\ j} \ . \end{aligned} \tag{5.5.7}$$

Here K appears Kac-Moody central charge, which must be integer in the real context at least.

Expressing the light likeness condition as quantum condition, one obtains an infinite series of conditions, which give the quantum counterparts of the Virasoro conditions

$$\begin{aligned} p_0^2 &= kL_0 \ , \\ L_n|phys\rangle &= 0 \ , \ n < 0 \ . \end{aligned} \tag{5.5.7}$$

k is some proportionality constant. One can solve these conditions by going to the transverse gauge in which physical states are created by oscillator operators orthogonal to an arbitrarily chosen light like vector. What quantization means physically is that zitterbewegung amplitudes are constrained by a Gaussian vacuum functional. A good guess motivated by the p-adic considerations is that the width of the ground state Gaussian is given by a p-adic length scale L_p: this is achieved if m^k is replaced with m^k/L_p in the general expression for $m^k(u)$. The experience with string models would suggests that vacuum functionals might be crucial for the understanding of graviton emission.

5.5.4 Zitterbewegung at the level of the modified Dirac action

At the level of the modified Dirac action zitterbewegung motion implies that the conserved momentum associated with CP_2 type extremal, besides being conserved and non-vanishing, is also time like. This means that zitterbewegung creates massive particles besides massless particles as well as off-mass-shell versions of both and Super Virasoro conditions imply the quantization of the mass squared spectrum.

This means that in quantum TGD Feynman diagrammatics is topologized in the sense that the lines of Feynman diagram correspond to CP_2 type extremals which in general performing zitterbewegung. The non-determinism of the CP_2 type extremals means that one obtains a sum over over all possible diagrams with vertices at arbitrary space-time locations just as in quantum field theory approach. What is so nice that the time-development operator associated with an individual line of the diagram is the exponent of the Hamiltonian operator identified as the Poincare energy associated with the modified Dirac action. This operator is that associated with a free theory and contains no nonlinear terms. Interactions result from absolute minimization of Kähler action. In particular, one gets rid of the divergences of the interacting quantum field theories by the topologization of the Feynman diagrammatics.

5.6 Black-hole-elementary particle analogy

String models have provided considerable insights into black hole thermodynamics by reducing it to ordinary thermodynamics for stringy black holes [19] although one still does not understand, which is the mechanism of the thermalization. In TGD context elementary particles are regarded as thermodynamical systems in p-adic sense. This is something new since the standard theories of particle physics describe elementary particles as pure quantum states. The resulting thermal description of the the particle massivation is extremely successful. The fact that one can associate a well defined entropy to an elementary particle, suggests an analogy between black holes and elementary particles and this analogy indeed exists in a quite precise form as will be found. It also leads to a partial explanation for the p-adic length scale hypothesis serving as the corner stone of the p-adic mass calculations. The identification of the CP_2 type extremal as a cognitive representation of elementary particle suggests that p-adic entropy characterizes information associated with a cognitive representation provided by CP_2 type extremal.

5.6.1 Generalization of the Hawking-Bekenstein law briefly

In TGD elementary particles are modelled as so called CP_2 type extremals, which are surfaces with a size of order Planck length having metric with Euclidian signature. These vacuum surfaces are isometric with CP_2 itself and have a one-dimensional, random light like curve as the M^4_+ projection. A natural candidate for the TGD:eish counterpart of the black hole horizon is the surface at which the Euclidian signature of the metric associated with the CP_2 type extremal is changed to the Minkowskian signature of the background space-time. The radius r of this surface is the crucial length scale for the topological condensation and the simplest guess is that it is of the order of the size of the CP_2 radius and hence of the fundamental p-adic length scale. The hope is that the generalization of the black hole thermodynamics, with r replacing the radius of the black hole horizon, could give this information.

p-Adic mass calculations indeed give the p-adic counterpart of the Hawking-Bekenstein formula $S \propto GM^2$ as an identity at p-adic level:

$$S_p = -\frac{1}{T_p}(M_p^2/m_0^2) \ ,$$

where $1/T_p = n$ is the the integer valued inverse of the p-adic temperature and the mass scale $m_0^2/3$ corresponds to unit p-adic number in the unit used. The peculiar looking sign of S_p does not have in the p-adic context the same significance as in real context since the real counterpart of S_p is positive. Although p-adic entropy and mass squared are linearly related, the real counterparts are not in such a simple relation. In case of massive particles the real counterpart of the entropy is in excellent approximation equal to $S = log(p)$ whereas the mass is of order $1/p$ (p is of order 10^{38} for electron!). For massless (or nearly massless) particles one has $S \leq log(p)/p$. The large difference between fermionic and photonic entropies does not favor pair annihilation and this suggests that matter antimatter asymmetry is generated thermodynamically. For instance, via the topological condensation of fermions and anti-fermions on different space-time sheets during the early cosmology.

The generalization of the Hawking-Bekenstein formula in the form of the area law $S = A/4G$ reads as

$$S = \frac{xA}{4l^2} \ ,$$

where the fundamental p-adic length scale $l \simeq 1.376 \cdot 10^4 \sqrt{G}$ replaces Planck length $\sqrt{G}$ and x is a numerical constant near unity. The radius of the elementary particle horizon is in an excellent approximation given by $r(p) = \sqrt{\frac{log(p)}{\pi x}} l$. Particles are thus surrounded by an Euclidian region of the space-time with radius r. Thus the fundamental p-adic length scale l of order CP_2 size has a direct geometric meaning. For instance, in the energy scales below $1/l$ the induced metric of the space-time becomes Euclidian and it might be possible to describe particle physics using Euclidian field theory: essentially QFT in a small deformation of CP_2 would be in question. It is encouraging, that l is also the length scale at which the standard model couplings become identical and super symmetry is expected to become manifest.

The p-adic length scale hypothesis stating that the primes p near prime powers of two are the physically most interesting p-adic primes, is the cornerstone of p-adic mass calculations but there is no really convincing argument for why should it be so. The proportionality of r to $\sqrt{log(p)}$ suggests an explanation for the p-adic length scale hypothesis. The point is that for $p \simeq 2^k$, k prime, one has $r \propto L(k)$ and if the numerical constant x is chosen to be $x = \frac{log(2)}{\pi}$, the radius of elementary particle horizon is in excellent approximation $r(p \simeq 2^k) = L(k)$. Note also that the area of the elementary particle horizon becomes quantized in multiples of prime. This suggests that the precise value of $p \simeq 2^k$ is such that this condition is satisfied optimally and that physics is k-adic below r and $p \simeq 2^k$-adic above r.

$M^4_+ \times CP_2$ allows the imbedding of Schwartshild metric in the region below Schwartchild radius but the imbedding fails for too small values of the radial variable [D3]. An interesting possibility is that black hole entropy is just the sum of the elementary particle entropies topologically condensed below the horizon. This would give $S_{TGD} \propto \sum m_i^2 < S_{GRT} \propto (\sum m_i)^2$. An interesting problem is related to the detailed definition of p-adic entropy: are the entropies of particles with same value of p additive as p-adic numbers or does the additivity hold true for the real counterparts of the p-adic entropies. A related question is whether it might be that also in case of black holes additivity holds true, not for the mass as it is usually assumed, but for the p-adic mass squared for a given p (in TGD inspired model of hadron this is true for quark masses). This could be understood as a result of strong gravitational interactions. The additivity with respect to mass squared would give an upper bound of order $10^{-4}/\sqrt{G}$ for the contribution of a given p-adic prime to the total mass. For instance, the total contribution of electrons to the mass would be always below this mass irrespective of the number of electrons!

5.6.2 In what sense CP_2 type extremals behave like black holes?

CP_2 type extremals are in some respects classically black hole like objects since their metric is Euclidian. When this kind of surface is glued to Minkowskian background there must exist a two-dimensional surface, where the signature of the induced metric changes from the Minkowskian $(1, -1, -1, -1)$ to the Euclidian $(-1, -1, -1, -1)$. On this surface, which could be called elementary particle horizon, the metric is degenerate and has the signature $(0, -1, -1, -1)$. Physically elementary particle horizon can be visualized as the throat of the wormhole feeding the elementary particle gauge fluxes to the background space-time. Of course, one cannot exclude the presence of several wormholes for a given space-time sheet.

This surface indeed behaves in certain respects like horizon. Time like geodesic lines cannot go through this surface. The reason is that the square of the four velocity associated with the geodesic is conserved:

$$v_\mu v^\mu = 1 \ , \ 0 \ \text{or} \ -1 \ ,$$

depending on whether the geodesic is time like, light like or space like. Clearly, a time like geodesic cannot enter from the external world to the interior of the CP_2 type extremal. If a space like geodesic starts from the interior of the CP_2 type extremal it can in principle continue as a space like geodesic into the exterior. These analogies should not be taken too seriously: it does not make sense to identify particles orbits as geodesics in these length scales shorter than the actual sizes of particle.

These analogies suggest that Hawking-Bekenstein formula $S = A/4G$ relating black hole entropy to the area of the black hole horizon, might have a generalization to the elementary particle context with the radius of the elementary particle horizon replacing the black hole horizon. The unit of the area need not be determined by Planck length $\sqrt{G}$, it could be replaced by the fundamental p-adic length scale $l \sim 10^4\sqrt{G}$: this length scale indeed replaces Planck length as a fundamental length scale in TGD.

5.6.3 Elementary particles as p-adically thermal objects?

In the p-adic mass calculations elementary particles were assumed to be thermal objects in the p-adic sense. What is new that energy is replaced with mass squared and the thermalization is believed to result from the interactions of a topologically condensed CP_2 type extremal with the background space-time surface of a much larger size. The thermalization mixes massless states with Planck mass states and gives rise to particle massivation. Super Virasoro invariance − abstracted from the Virasoro invariance of the CP_2 type extremals − together with the general symmetry considerations based on the symmetries of $M^4_+ \times CP_2$, leads to the realization of the mass squared operator essentially as the Virasoro generator L_0 in certain representations of the Super Virasoro algebra constructed using the representations of various Kac Moody algebras associated with Lorentz group, electro-weak group and color group.

$-L_0$ takes thus the role of a Hamiltonian in the partition function:

$$exp(-H/T) \rightarrow p^{L_0/T_p} \ ,$$

where T_p is the p-adic temperature, which by number theoretic reasons is quantized to $1/T_p = n$, n a positive integer. Mass squared is essentially the thermal expectation of L_0. The real mass squared is the real counterpart of the p-adic mass squared in the canonical identification $x = \sum x_n p^n \rightarrow \sum x_n p^{-n} \equiv x_R$ mapping p-adics to reals. Assuming that elementary particles correspond to p-adic primes near prime powers of two, one obtains excellent predictions, not only for the mass scales of elementary particles but also for the particle mass ratios. For instance, electron corresponds to the Mersenne prime $M_{127} = 2^{127} - 1$.

It should be noticed that the real counterpart of the p-adic inverse temperature $1/T_p$ is naturally defined as

$$(\frac{1}{T_p})_r = (\frac{1}{T_p})_R log(p) \ ,$$

where $log(p)$ factor results from the definition of Boltzmann weights as powers of p rather than power of e. The real counterpart T_r of T_p can be identified as

$$T_r = \frac{1}{nlog(p)} \ . \tag{5.6.1}$$

One might wonder about whether the sign of T_p should be taken as negative since positive exponent of L^0 appears in the Boltzman weights. The sign is correct; for the opposite sign T_r would be in good approximation equal to $\frac{1}{(p-n)log(p)}$, which is not consistent with the fact that physically temperature decreases when n increases.

As already explained, the new vision about p-adics and cognition forces to modify this early vision by interpreting CP_2 type extremals as cognitive representations of elementary particles rather than genuine elementary particles.

p-Adic mass squared

The thermal expectation of the p-adic mass squared operator is proportional to the thermal expectation of the Virasoro generator L_0:

$$\begin{aligned} M_p^2 &= k\langle L_0\rangle \ , \\ k &= 1 \ . \end{aligned} \tag{5.6.1}$$

The correct choice for the value of the rational number k is $k = 1$ as became clear in the recent reconstruction of the quantum TGD [F2].

The real mass squared M^2 is identified as

$$\begin{aligned} M^2 &= \frac{M_R^2 \pi^2}{l^2} \ , \\ l &\simeq 1.376 \cdot 10^4 \sqrt{G} \ , \end{aligned} \tag{5.6.1}$$

where l is the fundamental p-adic length scale and M_R^2 is the real counterpart of M_p^2 in the canonical identification. $\sqrt{G}$ is Planck length scale.

p-Adic entropy is proportional to p-adic mass squared

The definition of the p-adic entropy involves some number theory. The general definition

$$S = -p_n log(p_n) \ ,$$

in terms of the probabilities p_n of various states does not work as such since the e-based logarithm $log(p_n)$ does not exist p-adically. Since p-adic Boltzmann weights are integer powers of p it is natural to modify somehow the p-based logarithm $log_p(x)$ so that the resulting logarithm $Log_p(x)$ exists for any p-adic number and has the basic property

$$Log_p(xy) = Log_p(x) + Log_p(y) \ ,$$

guaranteing the additivity of the p-adic entropy for non-interacting systems. The definition satisfying these constraints is

$$Log_p(x = \sum_{n \geq n_0} x_n p^n) \equiv n_0 \ . \tag{5.6.2}$$

The lowest power in the expansion of x in powers of p fixes the value of the logarithm in the same way as it determines also the norm of the p-adic number. This leads to the definition of p-adic entropy as

$$S_p = -\sum_p p_n Log_p(p_n) \ . \tag{5.6.3}$$

In p-adic thermodynamics the p-adic probabilities have the general form

$$p_n = \frac{p^{L_0(n)/T_p}}{Z} \ .$$

Here $L_0(n)$ denotes the eigenvalue of the Virasoro generator L_0, which is integer. The partition function $Z = trace(p^{L_0/T_p})$ has unit p-adic norm if the ground state is massless, so that its p-adic logarithm vanishes in this case: $Log_p(Z) = 0$. This implies $Log_p(p_n) = Log_p(p^{L_0(n)/T_p}) = L_0(n)/T_p$ so that the p-adic entropy reduces to

$$S_p = \frac{1}{T_p} \langle L_0 \rangle \ , \tag{5.6.4}$$

ane hence that the p-adic mass squared and p-adic entropy are proportional to each other

$$S_p = -\frac{1}{kT_p} M_p^2 \ . \tag{5.6.5}$$

By noticing that the entropy for Schwartschild black hole is given by

$$S = 4\pi G M^2 \ , \qquad (5.6.6)$$

one finds that in the p-adic context the analog of the Hawking-Bekenstein formula indeed holds as an identity.

The proposed identification of the entropy is in accordance with the formula $dE = TdS$. In the p-adic context E should clearly be replaced by $\langle -L_0 \rangle$ and T by T_p. The differentials do not however make sense since the thermodynamical quantities are now discrete. Since only $\langle -L_0 \rangle$ and T_p appear as variables one could define

$$\langle -L_0 \rangle = T_p S_p \ .$$

This definition gives $S_p = -\frac{1}{kT_p} M_p^2$ and is in accordance with the standard definition of the Shannon entropy. The definition for the real counterpart of the p-adic entropy is

$$S = log(p) S_R \ .$$

The inclusion of $log(p)$-factor maximizes the resemblance with the usual Shannon entropy defined in terms of the e-based logarithm and makes it possible to compare the real counterpart of entropy with other kind of entropies.

The real counterparts of entropy and mass squared are not linearly related

Due to the delicacies related to the canonical identification, the real counterparts of entropy and mass squared differ drastically from each other and there is no simple relationship between the two quantities. The reason is that the vacuum expectation of $-L_0$ is of order $-np$ for particles having $T_p = 1$ and, essentially due to the presence of minus sign, one has $S_R(p) = 1$ in an excellent approximation, whereas the real counterpart of M_p^2 is of order n/p. For photon and other (nearly) massless bosons the entropy vanishes or is very small.

The fundamental difference in the thermal properties of fermions and massless bosons should have observable consequences. For instance, the annihilation of fermion-anti-fermion pair to massless particles means a considerable reduction of the p-adic entropy and would not be a favorable process thermodynamically. Thus the second law of thermodynamics would favor the presence of net fermion and anti-fermion number densities. For instance, fermions and anti-fermions could suffer a topological condensation on different space-time sheets to avoid annihilation during early cosmology or anti-fermions could even suffer topological evaporation as suggested in [D2, F6]. This in turn would lead to the generation of matter-antimatter asymmetry. It should be noticed that lare entropies are in accordance with the second law of thermodynamics.

Hawking-Bekenstein area formula in elementary particle context

Hawking-Bekenstein formula in the p-adic form $S_p \propto M_p^2$ holds true on basis of the previous considerations although ther are no hopes of deriving the area law from the first principles at this stage. Hawking-Bekenstein formula can be also written in the form

$$S = \frac{A}{4G} \ ,$$

relating black hole entropy to the area of the black hole horizon. One might hope that in the real context a generalization of the area law to the form

$$S = x\frac{A}{4L^2} \ ,$$

where L is some fundamental length scale analogous to the gravitational constant G and x is some numerical constant near unity, would hold true. Since the size of CP_2 defines the fundamental p-adic length scale and replaces $\sqrt{G}$ as a fundamental length scale in TGD, it is conceivable that L is of the order of the CP_2 size $l \sim 10^4\sqrt{G}$. The area in question would be most naturally the area of the elementary particle horizon, where the signature of the induced metric for the topologically condensed CP_2 type extremal changes from Euclidian to Minkowskian. It is well known that l is also the length scale at which the couplings of the standard model become identical and super-symmetry is expected to become manifest. This is what is expected since above cm energy $1/l$ one would have an Euclidian quantum field theory in CP_2.

The radius r of the elementary particle horizon is of order

$$r \simeq \sqrt{log(p)}L \ . \qquad (5.6.7)$$

This means that the # contacts connecting the CP_2 type extremal to the background space-time are surrounded by an Euclidian region with a size of order L.

It is interesting to look for the detailed form of the Hawking-Bekenstein law for elementary particles. One obtains the following general relationship

$$\begin{aligned} S &\equiv log(p)S_R = log(p)(\langle\frac{-L_0}{T_p}\rangle)_R == Xlog(p)M_R^2 = X \times log(p)\frac{l^2}{\pi^2}M^2 \ , \\ X &\equiv \frac{M_R^2}{S_R} \ . \end{aligned} \qquad (5.6.7)$$

For massive particles $X \sim p$ holds true. Hence the entropy is related by a factor $p \cdot 10^8$ to the corresponding black hole entropy:

$$\begin{aligned} S &= a^2 S_{BH} \ , \\ S_{BH} &= 4\pi G M^2 \\ a &= \sqrt{\frac{log(p)X}{4\pi^3}}\frac{l}{\sqrt{G}} \sim 10^4 \ , \\ l &\simeq 1.376 \cdot 10^4\sqrt{G} \ . \end{aligned} \qquad (5.6.5)$$

5.6.4 p-Adic length scale hypothesis and p-adic thermodynamics

The basic assumption of p-adic mass calculations is that physically interesting p-adic primes correspond to prime powers of two:

$$p \simeq 2^k \ , \ k \text{ prime} \ .$$

There are several arguments in favor of this hypothesis but no really convincing argument. The area law however leads to a very attractive, if not even convincing, explanation of the p-adic length scale hypothesis.

The proportionality of the elementary particle horizon radius to $\sqrt{log(p)}$ suggests quite attractive partial explanation for the p-adic length scale hypothesis. The point is that for $p \simeq 2^k$, k prime one has $r \propto L(k)$. Thus, if the numerical constant x is chosen suitably, it is possible to obtain very precisely

$$r(p \simeq 2^k) = L(k) \ .$$

The reason is that the p-adic entropy is in thermal equilibrium very near to its maximum value. The required value of the coefficient x is

$$x = \frac{log(2)}{\pi} \ . \qquad (5.6.6)$$

The requirement that r_F (r_B) is as near as possible to the appropriate p-adic length scale $L(k)$ $(L(k)\sqrt{p})$ fixes also the precise value of the p-adic prime $p \simeq 2^k$.

This hypothesis means that the area of the elementary particle horizon is quantized in the multiples of prime k:

$$A = kA_1 \ . \qquad (5.6.7)$$

The quantization law for the area has been proposed also in the context of the non-perturbative quantum gravity. A suggestive possibility is that physics is k-adic below the elementary particle horizon and $p \simeq 2^k$-adic above it. The appearance of an additional k-adic length scale suggests that for $p \simeq 2^k$ the degeneracy of the effective space-time surfaces is especially large due to the additional k-adic degeneracy and that the p-adic scattering amplitudes are be especially large for this reason. Hence the favored p-adic primes would emerge purely dynamically.

It must be noticed that k-adic fractality allows also more general primes of type $p \simeq 2^{k^n}$, where k is prime and n is integer. For these primes the radius of the elementary particle horizon is $\sqrt{k^{n-1}}L(k)$ and hence also a natural k-adic length scale. There are very few physically interesting length scales of this type. As the p-adic mass calculations show, the best fit to the neutrino mass squared differences is obtained for $p_\nu \simeq 2^{13^2=169}$ rather than $p \simeq 2^{167}$. The length scale $L(p_\nu)$ is also the natural length scale associated with the double cell layers appearing very frequently in bio-systems ($k = 167$ corresponds to the typical size of a cell)!

5.6.5 Black hole entropy as elementary particle entropy?

In TGD Schwartshild metric does not allow a global imbedding as a surface in $M^4_+ \times CP_2$. One can however find imbeddings, which extend also below the Schwartshild radius. This suggests that particles in the interior of the black hole are topologically condensed below the radius r_s. The problem is whether the single particle entropies are additive as real numbers or as p-adic numbers.

Additivity of real entropies?

Consider first the additivity as real numbers. With this assumption the sum for the real counterparts of the p-adic entropies of various particles gives a lower bound for the black hole entropy:

$$S = \sum_i S(i) = \sum_i km_i^2 \ .$$

This entropy is by a factor is $10^8 \cdot p$ larger than the corresponding black hole entropy so that black hole-elementary particle analogy does not work at quantitative level. For sufficiently large particle numbers elementary particle entropy becomes smaller than the black hole entropy, which behaves as $(\sum m_i)^2$. In case of protons $p = M_{107} = 2^{107} - 1$ the critical value of N would be roughly $N \sim 10^{32}$, which would mean black hole with a mass of order 100 kilograms.

Additivity of the p-adic entropies?

One can consider also a different definition of the black hole entropy. In p-adic thermodynamics the natural additive quantity for many particle systems is the Virasoro generator L_0 (mass squared essentially) rather than energy. The additivity works quite nicely for the TGD based model of a hadron as a bound state of quarks. Therefore one could consider the possibility that also for black holes the mass squared of elementary particles with same value of p-adic prime p is p-adically additive

$$(m_p^2)_R = (\sum_i m_p^2(i))_R \ \text{ rather than } \ m = \sum m_i \ .$$

Therefore for a black hole containing only particles with single value of the p-adic prime p, the Hawking-Bekenstein formula in the form

$$S_p \propto M_p^2$$

would hold true. For the real counterparts this proportionality does not hold.

When the particle number N exceeds p/n, the mass squared of the system reduces from its upper bound $10^{-4}/\sqrt{G}$ by a factor of order $1/\sqrt{p}$. Thus the mass of, say, the electrons inside black hole, is always below this upper bound irrespective of the number of the electrons!

If particles with several p-adic primes are present inside the black hole then the formula for the black hole entropy reads as

$$S = \sum_p S(p) = \sum_p k(p)M^2(p) \ ,$$

so that the proportionality to the total mass squared does not hold true except approximately (in the case that the mass is in good approximation given by the total mass of a particular particle species).

5.6.6 Why primes near prime powers of two?

The great challenge of TGD is to predict the p-adic prime associated with a given elementary particle. The problem decomposes into the following subproblems.

1. One must understand why there is a definite value of the p-adic prime associated with a given real region of space-time surface (in particular, the space-time time surface describing elementary particle) and how this prime is determined. The new view about p-adicity allows to understand the possibility to label elementary particles by p-adic primes if p-adic–real phase transitions occur already at elementary particle level or if real elementary particle regions are accompanied by p-adic space-time sheets possible providing some kind of a cognitive model of particle. The great question mark is the correlation of the p-adic prime characterizing the particle with the quantum numbers of the particle: is this correlation due to the intrinsic properties of the particle or perhaps a result of some kind of adaptation at elementary particle length scales. In the latter case sub-cosmologies with quite different elementary particle mass spectra are possible. On the other and, quantum self-organization does not allow too many final state patterns, so that elementary particle mass spectrum could be more or less a constant of Nature.

2. One must understand why quantum evolution by quantum jumps has led to a situation in which elementary particle like surfaces correspond to some preferred primes. It indeed seems that an evolution at elementary particle level is in question (how p-adic evolution follows from simple number theoretic consistency conditions is discussed in the [E6]. It seems that the degeneracy due to the p-adic space-time regions associated with the system must be counted as giving rise to different final states in a quantum jump between quantum histories. If the number $N_d(X^3)$ of the physically equivalent cognitive variants of the space-time surface is especially high, this particular physical state dominates over the other final states of the quantum jump. Highly cognitive systems are winners in the fight for survival. Thus in TGD framework evolution is also, and perhaps basically, evolution of cognition.

3. One should also understand why the primes $p \simeq 2^k$ near prime powers of two are favored physically and to predict the value of k for an elementary particle with given quantum numbers. The analogy between elementary particles and black holes suggests only a partial explanation for the prime powers of 2 and the real explanation should probably involve enhanced cognitive resources for these primes.

In order to formulate the argument supporting p-adic length scale hypothesis one must first describe the general conceptual background.

1. Configuration space of the 3-surfaces decomposes into regions D_P labelled by infinite p-adic primes. In each quantum jump localization of CH spinor field to single sector D_P must occur if localization in zero modes occurs. Quantum time development corresponds to a sequence of quantum jumps between quantum histories and the value of the infinite-p p-adic prime P characterizing the 3-surface associated with the entire universe increases in a statistical sense. This has natural interpretation as evolution. In a well defined sense the infinite prime characterizing infinitely large universe is a composite of finite p-adic primes characterizing various real regions (space-time sheets) of the space-time. The effective infinite-p p-adic topology associated with this infinite prime is very much like real topology since canonical identification mapping infinite number to its real counterpart just drops the infinitesimals of infinite-p p-adic number. Therefore real physics is an excellent approximation at this level. If the S-matrix is complex rational, the approximation is in fact exact. Note that real topology is quite possible also at the level of configuration space and configuration space might consist of both real and infinite-P p-adic regions.

2. The requirement that quantum jumps correspond to quantum measurements in the sense of QFT, implies that also localization in zero modes occurs in each quantum jump: localization could occur also in the length scale resolution defined by the p-adic length scale L_p. The strongest hypothesis suggested by the properties of thermodynamical spin glasses is that quantum jump occurs to a state localized around single maximum of the Kähler function.

3. This picture suggests that evolution has occurred already at the elementary particle level and selected preferred p-adic primes characterizing the space-time regions associated with the elementary particles. A crucial question is whether this evolution could have occurred for isolated elementary particles or whether the interaction of the elementary like space-time regions with the surrounding space-time has served as a selective pressure. It might well be that the latter option is the correct one. If this is the case, one can say that the winners in the fight for survival correspond to infinite primes, which are composites of preferred finite primes, perhaps the finite primes given by the p-adic length scale hypothesis.

4. In TGD framework evolution is also evolution of cognition and the most plausible guess is that p-adic non-determinism is what makes cognition possible. Of course, also the classical non-determinism of Kähler action is also present and also important. Perhaps one should call the space-time sheets of finite time duration made possible by this non-determinism as 'sensory space-time sheets' as opposed to p-adic space-time sheets. Certainly this non-determinism should be responsible for volition. In any case, the degenerate space-time sheets are not physically equivalent in this case as they are in case of the p-adic non-determinism. The number $N_d(X^3)$ of the p-adically degenerate and physically equivalent absolute minima $X^4(X^3)$ of Kähler action is the measure for the cognitive resources of the 3-surface. The basic idea is simple: if $N_d(X^3)$ is very large then quantum jumps lead with high probability to some degenerate physically equivalent maximum of the Kähler function associated with given value of p. One can see this also from the point of view of an elementary particle: the high cognitive degeneracy plus the possibility of p-adic–real phase transitions mean that the particle can adapt to the environment: the surviving elementary particles would be the most intelligent ones! What one should be able to show is that cognitive degeneracy is especially large for some preferred primes so that evolution selects these primes as the most intelligent ones.

In this conceptual framework one can develop more precise variants for arguments supporting the p-adic length scales hypothesis.

1. The simplest possibility is that single maximum of Kähler function is selected in the quantum jump. In this case the relative rate for quantum jumps to a given physical final state with fixed physical configuration is proportional to the p-adic cognitive degeneracy $N_d(N)$, where N denotes the infinite primes characterizing the interacting space-time surface associated with the final state. N decomposes into a product of infinite primes p and $N_d(N)$ decomposes decomposes into a product $N = \prod_P N_d(P)$ $N_d(N)$ is maximized if $N_d(P)$ is maximzes. The elementary systems for which $N_d(P)$ is especially large are winners.

2. The situation reduces to the level of finite p-adic primes if takes seriously the argument allowing to estimate the value of the gravitational constant. The argument was based on the assumption that P decomposes in a well defined sense into passive primes p_i and active prime p characterizing elementary particle: thus there would be the correspondence $P \leftrightarrow p$. This suggests that it is possible to understand the finite p-adic prime p associated with the elementary particle by restricting the consideration to the 3-surfaces describing topologically condensed elementary particles: that is, CP_2 type extremals glued to a space-time sheet with size of order Compton length. p-Adic cognitive degeneracy $N_d(p)$ should be especially high for p-adic primes predicted by the p-adic length scale hypothesis.

3. The interpretation of p-adic regions as cognitive regions suggests a more concrete explanation for the p-adic length scale hypothesis. The degeneracy due to p-adic non-determinism for the p-adic CP_2 type extremals presumably depends on the value of the p-adic prime characterizing the cognitive version of elementary particle. If p-adic–real phase transitions representing transformation of thought-to-action and viceversa are possible for CP_2 type extremals, one could understand the origin of the p-adic length scale hypothesis. p-Adic primes near prime powers of two are winners because the the degeneracy due to p-adic non-determinism is especially larger for them. The observed elementary particles would thus dominate in the Universe simply because the thoughts about them are winners in the fight for survival.

4. The black hole-elementary particle analogy suggests that the primes $p \simeq 2^k$, k prime, are especially interesting since the radius of the elementary particle horizon is the p-adic length

scale $L(k)$. This could be understood since k-adicity provides an additional cognitive degeneracy for the absolute minima of Kähler function coming from the region of size $L(k)$ surrounding a topologically condensed elementary particle and any # contact. This enhances the value of $N_d(p)$ further by a multiplicative factor $N_d(k)$ so that $N_d(P)$ becomes especially large.

5. These arguments do not yet tell how to deduce the prime k associated with a given elementary particle. Cognitive resources are measured by a negative on an negentropy type quantity proportional to $N_c = log(N_d(p))$. A natural guess is that N_c is dominated by a term proportional to $log(p)$: $N_c = A(p) + log(p)$. For $p \simeq 2^k$ one has an additional source of cognitive degeneracy which gives $N_c = log(k) + log(p)$ instead of $N_c = log(p)$ and these primes thus correspond to the local maxima of cognitive resources as a function of p. Quite generally, the larger the p, the more probable is its appearance as elementary particle prime (neglecting the constraints coming from, say, the cosmic temperature). Hence it seems that the p-adic evolution of a given elementary particle is frozen to some local maximum of $N_d(p(k))$, with $p(k)$ given by the p-adic length scale hypothesis.

6. Freezing can be understood if the transition probabilities $P(k \rightarrow k_1)$ are so small that further evolution by quantum jumps is impossible. A possible interpretation of the transition $k_i \rightarrow k_j$ is a p-adic phase transition changing the elementary particle horizon from radius L_{k_i} to L_{k_j} so that $P(k_i \rightarrow k_j)$ would describe the probability of this phase transition. For neutrinos the transition probabilities $P(k_i \rightarrow k_j)$ between different sectors allowed by the p-adic length scale hypothesis seem to be largest whereas for higher quark generations they seem to be smallest. Furthermore, k is smaller for higher generations. In particular, $P(k_i \rightarrow k_j)$ seems to be largest for spherical boundary topology. This suggests that the (phase) transition probabilities $P(k_i \rightarrow k_j)$ decrease as a function of the strength of the dominating particle interaction and of the genus of the particle (reflecting itself via the modular contribution to the particle mass increasing as a function of genus).

5.7 General vision about coupling constant evolution

Zero energy ontology, the construction of M-matrix as time like entanglement coefficients defining Connes tensor product characterizing finite measurement resolution in terms of inclusion of hyper-finite factors of type II_1, the realization that symplectic invariance of N-point functions provides a detailed mechanism eliminating UV divergences, and the understanding of the relationship between super-symplectic and super Kac-Moody symmetries: these are the pieces of the puzzle whose combination making possible a rather concrete vision about coupling constant evolution in TGD Universe and even a rudimentary form of generalized Feynman rules.

p-adic coupling constant evolution is discrete by p-adic length scale hypothesis justified by zero energy ontology. Discreteness means that continuous mass scale is replaced by mass scales coming as half octaves of CP_2 mass. One key question has been whether it is Kähler coupling strength α_K or gravitational coupling constant, which remains invariant under p-adic coupling constant evolution. Second problem relates to the value of α_K.

The realization that modified Dirac action assignable to Chern-Simons action for light-like 3-surfaces could be the fundamental variational principle initiated the process, which led to an answer to these and many other questions. The idea that some kind of Dirac determinant gives the vacuum functional identifiable as exponent of Kähler function in turn identifiable as Kähler action S_K for a preferred extremal came first. The basic challenges were to understand the conditions fixing this preferred extremal, how this information is feeded to the spectrum of generalized eigenvalues of the modified Dirac operator defined by C-S action, and how to define the Dirac determinant. A precise realization of the idea that light-like 3-surfaces can be regarded as spinorial shock waves provided a solution to these problems.

The most important outcome is a formula for Kähler coupling strength in terms of a calculable and manifestly finite Dirac determinant without any need for zeta function regularization. The formula fixes completely the number theoretic anatomy of Kähler coupling strength and of other gauge coupling strengths. When the formula for the gravitational constant involving Kähler coupling strength and the exponent of Kähler action for CP_2 type vacuum extremal - which remains still a conjecture -

is combined with the number theoretical results and with the constraints from the predictions of p-adic mass calculations, one ends up to an identification of Kähler coupling strength as fine structure constant at electron length scale characterized by p-adic prime M_{127}. Also the number theoretic anatomy of the ratio $R^2/\hbar G$, where R is CP_2 size, can be understood to high degree and a relationship between the p-adic evolutions of electromagnetic and color coupling strengths emerges.

5.7.1 General ideas about coupling constant evolution

Zero energy ontology

In zero energy ontology one replaces positive energy states with zero energy states with positive and negative energy parts of the state at the boundaries of future and past direct light-cones forming a causal diamond. All conserved quantum numbers of the positive and negative energy states are of opposite sign so that these states can be created from vacuum. "Any physical state is creatable from vacuum" becomes thus a basic principle of quantum TGD and together with the notion of quantum jump resolves several philosophical problems (What was the initial state of universe?, What are the values of conserved quantities for Universe, Is theory building completely useless if only single solution of field equations is realized?).

At the level of elementary particle physics positive and negative energy parts of zero energy state are interpreted as initial and final states of a particle reaction so that quantum states become physical events. Equivalence Principle would hold true in the sense that the classical gravitational four-momentum of the vacuum extremal whose small deformations appear as the argument of configuration space spinor field is equal to the positive energy of the positive energy part of the zero energy quantum state. Equivalence Principle is expected to hold true for elementary particles and their composites but not for the quantum states defined around non-vacuum extremals.

Does the finiteness of measurement resolution dictate the laws of physics?

The hypothesis that the mere finiteness of measurement resolution could determine the laws of quantum physics [C2] completely belongs to the category of not at all obvious first principles. The basic observation is that the Clifford algebra spanned by the gamma matrices of the "world of classical worlds" represents a von Neumann algebra [22] known as hyperfinite factor of type II_1 (HFF) [A9, C6, C2]. HFF [23, 28] is an algebraic fractal having infinite hierarchy of included subalgebras isomorphic to the algebra itself [29]. The structure of HFF is closely related to several notions of modern theoretical physics such as integrable statistical physical systems [50], anyons [44], quantum groups and conformal field theories[24, 39], and knots and topological quantum field theories [36, 35].

Zero energy ontology is second key element. In zero energy ontology these inclusions allow an interpretation in terms of a finite measurement resolution: in the standard positive energy ontology this interpretation is not possible. Inclusion hierarchy defines in a natural manner the notion of coupling constant evolution and p-adic length scale hypothesis follows as a prediction. In this framework the extremely heavy machinery of renormalized quantum field theory involving the elimination of infinities is replaced by a precisely defined mathematical framework. More concretely, the included algebra creates states which are equivalent in the measurement resolution used. Zero energy states are associated with causal diamond formed by a pair of future and past directed light-cones having positive and negative energy parts of state at their boundaries. Zero energy state can be modified in a time scale shorter than the time scale of the zero energy state itself.

On can imagine two kinds of measurement resolutions. The element of the included algebra can leave the quantum numbers of the positive and negative energy parts of the state invariant, which means that the action of subalgebra leaves M-matrix invariant. The action of the included algebra can also modify the quantum numbers of the positive and negative energy parts of the state such that the zero energy property is respected. In this case the Hermitian operators subalgebra must commute with M-matrix.

The temporal distance between the tips of light-cones corresponds to the secondary p-adic time scale $T_{p,2} = \sqrt{p}T_p$ by a simple argument based on the observation that light-like randomness of light-like 3-surface is analogous to Brownian motion. This gives the relationship $T_p = L_p^2/Rc$, where R is CP_2 size. The action of the included algebra corresponds to an addition of zero energy parts to either positive or negative energy part of the state and is like addition of quantum fluctuation below the time scale of the measurement resolution. The natural hierarchy of time scales is obtained as $T_n = 2^{-n}T$

since these insertions must belong to either upper or lower half of the causal diamond. This implies that preferred p-adic primes are near powers of 2. For electron the time scale in question is .1 seconds defining the fundamental biorhythm of 10 Hz.

M-matrix representing a generalization of S-matrix and expressible as a product of a positive square root of the density matrix and unitary S-matrix would define the dynamics of quantum theory [C2]. The notion of thermodynamical state would cease to be a theoretical fiction and in a well-defined sense quantum theory could be regarded as a square root of thermodynamics. M-matrix is identifiable in terms of Connes tensor product [28] and therefore exists and is almost unique. Connes tensor product implies that the Hermitian elements of the included algebra commute with M-matrix and hence act like infinitesimal symmetries. A connection with integrable quantum field theories is suggestive. The remaining challenge is the calculation of M-matrix and the needed machinery might already exist.

The tension is present also now. The connection with visions should come from the discretization in terms of number theoretic braids providing space-time correlate for the finite measurement resolution and making p-adicization in terms of number theoretic braids possible. Number theoretic braids give a connection with the construction of configuration space geometry in terms of Dirac determinant and with TGD as almost TQFT and with conformal field theory approach. The mathematics for the inclusions of hyper-finite factors of type II_1 is also closely related to that for conformal field theories including quantum groups relating closely to Connes tensor product and non-commutativity.

How do p-adic coupling constant evolution and p-adic length scale hypothesis emerge?

Zero energy ontology in which zero energy states have as imbedding space correlates causal diamonds for which the distance between the tips of future and past directed light-cones are power of 2 multiples of fundamental time scale ($T_n = 2^n T_0$) implies in a natural manner coupling constant evolution. A weaker condition would be $T_p = pT_0$, p prime, and would assign all p-adic time scales to the size scale hierarchy of CDs.

Could the coupling constant evolution in powers of 2 implying time scale hierarchy $T_n = 2^n T_0$ induce p-adic coupling constant evolution and explain why p-adic length scales correspond to $L_p \propto \sqrt{p}R$, $p \simeq 2^k$, R CP_2 length scale? This looks attractive but there is a problem. p-Adic length scales come as powers of $\sqrt{2}$ rather than 2 and the strongly favored values of k are primes and thus odd so that $n = k/2$ would be half odd integer. This problem can be solved.

1. The observation that the distance traveled by a Brownian particle during time t satisfies $r^2 = Dt$ suggests a solution to the problem. p-Adic thermodynamics applies because the partonic 3-surfaces X^2 are as 2-D dynamical systems random apart from light-likeness of their orbit. For CP_2 type vacuum extremals the situation reduces to that for a one-dimensional random light-like curve in M^4. The orbits of Brownian particle would now correspond to light-like geodesics γ_3 at X^3. The projection of γ_3 to a time=constant section $X^2 \subset X^3$ would define the 2-D path γ_2 of the Brownian particle. The M^4 distance r between the end points of γ_2 would be given $r^2 = Dt$. The favored values of t would correspond to $T_n = 2^n T_0$ (the full light-like geodesic). p-Adic length scales would result as $L^2(k) = DT(k) = D2^k T_0$ for $D = R^2/T_0$. Since only CP_2 scale is available as a fundamental scale, one would have $T_0 = R$ and $D = R$ and $L^2(k) = T(k)R$.

2. p-Adic primes near powers of 2 would be in preferred position. p-Adic time scale would not relate to the p-adic length scale via $T_p = L_p/c$ as assumed implicitly earlier but via $T_p = L_p^2/R_0 = \sqrt{p}L_p$, which corresponds to secondary p-adic length scale. For instance, in the case of electron with $p = M_{127}$ one would have $T_{127} = .1$ second which defines a fundamental biological rhythm. Neutrinos with mass around .1 eV would correspond to $L(169) \simeq 5$ μm (size of a small cell) and $T(169) \simeq 1. \times 10^4$ years. A deep connection between elementary particle physics and biology becomes highly suggestive.

3. In the proposed picture the p-adic prime $p \simeq 2^k$ would characterize the thermodynamics of the random motion of light-like geodesics of X^3 so that p-adic prime p would indeed be an inherent property of X^3. For $T_p = pT_0$ the above argument is not enough for p-adic length scale hypothesis and p-adic length scale hypothesis might be seen as an outcome of a process analogous to natural selection. Resonance like effect favoring octaves of a fundamental frequency might be in question. In this case, p would a property of CD and all light-like 3-surfaces inside it and also that corresponding sector of configuration space.

5.7.2 The bosonic action defining Kähler function as the effective action associated with the induced spinor fields

One could *define* the classical action defining Kähler function as the bosonic action giving rise to the divergences of the isometry currents. In this manner bosonic action, especially the value of the Kähler coupling strength, would come out as prediction of the theory containing no free parameters.

Thus the Kähler action S_B of preferred extremal of Käction defining Kähler function could be *defined* by the functional integral over the Grassmann variables for the exponent of the massless Dirac action. Formally the functional integral is defined as

$$\begin{aligned} exp(S_B(X^4)) &= \int exp(S_F)D\Psi D\bar{\Psi} \ , \\ S_F &= \bar{\Psi}\left[\hat{\Gamma}^\alpha D_\alpha^{\rightarrow} - D_\alpha^{\leftarrow}\hat{\Gamma}^\alpha\right]\Psi\sqrt{g} \ . \end{aligned} \tag{5.7.-1}$$

Formally the bosonic effective action is expressible as a logarithm of the fermionic functional determinant resulting from the functional integral over the Grassmann variables

$$\begin{aligned} S_B(X^4) &= log(det(D)) \ , \\ D &= \hat{\Gamma}^\alpha D_\alpha^{\rightarrow} \ . \end{aligned} \tag{5.7.-1}$$

Can one do without zeta function regularization?

The rigorous definition of the fermionic determinant has been already discussed in [A6]. The best one hope that the formal definition of the determinant as the the product of the generalized eigenvalues of D_{C-S} works as such. This is the case if the number of eigenvalues is finite; if the eigenvalues approach to constant which can be chosen to be equal to unity; or if the eigenvalues have approximate symmetry $\lambda \rightarrow 1/\lambda$.

1. Somewhat surprisingly the detailed construction of the eigenvalue spectrum discussed in [A6] shows that the number of eigenvalues is indeed finite and that eigenvalues are bounded from above. The basic idea of the construction is following. The eigenvalues correspond to the generalized eigenvalues of the modified Dirac operator D_{C-S} for Chern-Simons action at X_l^3. The modified Dirac equation for D_{C-S} does not however fix the eigenvalues but allows them to be arbitrary functions of the transversal coordinates of X_l^3. Therefore the data about preferred extremal of Kähler action can be feeded to the eigenvalue spectrum by assuming that spinor modes at X_l^3 can be also regarded as spinorial shock waves in the sense that they correspond to singular solutions of 4-D modified Dirac operator D_K assignable to Kähler action.

2. Since modified Dirac equation for D_K is equivalent with the conservation of super current, the shock wave property means that the super current is restricted to X_l^3 and thus has a vanishing normal component. In the case of wormhole throats the construction requires boundary conditions stating that there exist coordinates in which $J_{ni} = 0$ and $g_{ni} = 0$ at X_l^3 [A6]. Therefore classical gravitational field is effectively static at X_l^3 and the Maxwell field defined by the induced Kähler form has only the magnetic part in these coordinates.

3. The generalized eigenvalues of D_{C-S} appearing in Dirac determinant can be identified as eigenvalues of the transversal part of 3-D Dirac operator defined by the restriction of D_K to X_l^3 describing fermions in the electro-weak magnetic field associated with X_l^3. The physical analog is energy spectrum for Dirac operator in external magnetic field. The effective metric appearing in the modified Dirac operator corresponds to

$$\hat{g}^{\alpha\beta} = \frac{\partial L_K}{\partial h_\alpha^k}\frac{\partial L_K}{\partial h_\beta^l}h_{kl} \ ,$$

and vanishes at the boundaries of regions carrying non-vanishing Kähler magnetic field. Hence the shock waves must be localized to regions $X_{l,i}^3$ containing a non-vanishing Kähler magnetic field. Cyclotron states in constant magnetic field serve as a good analog for the situation and only a finite number of cyclotron states are possible since for higher cyclotron states the wave function -essentially harmonic oscillator wave function- would concentrate outside $X^3_{l,i}$.

4. A more precise argument goes as follows. Assume that it is induced Kähler magnetic field B_K that matters. The vanishing of the effective contravariant metric near the boundary of $X^3_{l,i}$ corresponds to an infinite effective mass for massive particle in constant magnetic field so that the counterpart for the cyclotron frequency scale eB/m reduces to zero. The radius of the cyclotron orbit is proportional to $1/\sqrt{eB}$ and approaches to infinity. Hence the required localization is not possible only for cyclotron states for which the cyclotron radius is below that the transversal size scale of $X^3_{l,i}$.

5. The eigenvalues of the modified Dirac operator vanish for the vacuum extremals but the Dirac determinant equals to one in this case since zero eigenvalues do not correspond to localized solutions and by definition do not contribute to it.

Formula for the Kähler coupling strength

The identification of exponent of Kähler function as Dirac determinant leads to a formula relating Kähler action for the preferred extremal to the Dirac determinant. The eigenvalues are proportional to $1/\alpha_K$ since the matrices $\hat{\Gamma}^\alpha$ have this proportionality. This gives the formula

$$exp(\frac{S_K(X^4(X^3))}{8\pi\alpha_K}) \quad = \quad \prod_i \lambda_i = \frac{\prod_i \lambda_{0,i}}{\alpha_K^N} \quad . \tag{5.7.0}$$

Here $\lambda_{0,i}$ corresponds to $\alpha_K = 1$. $S_K = \int J^* J$ is the reduced Kähler action.

For $S_K = 0$, which might correspond to so called massless extremals [D1] one obtains the formula

$$\alpha_K \quad = \quad (\prod_i \lambda_{0,i})^{1/N} \quad . \tag{5.7.1}$$

Thus for $S_K = 0$ extremals one has an explicit formula for α_K having interpretation as the geometric mean of the eigenvalues $\lambda_{0,i}$. Several values of α_K are in principle possible.

p-Adicization suggests that $\lambda_{0,i}$ are rational or at most algebraic numbers. This would mean that α_K is N:th root of this kind of number. S_K in turn would be

$$S_K \quad = \quad 8\pi\alpha_K log(\frac{\prod_i \lambda_{0,i}}{\alpha_K^N}) \quad . \tag{5.7.2}$$

so that S_K would be expressible as a product of the transcendental π, N:th root of rational, and logarithm of rational. This result would provide a general answer to the question about number theoretical anatomy of Kähler coupling strength and S_K. Note that S_K makes sense p-adically only if one adds π and its all powers to the extension of p-adic numbers. The exponent of Kähler function however makes sense also p-adically.

5.7.3 A revised view about coupling constant evolution

The development of the ideas related to number theoretic aspects has been rather tortuous and based on guess work since basic theory has been lacking.

1. The original hypothesis was that Kähler coupling strength is invariant under p-adic coupling constant evolution. Later I gave up this hypothesis and replaced it with the invariance of gravitational coupling since otherwise the prediction would have been that gravitational coupling strength is proportional to p-adic length scale squared. Second first guess was that Kähler coupling strength equals to the value of fine structure constant at electron length scale corresponding to Mersenne prime M_{127}. Later I replaced fine structure constant with electro-weak U(1) coupling strength at this length scale. The recent discussion returns back to the roots in both aspects.

2. The recent discussion relies on the progress made in the understanding of quantum TGD at partonic level [A6]. What comes out is an explicit formula for Kähler couplings strength in terms of Dirac determinant involving only a finite number of eigenvalues of the modified Dirac operator. This formula dictates the number theoretical anatomy of g_K^2 and also of other coupling constants: the most general option is that α_K is a root of rational. The requirement that the rationals involved are simple combined with simple experimental inputs leads to very powerful predictions for the coupling parameters.

3. A further simplification is due to the discreteness of p-adic coupling constant evolution allowing to consider only length scales coming as powers of $\sqrt{2}$. This kind of discretization is necessary also number theoretically since logarithms can be replaced with 2-adic logarithms for powers of 2 giving integers. This raises the question whether $p \simeq 2^k$ should be replaced with 2^k in all formulas as the recent view about quantum TGD suggests.

4. The prediction is that Kähler coupling strength α_K is invariant under p-adic coupling constant evolution and from the constraint coming from electron and top quark masses very near to fine structure constant so that the identification as fine structure constant is natural. Gravitational constant is predicted to be proportional to p-adic length scale squared and corresponds to the largest Mersenne prime (M_{127}), which does not correspond to a completely super-astronomical p-adic length scale. For the parameter R^2/G p-adicization program allows to consider two options: either this constant is of form e^q or 2^q: in both cases q is rational number. $R^2/G = exp(q)$ allows only M_{127} gravitons if number theory is taken completely seriously. $R^2/G = 2^q$ allows all p-adic length scales for gravitons and thus both strong and weak variants of ordinary gravitation.

5. A relationship between electromagnetic and color coupling constant evolutions based on the formula $1/\alpha_{em} + 1/\alpha_s = 1/\alpha_K$ is suggested by the induced gauge field concept, and would mean that the otherwise hard-to-calculate evolution of color coupling strength is fixed completely. The predicted value of α_s at intermediate boson length scale is correct.

It seems fair to conclude that the attempts to understand the implications of p-adicization for coupling constant evolution have begun to bear fruits.

Identifications of Kähler coupling strength and gravitational coupling strength

To construct an expression for gravitational constant one can use the following ingredients.

1. The exponent $exp(2S_K(CP_2))$ defining the value of Kähler function in terms of the Kähler action $S_K(CP_2)$ of CP_2 type extremal representing elementary particle expressible as

$$S_K(CP_2) = \frac{S_{K,R}(CP_2)}{8\pi\alpha_K} = \frac{\pi}{8\alpha_K} . \tag{5.7.3}$$

   Since CP_2 type extremals suffer topological condensation, one expects that the action is modified:

$$S_K(CP_2) \rightarrow a \times S_K(CP_2) . \tag{5.7.4}$$

   $a < 1$ conforms with the idea that a piece of CP_2 type extremal defining a wormhole contact is in question. One must however keep mind open in this respect.

2. The p-adic length scale L_p assignable to the space-time sheet along which gravitational interactions are mediated. Since Mersenne primes seem to characterized elementary bosons and since the Mersenne prime $M_{127} = 2^{127} - 1$ defining electron length scale is the largest non-super-astronomical length scale it is natural to guess that M_{127} characterizes these space-time sheets.

1. The formula for the gravitational constant

A long standing basic conjecture has been that gravitational constant satisfies the following formula

$$\begin{aligned} \hbar G &\equiv r\hbar_0 G = L_p^2 \times exp(-2aS_K(CP_2)) \ , \\ L_p &= \sqrt{p}R \ . \end{aligned} \quad (5.7.4)$$

Here R is CP_2 radius defined by the length $2\pi R$ of the geodesic circle. What was noticed before is that this relationship allows even constant value of G if a has appropriate dependence on p.

This formula seems to be correct but the argument leading to it was based on two erratic assumptions compensating each other.

1. I assumed that modulus squared for vacuum functional is in question: hence the factor $2a$ in the exponent. The interpretation of zero energy state as a generalized Feynman diagram requires the use of vacuum functional so that the replacement $2a \rightarrow a$ is necessary.

2. Second wrong assumption was that graviton corresponds to CP_2 type vacuum extremal- that is wormhole contact in the recent picture. This does allow graviton to have spin 2. Rather, two wormhole contacts represented by CP_2 vacuum extremals and connected by fluxes associated with various charges at their throats are needed so that graviton is string like object. This saves the factor $2a$ in the exponent.

The highly non-trivial implication to be discussed later is that ordinary coupling constant strengths should be proportional to $exp(-aS_K(CP_2))$.

The basic constraint to the coupling constant evolution comes for the invariance of g_K^2 in p-adic coupling constant evolution:

$$\begin{aligned} g_K^2 &= \frac{a(p,r)\pi^2}{log(pK)} \ , \\ K &= \frac{R^2}{\hbar G(p)} = \frac{1}{r}\frac{R^2}{\hbar_0 G(p)} \equiv \frac{K_0(p)}{r} \ . \end{aligned} \quad (5.7.4)$$

2. How to guarantee that g_K^2 is RG invariant and N:th root of rational?

Suppose that g_K^2 is N:th root of rational number and invariant under p-adic coupling constant evolution.

1. The most general manner to guarantee the expressibility of g_K^2 as N:th root of rational is guaranteed for both options by the condition

$$a(p,r) = \frac{g_K^2}{\pi^2} log(\frac{pK_0}{r}) \ . \quad (5.7.5)$$

That a would depend logarithmically on p and $r = \hbar/\hbar_0$ looks rather natural. Even the invariance of G under p-adic coupling constant evolution can be considered.

2. The condition

$$\frac{r}{p} < K_0(p) \ . \quad (5.7.6)$$

must hold true to guarantee the condition $a > 0$. Since the value of gravitational Planck constant is very large, also the value of corresponding p-adic prime must very large to guarantee this condition. The condition $a < 1$ is guaranteed by the condition

$$\frac{r}{p} > exp(-\frac{\pi^2}{g_K^2}) \times K_0(p) \ . \tag{5.7.7}$$

The condition implies that for very large values of p the value of Planck constant must be larger than $\hbar_0$.

3. The two conditions are summarized by the formula

$$K_0(p) \times exp(-\frac{\pi^2}{g_K^2}) < \frac{r}{p} < K_0(p) \tag{5.7.8}$$

characterizing the allowed interval for r/p. If G does not depend on p, the minimum value for r/p is constant. The factor $exp(-\frac{\pi^2}{g_K^2})$ equals to 1.8×10^{-47} for $\alpha_K = \alpha_{em}$ so that $r > 1$ is required for $p \geq 4.2 \times 10^{-40}$. $M_{127} \sim 10^{38}$ is near the upper bound for p allowing $r = 1$. The constraint on r would be roughly $r \geq 2^{k-131}$ and $p \simeq 2^{131}$ is the first p-adic prime for which $\hbar > 1$ is necessarily. The corresponding p-adic length scale is .1 Angstroms.

This conclusion need not apply to elementary particles such as neutrinos but only to the space-time sheets mediating gravitational interaction so that in the minimal scenario it would be gravitons which must become dark above this scale. This would bring a new aspect to vision about the role of gravitation in quantum biology and consciousness.

The upper bound for r behaves roughly as $r < 2.3 \times 10^7 p$. This condition becomes relevant for gravitational Planck constant GM_1M_2/v_0 having gigantic values. For Earth-Sun system and for $v_0 = 2^{-11}$ the condition gives the rough estimate $p > 6 \times 10^{63}$. The corresponding p-adic length scale would be of around $L(215) \sim 40$ meters.

4. p-Adic mass calculations predict the mass of electron as $m_e^2 = (5+Y_e)2^{-127}/R^2$ where $Y_e \in [0,1)$ parameterizes the not completely known second order contribution. Top quark mass favors a small value of Y_e (the original experimental estimates for m_t were above the range allowed by TGD but the recent estimates are consistent with small value Y_e [F4]). The range $[0,1)$ for Y_e restricts $K_0 = R^2/\hbar_0 G$ to the range $[2.3683, 2.5262] \times 10^7$.

5. The best value for the inverse of the fine structure constant is $1/\alpha_{em} = 137.035999070(98)$ and would correspond to $1/g_K^2 = 10.9050$ and to the range $(0.9757, 0.9763)$ for a for $\hbar = \hbar_0$ and $p = M_{127}$. Hence one can seriously consider the possibility that $\alpha_K = \alpha_{em}(M_{127}$ holds true. As a matter fact, this was the original hypothesis but was replaced later with the hypothesis that α_K corresponds to electro-weak $U(1)$ coupling strength in this length scale. The fact that M_{127} defines the largest Mersenne prime, which does not correspond to super-astrophysical length scale might relate to this co-incidence.

To sum up, the recent view about coupling constant evolution differs strongly from previous much more speculative scenarios. It implies that g_K^2 is root of rational number, possibly even rational, and can be assumed to be equal to e^2. Also $R^2/\hbar G$ could be rational. The new element is that G need not be proportional to p and can be even invariant under coupling constant evolution since the the parameter a can depend on both p and r. An unexpected constraint relating p and r for space-time sheets mediating gravitation emerges.

Are the color and electromagnetic coupling constant evolutions related?

Classical theory should be also able to say something non-trivial about color coupling strength α_s too at the general level. The basic observations are following.

1. Both classical color YM action and electro-weak U(1) action reduce to Kähler action.

2. Classical color holonomy is Abelian which is consistent also with the fact that the only signature of color that induced spinor fields carry is anomalous color hyper charge identifiable as an electro-weak hyper charge.

Suppose that α_K is a strict RG invariant. One can consider two options.

1. The original idea was that the sum of classical color action and electro-weak U(1) action is RG invariant and thus equals to its asymptotic value obtained for $\alpha_{U(1)} = \alpha_s = 2\alpha_K$. Asymptotically the couplings would approach to a fixed point defined by $2\alpha_K$ rather than to zero as in asymptotically free gauge theories.

   Thus one would have

$$\frac{1}{\alpha_{U(1)}} + \frac{1}{\alpha_s} = \frac{1}{\alpha_K} \ . \qquad (5.7.9)$$

   The relationship between $U(1)$ and em coupling strengths is

$$\begin{aligned} \alpha_{U(1)} &= \frac{\alpha_{em}}{cos^2(\theta_W)} \simeq \frac{1}{104.1867} \ , \\ sin^2(\theta_W)_{|10\ MeV} &\simeq 0.2397(13) \ , \\ \alpha_{em}(M_{127}) &= 0.00729735253327 \ . \end{aligned} \qquad (5.7.8)$$

   Here Weinberg angle corresponds to 10 MeV energy is reasonably near to the value at electron mass scale. The value $sin^2(\theta_W) = 0.2397(13)$ corresponding to 10 MeV mass scale [32] is used. Note however that the previous argument implying $\alpha_K = \alpha_{em}(M_{127})$ excludes $\alpha = \alpha_{U(1)}(M_{127})$ option.

2. Second option is obtained by replacing $U(1)$ with electromagnetic gauge $U(1)_{em}$.

$$\frac{1}{\alpha_{em}} + \frac{1}{\alpha_s} = \frac{1}{\alpha_K} . \qquad (5.7.9)$$

   Possible justifications for this assumption are following. The notion of induced gauge field makes it possible to characterize the dynamics of classical electro-weak gauge fields using only the Kähler part of electro-weak action, and the induced Kähler form appears only in the electromagnetic part of the induced classical gauge field. A further justification is that em and color interactions correspond to unbroken gauge symmetries.

The following arguments are consistent with this conclusion.

1. In TGD framework coupling constant is discrete and comes as powers of $\sqrt{2}$ corresponding to p-adic primes $p \simeq 2^k$. Number theoretic considerations suggest that coupling constants g_i^2 are algebraic or perhaps even rational numbers, and that the logarithm of mass scale appearing as argument of the renormalized coupling constant is replaced with 2-based logarithm of the p-adic length scale so that one would have $g_i^2 = g_i^2(k)$. g_K^2 is predicted to be N:th root of rational but could also reduce to a rational. This would allow rational values for other coupling strengths too. This is possible if $sin(\theta_W)$ and $cos(\theta_W)$ are rational numbers which would mean that Weinberg angle corresponds to a Pythagorean triangle as proposed already earlier. This would mean the formulas $sin(\theta_W) = (r^2 - s^2)/(r^2 + s^2)$ and $cos(\theta_W) = 2rs(r^2 + s^2)$.

2. A very strong prediction is that the beta functions for color and $U(1)$ degrees of freedom are apart from sign identical and the increase of $U(1)$ coupling compensates the decrease of the color coupling. This allows to predict the hard-to-calculate evolution of QCD coupling constant strength completely.

3. $\alpha(M_{127}) = \alpha_K$ implies that M_{127} defines the confinement length scale in which the sign of α_s becomes negative. TGD predicts that also M_{127} copy of QCD should exist and that M_{127} quarks should play a key role in nuclear physics [F8, F9]. Hence one can argue that color coupling strength indeed diverges at M_{127} (the largest not completely super-astrophysical Mersenne prime) so that one would have $\alpha_K = \alpha(M_{127})$. Therefore the precise knowledge of $\alpha(M_{127})$ in principle fixes the value of parameter $K = R^2/G$ and thus also the second order contribution to the mass of electron.

4. $\alpha_s(M_{89})$ is predicted to be $1/\alpha_s(M_{89}) = 1/\alpha_K - 1/\alpha(M_{89})$. $sin^2(\theta_W) = .23120$, $\alpha_{em}(M_{89}) \simeq 1/127$, and $\alpha_{U(1)} = \alpha_{em}/cos^2(\theta_W)$ give $1/\alpha_{U(1)}(M_{89}) = 97.6374$. $\alpha = \alpha_{em}$ option gives $1/\alpha_s(M_{89}) \simeq 10$, which is consistent with experimental facts. $\alpha = \alpha_{U(1)}$ option gives $\alpha_s(M_{89}) = 0.1572$, which is larger than QCD value. Hence $\alpha = \alpha_{em}$ option is favored.

Can one deduce formulae for gauge couplings?

The improved physical picture behind gravitational constant allows also to consider a general formula for gauge couplings.

1. The natural guess for the general formula would be as

$$g^2(p,r) = kg_K^2 \times exp[-a_g(p,r) \times S_K(CP_2)] \ . \tag{5.7.10}$$

here k is a numerical constant.

2. The condition
$g_K^2 = e^2(M_{127})$ fixes the value of k if it's value does not depend on the character of gauge interaction:

$$k = exp[a_{gr}(M_{127}, r=1) \times S_K(CP_2)] \ . \tag{5.7.11}$$

Hence the general formula reads as

$$g^2(p,r) = g_K^2 \times exp[(-a_g(p,r) + a_{gr}(M_{127}), r=1)) \times S_K(CP_2)] \ . \tag{5.7.11}$$

The value of $a(M_{127}, r=1)$ is near to its maximum value so that the exponential factor tends to increase the value of g^2 from e^2. The formula can reproduce α_s and various electro-weak couplings although it is quite possibile that Weinberg angle corresponds to a group theoretic factor not representable in terms of $a_g(p,r)$. The volume of the CP_2 type vacuum extremal would characterize gauge bosons. Analogous formula should apply also in the case of Higgs.

3. α_{em} in very long length scales would correspond to

$$e^2(p \rightarrow \infty, r=1) = e^2 \times exp[(-1 + a(M_{127}), r=1)) \times S_K(CP_2)] = e^2 x \ , \tag{5.7.11}$$

where x is in the range $[0.6549, 0.6609]$.

To sum up, the proposed formula would dictate the evolution of α_s from the evolution of the electro-weak parameters without any need for perturbative computations. Although the formula of proposed kind is encouraged by the strong constraints between classical gauge fields in TGD framework, it should be deduced in a rigorous manner from the basic assumptions of TGD before it can be taken seriously.

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Chapter 6

Fusion of p-Adic and Real Variants of Quantum TGD to a More General Theory

6.1 Introduction

The notion of p-adicization has for a long time been a somewhat obscure attempt to provide a theoretical justification for the successes of the p-adic mass calculations. The reduction of quantum TGD to a generalized number theory and the developments in TGD inspired theory of consciousness have however led to a better understanding what the p-adicization possibly means.

6.1.1 What p-adic physics means?

Contrary to the original expectations finite-p p-adic physics means the physics of the p-adic cognitive representations about real physics rather than 'real physics'. This forces to update the prejudices about what p-adicization means. The original hypothesis was that p-adicization is a strict one-to-one map from real to p-adic physics and this led to technical problems with symmetries.

The new vision about quantum TGD the notion of the p-adic space-time emerges dynamically and p-adic space-time regions are absolutely 'real' and certainly not 'p-adicized' in any sense. Furthermore, the new view also encourages the hypothesis that p-adic regions provide cognitive models for the real matter like regions becoming more and more refined in the evolutionary self-organization process by quantum jumps. p-Adic region can serve as a cognitive model for particle itself or for the external world. The model is defined by some cognitive map of real region to its p-adic counterpart. This cognitive map need not be unique. At the level of TGD inspired theory of consciousness the p-adicization becomes modelling of how cognition works.

In this conceptual framework the successes of the p-adic mass calculations can be understood only if p-adic mass calculations provide a model a 'cognitive model' of an elementary particle. The successes of the p-adic mass calculations, and also the fact that they rely on the fundamental symmetries of quantum TGD, encourages the idea that one could try to mimic Nature. Thus p-adic physics could be seen as an abstract mimicry for what Nature already does by constructing explicitly p-adic cognitive representations. This new view about p-adic physics allows much more flexibility since p-adicization can be interpreted as a cognitive map mapping real world physics to p-adic physics. In this view p-adicization need not and cannot be a unique procedure.

6.1.2 Number theoretic vision briefly

The number theoretic vision [E1, E2, E3] about the classical dynamics of space-time surfaces is now relatively detailed although it involves unproven conjectures inspired by physical intuition.

1. *Hyper-quaternions and octonions*

The original idea was that space-time surfaces could be regarded as four-surfaces in 8-D imbedding space with the property that the tangent spaces of these spaces can be locally regarded as 4- *resp.* 8-dimensional number fields of quaternions and octonions.

The difficulties caused by the Euclidian metric signature of the number theoretical norm have however forced to give up the original idea as such, and to introduce complexified octonions and quaternions resulting by extending quaternionic and octonionic algebra by adding imaginary units multiplied with $\sqrt{-1}$. This spoils the number field property but the notion of prime is not lost. The sub-space of hyper-quaternions *resp.* -octonions is obtained from the algebra of ordinary quaternions and octonions by multiplying the imaginary part with $\sqrt{-1}$. The transition is the number theoretical counterpart of the transition from Riemannian to pseudo-Riemannin geometry performed already in Special Relativity.

The problem is that $H = M^4 \times CP_2$ cannot be endowed with a hyper-octonionic manifold structure. Indeed, space-time surfaces are assumed to be hyper-quaternionic or co-hyper-quaternionic 4-surfaces of 8-dimensional Minkowski space M^8 identifiable as the hyper-octonionic space HO. Since the hyper-quaternionic sub-spaces of HO with fixed complex structure are labelled by CP_2, each (co)-hyper-quaternionic four-surface of HO defines a 4-surface of $M^4 \times CP_2$. One can say that the number-theoretic analog of spontaneous compactification occurs.

2. Space-time-surface as a hyper-quaternionic sub-manifold of hyper-octonionic imbedding space?

Space-time identified as a hyper-quaternionic sub-manifold of the hyper-octonionic space in the sense that the tangent space of the space-time surface defines a hyper-quaternionic sub-algebra of the hyper-octonionic tangent space of H at each space-time point, looks an attractive idea. Second possibility is that the tangent space-algebra of the space-time surface is either associative or co-associative at each point. One can also consider possibility that the dynamics of the space-time surface is determined from the requirement that space-time surface is algebraically closed in the sense that tangent space at each point has this property. Also the possibility that the property in question is associated with the normal space at each point of X^4 can be considered. Some delicacies are caused by the question whether the induced algebra at X^4 is just the hyper-octonionic product or whether the algebra product is projected to the space-time surface. If normal part of the product is projected out the space-time algebra closes automatically.

The first guess would be that space-time surfaces are hyper-quaternionic sub-manifolds of hyper-octonionic space $HO = M^8$ with the property that complex structure is fixed and same at all points of space-time surface. This corresponds to a global selection of a preferred octonionic imaginary unit. The automorphisms leaving this selection invariant form group $SU(3)$ identifiable as color group. The selections of hyper-quaternionic sub-space under this condition are parameterized by CP_2. This means that each 4-surface in HO defines a 4-surface in $M^4 \times CP_2$ and one can speak about number-theoretic analog of spontaneous compactification having of course nothing to do with dynamics. It would be possible to make physics in two radically different geometric pictures: HO picture and $H = M^4 \times CP_2$ picture.

For a theoretical physicists of my generation it is easy to guess that the next step is to realize that it is possible to fix the preferred octonionic imaginary at each point of HO separately so that local $S^6 = G_2/SU(3)$, or equivalently the local group G_2 subject to $SU(3)$ gauge invariance, characterizes the possible choices of hyper-quaternionic structure with a preferred imaginary unit. $G_2 \subset SO(7)$ is the automorphism group of octonions, and appears also in M-theory. This local choice has interpretation as a fixing of the plane of non-physical polarizations and rise to degeneracy which is a good candidate for the ground state degeneracy caused by the vacuum extremals.

$OH - -M^4 \times CP_2$ duality allows to construct a foliation of HO by hyper-quaternionic space-time surfaces in terms of maps $HO \rightarrow SU(3)$ satisfying certain integrability conditions guaranteing that the distribution of hyper-quaternionic planes integrates to a foliation by 4-surfaces. In fact, the freedom to fix the preferred imaginary unit locally extends the maps to $HO \rightarrow G_2$ reducing to maps $HO \rightarrow SU(3) \times S^6$ in the local trivialization of G_2. This foliation defines a four-parameter family of 4-surfaces in $M^4 \times CP_2$ for each local choice of the preferred imaginary unit. The dual of this foliation defines a 4-parameter famility co-hyper-quaternionic space-time surfaces.

Hyper-octonion analytic functions $HO \rightarrow HO$ with real Taylor coefficients provide a physically motivated ansatz satisfying the integrability conditions. The basic reason is that hyper-octonion analyticity is not plagued by the complications due to non-commutativity and non-associativity. Indeed, this notion results also if the product is Abelianized by assuming that different octonionic imaginary units multiply to zero. A good candidate for the HO dynamics is free massless Dirac action with Weyl condition for an octonion valued spinor field using octonionic representation of gamma matrices and coupled to the G_2 gauge potential defined by the tensor 7×7 tensor product of the imaginary parts of spinor fields.

The basic conjecture is that the absolute minima of Kähler action correspond to the hyper-quaternion analytic surfaces. This conjecture has several variants. It could be that only asymptotic behavior corresponds to hyper-quaternion analytic function but that that hyper-quaternionicity is general property of absolute minima. It could also be that maxima of Kähler function correspond to this kind of 4-surfaces. The encouraging hint is the fact that Hamilton-Jacobi coordinates appear naturally also in the construction of general solutions of field equations.

3. The notion of Kähler calibration

Calibration is a closed p-form, whose value for a given p-plane is not larger than its volume in the induced metric. What is important that if it is maximum for tangent planes of p-sub-manifold, minimal surface with smallest volume in its homology equivalence class results.

The idea of Kähler calibration is based on a simple observation. The octonionic spinor field defines a map $M^8 \rightarrow H = M^4 \times CP_2$ allowing to induce metric and Kähler form of H to M^8. Also Kähler action is well defined for the local hyper-quaternion plane.

The idea is that the non-closed 4-form associated the wedge product of unit tangent vectors of hyper-quaternionic plane in M^8 and saturating to volume for it becomes closed by multiplication with Kähler action density L_K. If L_K is minimal for hyper-quaternion plane, hyper-quaternionic manifolds define extremals of Kähler action for which the magnitudes of positive and negative contributions to the action are separately minimized.

This variational principle is not equivalent with the absolute minimization of Kähler action. Rather, Universe would do its best to save energy, being as near as possible to vacuum. Also vacuum extremals would become physically relevant (they carry non-vanishing density gravitational energy). The non-determinism of the vacuum extremals would have an interpretation in terms of the ability of Universe to engineer itself. The attractiveness of the number theoretical variational principle from the point of calculability of TGD would be that the initial values for the time derivatives of the imbedding space coordinates at X^3 at light-like 7-D causal determinant could be computed by requiring that the energy of the solution is minimized. This could mean a computerizable construction of Kähler function.

4. The representation of infinite hyper-octonionic primes as 4-surfaces

The discovery of infinite primes suggested strongly the possibility to reduce physics to number theory. The construction of infinite primes can be regarded as a repeated second quantization of a super-symmetric arithmetic quantum field theory. This hierarchy of second quantizations means an enormous generalization of physics to what might be regarded a physical counterpart for a hierarchy of abstractions about abstractions about.... The ordinary second quantized quantum physics corresponds only to the lowest level infinite primes. This hierarchy can be identified with the corresponding hierarchy of space-time sheets of the many-sheeted space-time.

One can even try to understand the quantum numbers of physical particles in terms of infinite primes. In particular, the hyper-quaternionic primes correspond four-momenta and mass squared is prime valued for them. The properties of 8-D hyper-octonionic primes motivate the attempt to identify the quantum numbers associated with CP_2 degrees of freedom in terms of these primes. Infinite primes can be mapped to polynomial primes and this observation allows to identify completely generally the spectrum of infinite primes.

This in turn led to the idea that it might be possible represent infinite primes (integers) geometrically as surfaces defined by the polynomials associated with infinite primes (integers). Obviously, infinite primes would serve as a bridge between Fock-space descriptions and geometric descriptions of physics: quantum and classical. Geometric objects could be seen as concrete representations of infinite numbers providing amplification of infinitesimals to macroscopic deformations of space-time surface. We see the infinitesimals as concrete geometric shapes!

Since the notion of prime makes sense for the complexified octonions, it makes sense also for the hyper-octonions. It is possible to assign to infinite prime of this kind a hyper-octonion analytic polynomial $P : OH \rightarrow OH$ and hence also a foliation of OH and $H = M^4 \times CP_2$ by hyper-quaternionic 4-surfaces and notion of Kähler calibration. Therefore space-time surface could be seen as a geometric counterpart of a Fock state. The assignment is not unique but determined only up to an element of the local octonionic automorphism group G_2 acting in HO and fixing the local choices of the preferred imaginary unit of the hyper-octonionic tangent plane. In fact, a map $HO \rightarrow S^6$ characterizes the choice since $SO(6)$ acts effectively as a local gauge group.

The construction generalizes to all levels of the hierarchy of infinite primes and produces also representations for integers and rationals associated with hyper-octonionic numbers as space-time surfaces. A close relationship with algebraic geometry results and the polynomials define a natural hierarchical structure in the space of 3-surfaces. By the effective 2-dimensionality naturally associated with infinite primes represented by real polynomials 4-surfaces are determined by data given at partonic 2-surfaces defined by the intersections of 3-D and 7-D light-like causal determinants. In particular, the notions of genus and degree serve as classifiers of the algebraic geometry of the 4-surfaces. The great dream is to prove that this construction yields the solutions to the absolute minimization of Kähler action.

6.1.3 p-Adic space-time sheets as solutions of real field equations continued algebraically to p-adic number field

The ideas about how p-adic topology emerges from quantum TGD have varied. The first belief was that p-adic topology is only an effective topology of real space-time sheets. This belief turned out to be not quite correct. p-Adic topology emerges also as a genuine topology of the space-time and p-adic regions could be identified as correlations for cognition and intentionality. This requires a generalization of the notion of number by gluing reals and various p-adic number fields together along common rationals. This in turn implies generalization of the notion of imbedding space. p-Adic transcendentals can be regarded as infinite numbers in the real sense and thus most points of the p-adic space-time sheets would be at infinite distance and real and p-adic space-time sheets would intersect in discrete set consisting of rational points. This view in which cognition and intentionality would be literally cosmic phenomena is in a sharp contrast with the often held belief that p-adic topology emerges below Planck length scale.

6.1.4 The notion of pinary cutoff

The notion of pinary cutoff is central for p-adic TGD and it should have some natural definition and interpretation in the new approach. The presence of p-adic pseudo constants implies that there is large number of cognitive representations with varying degrees of faithfulness. Pinary cutoff must serve as a measure for how faithful the p-adic cognitive representation is. Since the cognitive maps are not unique, one cannot even require any universal criterion for the faithfullness of the cognitive map. One can indeed imagine two basic criteria corresponding to self-representations and representations for external world.

1. The subset of rationals common to the real and p-adic space-time surface could define the resolution. In this case, the average distance between common rational points of these two surfaces would serve as a measure for the resolution. Pinary cutoff could be defined as the smallest number of pinary digits in expansions of functions involved above which the resolution does not improve. Physically the optimal resolution would mean that p-adic space-time surface, 'cognitive space-time sheet', has a maximal number of intersections with the real space-time surface for which it provides a self-representation. This purely algebraic notion of faithfullness does not respect continuity: two rational points very near in real sense could be arbitrary far from each other with respect to the p-adic norm.

2. One could base the notion of faithfulness on the idea that p-adic space-time sheet provides almost continuous map of the real space-time sheet belonging to the external world by the basic properties of the canonical identification. The real canonical image of the p-adic space-time sheet and real space-time sheet could be compared and some geometric measure for the nearness of these surfaces could define the resolution of the cognitive map and pinary cutoff could be defined in the same manner as above.

6.1.5 Program

These ideas lead to a rather well defined p-adicization program. Define precisely the concepts of the p-adic space-time and reduced configuration space, formulate the finite-p p-adic versions of quantum TGD and construct the p-adic variants of TGD. Of course, the aim is not to just construct p-adic version of the real quantum TGD but to understand how real and p-adic quantum TGD:s fuse together to form the full theory of physics and cognition.

The construction of the p-adic TGD necessitates the generalization of the basic tools of standard physics such as differential and integral calculus, the concept of Hilbert space, Riemannian geometry, group theory, action principles, probability and unitary concepts to p-adic context. Also new physical thinking and philosophy is needed and this long chapter is devoted to the description of the new elements. Before going to the detailed exposition it is appropriate to give a brief overall view of the basic mathematical tools.

6.2 p-Adic numbers and consciousness

The idea that p-adic physics provides the physics of cognition and intentionality has become more and more attractive during the 12 years or so that I have spent with p-adic numbers and I feel that it is good to add a summary about these ideas here.

6.2.1 p-Adic physics as physics of cognition

p-Adic physics began from p-adic mass calculations. The next step in the progress was the idea that p-adic physics serves as a correlate for cognition and this thread gradually led to the recent view requiring the generalization of the number concept.

Decomposition of space-time surface into p-adic and real regions as representation for matter-mind duality

Space-time surfaces contain genuinely p-adic and possibly even rational-adic regions so that no p-adicization is performed by Nature itself at at this level and it is enough to mimic the Nature. One manner to end up with the idea about p-adic space-time sheets is following.

Number theoretic vision leads to the idea that space-time surfaces can be associated with a hierarchy of polynomials to which infinite primes are mapped. It can happen that the components of quaternion are not always in algebraic extension of rationals but become complex. In this case the equations might however allow smooth solutions in some algebraic extension of p-adics for some values of prime p. It could also happen that real and p-adic roots exist simultaneously. In both cases the interpretation would be that the p-adic space-time sheets resulting as roots of the rational function provide self-representations for the real space-time sheets represented by real roots. This p-adicization would occur in the regions where some roots of the rational polynomial is complex or real roots exist also in the p-adic sense.

The dynamically generated p-adic space-time sheets could have a common boundary with the real surface in the following sense. At this surface a real root is transformed to a p-adic root and this surface corresponds to a boundary of catastrophe region in catastrophe theory. This boundary provides information about external real world very much in accordance with how nervous system receives information about the external world and makes possible cognitive representations about external world. Since the conditions defining the space-time surface expresses the vanishing of a derivative, the solution involves p-adic pseudo constants so that the cognitive representations are not unique and system can have more or less faithful cognitive representations about itself and about external world.

Rational points of the imbedding space and thus also of space-time surfaces are common to p-adics and reals and p-adic and real space-time surfaces differ only in that completion is different. This fixes the geometric interpretation of the cognitive maps involved with the p-adicization.

Different kinds of cognitive representations

At the level of the space-time surfaces and imbedding space p-adicization boils down to the task of finding a map mapping real space-time region to a p-adic space-time region. These regions correspond to definite regions of the rational imbedding space so that the map has a clear geometric interpretation at the level of rational physics.

The basic constraint on the map is that both real and p-adic space-time regions satisfy field equations: p-adic field equations make sense even if the integral defining the Kähler action does not exist p-adically. p-Adic nondeterminism makes possible this map when one allows finite pinary cutoff characterizing the resolution of the cognitive representation.

There are three basic types of cognitive representations which might be called self-representations and representations of the external world and the the map mediating p-adicization is different for these two maps.

1. The correspondence induced by the common rational points respects algebraic structures and defines self-representation. Real and p-adic space-time surfaces have a subset of rational points (defined by the resolution of the cognitive map) as common. The quality of the representation is defined by the resolution of the map and pinary cutoff for the rationals in pinary expansion is a natural measure for the resolution just as decimal cutoff is a natural measure for the resolution of a numerical model.

2. Canonical identification maps rationals to rationals since the periodic pinary expansion of a rational is mapped to a periodic expansion in the canonical identification. The rationals $q = m/n$ for which n is not divisible by p are mapped to rationals with p-adic norm not larger than unity. Canonical identification respects continuity. Real numbers with real norm larger than p are mapped to real numbers with norm smaller than one in canonical identification whereas reals with real norm in the interval $[1, p)$ are mapped to p-adics with p-adic norm equal to one. Obviously the generalization of the canonical identification can map the world external to a given space-time region into the interior of this region and provides an example of an abstract cognitive representation of the external world. Also now pinary cutoff serves as a natural measure for the quality of the cognitive map.

3. The basic problems of canonical identification is that it does not respect unitarity. For this reason it is not well suited for relating p-adic and real scattering amplitudes. The problem of the correspondence via direct rationals is that it does not respect continuity. A compromise between algebra and topology is achieved by using a modification of canonical identification $I_{R_p \to R}$ defined as $I_1(r/s) = I(r)/I(s)$. If the conditions $r \ll p$ and $s \ll p$ hold true, the map respects algebraic operations and also unitarity and various symmetries.

   This variant of canonical identification is not equivalent with the original one using the infinite expansion of q in powers of p since canonical identification does not commute with product and division. The variant is however unique in the recent context when r and s in $q = r/s$ have no common factors. For integers $n < p$ it reduces to direct correspondence.

   It seems that this option, the discovery of which took almost a decade, must be used to relate p-adic transition amplitudes to real ones and vice versa [F5]. In particular, real and p-adic coupling constants are related by this map. Also some problems related to p-adic mass calculations find a nice resolution when I_1 is used.

A fascinating possibility is that cognitive self-maps and maps of the external world at the level of human brain are basically realized by using these two basic types of mappings. Obviously canonical identification performed separately for all coordinates is the only possibility if this map is required to be maximally continuous.

p-Adic physics as a mimicry of p-adic cognitive representations

The success of the p-adic mass calculations suggests that one could apply the idea of p-adic cognitive representation even at the level of quantum TGD to build models which have maximal simplicity and calculational effectiveness. p-Adic mass calculations represent this kind of model: now canonical identification is performed for the p-adic mass squared values and can be interpreted as a map from cognitive representation back to real world.

The basic task is the construction of the cognitive self-map or a cognitive map of external world: the laws of p-adic physics define the cognitive model itself automatically. For the cognitive representations of external world involving some variant of canonical identification mapping the exterior of the imbedding space region inside this region. For self-representations situation is much more simpler. In practice, the direct modelling of p-adic physics without explicit construction of the cognitive map could give valuable information about real physics.

In the earlier approach based on phase preserving canonical identification to the mapping of real space-time surface to its p-adic counterpart led to the requirement about existence of unique (almost) imbedding space coordinates. In present case the selection of the quaternionic coordinates for the imbedding space is unique only apart from quaternion-analytic change of coordinates. This does not seem however pose any problems now. One must also remember that only cognitive representations are in question. These representations are not unique and selection of quaternionic coordinates might be even differentiate between different cognitive representations.

Since infinite primes serve as a bridge between classical and quantum, this map also assigns to a real Fock state associated with infinite prime its p-adic version identifiable as the ground state of a superconformal representation. Thus the map respects quantum symmetries automatically. If the construction of the states of the representation is a completely algebraic process, there are hopes of constructing the p-adic counterpart of S-matrix. If S-matrix is complex rational it can be mapped to its real counterpart. If the localization in zero modes occurs in each quantum jump the predictions of the theory could reduce to the integration in fiber degrees of freedom of CH reducible in turn to purely algebraic expressions making sense also p-adically.

6.2.2 Zero energy ontology, cognition, and intentionality

One could argue that conservation laws forbid p-adic-real phase transitions in practice so that cognitions (intentions) realized as real-to-padic (p-adic-to-real) transitions would not be possible. The situation changes if one accepts what might be called zero energy ontology [C1, C2].

Zero energy ontology classically

In TGD inspired cosmology [D5] the imbeddings of Robertson-Walker cosmologies are vacuum extremals. Same applies to the imbeddings of Reissner-Nordström solution [D3] and in practice to all solutions of Einstein's equations imbeddable as extremals of Kähler action. Since four-momentum currents define a collection of vector fields rather than a tensor in TGD, both positive and negative signs for energy corresponding to two possible assignments of the arrow of the geometric time to a given space-time surface are possible. This leads to the view that all physical states have vanishing net energy classically and that physically acceptable universes are creatable from vacuum.

The result is highly desirable since one can avoid unpleasant questions such as "What are the net values of conserved quantities like rest mass, baryon number, lepton number, and electric charge for the entire universe?", "What were the initial conditions in the big bang?", "If only single solution of field equations is selected, isn't the notion of physical theory meaningless since in principle it is not possible to compare solutions of the theory?". This picture fits also nicely with the view that entire universe understood as quantum counterpart 4-D space-time is recreated in each quantum jump and allows to understand evolution as a process of continual re-creation.

Zero energy ontology at quantum level

Also the construction of S-matrix [C2] leads to the conclusion that all physical states possess vanishing conserved quantum numbers. Furthermore, the entanglement coefficients between positive and negative energy components of the state define a unitary S-matrix. S-matrix thus becomes a property of the zero energy state and physical states code by their structure what is usually identified as quantum dynamics.

Also the transitions between zero energy states are possible but general arguments lead to the conclusion that the corresponding S-matrix is almost trivial. This finding, which actually forced the new view about S-matrix, is highly desirable since it explains why positive energy ontology works so well if one forgets effects related to intentional action.

At space-time level this would mean that positive energy component and negative energy component are at a temporal distance characterized by an appropriate p-adic time scale and the integer characterizing the value of Planck constant for the state in question. The scale in question would also characterize the geometric duration of quantum jump and the size scale of space-time region contributing to the contents of conscious experience. The interpretation in terms of a mini bang followed by a mini crunch suggests itself also.

Hyper-finite factors of type II_1 and new view about S-matrix

The representation of S-matrix as unitary entanglement coefficients would not make sense in ordinary quantum theory but in TGD the von Neumann algebra in question is not a type I factor as for quantum mechanics or a type III factor as for quantum field theories, but what is called hyper-finite factor of type II_1 [C6]. This algebra is an infinite-dimensional algebra with the almost defining, and at the first look very strange, property that the infinite-dimensional unit matrix has unit trace. The infinite dimensional Clifford algebra spanned by the configuration space gamma matrices (configuration space understood as the space of 3-surfaces, the "world of classical worlds") is indeed very naturally algebra of this kind since infinite-dimensional Clifford algebras provide a canonical representations for hyper-finite factors of type II_1.

The new view about quantum measurement theory

This mathematical framework leads to a new kind of quantum measurement theory. The basic assumption is that only a finite number of degrees of freedom can be quantum measured in a given measurement and the rest remain untouched. What is known as Jones inclusions $\mathcal{N} \subset \mathcal{M}$ of von Neumann algebras allow to realize mathematically this idea [C6]. $\mathcal{N}$ characterizes measurement resolution and quantum measurement reduces the entanglement in the non-commutative quantum space $\mathcal{M}/\mathcal{N}$. The outcome of the quantum measurement is still represented by a unitary S-matrix but in the space characterized by $\mathcal{N}$. It is not possible to end up with a pure state with a finite sequence of quantum measurements.

The obvious objection is that the replacement of a universal S-matrix coding entire physics with a state dependent unitary entanglement matrix is too heavy a price to be paid for the resolution of the above mentioned paradoxes. Situation could be saved if the S-matrices have fractal structure. The quantum criticality of TGD Universe indeed implies fractality. The possibility of an infinite sequence of Jones inclusions for hyperfinite type II_1 factors isomorphic as von Neumann algebras expresses this fractal character algebraically. Thus one can hope that the S-matrix appearing as entanglement coefficients is more or less universal in the same manner as Mandelbrot fractal looks more or less the same in all length scales and for all resolutions. Whether this kind of universality must be posed as an additional condition on entanglement coefficients or is an automatic consequence of unitarity in type II_1 sense is an open question.

The S-matrix for p-adic-real transitions makes sense

In zero energy ontology conservation laws do not forbid p-adic-real transitions and one can develop a relatively concrete vision about what happens in these kind of transitions. The starting point is the generalization of the number concept obtained by gluing p-adic number fields and real numbers along common rationals (expressing it very roughly). At the level of the imbedding space this means that p-adic and real space-time sheets intersect only along common rational points of the imbedding space and transcendental p-adic space-time points are infinite as real numbers so that they can be said to be infinite distant points so that intentionality and cognition become cosmic phenomena.

In this framework the long range correlations characterizing p-adic fractality can be interpreted as being due to a large number of common rational points of imbedding space for real space-time sheet and p-adic space-time sheet from which it resulted in the realization of intention in quantum jump. Thus real physics would carry direct signatures about the presence of intentionality. Intentional behavior is indeed characterized by short range randomness and long range correlations.

One can even develop a general vision about how to construct the S-matrix elements characterizing the process [C2]. The basic guideline is the vision that real and various p-adic physics as well as their hybrids are continuable from the rational physics. This means that these S-matrix elements must be characterizable using data at rational points of the imbedding space shared by p-adic and real space-time sheets so that more or less same formulas describe all these S-matrix elements. Note that also $p_1 \rightarrow p_2$ p-adic transitions are possible.

6.3 An overall view about p-adicization of TGD

In real context the coordinatization of manifold is regarded as a trivial problem. It took long time to realize that in p-adic context the proper treatment of coordinatization problem leads to deep insights about p-adic symmetries and about the origin og the p-adic length scales hypothesis. There are several approaches to the construction of the p-adic Riemann geometry. The most simple minded approach relies on a direct generalization of the real line element and to the proposed integral for p-adically analytic functions. A more refined approach relies on the general physical consistency conditions provided by quantum TGD and by the proposed definition of the Riemann integral.

6.3.1 p-Adic Riemannian geometry

p-Adic Riemann geometry is a direct formal generalization of the ordinary Riemann geometry. In the minimal generalization one does not try to define concepts like arch length and volume involving definite integrals but simply defines the p-adic geometry via the metric identified as a quadratic form in the tangent space of the p-adic manifold. One could also formally calculate p-adic arc lengths, areas, etc.. since canonical identification makes it possible to define p-adic Riemann integral. Also p-adic Fourier analysis could make possible to define the integrals in question. It seems however that these concepts are not needed in the formulation of QFT limit.

Group theoretical considerations dictate the p-adic counterpart of the Riemann geometry for $M^4_+ \times CP_2$ essentially uniquely. The most natural looking manner to define the p-adic counterpart of M^4_+ and CP_2 is by using a p-adic completion for a subset of rational points in coordinates which are preferred on physical basis. In case of M^4_+ linear Minkowski coordinates are an obvious choice. Rational CP_2 could be defined as a coset space $SU(3,Q)/U(2,Q)$ associated with complex rational unitary 3×3-matrices. CP_2 could be defined as coset space of complex rational matrices by choosing one point in each coset $SU(3,Q)/U(2,Q)$ as a complex rational 3×3-matrix representable in terms of Pythagorean phases and performing a completion for the elements of this matrix by multiplying the elements with the p-adic exponentials $exp(iu)$, $|u|_p < 1$ such that one obtains p-adically unitary matrix.

6.3.2 p-Adic imbedding space

It has become clear that the construction of both quantum TGD and p-adic QFT limit requires p-adicization of the imbedding space geometry. Also the fact that p-adic Poincare invariance throws considerable light to the p-adic length scale hypothesis suggests that p-adic geometry is really needed. The construction of the p-adic version of the imbedding space geometry and spinor structure relies on the symmetry arguments and to the generalization of the analytic formulas of the real case almost as such by requiring that the real counterpart for the length of the infinitesimal geodesic line segment is in the lowest order same as the corresponding real length. This approach leads to a highly nontrivial generalization of the symmetry concept and p-adic Poincare invariance throws light to the p-adic length scale hypothesis. An important delicacy is related to the identification of the fundamental p-adic length scale, which corresponds to the unit element of the p-adic number field and is mapped to the unit element of the real number field in the canonical identification.

The identification of the fundamental p-adic length scale

The fundamental p-adic length scale correponds to the p-adic unit $e = 1$ and is mapped to the unit of the real numbers in the canonical identification. The correct physical identification of the fundamental p-adic length scale is of crucial importance since the predictions of the theory for p-adic masses depend on the choice of this scale.

In TGD the 'radius' R of CP_2 is the fundamental length scale ($2\pi R$ is by definition the length of the CP_2 geodesics). In accordance with the idea that p-adic QFT limit makes sense only above length scales larger than the radius of CP_2 R is of same order of magnitude as the p-adic length scale defined as $l = \pi/m_0$, where m_0 is the fundamental mass scale and related to the 'cosmological constant' Λ ($R_{ij} = \Lambda s_{ij}$) of CP_2 by

$$m_0^2 = 2\Lambda \ . \tag{6.3.1}$$

The relationship between R and l is uniquely fixed:

$$R^2 = \frac{3}{m_0^3} = \frac{3}{2\Lambda} = \frac{3l^2}{\pi^2} \ . \tag{6.3.2}$$

Consider now the identification of the fundamental length scale.

1. One must use R^2 or its integer multiple, rather than l^2, as the fundamental p-adic length scale squared in order to avoid the appearance of the p-adically ill defined π:s in various formulas of CP_2 geometry.

2. The identification for the fundamental length scale as $1/m_0$ leads to difficulties.

   (a) The p-adic length for the CP_2 geodesic is proportional to $\sqrt{3}/m_0$. For the physically most interesting p-adic primes satisfying $p \ mod \ 4 = 3$ so that $\sqrt{-1}$ does not exist as an ordinary p-adic number, $\sqrt{3} = i\sqrt{-3}$ belongs to the complex extension of the p-adic numbers. Hence one has troubles in getting real length for the CP_2 geodesic.

   (b) If m_0^2 is the fundamental mass squared scale then general quark states have mass squared, which is integer multiple of $1/3$ rather than integer valued as in string models.

3. These arguments suggest that the correct choice for the fundamental length scale is as $1/R$ so that $M^2 = 3/R^2$ appearing in the mass squared formulas is p-adically real and all values of the mass squared are integer multiples of $1/R^2$. This does not affect the real counterparts of the thermal expectation values of the mass squared in the lowest p-adic order but the effects, which are due to the modulo arithmetics, are seen in the higher order contributions to the mass squared. As a consequence, one must identify the p-adic length scale l as

   $$l \equiv \pi R \ ,$$

   rather than $l = \pi/m_0$. This is indeed a very natural identification. What is especially nice is that this identification also leads to a solution of some longstanding problems related to the p-adic mass calculations. It would be highly desirable to have the same p-adic temperature $T_p = 1$ for both the bosons and fermions rather than $T_p = 1/2$ for bosons and $T_p = 1$ for fermions. For instance, black hole elementary particle analogy as well as the need to get rid of light boson exotics suggests this strongly. It indeed turns out possible to achieve this with the proposed identification of the fundamental mass squared scale.

p-Adic counterpart of M^4_+

The construction of the p-adic counterpart of M^4_+ seems a relatively straightforward task and should reduce to the construction of the p-adic counter part of the real axis with the standard metric. As already noticed, linear Minkowksi coordinates are physically and mathematically preferred coordinates and it is natural to construct the metric in these coordinates.

There are some quite interesting delicacies related to the p-adic version of the Poincare invariance. Consider first translations. In order to have imaginary unit needed in the construction of the ordinary representations of the Poincare group one must have $p \ mod \ 4 = 3$ to guarantee that $\sqrt{-1}$ does not exist as an ordinary p-adic number. It however seems that the construction of the representations is at least formally possible by replacing imaginary unit with the square root of some other p-adic number not existing as a p-adic number.

It seems that only the discrete group of translations allows representations consisting of orthogonal planewaves. p-Adic planewaves can be defined in the lattice consisting of the multiples of $x_0 = m/n$

consisting of points with p-adic norm not larger that $|x_0|_p$ and the points $p^n x_0$ define fractally scaled-down versions of this set. In canonical identification these sets corresponds to volumes scaled by factors p^{-n}.

A physically interesting question is whether the Lorentz group should contain only the elements obtained by exponentiating the Lie-algebra generators of the Lorentz group or whether also large Lorentz transformations, containing as a subgroup the group of the rational Lorentz transformations, should be allowed. If the group contains only small Lorentz transformations, the quantization volume of M^4_+ (say the points with coordinates m^k having p-adic norm not larger than one) is also invariant under Lorentz transformations. This means that the quantization of the theory in the p-adic cube $|m^k| < p^n$ is a Poincare invariant procedure unlike in the real case.

The appearance of the square root of p, rather than the naively expected p, in the expression of the p-adic length scale can be undertood if the p-adic version of M^4 metric contains p as a scaling factor:

$$\begin{aligned} ds^2 &= pR^2 m_{kl} dm^k dm^l \ , \\ R &\leftrightarrow 1 \ , \end{aligned} \tag{6.3.2}$$

where m_{kl} is the standard M^4 metric $(1,-1,-1,-1)$. The p-adic distance function is obtained by integrating the line element using p-adic integral calculus and this gives for the distance along the k:th coordinate axis the expression

$$s = R\sqrt{p} m^k \ . \tag{6.3.3}$$

The map from p-adic M^4 to real M^4 is canonical identification plus a scaling determined from the requirement that the real counterpart of an infinitesimal p-adic geodesic segment is same as the length of the corresponding real geodesic segment:

$$m^k \rightarrow \pi(m^k)_R \ . \tag{6.3.4}$$

The p-adic distance along the k:th coordinate axis from the origin to the point $m^k = (p-1)(1 + p + p^2 + ...) = -1$ on the boundary of the set of the p-adic numbers with norm not larger than one, corresponds to the fundamental p-adic length scale $L_p = \sqrt{p} l = \sqrt{p}\pi R$:

$$\sqrt{p}((p-1)(1+p+...))R \rightarrow \pi R \frac{(p-1)(1+p^{-1}+p^{-2}+...)}{\sqrt{p}} = L_p \ . \tag{6.3.4}$$

What is remarkable is that the shortest distance in the range $m^k = 1, ..m-1$ is actually $L/\sqrt{p}$ rather than l so that p-adic numbers in range span the entire R_+ at the limit $p \rightarrow \infty$. Hence p-adic topology approaches real topology in the limit $p \rightarrow \infty$ in the sense that the length of the discretization step approaches to zero.

CP_2 as a p-adic coset space

In case of CP_2 one can proceed by defining the p-adic counterparts of $SU(3)$ and $U(2)$ and using the identification $CP_2 = SU(3)/U(2)$. The p-adic counterpart of $SU(3)$ consists of all 3×3 unitary matrices satisfying p-adic unitarity conditions (rows/colums are mutually orthogonal unit vectors) or its suitable subgroup: the minimal subgroup corresponds to the exponentials of the Lie-algebra generators. If one allows algebraic extensions of the p-adic numbers, one obtains several extensions of the group. The extension allowing the square root of a p-adically real number is the most interesting one in this respect since the general solution of the unitarity conditions involves square roots.

The subgroup of $SU(3)$ obtained by exponentiating the Lie-algebra generators of $SU(3)$ normalized so that their nonvanishing elements have unit p-adic norm, is of the form

$$SU(3)_0 = \{x = exp(\sum_k it_k X_k) \ ; \ |t_k|_p < 1\} = \{x = 1 + iy \ ; \ |y|_p < 1\} \ . \tag{6.3.5}$$

The diagonal elements of the matrices in this group are of the form $1 + O(p)$. In order $O(p)$ these matrices reduce to unit matrices.

Rational $SU(3)$ matrices do not in general allow a representation as an exponential. In the real case all $SU(3)$ matrices can be obtained from diagonalized matrices of the form

$$h = diag\{exp(i\phi_1), exp(i\phi_2), exp(exp(-i(\phi_1 + \phi_2)\} \ . \tag{6.3.6}$$

The exponentials are well defined provided that one has $|\phi_i|_p < 1$ and in this case the diagonal elements are of form $1+O(p)$. For $p \ mod \ 4 = 3$ one can however consider much more general diagonal matrices

$$h = diag\{z_1, z_2, z_3\} \ ,$$

for which the diagonal elements are rational complex numbers

$$z_i = \frac{(m_i + in_i)}{\sqrt{m_i^2 + n_i^2}} \ ,$$

satisfying $z_1z_2z_3 = 1$ such that the components of z_i are integers in the range $(0, p-1)$ and the square roots appearing in the denominators exist as ordinary p-adic numbers. These matrices indeed form a group as is easy to see. By acting with $SU(3)_0$ to each element of this group and by applying all possible automorphisms $h \rightarrow ghg^{-1}$ using rational $SU(3)$ matrices one obtains entire $SU(3)$ as a union of an infinite number of disjoint components.

The simplest (unfortunately not physical) possibility is that the 'physical' $SU(3)$ corresponds to the connected component of $SU(3)$ represented by the matrices, which are unit matrices in order $O(p)$. In this case the construction of CP_2 is relatively straightforward and the real formalism should generalize as such. In particular, for $p \ mod \ 4 = 3$ it is possible to introduce complex coordinates ξ_1, ξ_2 using the complexification for the Lie-algebra complement of $su(2) \times u(1)$. The real counterparts of these coordinates vary in the range $[0, 1)$ and the end points correspond to the values of t_i equal to $t_i = 0$ and $t_i = -p$. The p-adic sphere S^2 appearing in the definition of the p-adic light cone is obtained as a geodesic submanifold of CP_2 ($\xi_1 = \xi_2$ is one possibility). From the requirement that real CP_2 can be mapped to its p-adic counterpart it is clear that one must allow all connected components of CP_2 obtained by applying discrete unitary matrices having no exponential representation to the basic connected component. In practice this corresponds to the allowance of all possible values of the p-adic norm for the components of the complex coordinates ξ_i of CP_2.

The simplest approach to the definition of the CP_2 metric is to replace the expression of the Kähler function in the real context with its p-adic counterpart. In standard complex coordinates for which the action of $U(2)$ subgroup is linear, the expression of the Kähler function reads as

$$\begin{aligned} K &= log(1 + r^2) \ , \\ r^2 &= \sum_i \bar{\xi}_i \xi_i \ . \end{aligned} \tag{6.3.6}$$

p-Adic logarithm exists provided r^2 is of order $O(p)$. This is the case when ξ_i is of order $O(p)$. The definition of the Kähler function in a more general case, when all possible values of the p-adic norm are allowed for r, is based on the introduction of a p-adic pseudo constant C to the argument of the Kähler function

$$K = log(\frac{1 + r^2}{C}) \ .$$

C guarantees that the argument is of the form $\frac{1+r^2}{C} = 1+O(p)$ allowing a well-defined p-adic logarithm. This modification of the Kähler function leaves the definition of Kähler metric, Kähler form and spinor connection invariant.

A more elegant manner to avoid the difficulty is to use the exponent $\Omega = exp(K) = 1 + r^2$ of the Kähler function instead of Kähler function, which indeed well defined for all coordinate values. In terms of Ω one can express the Kähler metric as

$$g_{k\bar{l}} = \frac{\partial_k \partial_{\bar{l}} \Omega}{\Omega} - \frac{\partial_k \Omega \partial_{\bar{l}} \Omega}{\Omega^2} \quad . \tag{6.3.7}$$

The p-adic metric can be defined as

$$s_{i\bar{j}} = R^2 \partial_i \partial_{\bar{j}} K = R^2 \frac{(\delta_{i\bar{j}} r^2 - \bar{\xi}_i \xi_j)}{(1+r^2)^2} \quad . \tag{6.3.7}$$

The expression for the Kähler form is the same as in the real case and the components of the Kähler form in the complex coordinates are numerically equal to those of the metric apart from the factor of i. The components in arbitrary coordinates can be deduced from these by the standard transformation formulas.

It is of considerable interest to find whether and how the concept of the geodesic line could generalize to p-adic context. This need not to be the case since the Riemannian metric could be regarded in p-adic context as a mere bilinear form defined in tangent space: this is how metric is understood also in case of Hilbert spaces. The simplest solutions of the geodesic equations do not contain any pseudo-constants and the analytic expressions for the geodesic lines are the same as in the real context. The length of a geodesic line could be defined by using p-adic integration and canonical identification. One can restrict the consideration to the geodesic submanifold S^2 with the induced metric

$$ds^2 = R^2 \frac{dz d\bar{z}}{(1+r^2)^2} \quad . \tag{6.3.8}$$

Under these assumptions the length of the geodesic segment $z = x \in (0, x_{max})$ extending from the North Pole to the Equator is

$$\frac{s}{R} = \int_0^{x_{max}} \frac{dx}{1+x^2} \quad , \tag{6.3.9}$$

$$\int \frac{dx}{1+x^2} = \arctan(x) = \frac{1}{2i} log(\frac{x-i}{x+i}) \quad . \tag{6.3.10}$$

$arctan(x)$ is well defined for $|x|_p < 1$. At the equator one has however $|x|_p \rightarrow \infty$ and one encounters the problem of defining the integral function properly. One possibility to proceed is by decomposing the integration interval two subintervals $[0, -p)$ and $[1, -p^{-N})$, $N \rightarrow \infty$ using ordering induced by canonical identification and to use the proposed integral formula. The first interval gives automatically a well defined result equal to $arctan(-p)$. The second integral gives zero on the lower boundary also zero on the upper boundary at the limit $N \rightarrow \infty$. Hence one has

$$\frac{s}{R} = arctan(-p) \quad . \tag{6.3.11}$$

The real counterpart of the geodesic line length is

$$(s_{tot})_R = (4s)_R = R(4arctan(-p))_R \leq R < 2\pi R \quad . \tag{6.3.12}$$

For the full geodesic line the length is smaller by a factor of order 2π for large values of p. In particular, the length of the full geodesic is shorter than the distance from the North Pole to the Equator! This is essentially due to the typical cancellation effects taking place in the p-adic summation.

6.3.3 Topological condensate as a generalized manifold

The ideas about how p-adic topology emerges from quantum TGD have varied. The first belief was that p-adic topology is only an effective topology of real space-time sheets. This belief turned out to be not quite correct. p-Adic topology emerges also as a genuine topology of the space-time and p-adic regions could be identified as correlates for cognition and intentionality. The vision about quantum TGD as a generalized number theory provides possible solutions to the basic problems associated with the precise definition of topological condensate.

Generalization of number concept and fusion of real and p-adic physics

The unification of real physics of material work and p-adic physics of cognition and intentionality leads to the generalization of the notion of number field. Reals and various p-adic number fields are glued along their common rationals (and common algebraic numbers too) to form a fractal book like structure. Allowing all possible finite-dimensional extensions of p-adic numbers brings additional pages to this "Big Book".

This generalization leads to a generalization of the notion of manifold as a collection of a real manifold and its p-adic variants glued together along common rationals. The precise formulation involves of course several technical problems. For instance, should one glue along common algebraic numbers and Should one glue along common transcendentals such as e^p? Are algebraic extensions of p-adic number fields glued together along the algebraics too?

This notion of manifold implies a generalization of the notion of imbedding space. p-Adic transcendentals can be regarded as infinite numbers in the real sense and thus most points of the p-adic space-time sheets would be at infinite distance and real and p-adic space-time sheets would intersect in a discrete set consisting of rational points. This view in which cognition and intentionality would be literally cosmic phenomena is in a sharp contrast with the often held belief that p-adic topology emerges below Planck length scale.

It took some time to end up with this vision. The first picture was based on the notion of real and p-adic space-time sheets glued together by using canonical identification or some of its variants but led to insurmountable difficulties since p-adic topology is so different from real topology. One can of course ask whether one can speak about p-adic counterparts of notions like boundary of 3-surface or genus of 2-surface crucial for TGD based model of family replication phenomenon. It seems that these notions generalize as purely algebraically defined concepts which supports the view that p-adicization of real physics must be a purely algebraic procedure.

How large p-adic space-time sheets can be?

Space-time region having finite size in the real sense can have arbitrarily large size in p-adic sense and vice versa. This raises a rather thought provoking questions. Could the p-adic space-time sheets have cosmological or even infinite size with respect to the real metric but have be p-adically finite? How large space-time surface is responsible for the p-adic representation of my body? Could the large or even infinite size of the cognitive space-time sheets explain why creatures of a finite physical size can invent the notion of infinity and construct cosmological theories? Could it be that pinary cutoff $O(p^n)$ defining the resolution of a p-adic cognitive representation would define the size of the space-time region needed to realize the cognitive representation?

In fact, the mere requirement that the neighborhood of a point of the p-adic space-time sheet contains points, which are p-adically infinitesimally near to it can mean that points infinitely distant from this point in the real sense are involved. A good example is provided by an integer valued point $x = n < p$ and the point $y = x + p^m$, $m > 0$: the p-adic distance of these points is p^{-m} whereas at the limit $m \to \infty$ the real distance goes as p^m and becomes infinite for infinitesimally near points. The points $n + y$, $y = \sum_{k>0} x_k p^k$, $0 < n < p$, form a p-adically continuous set around $x = n$. In the real topology this point set is discrete set with a minimum distance $\Delta x = p$ between neighboring points whereas in the p-adic topology every point has arbitrary nearby points. There are also rationals, which are arbitrarily near to each other both p-adically and in the real sense. Consider points $x = m/n$, m and n not divisible by p, and $y = (m/n) \times (1 + p^k r)/(1 + p^k s)$, $s = r + 1$ such that neither r or s is divisible by p and $k >> 1$ and $r >> p$. The p-adic and real distances are $|x - y|_p = p^{-k}$ and $|x - y| \simeq (m/n)/(r + 1)$ respectively. By choosing k and r large enough the points can be made arbitrarily close to each other both in the real and p-adic senses.

The idea about infinite size of the p-adic cognitive space-time sheets providing representation of body and brain is consistent with TGD inspired theory of consciousness, which forces to take very seriously the idea that even human consciousness involves cosmic length scales.

What determines the p-adic primes assignable to a given real space-time sheet?

The p-adic realization of the Slaving Principle suggests that various levels of the topological condensate correspond to real matter like regions and p-adic mind like regions labelled by p-adic primes p. The larger the length scale, the larger the value of p and the course the induced real topology. If the most interesting values of p indeed correspond Mersenne primes, the number of most interesting levels is finite: at most 12 levels below electron length scale: actually also primes near prime powers of two seem to be physically important.

The intuitive expectation is that the p-adic prime associated with a given real space-time sheet characterizes its effective p-adic topology. As a matter fact, several p-adic effective topologies can be considered and the attractive hypothesis is that elementary particles are characterized by integers defined by the product of these p-adic primes and the integers for particles which can have direct interactions possess common prime factors.

The intuitive view is that those primes are favored for with the p-adic space-time sheet obtained by an algebraic continuation has as many rational or algebraic space-time points as possible in common with the real space-time sheet. The rationale is that if the real space-time sheet is generated in a quantum jump in which p-adic space-time sheet is transformed to a real one, it must have a large number of points in common with the real space-time sheet if the probability amplitude for this process involves a sum over the values of an n-point function of a conformal field theory over all common n-tuples and vanishes when the number of common points is smaller than n.

6.3.4 Infinite primes, cognition and intentionality

Somehow it is obvious that infinite primes must have some very deep role to play in quantum TGD and TGD inspired theory of consciousness. What this role precisely is has remained an enigma although I have considered several detailed interpretations, one of them above.

In the following an interpretation allowing to unify the views about fermionic Fock states as a representation of Boolean cognition and p-adic space-time sheets as correlates of cognition is discussed. Very briefly, real and p-adic partonic 3-surfaces serve as space-time correlates for the bosonic super algebra generators, and pairs of real partonic 3-surfaces and their algebraically continued p-adic variants as space-time correlates for the fermionic super generators. Intentions/actions are represented by p-adic/real bosonic partons and cognitions by pairs of real partons and their p-adic variants and the geometric form of Fermi statistics guarantees the stability of cognitions against intentional action. It must be emphasized that this interpretation is not identical with the one discussed above since it introduces different identification of the space-time correlates of infinite primes.

Infinite primes very briefly

Infinite primes have a decomposition to infinite and finite parts allowing an interpretation as a many-particle state of a super-symmetric arithmetic quantum field theory for which fermions and bosons are labelled by primes. There is actually an infinite hierarchy for which infinite primes of a given level define the building blocks of the infinite primes of the next level. One can map infinite primes to polynomials and these polynomials in turn could define space-time surfaces or at least light-like partonic 3-surfaces appearing as solutions of Chern-Simons action so that the classical dynamics would not pose too strong constraints.

The simplest infinite primes at the lowest level are of form $m_B X/s_F + n_B s_F$, $X = \prod_i p_i$ (product of all finite primes). The simplest interpretation is that X represents Dirac sea with all states filled and $X/s_F + s_F$ represents a state obtained by creating holes in the Dirac sea. m_B, n_B, and s_F are defined as $m_B = \prod_i p_i^{m_i}$, $n_B = \prod_i q_i^{n_i}$, and $s_F = \prod_i q_i$, m_B and n_B have no common prime factors. The integers m_B and n_B characterize the occupation numbers of bosons in modes labelled by p_i and q_i and $s_F = \prod_i q_i$ characterizes the non-vanishing occupation numbers of fermions.

The simplest infinite primes at all levels of the hierarchy have this form. The notion of infinite prime generalizes to hyper-quaternionic and even hyper-octonionic context and one can consider the possibility that the quaternionic components represent some quantum numbers at least in the sense that one can map these quantum numbers to the quaternionic primes.

The obvious question is whether configuration space degrees of freedom and configuration space spinor (Fock state) of the quantum state could somehow correspond to the bosonic and fermionic parts of the hyper-quaternionic generalization of the infinite prime. That hyper-quaternionic (or possibly hyper-octonionic) primes would define as such the quantum numbers of fermionic super generators does not make sense. It is however possible to have a map from the quantum numbers labelling super-generators to the finite primes. One must also remember that the infinite primes considered are only the simplest ones at the given level of the hierarchy and that the number of levels is infinite.

Precise space-time correlates of cognition and intention

The best manner to end up with the proposal about how p-adic cognitive representations relate bosonic representations of intentions and actions and to fermionic cognitive representations is through the following arguments.

1. In TGD inspired theory of consciousness Boolean cognition is assigned with fermionic states. Cognition is also assigned with p-adic space-time sheets. Hence quantum classical correspondence suggets that the decomposition of the space-time into p-adic and real space-time sheets should relate to the decomposition of the infinite prime to bosonic and fermionic parts in turn relating to the above mention decomposition of physical states to bosonic and fermionic parts.

   If infinite prime defines an association of real and p-adic space-time sheets this association could serve as a space-time correlate for the Fock state defined by configuration space spinor for given 3-surface. Also spinor field as a map from real partonic 3-surface would have as a space-time correlate a cognitive representation mapping real partonic 3-surfaces to p-adic 3-surfaces obtained by algebraic continuation.

2. Consider first the concrete interpretation of integers m_B and n_B. The most natural guess is that the primes dividing $m_B = \prod_i p^{m_i}$ characterize the effective p-adicities possible for the real 3-surface. m_i could define the numbers of disjoint partonic 3-surfaces with effective p_i-adic topology and associated with with the same real space-time sheet. These boundary conditions would force the corresponding real 4-surface to have all these effective p-adicities implying multi-p-adic fractality so that particle and wave pictures about multi-p-adic fractality would be mutually consistent. It seems natural to assume that also the integer n_i appearing in $m_B = \prod_i q_i^{n_i}$ code for the number of real partonic 3-surfaces with effective q_i-adic topology.

3. Fermionic statistics allows only single genuinely q_i-adic 3-surface possibly forming a pair with its real counterpart from which it is obtained by algebraic continuation. Pairing would conform with the fact that n_F appears both in the finite and infinite parts of the infinite prime (something absolutely essential concerning the consistency of interpretation!).

   The interpretation could be as follows.

   (a) Cognitive representations must be stable against intentional action and fermionic statistics guarantees this. At space-time level this means that fermionic generators correspond to pairs of real effectively q_i-adic 3-surface and its algebraically continued q_i-adic counterpart. The quantum jump in which q_i-adic 3-surface is transformed to a real 3-surface is impossible since one would obtain two identical real 3-surfaces lying on top of each other, something very singular and not allowed by geometric exclusion principle for surfaces. The pairs of boson and fermion surfaces would thus form cognitive representations stable against intentional action.

   (b) Physical states are created by products of super algebra generators Bosonic generators can have both real or p-adic partonic 3-surfaces as space-time correlates depending on whether they correspond to intention or action. More precisely, m_B and n_B code for collections of real and p-adic partonic 3-surfaces. What remains to be interpreted is why m_B and n_B cannot have common prime factors (this is possible if one allows also infinite integers obtained as products of finite integer and infinite primes).

(c) Fermionic generators to the pairs of a real partonic 3-surface and its p-adic counterpart obtained by algebraic continuation and the pictorial interpretation is as fermion hole pair.

(d) This picture makes sense if the partonic 3-surfaces containing a state created by a product of super algebra generators are unstable against decay to this kind of 3-surfaces so that one could regard partonic 3-surfaces as a space-time representations for a configuration space spinor field.

4. Are alternative interpretations possible? For instance, could $q = m_B/m_B$ code for the effective q-adic topology assignable to the space-time sheet. That q-adic numbers form a ring but not a number field casts however doubts on this interpretation as does also the general physical picture.

Number theoretical universality of S-matrix

The discreteness of the intersection of the real space-time sheet and its p-adic variant obtained by algebraic continuation would be a completely universal phenomenon associated with all fermionic states. This suggests that also real-to-real S-matrix elements involve instead of an integral a sum with the arguments of an n-point function running over all possible combinations of the points in the intersection. S-matrix elements would have a universal form which does not depend on the number field at all and the algebraic continuation of the real S-matrix to its p-adic counterpart would trivialize. Note that also fermionic statistics favors strongly discretization unless one allows Dirac delta functions.

6.3.5 p-Adicization of second quantized induced spinor fields

Induction procedure makes it possible to geometrize the concept of a classical gauge field and also of the spinor field with internal quantum numbers. In the case of the electro-weak gauge fields induction means the projection of the H-spinor connection to a spinor connection on the space-time surface.

In the most recent formulation induced spinor fields appear only at the 3-dimensional light-like partonic 3-surfaces and the solutions of the modified Dirac equation can be written explicitly [C1, C2] as simple algebraic functions involving powers of the preferred coordinate variables very much like various operators in conformal field theory can be expressed as Laurent series in powers of a complex variable z with operator valued coefficients. This means that the continuation of the second quantized induced spinor fields to various p-adic number fields is a straightforward procedure. The second quantization of these induced spinor fields as free fields is needed to construct configuration space geometry and anti-commutation relation between spinor fields are fixed from the requirement that configuration space gamma matrices correspond to super-canonical generators.

The idea about rational physics as the intersection of the physics associated with various number fields inspires the hypothesis that induced spinor fields have only modes labelled by rational valued quantum numbers. Quaternion conformal invariance indeed implies that zero modes are characterized by integers. This means that same oscillator operators can define oscillator operators are universal. Powers of the quaternionic coordinate are indeed well-define in any number field provided the components of quaternion are rational numbers since p-adic quaternions have in this case always inverse.

6.3.6 Should one p-adicize at the level of configuration space?

If Duistermaat-Heckman theorem [20] holds true in TGD context, one could express configuration space functional integral in terms of exactly calculable Gaussian integrals around the maxima of the Kähler function defining what might be called reduced configuration space CH_{red}. The huge super-conformal symmetries raise the hope that the rest of S-matrix elements could be deduced using group theoretical considerations so that everything would become algebraic. If this optimistic scenario is realized, the p-adicization of CH_{red} might be enough to p-adicize all operations needed to construct the p-adic variant of S-matrix.

The optimal situation would be that S-matrix elements reduce to algebraic numbers for rational valued incoming momenta and that p-adicization trivializes in the sense that it corresponds only to different interpretations for the imbedding space coordinates (interpretation as real or p-adic numbers) appearing in the equations defining the 4-surfaces. For instance, space-time coordinates would correspond to preferred imbedding space coordinates and the remaining imbedding space coordinates could be rational functions of the latter with algebraic coefficients. Algebraic points in a given extension of rationals would thus be common to real and p-adic surfaces. It could also happen that there are no or very few common algebraic points. For instance, Fermat's theorem says that the surface $x^n + y^n = z^n$ has no rational points for $n > 2$.

This picture is probably too simple. The intuitive expectation is that ordinary S-matrix elements are proportional to a factor which in the real case involves an integration over the arguments of an n-point function of a conformal field theory defined at a partonic 2-surface. For p-adic-real transitions the integration should reduce to a sum over the common rational or algebraic points of the p-adic and real surface. Same applies to $p_1 \rightarrow p_2$ type transitions.

If this picture is correct, the p-adicization of the configuration space would mean p-adicization of CH_{red} consisting of the maxima of the Kähler function with respect to both fiber degrees of freedom and zero modes acting effectively as control parameters of the quantum dynamics. If CH_{red} is a discrete subset of CH ultrametric topology induced from finite-p p-adic norm is indeed natural for it. 'Discrete set in CH' need not mean a discrete set in the usual sense and the reduced configuration space could be even finite-dimensional continuum. Finite-p p-adicization as a cognitive model would suggest that p-adicization in given point of CH_{red} is possible for all p-adic primes associated with the corresponding space-time surface (maximum of Kähler function) and represents a particular cognitive representation about CH_{red}.

A basic technical problem is, whether the integral defining the Kähler action appearing in the exponent of Kähler function exists p-adically. Here the hypothesis that the exponent of the Kähler function is identifiable as a Dirac determinant of the modified Dirac operator defined at the light-like partonic 3-surfaces [A6] suggests a solution to the problem. By restricting the generalized eigen values of the modified Dirac operator to an appropriate algebraic extension of rationals one could obtain an algebraic number existing both in the real and p-adic sense if the number of the contributing eigenvalues is finite. The resulting hierarchy of algebraic extensions of R_p would have interpretation as a cognitive hierarchy. If the maxima of Kähler function assignable to the functional integral are such that the number of eigenvalues in a given algebraic extension is finite this hypothesis works.

If Duistermaat-Heckman theorem generalizes, the p-adicization of the entire configuration space would be un-necessary and it certainly does not look a good idea in the light of preceding considerations.

1. For a generic 3-surface the number of the eigenvalues in a given algebraic extension of rationals need not be finite so that their product can fail to be an algebraic number.

2. The algebraic continuation of the exponent of the Kähler function from CH_{red} to the entire CH would be analogous to a continuation of a rational valued function from a discrete set to a real or p-adic valued function in a continuous set. It is difficult to see how the continuation could be unique in the p-adic case.

6.4 p-Adic probabilities

p-Adic Super Virasoro representations necessitate p-adic QM based on the p-adic unitarity and p-adic probability concepts. The concept of a p-adic probability indeed makes sense as shown by [16]. p-Adic probabilities can be defined as relative frequencies N_i/N in a long series consisting of total number N of observations and N_i outcomes of type i. Probability conservation corresponds to

$$\sum_i N_i = N \ , \tag{6.4.1}$$

and the only difference as compared to the usual probability is that the frequencies are interpreted as p-adic numbers.

The interpretation as p-adic numbers means that the relative frequencies converge to probabilities in a p-adic rather than real sense in the limit of a large number N of observations. If one requires that probabilities are limiting values of the frequency ratios in p-adic sense one must pose restrictions on the possible numbers of the observations N if N is larger than p. For N smaller than p, the situation is similar to the real case. This means that for $p = M_{127} \simeq 10^{38}$, appropriate for the particle physics experiments, p-adic probability differs in no observable manner from the ordinary probability.

If the number of observations is larger than p, the situation changes. If N_1 and N_2 are two numbers of observations they are near to each other in the p-adic sense if they differ by a large power of p. A possible interpretation of this restriction is that the observer at the p:th level of the condensate cannot choose the number of the observations freely. The restrictions to this freedom come from the requirement that the sensible statistical questions in a p-adically conformally invariant world must respect p-adic conformal invariance.

6.4.1 p-Adic probabilities and p-adic fractals

p-Adic probalities are natural in the statistical description of the fractal structures, which can contain same structural detail with all possible sizes.

1. The concept of a structural detail in a fractal seems to be reasonably well defined concept. The structural detail is clearly fixed by its topology and p-adic conformal invariants associated with it. Clearly, a finite resolution defined by some power of p of the p-adic cutoff scale must be present in the definition. For example, p-adic angles are conformal invariants in the p-adic case, too. The overall size of the detail doesn't matter. Let us therefore assume that it is possible to make a list, possibly infinite, of the structural details appearing in the p-adic fractal.

2. What kind of questions related to the structural details of the p-adic fractal one can ask? The first thing one can ask is how many times i:th structural detail appears in a finite region of the fractal structure: although this number is infinite as a real number it might possess (and probably does so!) finite norm as a p-adic number and provides a useful p-adic invariant of the fractal. If a complete list about the structural details of the fractal is at use one can calculate also the total number of structural details defined as $N = \sum_i N_i$. This means that one can also define p-adic probability for the appearance of i:th structural detail as a relative frequency $p_i = N_i/N$.

3. One can consider conditional probabilities, too. It is natural to ask what is the probability for the occurrence of the structural detail subject to the condition that part of the structural detail is fixed (apart from the p-adic conformal transformations). In order to evaluate these probalities as relative frequencies one needs to look only for those structural details containing the substructure in question.

4. The evaluation of the p-adic probalities of occurrence can be done by evaluating the required numbers N_i and N in a given resolution. A better estimate is obtained by increasing the resolution and counting the numbers of the hitherto unobserved structural details. The increase in the resolution greatly increases the number of the observations in case of p-adic fractal and the fluctuations in the values of N_i and N increase with the resolution so that N_i/N has no well defined limit as a real number although one can define the probabilities of occurrence as a resolution dependent concept. In the p-adic sense the increase in the values of N_i and fluctuations are small and the procedure should converge rapidly so that reliable estimates should result with quite a reasonable resolution. Notice that the increase of the fluctuations in the real sense, when resolution is increased is in accordance with the criticality of the system.

5. p-Adic frequencies and probabilities define via the canonical correspondence real valued invariants of the fractal structure.

It must be emphasized that this picture can have practical applications only for small values of p, which could also be important in the macroscopic length scales. In elementary particle physics L_p is of the order of the Compton length associated with the particle and already in the first step CP_2 length scale is achieved and it is questionable whether it makes sense to continue the procedure below the length scale l. In particle physics context the renormalization is related to the the change of the reduction of the p-adic length scale L_p in the length scale hierarchy rather than p-adic fractality for a fixed value of p.

The most important application of the p-adic probability in this book is the description of the particle massivation based on p-adic thermodynamics. Instead of energy, Virasoro generator l is thermalized and in the low temperature phase temperature is quantized in the sense that the counterpart of the Boltzmann weight $exp(H/T)$ is $p^{L_0/T}$, where $T = 1/n$ from the requirement that Boltzmann weight exists (L_0 has integer spectrum). The surprising success of the mass calculations shows that p-adic probability theory is much more than a formal possibility.

6.4.2 Relationship between p-adic and real probabilities

There are uniqueness problems related to the mapping of p-adic probabilities to real ones. These problems find a nice resolution from the requirement that the map respects probability conservation. The implied modification of the original mapping does not change measurably the predictions for the masses of light particles.

How unique the map of p-adic probabilities and mass squared values are mapped to real numbers is?

The mapping of p-adic thermodynamical probabilities and mass squared values to real numbers is not completely unique.

1. Symplectic identification $I : \sum x_n p^n \rightarrow \sum x_n p^{-n}$ takes care of this mapping but does not respect the sum of probabilities so that the real images $I(p_n)$ of the probabilities must be normalized. This is a somewhat alarming feature.

2. The modification of the canonical identification mapping rationals by the formula $I(r/s) = I(r)/I(s)$ has appeared naturally in various applications, in particular because it respects unitarity of unitary matrices with rational elements with $r < p, s < p$. In the case of p-adic thermodynamic the formula $I(g(n)p^n/Z) \rightarrow I(g(n)p^n)/I(Z)$ would be very natural although Z need not be rational anymore. For $g(n) < p$ the real counterparts of the p-adic probabilities would sum up to one automatically for this option. One cannot deny that this option is more convincing than the original one. The generalization of this formula to map p-adic mass squared to a real one is obvious.

3. Options 1) and 2) differ dramatically when the $n = 0$ massless ground state has ground state degeneracy $D > 1$. For option 1) the real mass is predicted to be of order CP_2 mass whereas for option 2) it would be by a factor $1/D$ smaller than the minimum mass predicted by the option 1). Thus option 2) would predict a large number of additional exotic states. For those states which are light for option 1), the two options make identical predictions as far as the significant two lowest order terms are considered. Hence this interpretation would not change the predictions of the p-adic mass calculations in this respect. Option 2) is definitely more in accord with the real physics based intuitions and the main role of p-adic thermodynamics would be to guarantee the quantization of the temperature and fix practically uniquely the spectrum of the "Hamiltonian".

Under what conditions the mapping of p-adic ensemble probabilities to real probabilities respects probability conservation?

One can consider also a more general situation. Assume that one has an ensemble consisting of independent elementary events such that the number of events of type i is N_i. The probabilities are given by $p_i = N_i/N$ and $N = \sum N_i$ is the total number of elementary events. Even in the case that N is infinite as a real number it is natural to map the p-adic probabilities to their real counterparts using the rational canonical identification $I(p_i) = I(N_i)/I(N)$. Of course, N_i and N exist as well defined p-adic numbers under very stringent conditions only.

The question is under what conditions this map respects probability conservation. The answer becomes obvious by looking at the pinary expansions of N_i and N. If the integers N_i (possibly infinite as real integers) have pinary expansions having no common pinary digits, the sum of probabilities is conserved in the map. Note that this condition can assign also to a finite ensemble with finite number of a unique value of p.

This means that the selection of a basis for independent events corresponds to a decomposition of the set of integers labelling pinary digits to disjoint sets and brings in mind the selection of orthonormalized basis of quantum states in quantum theory. What is physically highly non-trivial that this "orthogonalization" alone puts strong constraints on probabilities of the allowed elementary events.

One can say that the probabilities define distributions of pinary digits analogous to non-negative probability amplitudes in the space of integers labelling pinary digits, and the probabilities of independent events must be orthogonal with respect to the inner product defined by point-wise multiplication in the space of pinary digits.

p-Adic thermodynamics for which Boltzman weights $g(E)exp(-E/T)$ are replaced by $g(E)p^{E/T}$ such that one has $g(E) < p$ and E/T is integer valued, satisfies this constraint. The quantization of E/T to integer values implies quantization of both T and "energy" spectrum and forces so called super conformal invariance in TGD applications, which is indeed a basic symmetry of the theory.

There are infinitely many ways to choose the elementary events and each choice corresponds to a decomposition of the infinite set of integers n labelling the powers of p to disjoint subsets. These subsets can be also infinite. One can assign to this kind of decomposition a resolution which is the poorer the larger the subsets involved are. p-Adic thermodynamics would represent the situation in which the resolution is maximal since each set contains only single pinary digit. Note the analogy with the basis of completely localized wave functions in a lattice.

How to map p-adic transition probabilities to real ones?

p-Adic variants of TGD, if they exist, give rise to S-matrices and transition probabilities P_{ij}, which are p-adic numbers.

1. The p-adic probabilities defined by rows of S-matrix mapped to real numbers using canonical identification respecting the $q = r/s$ decomposition of rational number or its appropriate generalization should define real probabilities.

2. The simplest example would simple renormalization for the real counterparts of the p-adic probabilities $(P_{ij})_R$ obtained by canonical identification (or more probably its appropriate modification).

$$
\begin{aligned}
P_{ij} &= \sum_{k\geq 0} P_{ij}^k p^k \ , \\
P_{ij} &\rightarrow \sum_{k\geq 0} P_{ij}^k p^{-k} \equiv (P_{ij})_R \ , \\
(P_{ij})_R &\rightarrow \frac{(P_{ij})_R}{\sum_j (P_{ij})_R} \equiv P_{ij}^R \ .
\end{aligned}
\tag{6.4.-1}
$$

The procedure converges rapidly in powers of p and resembles renormalization procedure of quantum field theories. The procedure automatically divides away one four-momentum delta function from the square of S-matrix element containing the square of delta function with no well defined mathematical meaning. Usually one gets rid of the delta function interpreting it as the inverse of the four-dimensional measurement volume so that transition rate instead of transition probability is obtained. Of course, also now same procedure should work either as a discrete or a continuous version.

3. Probability interpretation would suggest that the real counterparts of p-adic probabilities sum up to unity. This condition is rather strong since it would hold separately for each row and column of the S-matrix.

4. A further condition would be that the real counterparts of the p-adic probabilities for a given prime p are identical with the transition probabilities defined by the real S-matrix for real space-time sheets with effective p-adic topology characterized by p. This condition might allow to deduce all relevant phase information about real and corresponding p-adic S-matrices using as an input only the observable transition probabilities.

What it means that p-adically independent events are not independent in real sense?

A further condition would be that p-adic quantum transitions represent also in the real sense independent elementary events so that the real counterpart for a sum of the p-adic probabilities for a finite number of transitions equals to the sum of corresponding real probabilities. This condition is definitely too strong since only a single transition could correspond to a given p-adic norm of transition probability P_{ij} with i fixed.

The crucial question concerns the physical difference between the real counterpart for the sum of the p-adic transition probabilities and for the sum of the real counterparts of these probabilities, which are in general different:

$$(\sum_j P_{ij})_R \neq \sum_j (P_{ij})_R \quad . \tag{6.4.0}$$

The suggestion is that p-adic sum of the transition probabilities corresponds to the experimental situation, when one does not monitor individual transitions but using some common experimental signature only looks whether the transition leads to this set of the final states or not. When one looks each transition separately or effectively performs different experiment by considering only one transition channel in each experiment one must use the sum of the real probabilities. More precisely, the choice of the experimental signatures divides the set U of the final states to a disjoint union $U = \cup_i U_i$ and one must define the real counterparts for the transition probabilities P_{iU_k} as

$$\begin{aligned} P_{iU_k} &= \sum_{j\in U_k} P_{ij} \ , \\ P_{iU_k} &\rightarrow (P_{iU_k})_R \ , \\ (P_{iU_k})_R &\rightarrow \frac{(P_{iU_k})_R}{\sum_l (P_{iU_l})_R} \equiv P^R_{iU_k} \ . \end{aligned} \tag{6.4.-2}$$

The assumption means deep a departure from the ordinary probability theory. If p-adic physics is the physics of cognitive systems, there need not be anything mysterious in the dependence of the behavior of system on how it is monitored. At least half-jokingly one might argue that the behavior of an intelligent system indeed depends strongly on whether the boss is nearby or not. The precise definition for the monitoring could be based on the decomposition of the density matrix representing the entangled subsystem into a direct sum over the subspaces associated with the degenerate eigenvalues of the density matrix. This decomposition provides a natural definition for the notions of the monitoring and resolution.

The renormalization procedure is in fact familiar from standard physics. Assume that the labels j correspond to momenta. The division of momentum space to cells of a given size so that the individual momenta inside cells are not monitored separately means that momentum resolution is finite. Therefore one must perform p-adic summation over the cells and define the real probabilities in the proposed manner. p-Adic effects resulting from the difference between p-adic and real summations could be the counterpart of the renormalization effects in QFT. It should be added that similar resolution can be defined also for the initial states by decomposing them into a union of disjoint subsets.

6.4.3 p-Adic thermodynamics

The p-adic field theory limit as such is not expected to give a realistic theory at elementary particle physics level. The point is that particles are expected to be either massless or possess mass of order 10^{-4} Planck mass. The p-adic description of particle massivation described in the third part of the book shows that p-adic thermodynamics provides the proper formulation of the problem. What is thermalized is Virasoro generator L_0 (mass squared contribution is not included to L_0 so that states do not have fixed conformal weight). Temperature is quantized purely number theoretically in low temperature limit ($exp(H/kT) \rightarrow p^{L_0/T}$, $T = 1/n$): in fact, partition function does not even exist in high temperature phase. The extremely small mixing of massless states with Planck mass states implies massivation and predictions of the p-adic thermodynamics for the fermionic masses are in excellent agreement with experimental masses. Thermodynamic approach also explains the emergence of the length scale L_p for a given p-adic condensation level and one can develop arguments explaining why primes near prime powers of two are favored.

It should be noticed that rational p-adic temperatures $1/T = k/n$ are possible, if one poses the restriction that thermal probabilities are non-vanishing only for some subalgebra of the Super Virasoro algebra isomorphic to the Super Virasoro algebra itself. The generators L_{kn},G_{kn}, where k is a positive integer, indeed span this kind of a subalgebra by the fractality of the Super Virasoro algebra and $p^{L_0/T}$ is integer valued with this restriction.

One might apply thermodynamics approach should also in the calculation of S-matrix. What is is needed is thermodynamical expectation value for the transition amplitudes squared over incoming and outgoing states. In this expectation value 3-momenta are fixed and only mass squared varies.

6.4.4 Generalization of the notion of information

TGD inspired theory of consciousness, in particular the formulation of Negentropy Maximization Principle (NMP) in p-adic context, has forced to rethink the notion of the information concept. In TGD state preparation process is realized as a sequence of self measurements. Each self measurement means a decomposition of the sub-system involved to two unentangled parts. The decomposition is fixed highly uniquely from the requirement that the reduction of the entanglement entropy is maximal.

The additional assumption is that bound state entanglement is stable against self measurement. This assumption is somewhat ad hoc and it would be nice to get rid of it. The only manner to achieve this seems to be a generalized definition of entanglement entropy allowing to assign a negative value of entanglement entropy to the bound state entanglement, so that bound state entanglement would actually carry information, in fact conscious information (experience of understanding). This would be very natural since macro-temporal quantum coherence corresponds to a generation of bound state entanglement, and is indeed crucial for ability to have long lasting non-entropic mental images.

The generalization of the notion of number concept leads immediately to the basic problem. How to generalize the notion of entanglement entropy that it makes sense for a genuinely p-adic entanglement? What about the number-theoretically universal entanglement with entanglement probabilities, which correspond to finite extension of rational numbers? One can also ask whether the generalized notion of information could make sense at the level of the space-time as suggested by quantum-classical correspondence.

In the real context Shannon entropy is defined for an ensemble with probabilities p_n as

$$S = -\sum_n p_n log(p_n) \ . \tag{6.4.-1}$$

As far as theory of consciousness is considered, the basic problem is that Shannon entropy is always non-negative so that as such it does not define a genuine information measure. One could define information as a change of Shannon entropy and this definition is indeed attractive in the sense that quantum jump is the basic element of conscious experience and involves a change. One can however argue that the mere ability to transfer entropy to environment (say by aggressive behavior) is not all that is involved with conscious information, and even less so with the experience of understanding or moment of heureka. One should somehow generalize the Shannon entropy without losing the fundamental additivity property.

p-Adic entropies

The key observation is that in the p-adic context the logarithm function $log(x)$ appearing in the Shannon entropy is not defined if the argument of logarithm has p-adic norm different from 1. Situation changes if one uses an extension of p-adic numbers containing $log(p)$: the conjecture is that this extension is finite-dimensional. One might however argue that Shannon entropy should be well defined even without the extension.

p-Adic thermodynamics inspires a manner to achieve this. One can replace $log(x)$ with the logarithm $log_p(|x|_p)$ of the p-adic norm of x, where log_p denotes p-based logarithm. This logarithm is integer valued ($log_p(p^n) = n$), and is interpreted as a p-adic integer. The resulting p-adic entropy

$$\begin{aligned} S_p &= \sum_n p_n k(p_n) \ , \\ k(p_n) &= -log_p(|p_n|) \ . \end{aligned} \tag{6.4.-1}$$

is additive: that is the entropy for two non-interacting systems is the sum of the entropies of composites. Note that this definition differs from Shannon's entropy by the factor $log(p)$. This entropy vanishes identically in the case that the p-adic norms of the probabilities are equal to one. This means that it is possible to have non-entropic entanglement for this entropy.

One can consider a modification of S_p using p-adic logarithm if the extension of the p-adic numbers contains $log(p)$. In this case the entropy is formally identical with the Shannon entropy:

$$S_p = -\sum_n p_n log(p_n) = -\sum_n p_n \left[-k(p_n)log(p) + p^{k_n} log(p_n/p^{k_n}\right] \ . \tag{6.4.0}$$

It seems that this entropy cannot vanish.

One must map the p-adic value entropy to a real number and here canonical identification can be used:

$$\begin{aligned} S_{p,R} &= (S_p)_R \times log(p)) \ , \\ (\sum_n x_n p^n)_R &= \sum_n x_n p^{-n} \ . \end{aligned} \tag{6.4.0}$$

The real counterpart of the p-adic entropy is non-negative.

Number theoretic entropies and bound states

In the case that the probabilities are rational or belong to a finite-dimensional extension of rationals, it is possible to regard them as real numbers or p-adic numbers in some extension of p-adic numbers for any p. The visions that rationals and their finite extensions correspond to islands of order in the seas of chaos of real and p-adic transcendentals suggests that states having entanglement coefficients in finite-dimensional extensions of rational numbers are somehow very special. This is indeed the case. The p-adic entropy entropy $S_p = -\sum_n p_n log_p(|p_n|)log(p)$ can be interpreted in this case as an ordinary rational number in an extension containing $log(p)$.

What makes this entropy so interesting is that it can have also negative values in which case the interpretation as an information measure is natural. In the real context one can fix the value of the value of the prime p by requiring that S_p is maximally negative, so that the information content of the ensemble could be defined as

$$I \equiv Max\{-S_p, \ p \ \text{prime}\} \ . \tag{6.4.1}$$

This information measure is positive when the entanglement probabilities belong to a finite-dimensional extension of rational numbers. Thus kind of entanglement is stable against NMP, and has a natural interpretation as bound state entanglement. The prediction would be that the bound states of real systems form a number theoretical hierarchy according to the prime p and and dimension of algebraic extension characterizing the entanglement.

Number theoretically state function reduction and state preparation could be seen as information generating processes projecting the physical states from either real or p-adic sectors of the state space to their intersection. Later an argument that these processes have a purely number theoretical interpretation will be developed based on the generalized notion of unitarity allowing the U-matrix to have matrix elements between the sectors of the state space corresponding to different number fields.

Number theoretic information measures at the space-time level

Quantum classical correspondence suggests that the notion of entropy should have also space-time counterpart. Entropy requires ensemble and both the p-adic non-determinism and the non-determinism of Kähler action allow to define the required ensemble as the ensemble of strictly deterministic regions of the space-time sheet. One can measure various observables at these space-time regions, and the frequencies for the outcomes are rational numbers of form $p_k = n(k)/N$, where N is the number of strictly deterministic regions of the space-time sheet. The number theoretic entropies are well defined and negative if p divides the integer N. Maximum is expected to result for the largest prime power factor of N. This would mean the possibility to assign a unique prime to a given real space-time sheet.

The classical non-determinism resembles p-adic non-determinism in the sense that the space-time sheet obeys effective p-adic topology in some length and time scale range is consistent with this idea since p-adic fractality suggests that N is power of p.

6.5 p-Adic Quantum Mechanics

An interesting question is whether p-adic quantum mechanics might exist in some sense. The purely formal generalizations of the ordinary QM need not be very interesting physically and the following considerations describe p-adic QM as a limiting case of the p-adic field theory limit of TGD to be constructed later. This particular p-adic QM is based on the p-adic Hilbert-space, p-adic unitarity and p-adic probability concepts whereas the physical interpretation is based on the correspondence between the p-adic and real probabilities given by the canonical correspondence. p-Adic QM is expected to apply below the p-adic length scale $L_p = \sqrt{p}l$ and above L_p ordinary QM should work, when length scale resolution L_p is used. Although one can define p-adic Schrödinger equation formally without any difficulty it is not at all obvious whether, or even too plausible, that it emerges from the p-adic QFT limit of TGD.

6.5.1 p-Adic modifications of ordinary Quantum Mechanics

One can consider several modifications of the ordinary quantum mechanics depending on what kind of p-adicizations one is willing to make.

p-Adicization in dynamical degrees of freedom

The minimal alternative is to replace time- and spatial coordinates with their p-adic counterparts so that the space time is a Cartesian power of R_p. A more radical possibility is to replace the 3-space with a 3-dimensional algebraic extension of the p-adic numbers. This means that space time is replaced with a Cartesian product of R_p and its 3-dimensional extension. The most radical possibility, suggested by the relativistic considerations, is a four-dimensional algebraic extension treating space and time degrees of freedom in an equal position: this alternative is encountered in the formulation of the p-adic field theory limit of TGD.

In practice the formulation of the quantum theory involves an action principle defining the so called classical theory and this is defined by using the integral of the the action density. These integrals certainly exists as real quantities and are defined by the Haar measure for the p-adic numbers. Algebraic continuation of real integrals seems to be the only reasonable manner to defined these integrals.

p-Adicization at Hilbert space level

One can imagine essentially two different manners to p-adicize Hilbert space.

1. The first approach, followed in [17], is to keep Schrö-dinger amplitudes complex. In this case it is better to consider a Cartesian power of R_p instead of an algebraic extension as a coordinate space. The canonical identification allows to replace the expressions of the coordinate and momentum operators via their p-adic counterparts. For example, $x \times \Psi$ is replaced with $x \times_p \Psi$, where p-adic multiplication rule is used. Derivative corresponds to a p-adic derivative. It was the lack of the canonical identification replacement, which forced to give up the straightforward generalization of standard QM in the approach followed in [17, 9]. What this approach effects,

is the replacement of the ordinary continuity and differentiability and concepts with the p-adic differentiability and the approach looks rather reasonable manner to construct a fractal quantum mechanics. This approach however is not applicable in the present context.

2. A more radical approach uses Schrödinger amplitude with values in some complex extension, say a square root allowing extension of the p-adic numbers. p-Adic inner product implies that the ordinary unitarity and probability concepts are replaced with there p-adic counterparts. This approach looks natural for various reasons. The representation theory for the Lie-groups generalizes to p-adic case and the replacement implies certain mathematical elegance since p-analyticity and the realization of the p-adic conformal invariance becomes possible. It will be found that p-adic valued inner product is the natural inner product for the quantized harmonic oscillator and for Super Virasoro representations. The concept of the p-adic probability makes sense as first shown by [16]. The physical interpretation of the theory is however always in terms of the real numbers and the canonical identification provides the needed tool to map the predictions of the theory to real numbers. That physical observables are always real numbers is suggested by the success of the p-adic mass calculations. p-Adic probabilities can be mapped to real probabilities and in the last chapter of the third part of the book it is shown that this correspondence predicts genuinely novel physical effects.

The p-adic representations of the Super Virasoro algebra to be used are defined in the p-adic Hilbert space and everything is well defined at algebraic level if 4- ($p>2$) or 8- ($p=2$) dimensional algebraic extension allowing square roots is used. Unitarity concept generalizes in a straightforward manner to the p-adic context and the elements of the S-matrix should have values in the same extension of the p-adic numbers. The requirement that the squares of S-matrix elements are p-adically real numbers gives strong constraints on the S-matrix elements since the quantities $S(m,n)\bar{S}(m,n)$ in general belong to the 4- (2-) dimensional real subspace $x+\theta y+\sqrt{p}z+\sqrt{p}\theta u$ of the 8- (4-) dimensional extension and p-adic reality implies the conditions:$y=z=..=u=0$. Reality conditions can be solved always since the solution involves only square roots of rational functions. What is exciting is that space time and imbedding space dimensions for the extension allowing square roots are forced by the quantum mechanical probability concept, by p-adic group theory and by the p-adic Riemannian Geometry.

The existence of the p-adic valued definite integral is crucial concerning the practical construction of the p-adic Quantum Mechanics.

1. In the ordinary wave mechanics the inner product involves an integration over the configuration space degrees of freedom. This inner product can be generalized to the p-adic integral of $\bar{\Psi}_1\Phi_2$ over the 3-space using p-adic valued integration defined in the first chapter, which works for all analytic functions and also for p-adic counterparts of the plane waves (nonanalytic functions).

2. The perturbative formulation QM in terms of the time development operator

$$U(t) = P(exp(i\int exp(\int dt\ V)) \ , \tag{6.5.1}$$

generalizes to the p-adic context. In particular, the concept of the time ordered product $P(...)$ appearing in the definition of the time development operator generalizes since the canonical identification induces ordering for the values of the p-adic time coordinate: $t_1<t_2$ if $(t_1)_R<(t_2)_R$ holds true. Non-trivialities are related to the p-adic existence of the time development operator: for sufficiently larger values of the time coordinate, the exponent appearing in the time development operator does not exist p-adically and this implies infrared cutoff time and length scale in the p-adic QM.

One can define the action of the time development operator for longer time intervals only if one makes some restrictions on the physical states appearing in the matrix elements. This could explain color confinement number theoretically. For sufficiently long time intervals the color interaction part of the interaction Hamiltonian is so large for colored states that p-adic time development operator fails to exist number theoretically and one must restrict the physical states to be color singlets.

The generalization of the p-adic formula for Riemann integral [E4] suggests an exact formula for the time ordered product. The first guess is that one simply forms the product

$$\begin{aligned} Pexp(i\int_0^t Hdt) &\equiv P\prod_n exp\left[iV(t(n))\Delta t(n)\right] \ , \\ \Delta t(n) &= t_+(n) - t_-(n) = (1+p)p^{m(n)} \ , \end{aligned} \tag{6.5.1}$$

to obtain the value of the time ordered product for time values t having finite number of pinary digits. The product is over all points $t(n)$ having finite number of pinary digits and $m(n)$ is the highest pinary digit in the expansion of $t(n)$ and $t_\pm(n)$ denote the two p-adic images of the real coordinate $t(n)_R$ under canonical identification. $\Delta t(n)$ corresponds to the difference of the p-adic time coordinates, which are mapped to the same value of the real time coordinate in canonical identification so that one can regard the time ordered product as a limiting case in which real time coordinate differences are exactly zero in the time ordered product.

The time ordering of the product is induced by canonical identification from real time ordering. This time development operator is defined for time values with finite number of pinary digits only and defines p-adic pseudoconstant. The hope is that the inherent non-determinism of the p-adic differential equations, implied by the existence of the p-adic pseudo constants, makes it possible to continue this function to a p-adically differentiable function of the p-adic time coordinate satisfying the counterpart of the Schrödinger equation for the time development operator.

Not surprisingly, number theoretical problems are encountered also now: the exponential $exp\left[iV(t(n))\Delta t(n)\right]$ need not exist p-adically. The possibility of p-adic pseudo constants suggests that one could simply drop off the troublesome exponentials: this has far reaching physical consequences [F5].

6.5.2 p-Adic inner product and Hilbert spaces

Concerning the physical applications of algebraically extended p-adic numbers the problem is that p-adic norm is not in general bilinear in its arguments and therefore it does not define inner product and angle. One can however consider a generalization of the ordinary complex inner product $\bar{z}z$ to a p-adic valued inner product. It turns out that p-adic quantum mechanics in the sense as it is used in p-adic TGD can be based on this inner product.

The algebraic generalization of the ordinary Hilbert space inner product is bilinear and symmetric, defines p-adic valued norm. The norm can however for non-vanishing states. This inner product leads to p-adic generalization of unitarity and probability concept. The solution of the unitarity condition $\sum_k S_{mk}\bar{S}_{nk} = \delta(m,n)$ involves square root operations and therefore the minimal extension for the Hilbert space is 4-dimensional in $p > 2$ case and 8-dimensional in $p = 2$ case. Of course, extensions of arbitrary dimension are allowed.

The inner product associated with a minimal extension allowing square root near real axis provides a natural generalization of the real and complex Hilbert spaces respectively. Instead of real or complex numbers, a square root allowing algebraic extension extension appears as the multiplier field of the Hilbert space and one can understand the points of Hilbert space as infinite sequences $(Z_1, Z_2, ..., Z_n, ...)$, where Z_i belongs to the extension. The inner product $\sum_k \langle Z_k^1, Z_2^k\rangle$ is completely analogous to the ordinary Hilbert space inner product.

The generalization of the the Hilbert space of square integrable functions to a p-adic context is far from trivial since definite integral in in general ill defined procedure. Second problem is posed by the fact that p-adic counterparts of say oscillator operator wave functions do not exist in the entire p-adic variant of the configuration space. Algebraic definition of the inner product by using the rules of Gaussian integration provides a possible solution to the problem.

For Fock space generated by anti-commuting fermionic and commuting bosonic oscillator operators the p-adic counterpart exists naturally and it seems that Fock spaces can be seen as universal Hilbert spaces with rational coefficients identifiable as subspaces of both real Fock space and of all p-adic Fock spaces.

6.5.3 p-Adic unitarity and p-adic cohomology

p-Adic unitarity and probability concepts lead to highly nontrivial conclusions concerning the general structure of the p-adic S-matrix. The most general S-matrix is a product of a complex rational (extended rationals are also possible) unitary S-matrix S_Q and a genuinely p-adic S-matrix S_p which deviates only slightly from unity

$$\begin{aligned} S &= 1+i\sqrt{p}T \ , \\ T &= O(p^0) \ . \end{aligned} \tag{6.5.1}$$

for $p \ mod\ 4 = 3$ allowing imaginary unit in its four-dimensional algebraic extension. In perturbative context one expects that the p-adic S-matrix differs only slightly from unity. Using the form $S = 1+iT$, $T = O(p^0)$ one would obtain in general transition rates of order inverse of Planck mass and theory would have nothing to do with reality. Unitarity requirement implies iterative expansion of T in powers of p and the few lowest powers of p give excellent approximation for the physically most interesting values of p.

The unitarity condition implies that the moduli squared of the matrix T in $S = 1+iT$ are of order $O(p^{1/2})$ if one assumes a four-dimensional p-adic extension allowing square root for the ordinary p-adic numbers and one can write

$$\begin{aligned} S &= 1+i\sqrt{p}T \ , \\ i(T-T^{\dagger}) &+ \sqrt{p}TT^{\dagger} = 0 \ . \end{aligned} \tag{6.5.1}$$

This expression is completely analogous to the ordinary one since $i\sqrt{p}$ is one of the units of the four-dimensional algebraic extension. Unitarity condition in turn implies a recursive solution of the unitary condition in powers of p:

$$\begin{aligned} T &= \sum_{n\geq 0} T_n p^{n/2} \ , \\ T_n - T_n^{\dagger} &= \frac{1}{i}\sum_{k=0,..,n-1} T_{n-1-k}T_k^{\dagger} \ . \end{aligned} \tag{6.5.1}$$

If algebraic extension is not allowed then the expansion is in powers of p instead of $\sqrt{p}$. Note that the real counterpart of the series converges extremely rapidly for physically interesting primes (such as $M_{127} = 2^{127}-1$).

In the p-adic context S-matrix $S = 1+T$ satisfies the unitarity conditions

$$T + T^{\dagger} = -TT^{\dagger} \tag{6.5.2}$$

if the conditions

$$\begin{aligned} T &= T^{\dagger} \ , \\ T^2 &= 0 \ . \end{aligned} \tag{6.5.2}$$

defining what might be called p-adic cohomology, are satisfied [C2]. In the real context these conditions are not possible to satisfy as is clear from the fact that the total scattering rate from a given state, which is proportional to T^2_{mm} vanishes.

p-Adic cohomology defines a symmetry analogous to BRST symmetry: if T satisfies unitarity conditions and T_0 satisfies the conditions

$$\begin{aligned} &T_0 = T_0^{\dagger} \ , \qquad T_0^2 = 0 \ , \\ &\{T_0, T\} = T_0T + TT_0 = 0 \ , \end{aligned} \tag{6.5.3}$$

unitary conditions are satisfies also by the matrix $T_1 = T + T_0$. The total scattering rates are same for T and T_1.

6.5.4 The concept of monitoring

The relationship between p-adic and real probabilities involves the hypothesis that real transition probabilities depend on the cognitive resolution. Cognitive resolution is defined by the decomposition of the state space H into direct sum $H = \oplus H_i$ so that the experimental situation cannot differentiate between different states inside H_i. Each resolution defines different real transition probabilities unlike in ordinary quantum mechanics. Physically this means that the arrangement, where each state in H_i is monitored separately differs from the situation, when one only looks whether the state belongs to H_i. One can say that monitoring affects the behavior of a p-adic subsystem. Of course, these exotic effects relate to the physics of cognition rather than real physics.

Standard probability theory, which also lies at the root of the standard quantum theory, predicts that the probability for a certain outcome of experiment does not depend on how the system is monitored. For instance, if system has N outcomes $o_1, o_2, ...o_N$ with probabilities $p_1, ..., p_N$ then the probability that o_1 or o_2 occurs does not depend on whether common signature is used for o_1 and o_2 or whether observer also detects which of these outcomes occurs. The crucial signature of p-adic probability theory is that monitoring affects the behavior of the system.

Physically monitoring is represented by quantum entanglement [H2], and differentiates between two eigen states of the density matrix only provided the eigenvalues of the density matrix are different. If there are several degenerate eigenvalues, quantum jump occurs to any state in the eigen space and one can predict only the total probability for the quantum jump into this eigen space: the real probabilities for jumps into individual states are obtained by dividing total real probability by the degeneracy factor. Hence the p-adic probability for a quantum jump to a given eigenspace of density matrix is p-adic sum of probabilities over the eigen states belonging to this eigenspace:

$$P_i = \frac{(n(i)P(i))_R}{\sum_j (n(j)P(j))_R} \quad .$$

Here n_i are dimensions of various eigenspaces.

If the degeneracy of the eigenvalues is removed by an arbitrary small perturbation, the total probability for the transition to the same subspace of states becomes the sum for the real counterparts of probabilities and one has in good approximation:

$$P^R = \frac{n(i)P(i)_R}{[\sum_{j\neq i}\sum_j (n(j)P(j))_R + n(i)P(i)_R]} \quad .$$

Rather dramatic effects could occur. Suppose that that the entanglement probability $P(i)$ is of form $P(i) = np$, $n \in \{0, p-1\}$ and that n is large so that $(np)_R = n/p$ is a considerable fraction of unity. Suppose that this state becomes degenerate with a degeneracy m and $mn > p$ as integer. In this kind of situation modular arithmetics comes into play and $(mnp)_R$ appearing in the real probability $P(1\ or\ 2)$ can become very small. The simplest example is $n = (p+1)/2$: if two states i and j have *very nearly equal but not identical* entanglement probabilities $P(i) = (p+1)p/2+\epsilon$, $P(j) = (p+1)p/2-\epsilon$, monitoring distinguishes between them for arbitrary small values of ϵ and the total probability for the quantum jump to this subspace is in a good approximation given by

$$\begin{aligned} P(1\ or\ 2) &\simeq \frac{x}{\left[\sum_{k\neq i,j}(P_k)_R + x\right]} \quad , \\ x &= 2\left[(p+1)p/2\right]_R \quad . \end{aligned} \tag{6.5.3}$$

and is rather large. For instance, for Mersenne primes $x \simeq 1/2$ holds true. If the two states become degenerate then one has for the total probability

$$\begin{aligned} P(1\ or\ 2) &\simeq \frac{x}{\left[\sum_{k\neq i,j}(P_k)_R + x\right]} \quad , \\ x &= \frac{1}{p} \quad . \end{aligned} \tag{6.5.3}$$

The order of magnitude for $P(1\ or\ 2)$ is reduced by a factor of order $1/p$!

Since p-adicity is essential for the exotic effects related to monitoring, the exotic phenomena of monitoring should be related to the quantum physics of cognition rather than real quantum physics. A test for quantum TGD would be provided by the study of the dependence of the transition rates of quantum systems on the resolution of monitoring defined by the dimensions of the degenerate eigenspaces of the subsystem density matrix. One could even consider the possibility of measuring the value of the p-adic prime in this manner. The behavior of living systems is known to be sensitive to monitoring and an exciting possibility is that this sensitivity, if it really can be shown to have statistical nature, could be regarded as a direct evidence for TGD inspired theory of consciousness. Note that the mapping of the physical quantities to entanglement probabilities could provide an ideal manner to compare physical quantities with huge accuracy! Perhaps bio-systems have invented this possibility before physicists and this could explain the miraculous accuracy of biochemistry in realizing genetic code. The measurement of the monitoring effect could provide a manner to determine the value of p_i for each p-adic region of space-time.

6.5.5 p-Adic Schrödinger equation

The emergence of the p-adic infrared cutoff

The experience with the construction of the p-adic counterpart of the standard model shows that p-adic quantum theory involves in practice infrared cutoff length scale in both time and spatial directions. The cutoff length scale comes out purely number theoretically. In the time like direction the cutoff length scale comes out from the exponent of the time ordered integral: p-adic exponent function $exp(x)$ does not exist unless the p-adic norm of the argument is smaller than one and this in turn means that $P(exp(i\int_0^t Vdt))$ does not exist for too larger values of time argument. A more concrete manner to see this is to consider time dependence for the eigenstates of Hamiltonian: the exponent $exp(iEt)$ exists only for $|Et|_p < 1$. The necessity of the spatial cutoff length scale is seen by considering concrete examples. For instance, the p-adic counterparts of the harmonic oscillator Gaussian wavefunctions are defined only in a finite range of the argument. As far as the definition of exponent function is considered one must keep in mind that the formal exponent function does not have the usual periodicity properties. The definition as a p-adic plane wave gives the needed periodicity properties but also in this case the infrared cutoff is necessary.

One should be able to construct also global solutions of the p-adic Schrödinger equation. The concept of p-adic integration constant might make this possible: by multiplying the solution of thhe Schrödinger equation with a constant depending on a finite number of the pinary digits, one can extend the solution to an arbitrary large region of the space time. What one cannot however avoid is the decomposition of the space time into disjoint quantization volumes.

One of the original motivation to introduce p-adic numbers was to introduce ultraviolet cutoff as a p-adic cutoff but, as the considerations of the second part of the book show, UV divergences are absent in the p-adic case and short distance contributions to the loops are negligibly small so that the mere p-adicization eliminates automatically UV divergences. Rather, it seems that the length scale L_p serves as an infrared cutoff and, if a length scale resolution rougher than L_p is used, ordinary real theory should work. Only in the length scales $L \leq L_p$ should the p-adic field theory and Quantum Mechanics be useful. The applicability of the real QM for length scale resolution $L \geq L_p$ is in accordance with the fact that the real continuity implies p-adic contintuity.

Formal p-adicization of the Schrödinger equation

The formal p-adic generalization of the Schrödinger equation is of the following general form

$$\theta \frac{d\Psi}{dt} = H\Psi \ , \tag{6.5.4}$$

where H is in some sense Hermitian operator. If Schödinger amplitudes are complex values θ can be taken to be imaginary unit i. The same identification is possible if Ψ possesses values in the extension of p-adic allowing square root and the condition $p\ mod\ 4 = 3$ or $p = 2$ guaranteing that $\sqrt{-1}$ does not exist as an ordinary p-adic number, is satisfied. For $p\ mod\ 4 = 1$ the situation is more complicated since imaginary unit i does not in general belong to the generators of the minimal extension allowing a square root. An open problem is whether one could replace θ appearing in the quadratic extension and define complex conjugation as the operation $\theta \rightarrow -\theta$. The analogy with the ordinary quantum mechanics suggests the form

$$H = -\frac{\nabla^2}{2m} + V \ , \tag{6.5.4}$$

for the Hamiltonian in $p \ mod \ 4 = 3$ case. In the complex case ∇^2 is obtained by replacing the ordinary derivatives with the p-adic derivatives and V is a p-adically differentiable function of the coordinates typically obtained from a p-analytic function via the canonical identication.

Although the formal p-adicization is possible, it is not at all obvious whether one can get anything physically interesting from the straightforward p-adicization of the Schrödinger equation. The study of the the p-adic hydrogen atom shows that formal p-adicization need not have anything to do with physics. For instance, Coulomb potential contains a factor $1/4\pi$ not existing p-adically, the energy eigenvalues depend on π and the straightforward p-adic counterparts of the exponentially decreasing wave functions are not exponentially decreasing functions p-adically and do not even exist for sufficiently large values of the argument r. It seems that a more realistic manner to define the p-adic Schrödinger equation is as limiting case of the p-adic field theory. Of course, it might also be that p-adic Schrödinger equation does not make sense. A more radical solution of the problems is the allowance of finite-dimensional extensions of p-adic numbers allowing also transcendental numbers.

p-Adic harmonic oscillator

The formal treatment of the p-adic oscillator using oscillator operator formalism is completely analogous to that of the ordinary harmonic oscillator. The only natural inner product is the p-adic valued one. That the treatment is correct is suggested by the fact that it is purely algebraic involving only the p-adic counter part of the oscillator algebra. The matrix elements of the oscillator operators $a^\dagger$ and a involve square roots and they exist provided the minimal extension allowing square roots appears as a coefficient ring of the Hilbert space. If two-dimensional quadratic extension not containing $\sqrt{p}$ is used occupation number must be restricted to the range $[0, p-1]$. If the Hilbert space inner product based on non-degenerate p-adic inner product $Z_c Z + \hat{Z}_c \hat{Z}$ the extension implies a characteristic degeneracy of states with complex amplitudes related to the conjugation $\sqrt{p} \rightarrow -\sqrt{p}$. 2-adic and p-adic cases differ in radical manner since the dimensions of the extension are 4 for $p > 2$ and 8 for $p = 2$. Since the representations of the Kac Moody and Super Virasoro algebras are based on oscillator operators this means that there is deep difference between $p = 2$ and $p > 2$ p-adic conformal field theories.

The p-adic energy eigen values are $E_n = (n + 1/2)\omega_0$ and their real counterparts form a quasi-continuous spectrum in the interval $(2, 4)$ for $p = 2$ and $(1, p)$ for $p > 2$! If p is very large (of order 10^{38} in the TGD:eish applications) the small quantum number limit $n < p$ gives the quantum number spectrum of the ordinary quantum mechanics. The occupation numbers $n > p$ have no counterpart in the conventional quantum theory and it seems that the classical theory with a quasi-continuous spectrum but with energy cutoff $p\omega_0$ is obtained at the limit of the arbitrarily large occupation numbers. The limit $p \rightarrow \infty$ gives essentially the classical theory with no upper bound for the energy.

The results suggests the idea that p-adic QM might be somewhere halfway between ordinary QM and classical mechanics. This need not however be the case as the study of the p-adic thermodynamics suggests. p-Adic thermodynamics allows a low temperature phase $exp(E_n/T) \equiv p^{n/T_k}$, $T_k = 1/k$, with quantized value of temperature. In this phase the probabilities for the energy eigenstates E_n, $n = \sum_k n_k p^k$ are extremely small except for the smallest values of n so that low temperature thermodynamics does not allow the effective energy continuum. One might argue that situation changes in the high temperature phase. The problem is that p-adic thermodynamics for the harmonic oscillator allows only formally high temperature phase $T = t_0\omega_0/p^k$, $k = 1, 2, ...$, $|t_0| = 1$. The reason is that Boltzmann weights $exp(-E_n/T) = exp(np^k/t_0)$ have p-adic norm equal to 1 so that the sum of probabilities giving free energy converges only formally. If one accepts the formal definition of the free energy as $exp(F) \equiv 1/(1 - exp(-E_0/T))$ then the real counterpart of the energy spectrum indeed becomes continuum also in the thermodynamic sense.

Consider next what a more concrete treatment using Schrödinger equation gives. The p-adic counterpart of the Schrödinger equation is formally the same as the ordinary Schrödinger equation. Ψ is assumed to have values in a minimal extension of p-adic numbers allowing square root and possessing imaginary unit so that the condition $p \ mod \ 4 = 3$ or $p = 2, 3$ must hold true. For the energy momentum eigenstates the equation reduces to

$$(-\frac{d^2}{dy^2} + y^2)\Psi = 2e\Psi \ , \tag{6.5.5}$$

where the dimensionless variables $y = \sqrt{\omega}x$ and $e = \frac{E}{\omega}$ have been introduced. This transformation makes sense provided ω possesses p-adic square root.

The solution ansatz to this equation can be written in the general form $\Psi = exp(-y^2/2)H_{e-1/2}(y)$, where H is the p-adic counter part of a Hermite polynomial. The first thing to notice is that vacuum wave function does not converge in a p-adic sense for all values of y. A typical term in series is of the form $X_n = \frac{y^{2n}}{2^n n!}$. In ordinary situation the factors, in particular $n!$, in the numerator imply convergence but in present case the situation is exactly the opposite.

In 2-adic case both the factor 2^n and the factor $n!$ in the denominator cause troubles whereas for $p > 2$ the p-adic norm of 2^n is equal to one. $n!$ gives at worst the power 2^{n-1} to the 2-adic norm. Therefore the 2-adic norm of X_n behaves as $N(X_n) \simeq |y_2|^{2n}2^n2^{n-1}$. The convergence is therefore achieved for $|y|_2 \leq 1/4$ only. For $p > 2$ the convergence is achieved for $|y|_p \leq 1/p$. One can continue the oscillator Gaussian to a globally defined function of y by observing that the scaling $y \rightarrow y/\sqrt{2}$ corresponds to taking a square root of the oscillator Gaussian and this square root exists if minimal quadratic extension allowing square root is used. In the usual situation the function $H_e(y)$ must be polynomial since otherwise it behaves as $exp(y^2)$ and does not converge: this implies the quantization of energy also now.

The inner product, which should orthogonalize the states is the p-adic valued inner product based on the p-adic generalization of the definite integral. The generalizations of the analytic formulas encountered in the real case should hold true also now. The guess motivated by the formal treatment is that p-adic energies are quantized according to the usual formula and classical energies form a continuum below the upper bound $e_R \leq 4$ in 2-adic case and $e_R =\leq p$ in p-adic case. In fact, the mere requirement $|e|_p \leq 1$ implies that energy is quantized according to the formula $e = n + 1/2$ in p-adic case.

p-Adic fractality in the temporal domain

The assumption that p-adic physics gives faithful cognitive representation of the real physics leads to highly nontrivial predictions, the most important prediction being p-adic fractality with long range temporal correlations and microtemporal chaos.

In p-adic context the diagonalization of the Hamiltonian for N-dimensional state space in general requires N-dimensional algebraic extension of p-adic numbers even when the matrix elements of the Hamiltonian are complex rational numbers. TGD as a generalized number theory vision allows all algebraic extensions of p-adic numbers so that this is not a problem. The necessity to decompose p-adic Hamiltonian to a complex rational free part and p-adically small interaction part could provide the fundamental reason for why Hamiltonians have the characteristic decomposition into free and interaction parts. Of course, it might be that Hamiltonian formalism does not make sense in the p-adic context and should be replaced with the approach based on Lagrangian formalism: at least in case of p-adic QFT limit of TGD this approach seems to be more promising. One could also argue that the very fact that p-adic physics provides a cognitive representations of TGD based physics gives a valuable guide to the real physics itself, and that one should try to identify the constraints on real physics from the requirement that its p-adic counterpart exists. The following discussion is motivated by this kind of attitude.

The emergence of various dynamical time scales is a very general phenomenon. For instance, it seems that strong and weak interactions correspond to different time scales in well defined sense and that it is a good approximation to neglect strong interaction in weak time scales and vice versa. p-Adic framework gives hopes of finding a more precise formulation for this heuristics using number theoretical ideas. The basic observation is that the time ordered exponential of a given interaction Hamiltonian exists only over a finite time interval of length $T_p(n) = p^n L_p$. This suggests that one should distinguish between the time developments associated with various p-adic time scales $T_n = p^n L_p/c$: obviously temporal fractality would be in question.

More concretely, the p-adic exponential $exp(iH\Delta t)$ of the free Hamiltonian exists p-adically only if one assumes that Δt is a small rational proportional to a positive power of p: $\Delta t \propto p^n$. Of course, this restriction to the allowed values of Δt might be interpreted as a failure of the cognitive representation rather than a real physical effect. Alternatively, one might argue that the emergence of the p-adic time scales is a real physical effect and that one must define a separate S-matrix for each p-adic time scale $\Delta t \propto p^n$. Thus p-adic S-matrices for time intervals that differ from each other by arbitrarily long real time interval could be essentially identical. This would mean extremely precise fractal long range correlations and chaos in short time scales also at the level of real physics. This is certainly a testable and rather dramatic prediction in sharp contrast with standard physics views. $1/f$ noise could be seen as one manifestation of these long range correlations.

What would distinguish between different times scales would be different decomposition of the Hamiltonian to free and interaction parts to achieve interaction part which is p-adically small in the time scale involved. For instance, it could be possible to understand color confinement in this manner: in quark gluon plasma phase below the length scale L_p many quark states without any constraints on color are the natural state basis whereas above the length scale L_p physical states must be color singlets since otherwise time evolution operator does not exist.

In case of the cognitive representations of the external world canonical identification maps long external time and length scales to short internal time and length scales and vice versa. Thus p-adic fractality of the cognitive dynamics induces at the level of cognitive representation order in short length and time scales and chaos in long length and time scales: this is of course natural since sensory information comes mainly from the nearby spatiotemporal regions of the system. For self-representations there is chaos in short time scales and fractal long range correlations (so that our temptation to see our life as a coherent temporal pattern would not be self deception!). This kind of fractality is of course absolutely essential in order to understand bio-systems as intentional systems able to plan their future behavior. This prediction is about behaviorial patterns of cognitive systems and also testable.

One can get a more quantitative grasp on this idea by studying the time development operator associated with a diagonalizable Hamiltonian. If the eigenvalues E_n of the diagonalized Hamiltonian have p-adic norms $|E_n|_p \leq p^{-m}$, the time evolution determined by this Hamiltonian is defined at most over a time interval of length norm $T_p(m) = p^{m-1}L_p$ since for time intervals longer than this the eigenvalues $exp(iE_nt)$ of $exp(iHt)$ do not exists as a p-adic numbers for all energy eigenstates. Thus one must restrict the time evolution to time scale $t \leq p^{m-1}L_p$: this is consistent with a p-adic hierarchy of interaction time scales.

An alternative approach is based on the requirement that the complex phase factors $exp(iET)$ for the eigenstates of the diagonal part of the Hamiltonian are complex rational phases forming a multiplicative group. This means that one can map the phase factors $exp(iET)$ directly to their p-adic counterparts as complex rational numbers. With suitable constraints on the energy spectrum this makes sense if the interaction time T is quantized so that it is proportional to a power of p. The decomposition of the Hamiltonian to free and interacting parts could be done in such a manner that the exponential of Hamiltonian decomposes to a product of diagonal part representable as complex rational phases and interaction part which is of higher order in p so that ordinary exponential exists for sufficiently small values of interaction time. This decomposition depends on the p-adic time scale.

How to define time ordered products?

In perturbation theory one must deal with the p-adic counterpart of the time ordered exponential $\prod_n Pexp\left[\int_0^t H_{int}(n)dt\right]$ appearing in the definition of the time development operator. In the case of a nondiagonal, time dependent interaction Hamiltonian the very definition of the p-adic counterpart of the time ordered integral is far from obvious since p-adic numbers do not allow natural ordering. Perhaps the simplest possibility is based on Fourier analysis based on the use of Pythagorean phases. This automically involves the introduction of a time resolution $\Delta t = q = m/n$ and discretization of the time coordinate. Depending on the p-adic norm of Δt one obtains a hierarchy of S-matrices corresponding to different p-adic fractalities. Time ordering would be naturally induced from the ordering of ordinary integers since only the integer multiples of Δt are involved in the discretized version of integral defined by the inner product for the Pythagorean plane waves. The requirement that all time values have same p-adic norm implies $T = n\Delta t$, $n = 0, ..., p-1$. If one assumes that long range fractal temporal order is present one can also allow time intervals $T = n\delta t + mp^k$ which correspond to arbitrarily long real time intervals.

p-Adic particle stability is not equivalent with real stability

It is natural to require that single hadron states are eigenstates for that part of the total Hamiltonian, which consists of the kinetic part of the Hamiltonian. If this the case, one can require that the effect of $exp(iH_0t)$ is just a multiplication by the factor $exp(iEt)$. The fact that particles are not stable against decay to many-particle states suggests that E must be complex. Generalizing the construction of the p-adic planewaves one could define this prefactor for all values of time even in this case. One can however criticize this approach: the introduction of the decay width as imaginary part of E is is category error since decay width characterizes the statistical aspects of the dynamics associated with quantum jumps rather than the dynamics of the Schrödinger equation.

p-Adic unitarity concept suggests a more elegant description. The truncated S-matrix describing the transitions $H_p \rightarrow H_p$ is unitary despite the fact that the transitions between different sectors are possible. This makes sense because the total p-adic transition probability from H_p to H_q, $q \neq p$, vanishes by generalized unitarity conditions. Generalizing, the p-adic representations of elementary particles and even hadrons would p-adically stable in the sense that the total p-adic decay probability would vanishes for them. One could also say that in absence of monitoring p-adic cognitive representation of particle would be stable. This picture is consistent with the notion of p-adic cohomology reducing unitarity conditions for S-matrix $S = 1 + iT$ to the conditions $T = T^\dagger$ and $T^2 = 0$. Of course, it would apply only at the level of cognitive physics.

6.6 Generalized Quantum Mechanics

One can consider two generalizations of quantum mechanics to a fusion of p-adic and real quantum mechanics.

1. For the first generalization the guiding principle for the generalization of quantum mechanics is that quantum mechanics in a given number field is obtained as an algebraic continuation of the quantum mechanics in the field of rational numbers common to all number fields or in finite-dimensional extensions of rational numbers. This means that U-matrices U_F for transitions from H_Q to H_F, where F refers to various completions of rationals, are obtained as algebraic continuations of the unitary U-matrix U_Q for H_Q.

   The variant of the canonical identification I mapping rationals as $r/s \rightarrow I(r)/I(s)$ is the most natural relationship between real and p-adic U-matrices since it is a compromise between topology and algebra mapping rationals to rationals in a continuous manner and respecting rational unitarity assuming that the matrix elements of U do not involve integers $n > p - 1$. At the limit $p \rightarrow \infty$ unitarity is possible for all rational matrices U. This argument applies also for the extensions of rationals. The generalization means enormously strong algebraic constraints on the form of the U-matrix, especially so for small values of p.

2. A more radical option is that transitions from rational Hilbert space H_Q to the Hilbert spaces H_F associated with different number fields occur. This requires that U-process is followed by a process analogous to a state function reduction and preparation takes care that the resulting states become states in H_Q: this is what makes this generalization of a special interest. In this case one can speak about total scattering probability from H_Q to H_F. The U-matrices U_F are not anymore mere analytic continuations of U_Q. A possible interpretation of the unitary process $H_Q \rightarrow H_F$ is as generation of intention whereas the reduction and preparation means the transformation of the intention to action.

   The assumption that H_Q allows an algebraic continuation to the spaces H_F is probably too strong an idealization in p-adic and even in the real case. For instance, one cannot allow all rational valued momenta in p-adic case for the simple reason that the continuation to the p-adic case involves always some momentum cutoff if the extension of p-adics remains finite. Even in the real case the summation over all rational momenta in the unitarity conditions of U-matrix fails to make sense and cutoff is needed. A hierarchy of cutoffs suggests itself and has a natural interpretation as number theoretical hierarchy of extensions of p-adics.

In order to avoid un-necessary complications the following formal discussion however uses H_Q as a universal Hilbert space contained by the various state spaces H_F.

6.6.1 Quantum mechanics in H_F as a algebraic continuation of quantum mechanics in H_Q

The rational Hilbert space H_Q is representable as the set of sequences of real or complex rationals of which only finite number are non-vanishing. Real and p-adic Hilbert spaces are obtained as the numbers in the sequences to become real or p-adic numbers and no limitations are posed to the number of non-vanishing elements. All these Hilbert spaces have rational Hilbert space H_Q as a common sub-space. Also momenta and other continuous quantum numbers are replaced by a discrete value set. Superposition principle holds true only in a restricted sense, and state function reduction and preparation leads always to a final state which corresponds to a state in H_Q. This picture differs from the earlier one in which p-adic and real Hilbert spaces were assumed to form a direct sum.

The notion of unitarity generalizes. Contrary to the earlier beliefs, U-matrix does not possess matrix elements between different number fields but between rational Hilbert space and Hilbert spaces associated with various completions of rationals. This makes sense since the final state of the quantum jump (and thus the initial state of the unitary process, is always in H_Q.

The U-matrix is a collection of matrices U_F having matrix elements in the number field F. U_F maps H_Q to H_F. Each of these U-matrices is unitary. Also U_Q is unitary and U_F is obtained by algebraic continuation in the quantum numbers labelling the states of U_Q to U_F.

Hermitian conjugation makes sense since the defining condition

$$\langle \alpha_F | U n_Q \rangle = \langle U^\dagger \alpha_F | n_Q \rangle \ . \tag{6.6.1}$$

allows to interpret $|n_Q\rangle$ also as an element of H_F. If U would map different completed number fields to each other, hermiticity conditions would not make sense.

The hermitian conjugate of U-matrix maps H_F to H_Q so that $UU^\dagger$ *resp.* $U^\dagger U$ maps H_F *resp.* H_Q to itself. This means that there are two independent unitarity conditions

$$\begin{aligned} U_F U_F^\dagger &= Id_F \ , \\ U_F^\dagger U_F &= Id_Q \ . \end{aligned} \tag{6.6.1}$$

One can write $U = P_Q + T_F$ and $U^\dagger = P_Q + T_F^\dagger$, where P_Q refers to the projection operator to H_Q. This gives

$$\begin{aligned} T_F + T_F^\dagger &= -T_F T_F^\dagger \ , \\ P_Q T_F + T_F^\dagger P_Q &= -T_F^\dagger T_F \ . \end{aligned} \tag{6.6.1}$$

It is convenient to introduce the notations $T_Q = P_Q T_F$ and $T_Q^\dagger = T_F^\dagger P_Q$ with analogous notations for U and $U^\dagger$. The first condition, when multiplied from both sides by P_Q, gives together with the second equation unitarity conditions for T_Q

$$\begin{aligned} T_Q + T_Q^\dagger &= -T_Q T_Q^\dagger \ , \\ T_Q + T_Q^\dagger &= -T_F^\dagger T_F \ . \end{aligned} \tag{6.6.1}$$

This means that the restriction of the U-matrix to H_Q is unitary.

The difference between the right hand sides of the equation should vanish. The understanding of how this happens requires more delicate considerations. For instance, in the case of $F = C$ continuous sum over indices appears at the right hand side coming from four-momenta labelling the states. The restrictions of quantum numbers to Q and its subsets could be a process analogous to the momentum cutoff of quantum field theories. The continuation from discrete integer valued labels of, say discrete momenta, to continuous values is performed routinely in various physical models routinely, and it would seem that this process has cognitive and physical counterparts. This picture conforms with the vision that the rational (or extended rational) U-matrix U_Q gives the U-matrices U_F by an algebraic continuation in the quantum numbers labelling the states (say 4-momenta).

6.6.2 Could U_F describe dispersion from H_Q to the spaces H_F?

One can also consider a more general situation in which the states in H_Q can be said to disperse to the sectors H_F. In this case one can write

$$T = \text{"}\sum_F\text{"}T_F \ . \tag{6.6.2}$$

Here the sum has only a symbolic meaning since different number fields are in question and an actual summation is not possible. The T-matrix T_Q is the sum of the restrictions of T_F to H_Q and is the sum of rational valued T-matrices: $T_Q = \sum_F P_Q T_F$.

The T-matrices T_F are not anymore obtainable by algebraic continuation from same T-matrix T_Q. The unitarity conditions

$$\sum_F (P_Q T_F + T_F^{\dagger} P_Q) = -\sum_F T_F^{\dagger} T_F \tag{6.6.3}$$

make sense only if they are satisfied separately for each T_F, exactly as in the previous case. T

The diagonal elements

$$T_F^{mm} + \overline{T}_F^{mm} = \sum_{\alpha} T_F^{m\alpha}\overline{T}_F^{m\alpha} = \sum_r T_F^{mr}\overline{T}_F^{mr}$$

give essentially total scattering probabilities from the state $|m\rangle$ of H_Q to the sector H_F, and must be rational (or extended rational) numbers. One can therefore say that each U-process leads with a definite probability to a particular sector of the state space.

The fact that states which are superpositions of states in different spaces H_F does not make sense mathematically, forces the occurrence of a process, which might be regarded as a number theoretical counterpart of state function reduction and preparation. First a sector H_F is selected with probability p_F. Then F-valued (in particular complex valued) entanglement in H_F is reduced by state reduction and preparation type processes to a rational or extended rational entanglement having interpretation as bound state entanglement. It would be natural to assume that Negentropy Maximization Principle governs this process. Obviously the possibility to reduce state function reduction to number theory forces to consider quite seriously the proposed option.

6.6.3 Do state function reduction and state-preparation have number theoretical origin?

The foregoing considerations support the view that state function reduction and state preparation are number theoretical necessities so that there would be a deep connection between number theory and free will. One could even say that free will is a number theoretic necessity. The resulting more unified view provides the reason why for state function reduction, and preparation and allows to generalize previous views developed gradually by physics and consciousness inspired educated guess work.

Negentropy Maximization Principle as variational principle of cognition

It is useful to discuss the original view about Negentropy Maximization Principle (NMP) before considering the possible generalization of NMP inspired by the number theoretic vision.

NMP was originally motivated by the need to construct a TGD based quantum measurement theory. Gradually it however became clear that standard quantum measurement theory more or less follows from the assumption that the world of conscious experience is classical: this meant that NMP became a principle governing only state preparation.

State function reduction is achieved if a localization in zero modes occurs in each quantum jump, and if U matrix in zero modes corresponds to a flow in some orthogonal basis for the configuration space spinor fields in the quantum fluctuating fiber degrees of freedom of the configuration space. The requirement that U-matrix induces effectively a flow in zero modes is consistent with the effective classicality of the zero modes requiring that quantum evolution causes no dispersion. The one-one correlation between preferred quantum state basis in quantum fluctuating degrees of freedom and zero modes implies nothing but a one-one correspondence between quantum states and classical variables crucial for the interpretation of quantum theory. It seems that number theoretical vision forces to generalize this view, and to raise NMP to a completely general principle applying also to the state function reduction as the original proposal indeed was.

In its original form NMP governs the dynamics of self measurements and thus applies to the quantum jumps reducing the entanglement between quantum fluctuating degrees of freedom for given values of zero modes. Self measurements reduce the entanglement only between subsystems in quantum fluctuating degrees of freedom since they occur after the localization in the zero modes. Self measurement is repeated again and again for the unentangled subsystems resulting in each self measurement. This cascade of self measurements leads to a state possessing only extended rational entanglement identifiable as bound state entanglement and having negative number theoretic entanglement entropy. This process should be equivalent with the state preparation process assumed to be performed by a conscious observer in standard quantum measurement theory.

NMP states that the self measurement can be regarded as a quantum measurement of the subsystem's density matrix reducing the counterpart of the entanglement entropy of some subsystem to a smaller value, and that this occurs for the subsystem for which the reduction of the entanglement entropy is largest among all subsystems of the p-adic self. Inside each self NMP fixes some subsystem which is quantum measured in the quantum jump. One could perhaps say that self measurements make possible quantum level self repair since they allow the system in self state to fight against thermalization which results from the generation of unbound entanglement between subsystem-complement pairs.

NMP and number theory

The requirement the universe of conscious experience is classical is one manner to justify quantum jump. This hypothesis could be replaced by a postulate that state function reduction and preparation project quantum states to a definite number field and that only extended rational entanglement identifiable as bound state entanglement is stable. This is consistent with NMP since it is possible to assign to an extended rational entanglement a non-negative number theoretic negentropy as the maximum over entropies defined by various p-adic entropies $S_p = -\sum p_k log(|p_k|_p)$.

The unitary process U would thus start from a product ofbound states for which entanglement coefficient are extended rationals, and would lead to a formal superposition of states belonging to different number fields. Both state function reduction and state preparation would begin with a localization to a definite number field. This localization would be followed by a self measurement cascade reducing the entanglement to extended rational entanglement.

This vision forces to challenge the earlier views about state function reduction.

1. There is no good reason for why NMP could not be applied to both state function reduction and preparation.

2. If the entanglement between zero modes and quantum fluctuating degrees of freedom involves only discrete values of zero modes, the problems caused by the fact that no well-defined functional integral measure over zero modes exists, find an automatic resolution. Since extended rational entanglement possesses negative entanglement entropy, it is stable also against reduction if NMP applies completely generally. A discrete entanglement involving transcendentals not contained to any *finite* extension of any p-adic number field is unstable and reduced.

3. The quantum measurement lasts for a time determined by the life-time of the bound state entanglement between zero modes and quantum fluctuating degrees of freedom. Physical considerations of course support the view that it takes more than single quantum jump (10^{-39} seconds of psychological time) for the state function reduction to take place. The notion of zero mode-zero mode bound state entanglement seems however to be self-contradictory. If join along boundaries bonds are space-time correlates for the bound state entanglement, their formation should transform roughly half of the zero modes associated with the two space-time sheets to quantum fluctuating degrees of freedom.

4. If p-adic length scale hierarchy has as its counterpart a hierarchy of state function reduction and preparation cascades, one must accept the quantum parallel occurrence of state function

reduction and preparation processes in the parallel quantum universes corresponding to different p-adic length scales. This picture provides a justification for the modelling of hadron as a quantum system in long length and time scales and as a dissipative system consisting of quarks and gluons in shorter length and time scales. The bound state entanglement between subsystems of entangled systems having as a space-time correlate join along boundaries bonds connecting subsystem space-time sheets, is a second important implication of the new sub-system concept, and plays a central role in TGD inspired theory of consciousness.

6.7 Generalization of the notion of configuration space

The only manner to possibly p-adicize the notion of the configuration space is provided by the algebraic continuation from a subset of rational configuration space consisting of points for which a finite number of coordinates are non-vanishing and rational values. The representability of the configuration space as a union of symmetric spaces means an enormous simplification since everything reduces to a single point, most naturally the maximum of Kähler function for given zero modes, but there are still several challenges involved.

1. One must construct p-adic counterparts of Kähler function, Kähler metric and Kähler form. There are hopes to achieve this if it is possible to assign to each real space-time sheet a p-adic space-time sheet and identifye the value of p-adic Kähler function as that of the real Kähler function in the case that the values of the real Kählr function $K(X^3)$ values belongs to a finite-dimensional extension of rationals for rational argument. This assignment need not be unique and an entire hierarchy of assignments labelled by the dimension of p-adic numbers involved. The higher the dimension the shorter the pinary cutoff.

2. If Kähler action is rational function in a generalized sense the continuation at rational points is in principle trivial. Also the exponent of Kähler function defining vacuum functional should have continuation to the p-adic context.

3. The continuation of the configuration space Kähler metric and Kähler form reduce to the algebraic continuation of the configuration space Hamiltonians and corresponding super charges. If these define rational or algebraic functions in generalized sense also this continuation might be possible.

4. Also the p-adic variant of the configuration space functional integral must be constructed. Here symmetric space structure gives hopes that Gaussian integral of free field theories generalizes to a functional integral around maxima of Kähler function. It is essential that free field theory situation prevails since only in this case one has control over the extended rationality of the resulting expressions for S-matrix elements.

6.7.1 p-Adic counterparts of configuration space Hamiltonians

One must continue the δM^4_+ local CP_2 Hamiltonians appearing in the integrals defining configuration space Hamiltonians to various p-adic sectors. CP_2 harmonics are homogeneous polynomials with rational coefficients and do not therefore produce any trouble since normalization factors involve only square roots. The p-adicization of δM^4_+ function basis defining representations of Lorentz group involves more interesting aspects.

p-Adicization of representations of Lorentz group

In the light cone geometry Poincare invariance is strictly speaking broken to Lorentz invariance with respect to the dip of the light cone and at least cosmologically a more natural basis is characterized by the eigenvalues of angular momentum and boost operator in a given direction. The eigenvalue spectrum of the boost operator is continuous without further conditions. One can study these conditions in the realization of the unitary representations of Lorenz group as left translations in the Lorentz group itself by utilizing homogenous functions of four complex variables z^1, z^2, z^3, z^4 satisfying the constraint $z_1z_4 - z_2z_3 = 1$ expressing the fact that they correspond to the homogenous coordinates of the Lorentz group defined by that matrix elements of the SL(2,C) matrix

$$\begin{pmatrix} z_1 & z_3 \\ z_2 & z_4 \end{pmatrix} .$$

The function basis consists of

$$\begin{array}{rcl} f^{a_1,a_2,a_3,a_4}(z_1,z_2,z_3,z_4) & = & z_1^{a_1} z_2^{a_2} z_3^{a_3} z_4^{a_4} \ , \\ & & \\ a_1 = m_1 + i\alpha, & & a_2 = m_2 - i\alpha \ , \\ a_3 = m_3 - i\alpha, & & a_4 = m_4 + i\alpha \ , \\ m_1 + m_2 = M \ , & & m_3 + m_4 = M \ . \end{array}$$

The action of Lorentz transformation is given by

$$\begin{pmatrix} z_1 & z_3 \\ z_2 & z_4 \end{pmatrix} \rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z_1 & z_3 \\ z_2 & z_4 \end{pmatrix} . \tag{6.7.1}$$

and unimodular ($ad - bc = 1$). Lorentz transformation preserves the imaginary parts $i\alpha$ of the complex degrees $d_i = m \pm i\alpha$ of $z_k^{\pm i\alpha + m_k}$ (as can be seen by using binomial series representations for the transformed coordinates). Also the sums $m_1 + m_2 = M$ and $m_3 + m_4 = M$ are Lorentz invariants. Hence the representation is characterized by the the pair (α, M). M corresponds to the minimum angular momentum for the $SU(2)$ decomposition of the representation.

The imaginary parts $i\alpha$ of the complex degrees correspond to the eigen values of Lorentz boost in the direction of the quantization axis of angular momentum. The eigen functions are proportional to the factor

$$\begin{array}{l} \rho_1^{i2\alpha} \rho_2^{-i2\alpha} \rho_3^{-i2\alpha} \rho_4^{i2\alpha} \ , \\ \rho_i = \sqrt{z_i \overline{z}_i} \ . \end{array}$$

Since one can write $\rho^{i2\alpha} = e^{i2log(\rho)\alpha}$, these are nothing but the logarithmic plane waves. The value set of $\alpha \geq 0$ is continuous in the real context.

The requirement that the logarithmic plane waves are continuable to p-adic number fields and exist p-adically for rational values of $\rho_i = m/n$, quantizes the values of α. This condition is satisfied if the quantities $p^{i2\alpha_i} = e^{i2log(p)\alpha_i}$ exist p-adically for any prime. As shown in [E8], there seems to be no number theoretical obstructions for the simplest hypothesis $log(p) = q_1(p)exp\left[q_2(p)\right]/\pi$, with $q_2(p_1) \neq q_2(p_2)$ for all pairs of primes. The existence of p^{iy} in a finite-dimensional extension would require that α_i is proportional to π by a coefficient which for a given prime p_1 has sufficiently small p-adic norm so that the exponent can be expanded in powers series.

Obviously p-adicization gives strong quantization conditions. There is also a second possibility. As discussed in the same chapter, the allowance of infinite primes changes the situation. Let $X = \prod p_i$ be the product of all finite primes. $1+X$ is the simplest infinite prime and the ratio $Y = X/(1+X)$ equals to 1 in real sense and has p-adic norm $1/p$ for all finite primes. If one allows α to be proportional to a power Y, then the p-adic norm of α can be so small for all primes that the expansion converges without further conditions. Infinite primes will be discussed later in more detail.

Exactly similar exponents (p^{iy}) appear in the partition function decomposition of the Riemann Zeta, and the requirement that these quantities exist in a finite algebraic extension of p-adic numbers for the zeros $z = 1/2 + iy$ of ζ requires that $e^{ilog(p)y}$ is in a finite-dimensional extension involving algebraic numbers and e. One could argue that for the extensions of p-adics the zeros of Zeta define a universal spectrum of the eigen values of the Lorentz boost generator. This might have implications in hadron physics, where the so called rapidity distribution correspond to the distributions of the particles with respect to the variable characterizing finite Lorentz boosts.

Although the realization of the using the functions in Lorentz group differs from the discussed one, the conclusion is same also for them, in particular for the representation realized at the boundary of the light cone which is one of the homogenous spaces associated with Lorentz group.

Function basis of δM^4_+

One can consider two function basis for δM^4_+ and both function basis allow continuation to p-adic values under similar conditions.

1. Spherical harmonic basis

The first basis consists of functions $Y^l_m \times (r_M/r_0)^{n/2+i\rho}$, $n = -2, -1, 0, ...$ For $n = -2$ these functions define a unitary representation of Lorentz group. The spherical harmonics Y^l_m require a finite-dimensional algebraic extension of p-adic numbers. Radial part defines a logarithmic wave $exp[i\rho log(r_M/r_0)]$ and the existence of this for finite-dimensional extension of p-adic numbers for rational values ρ and r_M is guaranteed by $log(p) = q_1 exp(q_2)/\pi$ ansatz under the conditions already discussed.

2. Basis consisting of eigen functions of angular momentum and boost

Another function basis of δM^4_+ defining a non-unitary representation of Lorentz group and of conformal algebra consists of eigen states of rotation generator and Lorentz boost and is given by

$$f_{m,n,k} = e^{im\phi}\frac{\rho^{n-k}}{(1+\rho^2)^k} \times (\frac{r_M}{r_0})^k \ . \tag{6.7.2}$$

$n = n_1 + in_2$ and $k = k_1 + ik_2$ are in general complex numbers. The condition

$$n_1 - k_1 \geq 0$$

is required by regularity at the origin of S^2. The requirement that the integral over S^2 defining norm exists (the expression for the differential solid angle is $d\Omega = \frac{\rho}{1+\rho^2)^2} d\rho d\phi$) implies

$$n_1 < 3k_1 + 2 \ .$$

From the relationship $(cos(\theta), sin(\theta)) = (\rho^2 - 1)/(\rho^2 + 1), 2\rho/(\rho^2 + 1))$ one can conclude that for $n_2 = k_2 = 0$ the representation functions are proportional to f $sin(\theta)^{n-k}(cos(\theta) - 1)^{n-k}$. Therefore they have in their decomposition to spherical harmonics only spherical harmonics with angular momentum $l < 2(n - k)$. This suggests that the condition

$$|m| \leq 2(n - k) \tag{6.7.3}$$

is satisfied quite generally.

The emergence of the three quantum numbers (m, n, k) can be understood. Light cone boundary can be regarded as a coset space $SO(3, 1)/E^2 \times SO(2)$, where $E^2 \times SO(2)$ is the group leaving the light like vector defined by a particular point of the light cone invariant. The natural choice of the Cartan group is therefore $E^2 \times SO(2)$. The three quantum numbers (m, n, k) have interpretation as quantum numbers associated with this Cartan algebra. The representations of the Lorentz group are characterized by half-integer valued parameter $l_0 = m/2$ and complex parameter l_1. Thus k_2 and n_2, which are Lorentz invariants, might not be independent parameters, and the simplest option is $k_2 = n_2$.

It is interesting to compare the representations in question to the unitary representations of Lorentz group discussed in [19].

1. The unitary representations discussed in [19] are characterized by are constructed by deducing the explicit representations for matrix elements of the rotation generators J_x, J_y, J_z and boost generators L_x, L_y, L_z by decomposing the representation into series of representations of $SU(2)$ defining the isotropy subgroup of a time like momentum. Therefore the states are labelled by eigenvalues of J_z. In the recent case the isotropy group is $E^2 \times SO(2)$ leaving light like point invariant. States are therefore labelled by three different quantum numbers.

2. The representations of [19] are realized the space of complex valued functions of complex coordinates ξ and $\overline{\xi}$ labelling points of complex plane. These functions have complex degrees $n_+ = m/2 - 1 + l_1$ with respect to ξ and $n_- = -m/2 - 1 + l_1$ with respect to $\overline{\xi}$. l_0 is complex number in the general case but for unitary representations of main series it is given by $l_1 = i\rho$ and for the representations of supplementary series l_1 is real and satisfies $0 < |l_1| < 1$. The main series representation is derived from a representation space consisting of homogenous functions of variables z^0, z^1 of degree n_+ and of $\overline{z}^0$ and $\overline{z}^1$ of degrees $n_\pm$. One can separate express these

functions as product of $(z^1)^{n^+}$ $(\overline{z}^1)^{n_-}$ and a polynomial of $\xi = z^1/z^2$ and $\overline{\xi}$ with degrees n_+ and n_-. Unitarity reduces to the requirement that the integration measure of complex plane is invariant under the Lorentz transformations acting as Moebius transformations of the complex plane. Unitarity implies $l_1 = -1 + i\rho$.

3. For the representations at δM^4_+ unitarity reduces to the requirement that the integration measure of $r_M^2 d\Omega dr_M/r_M$ of δM^4_+ remains invariant under Lorentz transformations. The action of Lorentz transformation on the complex coordinates of S^2 induces a conformal scaling which can be compensated by an S^2 local radial scaling. At least formally the function space of δM^4_+ thus defines a unitary representation. For the function basis f_{mnk} $k = -1 + i\rho$ defines a candidate for a unitary representation since the logarithmic waves in the radial coordinate are completely analogous to plane waves. This condition would be completely analogous to the vanishing of conformal weight for the physical states of super conformal representations. The problem is that for $k_1 = -1$ guaranteing square integrability in S^2 implies $-2 < n_1 < -2$ so that unitarity in this sense is not possible.

   There is no deep reason against non-unitary representations and symmetric space structure indeed requires that k_1 is half-integer valued. First of all, configuration space spinor fields are analogous to ordinary spinor fields in M^4, which also define non-unitary representations of Lorentz group. Secondly, if 3-surfaces at the light cone boundary are finite-sized, the integrals defined by f_{mnk} over 3-surfaces Y^3 are always well-defined. Thirdly, the continuous spectrum of k_2 could be transformed to a discrete spectrum when k_1 becomes half-integer valued.

Logarithmic waves and possible connections with number theory and fundamental physics

Logarithmic plane waves labelled by eigenvalues of the scaling momenta appear also in the definition of the Riemann Zeta defined as $\zeta(z) = \sum_n n^{-z}$, n positive integer [E8]. Riemann Zeta is expressible as a product of partition function factors $1/(1 + p^{-x-iy})$, p prime and the powers n^{-x-iy} appear as summands in Riemann Zeta. Riemann hypothesis states that the non-trivial zeros of Zeta reside at the line $x = 1/2$. There are indeed intriguing connections. $Log(p)$ corresponds now to the $log(r_M/r_{min})$ and -x-iy corresponds to the scaling momentum $k_1 + ik_2$ so that the special physical role of the conformal weights $k_1 = 1/2 + iy$ corresponds to Riemann hypothesis. The appearance of powers of p in the definition of the Riemann Zeta corresponds to p-adic length scale hypothesis, ($r_M/r_0 = p$ in ζ and corresponds to a secondary p-adic length scale).

The assumption that the logarithmic plane waves are algebraically continuable from the rational points $r_M/r_{min} = m/n$ to p-adic plane waves using a finite-dimensional extension of p-adic numbers leads to the $log(p) = q_1 exp(q_2)/\pi$ ansatz. Similar hypothesis is inspired by the hypothesis that Riemann Zeta is a universal function existing simultaneously in all number fields. This inspires several interesting observations.

1. p-adic length scale hypothesis stating that $r_{max}/r_{min} = p^n$ is consistent with the number theoretical universality of the logarithmic waves. The universality of Riemann Zeta inspires the hypothesis that the zeros of Riemann Zeta correspond to rational numbers and to preferred values $k_1 + ik_2$ of the scaling momenta appearing in the logarithmic plane waves. In the recent context the most general hypothesis would be that the allowed momenta k_2 correspond to the linear combinations of the zeros of Riemann Zeta with integer coefficients.

2. Hardmuth Mueller [48] claims on basis of his observations that gravitational interaction involves logarithmic radial waves for which the nodes come as $r/r_{min} = e^n$. This is true if the the scaling momenta k_2 satisfy the condition $k_2/\pi \in Z$. Perhaps Mueller's logarithmic waves really could be seen as a direct signature of the fundamental symmetries of the configuration space. In particular, this would require $r_{max}/r_{min} = e^m$.

3. The special role of Golden Mean $\Phi = (1 + \sqrt{5})/2$ in Nature could be understood if also $log(\Phi) = q_1 exp(q_2)/\pi$ or more general ansatz holds true. This would imply that the nodes of logarithmic waves can correspond also to the powers of Φ.

   One could of course argue that the number theory at the moment of Big Bang cannot have strong effects on what is observed in laboratory. This might be the case. On the other hand, the non-determinism of the Kähler action however strongly suggests that the construction of the configuration space geometry involves all possible light like 3-surfaces of the future light cone so that logarithmic waves would appear in all length scales. Be as it may, it would be amazing if such an abstract mathematical structure as configuration space geometry would have direct implications to cosmology and to the physics of living systems.

6.7.2 Configuration space integration

Assuming that U-matrix exists simultaneously in all number fields (allowing finite-dimensional extensions of p-adics), the immediate question is whether also the construction procedure of the real S-matrix could have a p-adic counterpart for each p, and whether the mere requirement that this is the case could provide non-trivial intuitions about the general structure of the theory. Not only the configuration space but also Kähler function and its exponent, Kähler metric, and configuration space functional integral should have p-adic variants. In the following this challenge is discussed in a rather optimistic number theoretic mood using the ideas stimulated by the connections between number theory and cognition.

Does symmetric space structure allow algebraization of configuration space integration?

The basic structure is the rational configuration space whose points have rational valued coordinates. This space is common to both real and p-adic variants of the configuration space. Therefore the construction of the generalized configuration space as such is not a problem.

The assumption that configuration space decomposes into a union of symmetric spaces labeled by zero modes means that the left invariant metric for each space in the union is dictated by isometries. It should be possible to interpret the matrix elements of the configuration space metric in the basis of properly normalized isometry currents as p-adic numbers in some finite extension of p-adic numbers allowing perhaps also some transcendentals. Note that the Kähler function is proportional to the inverse of Kähler coupling strength α_K which depends on p-adic prime p, and does seem to be a rational number if one takes seriously various arguments leading to the hypothesis $1/\alpha_K = klog(K^2)$, $K^2 = p \times 2 \times 3 \times 5.. \times 23$, and $k = \pi/4$ or $k = 137/107$ for the two alternative options discussed in [E8]. If so then the most general transcendentals required and allowed in the extensions used correspond to roots of polynomials with coefficients in an extension of rationals by e and algebraic numbers. As already discussed, infinite primes might provide the ultimate solution to the problem of continuation.

The continuation of the exponent of Kähler function and of configuration space spinor fields to p-adic sectors would require some selection of a subset of points of the rational configuration space. On the other hand, the minimum requirement is that it is possible to define configuration space integration in the p-adic context. The only manner to achieve this is by defining configuration space integration purely algebraically by perturbative expansion. For free field theory Gaussian integrals are in question and one can calculate them trivially. The Gaussian can be regarded as a Kähler function of a flat Kähler manifold having maximal translational and rotational symmetries. Physically infinite number of harmonic oscillators are in question. The origin of the symmetric space is preferred point as far as Kähler function is considered: metric itself is invariant under isometries.

Algebraization of the configuration space functional integral

Configuration space is a union of infinite-dimensional symmetric spaces labelled by zero modes. One can hope that the functional integral could be performed perturbatively around the maxima of the Kähler function. In the case of CP_2 Kähler function has only single maximum and is a monotonically decreasing function of the radial variable r of CP_2 and thus defines a Morse function. This suggests that a similar situation is true for all symmetric spaces and this might indeed be the case. The point is that the presence of several maxima implies also saddle points at which the matrix defined by the second derivatives of the Kähler function is not positive definite. If the derivatives of type $\partial_K \partial_L K$ and $\partial_{\overline{K}} \partial_{\overline{L}} K$ vanish at the saddle point (this is the crucial assumption) in some complex coordinates holomorphically related to those in which the same holds true at maximum, the Kähler metric is not positive definite at this point. On the other hand, by symmetric space property the metric should be isometric with the positive define metric at maxima so that a contradiction results.

If this argument holds true, for given values of zero modes Kähler function has only one maximum, whose value depends on the values zero modes. Staying in the optimistic mood, one could go on to guess that the Duistermaat-Heckman theorem [20] generalizes and the functional integral is simply the exponent of the Kähler function at the maximum (due to the compensation of Gaussian and metric determinants). Even more, one could bravely guess that for configuration space spinor fields belonging to the representations of symmetries the inner products reduces to the generalization of correlation functions of Gaussian free field theory. Each configuration space spinor field would define a vertex from which lines representing the propagators defined by the contravariant configuration space metric in isometry basis emanate.

If this optimistic line of reasoning makes sense, the definition of the p-adic configuration space integral reduces to a purely algebraic one. What is needed is that the contravariant Kähler metric fixed by the symmetric space-property exists and that the exponent of the maximum of the Kähler function exists for rational values of zero modes or subset of them if finite-dimensional algebraic extension is allowed. This would give could hopes that the U-matrix elements resulting from the configuration space integrals would exist also in the p-adic sense.

Is the exponential of the Kähler function rational function?

The simplest possibility that one can imagine are that the exponent e^{2K} of Kähler function appearing in the configuration space inner products is a rational or at most a simple algebraic function existing in a finite-dimensional algebraic extension of p-adic numbers.

The exponent of the CP_2 Kähler function is a rational function of the standard complex coordinates and thus rational-valued for all rational values of complex CP_2 coordinates. Therefore one is lead to ask whether this property holds true quite generally for symmetric spaces and even in the infinite-dimensional context. If so, then the continuation of the vacuum functional to the p-adic sectors of the configuration space would be possible in the entire configuration space. Also the spherical harmonics of CP_2 are rational functions containing square roots in normalization constants. That also configuration space spinor fields could use rational functions containing square roots as normalization constant as basic building blocks would conform with general number theoretical ideas as well as with the general features of harmonic oscillator wave functions.

The most obvious manner to realize this idea relies on the restriction of light-like 3-surfaces X_l^3 to those representable in terms of polynomials or rational functions with rational or at most algebraic coefficients serving as natural preferred coordinates of the configuration space. This of course requires identification of preferred coordinates also for H. This would lead to a hierarchy of inclusions for sub-configuration spaces induced by algebraic extensions of rationals.

The presence of cutoffs for the degrees of polynomials involved makes the situation finite-dimensional and give rise to a hierarchy of inclusions also now. These inclusion hierarchies would relate naturally also to hierarchies of inclusions for hyperfinite factors of type II_1 since the spinor spaces associated with these finite-D versions of WCW would be finite-dimensional. Hyper-finiteness means that this kind of cutoff can give arbitrarily precise approximate representation of the infinite-D situation.

This vision is supported by the recent understanding related to the definition of exponent of Kähler function as Dirac determinant [A6]. The number of eigenvalues involved is necessarily finite, and if the eigenvalues of D_{C-S} are algebraic numbers for 3-surfaces X_l^3 for which the coefficients characterizing the rational functions defining X_l^3 are algebraic numbers, the exponent of Kähler function is algebraic number.

The general number theoretical conjectures implied by p-adic physics and physics of cognition and intention support also this conjecture. Although one must take these arguments with a big grain of salt, the general idea might be correct. Also the elements of the configuration space metric would be rational functions as is clear from the fact that one can express the second derivatives of the Kähler function in terms of $F = exp(K)$ as

$$\partial_K \partial_{\overline{L}} K = \frac{\partial_K \partial_{\overline{L}} F}{F} - \frac{\partial_K F \partial_{\overline{L}} F}{F^2} \quad .$$

Coupling constant evolution and number theory

The coupling constant evolution associated with the Kähler action might be at least partially understood number-theoretically.

A given space-time sheet is connected by wormhole contacts to the larger space-time sheets. The induced metric within the wormhole contact has an Euclidian signature so that the wormhole contact is surrounded by elementary particle horizons at which the metric is degenerate so that the horizons are metrically effectively 2-dimensional giving rise to quaternion conformal invariance. Because of the causal horizon it would seem that Kähler coupling strength can depend on the space-time sheet via the p-adic prime characterizing it. If so the exponent of the Kähler function would be simply the product of the exponents for the space-time sheets and one would have finite-dimensional extension as required.

If the exponent of the Kähler function is rational function, also the components of the contravariant Kähler metric are rational functions. This would suggest that one function of the coupling constant evolution is to keep the exponent rational.

From the point of view of p-adicization the ideal situation results if Kähler coupling strength is invariant under the p-adic coupling constant evolution as I believed originally. For a long time it however seemed that this option cannot be realized since the prediction $G = L_p^2 exp(-2S_K(CP_2))$ for the gravitational coupling constant following from dimensional considerations alone implies that G increases without limit as a function of p-adic length scale if α_K is RG invariant. If one however assumes that bosonic space-time sheets correspond to Mersenne primes, situation changes since M_{127} defining electron length scale is the largest Mersenne prime for which p-adic length scale is not super-astronomical and thus excellent candidate for characterizing gravitonic space-time sheets. There is much stronger motivation for this hypothesis coming from the fact that a nice picture about evolution of electro-weak and color coupling strengths emerges just from the physical interpretation of the fact that classical color action and electro-weak $U(1)$ action are proportional to Kähler action [C6].

The recent progress in the understanding of the definition of the exponent of Kähler function as Dirac determinant [A6] leads to rather detailed picture about the number theoretic anatomy of α_K and other coupling constant strengths as well as the number theoretic anatomy of $R^2/\hbar G$ [C4]. By combining these results with the constraints coming from p-adic mass calculations one ends up to rather strong predictions for α_K and $R^2/\hbar G$.

Consistency check in the case of CP_2

It is interesting to look whether this vision works or fails in a simple finite-dimensional case. For CP_2 the Kähler function is given by $K = -log(1 + r^2)$. This function exists if an extension containing the logarithms of primes is used. $log(1+x)$, $x = O(p)$ exists as an ordinary p-adic number and a logarithm of $log(m)$, $m < p$ such that the powers of m span the numbers $1, ..., p-1$ besides $log(p)$ should be introduced to the extension in order that logarithm of any integer and in fact of any rational number exists p-adically. Also logarithms of roots of integers and their products would exist. The problem is however that the powers of $log(m)$ and $log(p)$ would generate an infinite-dimensional extension since finite-dimensional extension leads to a contradiction as shown in [E8].

The exponent of Kähler function as well as Kähler metric and Kähler form have rational-valued elements for rational values of the standard complex coordinates for CP_2. The exponent of the Kähler function is $1/(1 + r^2)$ and exists as a rational number at 3-spheres of rational valued radius. The negative of the Kähler function has a single maximum at $r = 0$ and vanishes at the coordinate singularity $r \to \infty$, which corresponds to the geodesic sphere S^2.

If one wants to cognize about geodesic length, areas of geodesic spheres, and about volume of CP_2, π must be introduced to the extension of p-adics and means infinite-dimensional extension by the arguments of [E8]. The introduction of π is not however necessary for introducing of spherical coordinates if one expresses everything in terms of trigonometric functions. For ordinary spherical coordinates this means effectively replacing θ and ϕ by $u = \theta/\pi$ and $v = \phi/2\pi$ as coordinates. By allowing u and v to have a finite number of rational values requires only the introduction of a finite-dimensional algebraic extension in order to define cosines and sines of the angle variables at these values. What seems clear is that the evolution of cognition as the emergence of higher-dimensional extensions corresponds quite concretely to the emergence of finer discretizations.

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Part III

RELATED TOPICS

Chapter 7

Category Theory, Quantum TGD, and TGD Inspired Theory of Consciousness

7.1 Introduction

Goro Kato has proposed an ontology of consciousness relying on category theory [22, 23]. Physicist friendly summary of the basic concepts of category theory can be found in [20]) whereas the books [24, 25] provide more mathematically oriented representations. Category theory has been proposed as a new approach to the deep problems of modern physics, in particular quantization of General Relativity. To mention only one example, C. J. Isham [20] has proposed that topos theory could provide a new approach to quantum gravity in which space-time points would be replaced by regions of space-time and that category theory could geometrize and dynamicize even logic by replacing the standard Boolean logic with a dynamical logic dictated by the structure of the fundamental category purely geometrically [19].

Although I am an innocent novice in this field and know nothing about the horrible technicalities of the field, I have a strong gut feeling that category theory might provide the desired systematic approach to quantum TGD proper, the general theory of consciousness, and the theory of cognitive representations [H8].

7.1.1 Category theory as a purely technical tool

Category theory could help to disentangle the enormous technical complexities of the quantum TGD and to organize the existing bundle of ideas into a coherent conceptual framework. The construction of the geometry of the configuration space ("world of classical worlds")[A1, A2, B1, B2, B3], of classical configuration space spinor fields [A6], and of S-matrix [C2] using a generalization of the quantum holography principle are especially natural applications. Category theory might also help in formulating the new TGD inspired view about number system as a structure obtained by "gluing together" real and p-adic number fields and TGD as a quantum theory based on this generalized notion of number [A1, A2, E1, E2, E3].

7.1.2 Category theory based formulation of the ontology of TGD Universe

It is interesting to find whether also the ontology of quantum TGD and TGD inspired theory of consciousness based on the trinity of geometric, objective and subjective existences [K1] could be expressed elegantly using the language of the category theory.

There are indeed natural and non-trivial categories involved with many-sheeted space-time and the geometry of the configuration space ("the world of classical worlds"); with configuration space spinor fields; and with the notions of quantum jump, self and self hierarchy. Functors between these categories could express more precisely the quantum classical correspondences and self-referentiality of quantum states allowing them to express information about quantum jump sequence.

i) Self hierarchy has a structure of category and corresponds functorially to the hierarchical structure of the many-sheeted space-time.
ii) Quantum jump sequence has a structure of category and corresponds functorially to the category formed by a sequence of maximally deterministic regions of space-time sheet.
iii) Even the quantum jump could have space-time correlates made possible by the generalization of the Boolean logic to what might be space-time correlate of quantum logic and allowing to identify space-time correlate for the notion of quantum superposition.
iv) The category of light cones with inclusion as an arrow defining time ordering appears naturally in the construction of the configuration space geometry and realizes the cosmologies within cosmologies scenario. In particular, the notion of the arrow of psychological time finds a nice formulation unifying earlier two different explanations.

7.1.3 Other applications

One can imagine also other applications.

1. Categories posses inherent logic [19] based on the notion of sieves relying on the notion of presheaf which generalizes Boolean logic based on inclusion. In TGD framework inclusion is naturally replaced by topological condensation and this leads to a two-valued logic realizing space-time correlate of quantum logic based on the notions of quantum sieve and quantum topos.

   This suggests the possibility to geometrize the logic of both geometric, objective and subjective existences and perhaps understand why ordinary consciousness experiences the world through Boolean logic and Zen consciousness experiences universe through logic in which the law of excluded middle is not true. Interestingly, the p-adic logic of cognition is naturally 2-valued whereas the real number based logic of sensory experience allows excluded middle (is the person at the door in or out, in and out, or neither in nor out?). The quantum logic naturally associated with spinors (in the "world of classical worlds") is consistent with the logic based on quantum sieves.

2. Simple Boolean logic of right and wrong does not seem to be ideal for understanding moral rules. Same applies to the beauty-ugly logic of aesthetic experience. The logic based on quantum sieves would perhaps provide a more flexible framework.

3. Cognition is categorizing and category theory suggests itself as a tool for understanding cognition and self hierarchies and the abstraction processes involved with conscious experience. Here the new elements associated with the ontology of space-time due to the generalization of number concept would be central. Category theory could be also helpful in the modelling of conscious communications, in particular the telepathic communications based on sharing of mental images involving the same mechanism which makes possible space-time correlates of quantum logic and quantum superposition.

7.2 What categories are?

In the following the basic notions of category theory are introduced and the notion of presheaf and category induced logic are discussed.

7.2.1 Basic concepts

Categories [24, 25, 20] are roughly collections of objects A, B, C... and morphisms $f(A \rightarrow B)$ between objects A and B such that decomposition of two morphisms is always defined. Identity morphisms map objects to objects. Topological/linear spaces form a category with continuous/linear maps acting as morphisms. Also algebraic structures of a given type form a category: morphisms are now homomorphisms. Practically any collection of mathematical structures can be regarded as a category. Morphisms can can be very general: for instance, partial ordering $a \leq b$ can define morphism $f(A \rightarrow B)$.

Functors between categories map objects to objects and morphisms to morphisms so that a product of morphisms is mapped to the product of the images and identity morphism is mapped to identity morphism. Group representation is example of this kind of a functor: now group action in group is mapped to a linear action at the level of the representations. Commuting square is an easy visual manner to understand the basic properties of a functor, see Fig. 7.2.1.

The product $C = AB$ for objects of categories is defined by the requirement that there are projection morphisms π_A and π_B from C to A and B and that for any object D and pair of morphisms $f(D \rightarrow A)$ and $g(D \rightarrow B)$ there exist morphism $h(D \rightarrow C)$ such that one has $f = \pi_A h$ and $g = \pi_B h$. Graphically (see Fig. 7.2.1) this corresponds to a square diagram in which pairs A,B and C,D correspond to the pairs formed by opposite vertices of the square and arrows DA and DB correspond to morphisms f and g, arrows CA and CB to the morphisms π_A and π_B and the arrow h to the diagonal DC.

Examples of product categories are Cartesian products of topological and linear spaces, of differentiable manifolds, groups, etc. Also tensor products of linear spaces satisfies these axioms. One can define also more advanced concepts such as limits and inverse limits. Also the notions of sheafs, presheafs, and topos are important.

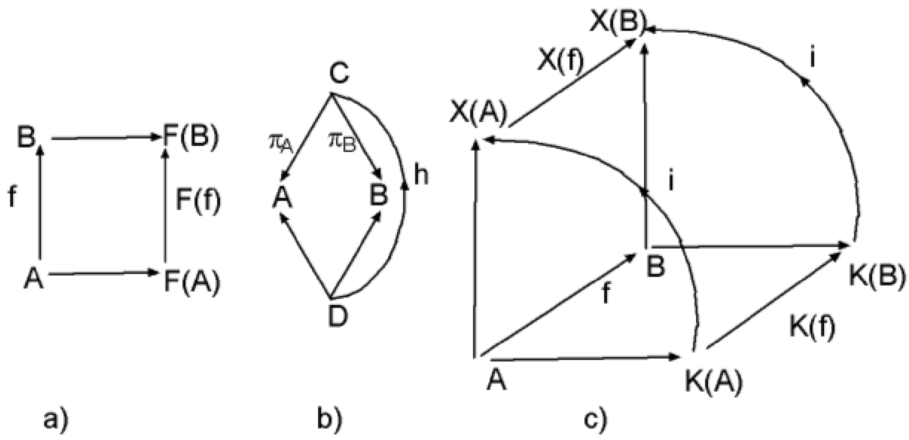

Figure 7.1: Commuting diagram associated with the definition of a) functor, b) product of objects of category, c) presheaf K as sub-object of presheaf X ("two pages of book".)

7.2.2 Presheaf as a generalization for the notion of set

Presheafs can be regarded as a generalization for the notion of set. Presheaf is a functor X that assigns to any object of a category **C** an object in the category **Set** (category of sets) and maps morphisms to morphisms (maps between sets for **C**). In order to have a category of presheafs, also morphisms between presheafs are needed. These morphisms are called natural transformations $N : X(A) \rightarrow Y(A)$ between the images $X(A)$ and $Y(A)$ of object A of **C**. They are assumed to obey the commutativity property $N(B)X(f) = Y(f)N(A)$ which is best visualized as a commutative square diagram. Set theoretic inclusion $i : X(A) \subset Y(A)$ is obviously a natural transformation.

An easy manner to understand and remember this definition is commuting diagram consisting of two pages of book with arrows of natural transformation connecting the corners of the pages: see Fig. 7.2.1.

As noticed, presheafs are generalizations of sets and a generalization for the notion of subset to a sub-object of presheaf is needed and this leads to the notion of topos [19, 20]. In the classical set theory a subset of given sets X can be characterized by a mapping from set X to the set $\Omega = \{true, false\}$ of Boolean statements. Ω itself belongs to the category **C**. This idea generalizes to sub-objects whose objects are collections of sets: Ω is only replaced with its Cartesian power. It can be shown that in the case of presheafs associated with category **C** the sub-object classifier Ω can be replaced with a more general algebra, so called Heyting algebra [20, 19] possessing the same basic operations as Boolean algebra (and, or, implication arrow, and negation) but is not in general equivalent with any

Boolean algebra. What is important is that this generalized logic is inherent to the category **C** so that many-valued logic ceases to be an ad hoc construct in category theory.

In the theory of presheafs sub-object classifier Ω, which belongs to **Set**, is defined as a particular presheaf. Ω is defined by the structure of category **C** itself so that one has a geometrization of the notion of logic implied by the properties of category. The notion of sieve is essential here. A sieve for an object A of category **C** is defined as a collection of arrows $f(A \rightarrow ...)$ with the property that if $f(A \rightarrow B)$ is an arrow in sieve and if $g(B \rightarrow C)$ is any arrow then $gf(A \rightarrow C)$ belongs to sieve.

In the case that morphism corresponds to a set theoretic inclusion the sieve is just either empty set or the set of all sets of category containing set A so that there are only two sieves corresponding to Boolean logic. In the case of a poset (partially ordered set) sieves are sets for which all elements are larger than some element.

7.2.3 Generalized logic defined by category

The presheaf $\Omega : \mathbf{C} \rightarrow \mathbf{Set}$ defining sub-object classifier and a generalization of Boolean logic is defined as the map assigning to a given object A the set of all sieves on A. The generalization of maps $X \rightarrow \Omega$ defining subsets is based on the the notion of sub-object K. K is sub-object of presheaf X in the category of presheaves if there exist natural transformation $i : K \rightarrow X$ such that for each A one has $K(A) \subset X(A)$ (so that sub-object property is reduced to subset property).

The generalization of the map $X \rightarrow \Omega$ defining subset is achieved as follows. Let K be a sub-object of X. Then there is an associated characteristic arrow $\chi^K : X \rightarrow \Omega$ generalizing the characteristic Boolean valued map defining subset, whose components $\chi^K_A : X(A) \rightarrow \Omega(A)$ in **C** is defined as

$$\chi^K_A(x) = \{f(A \rightarrow B) | X(f)(x) \in K(B)\} \ .$$

By using the diagrammatic representation of Fig. 7.2.1 for the natural transformation i defining sub-object, it is not difficult to see that by the basic properties of the presheaf K $\chi^K_A(x)$ is a sieve. When morphisms f are inclusions in category Set, only two sheaves corresponding to all sets containing X and empty sheaf result. Thus binary valued maps are replaced with sieve-valued maps and sieves take the role of possible truth values. What is also new that truths and logic are in principle context dependent since each object A of **C** serves as a context and defines its own collection of sieves.

The generalization for the notion of point of set X exists also and corresponds to a selection of single element γ_A in the set $X(A)$ for each A object of **C**. This selection must be consistent with the action of morphisms $f(A \rightarrow B)$ in the sense that the matching condition $X(f)(\gamma_A) = \gamma_B$ is satisfied. It can happen that category of presheaves has no points at all since the matching condition need not be satisfied globally.

It turns out that TGD based notion of subsystem leads naturally to what might be called quantal versions of topos, presheaves, sieves and logic.

7.3 Category theory and consciousness

Category theory is basically about relations between objects, rather than objects themselves. Category theory is not about Platonic ideas, only about relations between them. This suggests a possible connection with TGD and TGD inspired theory of consciousness where the sequences quantum jumps between quantum histories defining selves have a role similar to morphisms and quantum states themselves are like Platonic ideas not conscious as such. Also the fact that it is not possible to write any formula for the contents of conscious experience although one can say a lot about its general structure bears a striking similarity to the situation in category theory.

7.3.1 The ontology of TGD is tripartistic

The ontology of TGD involves a trinity of existences.

1. Geometric existence or existence in the sense of classical physics. Objects are 3-surfaces in 8-D imbedding space, matter as res extensa. Quantum gravitational holography assigns to a 3-surface X^3 serving as a causal determinant space-time sheet $X^4(X^3)$ defining the classical

physics associated with X^3 as a generalization of Bohr orbit. X^3 can be seen as a 3-D hologram representing the information about this 4-D space-time sheet

The geometry of configuration space of 3-surfaces, "the world of classical worlds" corresponds to a higher level geometric existence serving as the fixed arena for the quantum dynamics. The basic vision is that the existence requirement for Kähler geometry in the infinite-dimensional context fixes the infinite-dimensional geometric existence uniquely.

2. Quantum states defined as classical spinor fields in the world of classical worlds, and provide the quantum descriptions of possible physical realities that the probably never-reachable ultimate theory gives as solutions of field equations. The solutions *are* the objective realities in the sense of quantum theory: theory and theory about world are one and the same thing: there is no separate 'reality' behind the solutions of the field equations.

3. Subjective existence corresponds to quantum jumps between the quantum states identified as moment of consciousness. Just as quantum numbers characterize physical states, the increments of quantum numbers in quantum jump are natural candidates for qualia, and this leads to a concrete quantum model for sensory qualia and sensory perception [K3].

Quantum jump has a complex anatomy: counterpart for the unitary U process of Penrose followed by a counterpart of the state function reduction followed by a counterpart of the state preparation process yielding a classical state in Boolean and geometrical sense. State function preparation and reduction are nondeterministic processes and preparation is analogous to analysis since it decomposes at each step the already existing unentangled subsystems to unentangled subsystems if possible.

Quantum jump is the elementary particle of consciousness and selves are like atoms, molecules,... built from these. Self is by definition a system able to not develop bound state quantum entanglement with environment and loses consciousness when this occurs. Selves form a hierarchy very much analogous to the hierarchy of states formed from elementary particles. Self experiences its sub-selves as mental images. Selves form objects of a category in which arrows connect sub-selves to selves.

Macro-temporal and macroscopic quantum coherence corresponds to the formation of bound states [K2]: in this process state function reduction and preparation effective cease in appropriate degrees of freedom. In TGD framework one can assign to bound state entanglement negative entropy identifiable as a genuine measure for information [H2]. The bound state entanglement stable against state function preparation would thus serve as a correlate for the experience of understanding, and one could compare quantum jump to a brainstorm followed by an analysis leading to an experience of understanding.

Quantum classical correspondence relates the three levels of existence to each other. It states that both quantum states and quantum jump sequences have space-time correlates. This is made possible by p-adic and classical non-determinism, which are characteristic features of TGD space-time. p-Adic non-determinism makes it possible to map quantum jump sequences to p-adic space-time sheets: this gives rise to cognitive representations. The non-determinism of Kähler action makes possible symbolic sensory representations of quantum jump sequences of which language is the basic example.

The natural identification of the correlates of quantum states is as maximal deterministic regions of space-time sheet. The final states of quantum jump define a sequence of quantum states so that quantum jump sequence (contents of consciousness) has the decomposition of space-time sheet to maximal deterministic regions as a space-time correlate. Thus space-time surface can be said to define a symbolic (and unfaithful) representation for the contents of consciousness. Since configuration space spinor field is defined in the world of classical worlds, this means that quantum states carry information about quantum jump sequence and self reference becomes possible. System can become conscious about what it *was* (not "is") conscious of.

The possibility to represent quantum jump sequences at space-time level is what makes possible practical mathematics, cognition, and symbolic representations. The generation of these representations in turn means generation of reflective levels of consciousness and thus explains self-referential nature of consciousness. This feedback makes also possible the evolution of mathematical consciousness: mathematician without paper and pencil (or computer keyboard!) cannot do very much.

Category theory might help to formulate more precisely the quantum classical correspondence and self referentiality as structure respecting functors from the categories associated with subjective existence to the categories of quantum and classical existence and from the category of quantum existence to that of classical existence.

7.3.2 The new ontology of space-time

Classical worlds are space-time surfaces and have much richer ontology than the space-time of general relativity. Space-time is many-sheeted possessing a hierarchy of parallel space-time sheets topologically condensed at larger space-time sheets and identifiable as geometric correlates for physical objects in various length scales (see Fig. 7.3.3). Topological field quantization allows to assign to any material system "field body": this has important implications for quantum biology in TGD Universe [K1].

TGD leads to a generalization of the notion of real numbers obtained by gluing real number field and p-adic number fields R_p, labelled by primes $p = 2, 3, 5, ...$ and their extensions together along common rationals (very roughly) to form a "book like" structure [A1, A2, E1, E3, K1]. p-Adic space-time sheets are interpreted as space-time correlates of cognition and intentionality. The transformation of intention to action corresponds to a quantum jump replacing p-adic space-time sheet with a real one.

The p-adic notion of distance differs dramatically from its real counterpart. Two rationals infinitesimally near p-adically are infinitely distance in real sense. This means that p-adic space-time sheets have literally infinite size in the real sense and cognition and intentionality cannot be localized in brain. Biological body serves only as a sensory receptor and motor instrument utilizing symbolic representations built by brain.

The notion of infinite numbers (primes, rationals, reals, complex numbers and also quaternions and octonions)[E3] inspired by TGD inspired theory of consciousness leads to a further generalization. One can form ratios of infinite rationals to get ordinary rational numbers in the real sense and division by its inverse gives numbers which are units in the real sense but not in various p-adic senses ($p = 2, 3, 5, ...$).

This means that each space-time point is infinitely structured (note also that configuration space points are 3-surfaces and infinitely structure too!) but this structure is not seen at the level of real physics. The infinite hierarchy of infinite primes implies that single space-time point is in principle able to represent the physical quantum state of the entire universe in its structure cognitively. There are several interpretations: space-time points are algebraic holograms realizing Brahman=Atman identity; the Platonia of mathematical ideas resides at every space-time point, space-time points are the monads of Leibniz or the nodes of Indra's web...

One might hope that category theory could be of help in formulating more precisely this intuitive view about space-time which generalizes also to the other two levels of ontology.

7.3.3 The new notion of sub-system and notions of quantum presheaf and "quantum logic" of sub-systems

TGD based notion subsystem differs from the standard one already at the classical level [H2]. The relationship of having wormhole contacts to a larger space-time sheet would correspond to the basic morphism and would correspond to inclusion in category Set. Note that same space-time sheet can have wormhole contacts to several larger space-time sheets (see Fig. 7.3.3). The wormhole contacts are surrounded by light like 3-surfaces somewhat analogous to black hole horizons. They act as causal determinants and define 3-dimensional quantum gravitational holograms. Also other causal determinants are possible but light-likeness seems to a common feature of them.

Subsystem does not correspond to a mere subset geometrically as in standard physics and the functors mapping quantum level to space-time level are not maps to the category of sets but to that of space-time sheets, and thus pre-sheafs are replaced with what might be called quantum pre-sheafs. Boolean algebra and also Heyting algebra are replaced with their quantum variants.

1. The set theoretic inclusion $\subset$ in the definition of Heyting algebra is replaced by the arrow $A \to B$ representing a sequence of topological condensations connecting the space-time sheet A to B. The arrow from A to B is possible only if A is smaller than B, more precisely: if the p-adic prime $p(A)$ characterizing A is larger (or equal) than $p(B)$. The relation $\in$ of being a point of the space-time sheet A is not utilized at all.

2. Sieves at A are defined, not in terms of arrow sequences $f(A \to B)$, but as arrow sequences $f(B \to A)$: the wormhole contact roads leading from sheet B down to A. If there is a road from B to A then all roads to $C \to B$ combine with roads $B \to A$ to give roads $C \to A$ and thus define elements of the sieve.

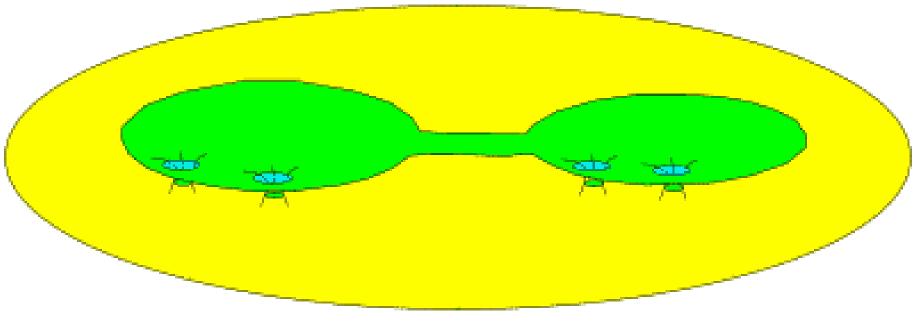

Figure 7.2: a) Wormhole contacts connect interiors space-time parallel space-time sheets (at a distance of about 10^4 Planck lengths) and join along boundaries bonds of possibly macroscopic size connect boundaries of space-time sheets. b) Wormhole contacts connecting space-time sheet to several space-time sheets could represent space-time correlate of quantum superposition. c) Space-time correlate for bound state entanglement making possible sharing of mental images.

3. X is quantum presheaf if it is a functor from the a category C to the category of space-time sheets. A sub-object of X is presheaf K such that for every A there is a road from $K(A)$ to $X(A)$.

4. Let K be a sub-object of the pre-sheaf X. The elements of the corresponding quantum Heyting algebra at A are defined as the collections of roads $f(B, A)$ leading via $K(A)$ to $K(X)$. This collection is either empty or contains all the roads via $K(A)$ to $K(X)$. A two-valued logic results trivially.

5. The difference with respect to Boolean logic comes from the fact space-time sheet can condense simultaneously to several disjoint space-time sheets whereas a given set cannot be a subset of two disjoint sets (see Fig. 7.3.3).

One can ask whether this property of "quantum logic" allows a space-time correlate even for the superposition of orthogonal quantum states as simultaneous topological condensation at several space-time sheets. This interpretation would be consistent with the hypothesis that bound state entanglement has the formation of join along boundaries bonds (JABs) as a space-time correlate. Topologically condensed JAB-connected space-time sheets could indeed condense simultaneously on several space-time sheets. It however seems that this interpretation is not consistent with quantum superpositions.

The new notion of sub-system at space-time level forces to modify the notion of sub-system at quantum level. The subsystem defined by a smaller space-time sheet is not describable as a simple tensor factor but the relation is given by the morphism representing the property of being sub-system. In the chapter "Was von Neumann Right After All" [C6] a mathematical formulation for this relationship is proposed in terms of so called Jones inclusions of von Neumann algebras of type II_1, which seem to provide the proper mathematical framework for quantum TGD. Wormhole contacts would represent space-time correlate for inclusion as a generalized tensor factor rather than inclusion as a direct summand as in quantum superposition.

Space-time correlate for ordinary quantum logic

The proposed "quantum logic" for subsystems based on topological condensation by the formation of wormhole contacts does not seem to correspond to the formation of quantum superpositions and the usual quantum logic. The most non-intuitive aspect of quantum logic is represented by the quantum superposition of mutually exclusive options represented by orthogonal quantum states.

In the double-slit experiment this corresponds to the possibility of single photon to travel along the paths going through the two slits simultaneously and to interfere on the screen. In TGD framework this would correspond quite literally to the decay of the 3-surface describing photon to two pieces which travel through the slits and fuse together before the screen. More generally, the space-time correlate for this aspect of quantum logic would be splitting of 3-surface to several pieces. In string models where the splitting of string means creation of 2-particle state (2-photon state in the case of double slit experiment), which at state space-level corresponds to a tensor product product state. Therefore the ontologies of string models and TGD differ in a profound manner.

In quantum measurement the projection to an eigen state of observables means that a quantum jump in which all branches except one become vacuum extremal occurs. What is also new that by the classical non-determinism space-time surface can also represent a quantum jump sequence. For instance, the states before and after the reduction correspond to space-time regions. This picture allows to understand the recent findings of Afshar [39, K1], which challenge Copenhagen interpretation.

7.3.4 Does quantum jump allow space-time description?

Quantum jump consists of a unitary process, state function reduction and state preparation. The geometrical realization of "quantum logic" suggests that simultaneous topological condensation to several space-time sheets could be a space-time correlate for the maximally entangled superposition of quantum states created in the U-process. Quantal multi-verse states would functorially correspond to classical multi-verse states: something which obviously came in my mind for long time ago but seemed stupid. State function reduction would lead to the splitting of the wormhole contacts and as a result maximally reduced state would result: one cannot however exclude bound state entanglement due to interactions mediated by wormhole contacts.

State function preparation would correspond to a sequence of splittings for join along boundaries bonds serving as prerequisites for entanglement in the degrees of freedom associated with second quantized induced spinor fields at space-time sheets. An equivalent process is the decay of 3-sheet to two pieces interpretable as de-coherence. For instance, the splitting of photon beam in the modified double slit experiment by Afshar [39, K1], which challenges the existing interpretations of quantum theory and provides support for TGD based theory of quantum measurement relying on classical non-determinism, would correspond to this process.

State preparation yields states in which no dissipation occurs. The space-time correlates are asymptotic solutions of field equations for which classical counterpart of dissipation identified as Lorentz 4-force vanishes: this hypothesis indeed leads to very general solutions of field equations [D1]. The non-determinism at quantum level would correspond to the non-determinism for the evolution of induced spinor fields at space-time level.

7.3.5 Brief summary of the basic categories relating to the self hierarchy

Category theory suggests the identification of space-time sheets as basic objects of the space-time category. Space-time sheets are natural correlates for selves and the arrow describing sub-self property is mapped to the arrow of being topologically condensed space-time sheet. Category theoretically this would mean the existence of a functor from the the category defined by self hierarchy to the hierarchy of space-time sheets.

The highly non-trivial implication of the new notion of sub-system is that same sub-self can be sub-self of several selves: mental images can be shared so that consciousness would not be so private as usually believed. Sharing involves also fusion of mental images. Sub-selves of different selves form a bound state and fuse to single sub-self giving rise to stereo consciousness (fusion of right and left visual fields is the basic example).

The formation of join along boundaries bonds connecting the boundaries of a sub-self space-time sheets is the space-time correlate for this process. The ability of subsystems to entangle when systems remain un-entangled is completely new and due to the new notion of subsystem (subsystem is separated by elementary particle horizon from system). Sharing of mental images and the possibility of time-like entanglement also possible telepathic quantum communications: for instance, TGD based model of episodal memories relies on this mechanism [K1].

The hierarchy of space-time sheets functorially replicates itself at the level of quantum states and of subjective existence. Quantum states have a hierarchical structure corresponding to the decomposition of space-time to space-time sheets. The sequence of quantum jumps decomposes into parallel sequences of quantum jumps occurring at different parallel space-time sheets characterized by p-adic length scales. The possibility of quantum parallel dissipation (quarks inside hadrons) is one important implication: although dissipation and de-coherence occur in short length and time scales, quantum coherence is preserved in longer length and time scales. This is of utmost importance for understanding how wet and hot brain can be macroscopic quantum system [K2].

The self hierarchy has also counterpart at the level of Platonia made possible by infinitely structured points of space-time. The construction of infinite primes is analogous to a repeated second quantization of an arithmetic quantum field theory such that the many particle states of previous level representing infinite primes at that level become elementary particles at the next level of construction. This hierarchy reflect itself as the hierarchy of units and as a hierarchy of levels of mathematical consciousness.

The steps in quantum jump, or equivalently the sequence of final states of individual steps would define the objects of the category associated with the quantum jump. The first step would be the formation of a larger number of wormhole contacts during U process followed by their splitting to minimum in the state function reduction. Formation and splitting of contacts would define arrows now. During the state preparation each decay to separate 3-sheets would define arrow from connecting initial state to both final states.

7.3.6 The category of light cones, the construction of the configuration space geometry, and the problem of psychological time

Light-like 7-surfaces of imbedding space are central in the construction of the geometry of the world of classical worlds. The original hypothesis was that space-times are 4-surfaces of $H = M^4_+ \times CP_2$, where M^4_+ is the future light cone of Minkowski space with the moment of big bang identified as its boundary $\delta H = \delta M^4_+ \times CP_2$: "the boundary of light-cone". The naive quantum holography would suggest that by classical determinism everything reduces to the light cone boundary. The classical non-determinism of Kähler action forces to give up this naive picture which also spoils the full Poincare invariance.

The new view about energy and time forces to conclude that space-time surfaces approach vacua at the boundary of the future light cone. The world of classical worlds, call it CH, would consist of classical universes having a vanishing inertial 4-momentum and other conserved quantities and being created from vacuum: big bang would be replaced with a "silent whisper amplified to a big bang". The net gravitational mass density can be non-vanishing since gravitational momentum is difference of inertial momenta of positive and negative energy matter: Einstein's Equivalence Principle is exact truth only at the limit when the interaction between positive and negative energy matter can be neglected [D5].

Poincare invariant theory results if one replaces CH with the union of its copies $CH(a)$ associated with the light cones $M^4_+(a)$ with a specifying the position of the dip of $M^4_+(a)$ in M^4. Also past directed light-cones $M^4_-(a)$ are allowed. The unions and intersections of the light cones with inclusion as a basic arrow would form category analogous to the category Set with inclusion defining the arrow of time. This category formalizes the ideas that cosmology has a fractal Russian doll like structure, that the cosmologies inside cosmologies are singularity free, and that cosmology is analogous to an organic evolution and organic evolution to a mini cosmology [D5].

The view also unifies the proposed two explanations for the arrow of psychological time [K1].

1. The mind like space-time sheets representing conscious self drift quantum jump by quantum jump towards geometric future whereas the matter like space-time sheets remain stationary. The self of the organism presumably consisting mostly of topological field quanta, would be like a passenger in a moving train seeing the changing landscape. The organism would be a mini cosmology drifting quantum jump to the geometric future. Also selves living in the reverse direction of time are possible.

2. Psychological time corresponds to a phase transition front in which intentions represented by p-adic space-time sheets transform to actions represented by real space-time sheets moving to the direction of geometric future. The motion would be due to the drift of $M^4_+(a)$. The very fact that the mini cosmology is created from vacuum, implies that space-time sheets of both negative and positive field energy are abundantly generated as realizations of intentions. The intentional resources are richest near the boundary of $M^4_+(a)$ and depleted during the ageing with respect to subjective time as asymptotic self-organization patterns are reached. Interestingly,

mini cosmology can be seen as a fractally scaled up variant of quantum jump. The realization of intentions as negative energy signals (phase conjugate light) sent to the geometric past and inducing a positive energy response (say neural activity) is consistent with the TGD based models for motor action and long term memory [K1].

7.4 More precise characterization of the basic categories and possible applications

In the following the categories associated with self and quantum jump are discussed in more precise manner and applications to communications and cognition are considered.

7.4.1 Intuitive picture about the category formed by the geometric correlates of selves

Space-time surface $X^4(X^3)$ decomposes into regions obeying either real or p-adic topology and each region of this kind corresponds to an unentangled subsystem or self lasting at least one quantum jump. By the localization in the zero modes these decompositions are equivalent for all 3-surfaces X^3 in the quantum superposition defined by the prepared configuration space spinor fields resulting in quantum jumps. There is a hierarchy of selves since selves can contain sub-selves. The entire space-time surface $X^4(X^3)$ represents the highest level of the self hierarchy.

This structure defines in a natural manner a category. Objects are all possible sub-selves contained in the self hierarchy: sub-self is set consisting of lower level sub-selves, which in turn have a further decomposition to sub-selves, etc... The naive expectation is that geometrically sub-self belongs to a self as a subset and this defines an inclusion map acting as a natural morphism in this category. This expectation is not quite correct. More natural morphisms are the arrows telling that self as a set of sub-selves contains sub-self as an element. These arrows define a structure analogous to a composite of hierarchy trees.

To be more precise, for a single space-time surface $X^4(X^3)$ this hierarchy corresponds to a subjective time slice of the self hierarchy defined by a single quantum jump. The sequence of hierarchies associated with a sequence of quantum jumps is a natural geometric correlate for the self hierarchy. This means that the objects are now sequences of submoments of consciousness. Sequences are not arbitrary. Self must survive its lifetime although sub-selves at various levels can disappear and reappear (generation and disappearance of mental images). Geometrically this means typically a phase transition transforming real or p_1-adic to p_2-adic space-time region with same topology as the environment. Also sub-selves can fuse to single sub-self. The constraints on self sequences must be such that it takes these processes into account. Note that these constraints emerge naturally from the fact that quantum jumps sequences define the sequences of surfaces $X^4(X^3)$.

By the rich anatomy of the quantum jump there is large number of quantum jumps leading from a given initial quantum history to a given final quantum history. One could envisage quantum jump also as a discrete path in the space of configuration space spinor fields leading from the initial state to the final state. In particular, for given self there is an infinite number of closed elementary paths leading from the initial quantum history back to the initial quantum history and these paths in principle give all possible conscious information about a given quantum history/idea: kind of self morphisms are in question (analogous to, say, group automorhisms). Information about point of space is obtained only by moving around and coming back to the point, that is by studying the surroundings of the point. Self in turn can be seen as a composite of elementary paths defined by the quantum jumps. Selves can define arbitrarily complex composite closed paths giving information about a given quantum history.

7.4.2 Categories related to self and quantum jump

The categories defined by moments of consciousness and the notion of self

Since quantum jump involves state reduction and the sequence of self measurement reducing all entanglement except bound state entanglement, it defines a hierarchy of unentangled subsystems allowing interpretation as objects of a category. Arrows correspond to subsystem-system relationship and the two subsystems resulting in self measurement to the system. What subsystem corresponds mathematically is however not at all trivial and the naive description as a tensor factor does not work. Rather, a definition relying on the notion of p-adic length scale cutoff identified as a fundamental aspect of nature and consciousness is needed.

It is not clear what the statement that self corresponds to a subsystem which remains unentangled in subsequent quantum jump means concretely since subsystem can certainly change in some limits. What is clear that bound state entanglement between selves means a loss of consciousness. Category theory suggests that there should exist a functor between categories defined by two subsequent moments of consciousness. This functor maps submoments of consciousness to submoments of consciousness and arrows to arrows. Two subsequent submoments of consciousness belong to same sub-self is the functor maps the first one to the latter one. Thus category theory would play essential role in the precise definition of the notion of self.

The sequences of moments of consciousness form a larger category containing sub-selves as sequences of unentangled subsystems mapped to each other by functor arrows functoring subsequent quantum jumps to each other.

What might then be the ultimate characterizer of the self-identity? The theory of infinite primes suggests that space-time surface decomposes into regions labelled by finite p-adic primes. These primes must label also real regions rather than only p-adic ones, and one could understand this as resulting from a resonant transformation of intention to action. A p-adic space-time region characterized by prime p can transform to a real one or vice versa in quantum jump if the sizes of real and p-adic regions are characterized by the p-adic length scale L_p (or n-ary p-adic length scale $L_p(n)$. One can also consider the possibility that real region is accompanied by a p-adic region characterized by a definite prime p and providing a cognitive self-representation of the real region.

If this view is correct, the p-adic prime characterizing a given real or p-adic space-time sheet is the ultimate characterizer of the self-identity. Self identity is lost in bound state entanglement with another space-time sheet (at least when a space-time sheet with smaller value of the p-adic prime joins by join along boundaries bond to a one with a higher value of the p-adic prime). Self identity is also lost if a space-time sheet characterized by a given p-adic prime disappears in quantum jump.

The category associated with quantum jump sequences

There are several similarities between the ontologies and epistemologies of TGD and of category theory. Conscious experience is always determined by the discrete paths in the space of configuration space spinor fields defined by a quantum jump connecting two quantum histories (states) and is never determined by single quantum history as such (quantum states are unconscious). Also category theory is about relations between objects, not about objects directly: self-morphisms give information about the object of category (in case of group composite paths would correspond to products of group automorphisms). Analogously closed paths determined by quantum jump sequences give information about single quantum history. The point is however that it is impossible to have direct knowledge about the quantum histories: they are not conscious.

One can indeed define a natural category, call it **QSelf**, applying to this situation. The objects of the category **QSelf** are initial quantum histories of quantum jumps and correspond to prepared quantum states. The discrete path defining quantum jump can be regarded as an elementary morphism. Selves are composites of elementary morphisms of the initial quantum history defined by quantum jumps: one can characterize the morphisms by the number of the elementary morphisms in the product. Trivial self contains no quantum jumps and corresponds to the identity morphism, null path. Thus the collection of all possible sequences of quantum jumps, that is collections of selves allows a description in terms of category theory although the category in question is not a subcategory of the category **Set**.

Category **QSelf** does not possess terminal and initial elements (for terminal (initial) element T there is exactly one arrow $A \rightarrow T$ ($T \rightarrow A$) for every A: now there are always many paths between quantum histories involved).

7.4.3 Communications in TGD framework

Goro Kato identifies communications between conscious entities as natural maps between them whereas in TGD natural maps bind submoments of consciousness to selves. In TGD framework quantum measurement and the sharing of mental images are the basic candidates for communications. The problem is that the identification of communications as sharing of mental images is not consistent with the naive view about subsystem as a tensor factor. Many-sheeted space-time however forces length scale dependent notion of subsystem at space-time level and this saves the situation.

What communications are?

Communication is essentially generation of desired mental images/sub-selves in receiver. Communication between selves need not be directly conscious: in this case communication would generate mental images at some lower level of self hierarchy of receiver: for instance generate large number of sub-sub-selves of similar type. This is like communications between organizations. Communication can be also vertical: self can generate somehow sub-self in some sub-sub....sub-self or sub-sub...sub-self can generate sub-self of self somehow. This is communication from boss to the lower levels organization or vice versa.

These communications should have direct topological counterparts. For instance, the communication between selves could correspond to an exchange of mental image represented as a space-time region of different topology inside sender self space-time sheet. The sender self would simply throw this space-time region to a receiver self like a ball. This mechanism applies also to vertical communications since the ball could be also thrown from a boss to sub...sub-self at some lower level of hierarchy and vice versa.

The sequence of space-time surfaces provides a direct topological counterpart for communication as throwing balls representing sub-selves. Quantum jump sequence contains space-time surfaces in which the regions corresponding to receiver and sender selves are connected by a join along boundaries bond (perhaps massless extremal) representing classically the communication: during the communication the receiver and sender would form single self. The cartoon vision about rays connecting the eyes of communication persons would make sense quite concretely.

More refined means of communication would generate sub-selves of desired type directly at the end of receiver. In this case it is not so obvious how the sequence $X^(X^3)$ of space-time surfaces could represent communication. Of course, one can question whether communication is really what happens in this kind of situation. For instance, sender can affect the environment of receiver to be such that receiver gets irritated (computer virus is good manner to achieve this!) but one can wonder whether this is real communication.

Communication as quantum measurement?

Quantum measurement generates one-one map between the states of the entangled systems resulting in quantum measurement. Both state function reduction and self measurement give rise to this kind of map. This map could perhaps be interpreted as quantum communication between unentangled subsystems resulting in quantum measurement. For the state reduction process the space-time correlates are the values of zero modes. For state preparation the space-time correlates should correspond to classical spinor field modes correlating for the two subsystems generated in self measurement.

Communication as sharing of mental images

It has become clear that the sharing of mental images induced by quantum entanglement of sub-selves of two separate selves represents genuine conscious communication which is analogous telepathy and provides general mechanism of remote mental interactions making possible even molecular recognition mechanisms.

1. The sharing of mental images is not possible unless one assumes that self hierarchy is defined by using the notion of length scale resolution defined by p-adic length scale. The notion of scale of resolution is indeed fundamental for all quantum field theories (renormalization group invariance) for all quantum field theories and without it the practical modelling of physics would not be possible. The notion reflects directly the length scale resolution of conscious experience. For a given sub-self of self the resolution is given by the p-adic length scale associated with the sub-self space-time sheet.

2. Length scale resolution emerges naturally from the fact that sub-self space-time sheets having Minkowskian signature of metric are separated from the one representing self by wormhole

contacts with Euclidian signature of metric. The signature of the induced metric changes from Minkowskian signature to Euclidian signature at 'elementary particle horizons' surrounding the throats of the wormhole contacts and having degenerate induced metric. Elementary particle horizons are thus metrically two-dimensional light like surfaces analogous to the boundary of the light cone and allow conformal invariance. Elementary particle horizons act as causal horizons. Topologically condensed space-time sheets are analogous to black hole interiors and due to the lack of the causal connectedness the standard description of sub-selves as tensor factors of the state space corresponding to self is not appropriate.

Hence systems correspond, not to the space-time sheets plus entire hierarchy of space-time sheets condensed to it, but rather, to space-time sheets with holes resulting when the space-time sheets representing subsystems are spliced off along the elementary particle horizons around wormhole contacts. This does not mean that all information about subsystem is lost: subsystem space-time sheet is only replaced by the elementary particle horizon. In analogy with the description of the black hole, some parameters (mass, charges,...) characterizing the classical fields created by the sub-self space-time sheet characterize sub-self.

One can say that the state space of the system contains 'holes'. There is a hierarchy of state spaces labelled by p-adic primes defining length scale resolutions. This picture resolves a long-standing puzzle relating to the interpretation of the fact that particle is characterize by both classical and quantum charges. Particle cannot couple simultaneously to both and this is achieved if quantum charge is associated with the lowest level description of the particle as CP_2 extremal and classical charges to its description at higher levels of hierarchy.

3. The immediate implication indeed is that it is possible to have a situation in which two selves are unentangled although their sub-selves (mental images) are entangled. This corresponds to the fusion and sharing of mental images. The sharing of the mental images means that union of disjoint hierarchy trees with levels labelled by p-adic primes p is replaced by a union of hierarchy threes with horizontal lines connecting subsystems at the same level of hierarchy. Thus the classical correspondence defines a category of presheaves with both vertical arrows replaced by sub-self-self relationship, horizontal arrows representing sharing of mental images, and natural maps representing binding of submoments of consciousness to selves.

Comparison with Goro Kato's approach

It is of interest to compare Goro Kato's approach with TGD approach. The following correspondence suggests itself.

1. In TGD each quantum jumps defines a category analogous to the Goro Kato's category of open sets of some topological space but set theoretic inclusion replaced by topological condensation. The category defined by a moment of consciousness is dynamical whereas the category of open sets is non-dynamical.

2. The assignment of a 3-surface acting as a causal determinant to each unentangled subsystem defined by a moment of consciousness defines a unique "quantum presheaf" which is the counterpart of the presheaf in Goro Kato's theory. The conscious entity of Kato's theory corresponds to the classical correlate for a moment of consciousness.

3. Natural maps between the causal determinants correspond to the space-time correlates for the functor arrows defining the threads connecting submoments of consciousness to selves. In Goro Kato's theory natural maps are interpreted as communications between conscious entities. The sharing of mental images by quantum entanglement between subsystems of unentangled systems defines horizontal bi-directional arrows between subsystems associated with same moment of consciousness and is counterpart of communication in TGD framework. It replaces the union of disjoint hierarchy trees associated with various unentangled subsystems with hierarchy trees having horizontal connections defining the bi-directional arrows. The sharing of mental images is not possible if subsystem is identified as a tensor factor and thus without taking into account length scale resolution.

7.4.4 Cognizing about cognition

There are close connections with basic facts about cognition.

1. Categorization means classification and abstraction of common features in the class formed by the objects of a category. Already quantum jump defines category with hierarchical structure and can be regarded as consciously experienced analysis in which totally entangled entire universe $U\Psi_i$ decomposes to a product of maximally unentangled subsystems. The sub-selves of self are like elements of set and are experienced as separate objects whereas sub-sub-selves of sub-self self experiences as an average: they belong to a class or category formed by the sub-self. This kind of averaging occurs also for the contributions of quantum jumps to conscious experience of self.

2. The notions of category theory might be useful in an attempt to construct a theory of cognitive structures since cognition is indeed to high degree classification and abstraction process. The sub-selves of a real self indeed have p-adic space-time sheets as geometric correlates and thus correspond to cognitive sub-selves, thoughts. A meditative experience of empty mind means in case of real self the total absence of thoughts.

3. Predicate logic provides a formalization of the natural language and relies heavily on the notion of n-ary relation. Binary relations $R(a,b)$ corresponds formally to the subset of the product set $A \times B$. For instance, statements like 'A does something to B' can be expressed as a binary relation, particular kind of arrow and morphism ($A \leq B$ is a standard example). For sub-selves this relation would correspond to a dynamical evolution at space-time level modelling the interaction between A and B. The dynamical path defined by a sequence of quantum jumps is able to describe this kind of relationships too at level of conscious experience. For instance, 'A touches B' would involve the temporary fusion of sub-selves A and B to sub-self C.

7.5 Logic and category theory

Category theory allows naturally more general than Boolean logics inherent to the notion of topos associated with any category. Basic question is whether the ordinary notion of topos algebra based on set theoretic inclusion or the notion of quantum topos based on topological condensation is physically appropriate. Starting from the quasi-Boolean algebra of open sets one ends up to the conclusion that quantum logic is more natural. Also configuration space spinor fields lead naturally to the notion of quantum logic.

7.5.1 Is the logic of conscious experience based on set theoretic inclusion or topological condensation?

The algebra of open sets with intersections and unions and complement defined as the interior of the complement defines a modification of Boolean algebra having the peculiar feature that the points at the boundary of the closure of open set cannot be said to belong to neither interior of open set or of its complement. There are two options concerning the interpretation.

1. 3-valued logic could be in question. It is however not possible to understand this three-valuedness if one defines the quasi-Boolean algebra of open sets as Heyting algebra. The resulting logic is two-valued and the points at boundaries of the closure do not correspond neither to the statement or its negation. In p-adic context the situation changes since p-adic open sets are also closed so that the logic is strictly Boolean. That our ordinary cognitive mind is Boolean provides a further good reason for why cognition is p-adic.

2. These points at the boundary of the closure belong to both interior and exterior in which case a two-valued "quantum logic" allowing superposition of opposite truth values is in question. The situation is indeed exactly the same as in the case of space-time sheet having wormhole contacts to several space-time sheets.

The quantum logic brings in mind Zen consciousness [41] (which I became fascinated of while reading Hofstadter's book "Gödel, Escher,Bach" [21]) and one can wonder whether selves having real space-time sheets as geometric correlates and able to live simultaneously in many parallel worlds correspond to Zen consciousness and Zen logic. Zen logic would be also logic of sensory experience whereas cognition would obey strictly Boolean logic.

The causal determinants associated with space-time sheets correspond to light like 3-surfaces which could elementary particle horizons or space-time boundaries and possibly also 3-surfaces separating two maximal deterministic regions of a space-time sheet. These surfaces act as 3-dimensional quantum holograms and have the strange Zen property that they are neither space-like nor time-like so that they represent both the state and the process. In the TGD based model for topological quantum computation (TQC) light-like boundaries code for the computation so that TQC program code would be equivalent with the running program [E9].

7.5.2 Do configuration space spinor fields define quantum logic and quantum topos

I have proposed already earlier that configuration space spinor fields define what might be called quantum logic. One can wonder whether configuration space spinors could also naturally define what might be called quantum topos since the category underlying topos defines the logic appropriate to the topos. This question remains unanswered in the following: I just describe the line of though generalizing ordinary Boolean logic.

Finite-dimensional spinors define quantum logic

Spinors at a point of an $2N$-dimensional space span 2^N-dimensional space and spinor basis is in one-one correspondence with Boolean algebra with N different truth values (N bits). 2N=2-dimensional case is simple: Spin up spinor= true and spin-dow spinor=false. The spinors for $2N$-dimensional space are obtained as an N-fold tensor product of 2-dimensional spinors (spin up,spin down): just like in the case of Cartesian power of Ω.

Boolean spinors in a given basis are eigen states for a set N mutually commuting sigma matrices providing a representation for the tangent space group acting as rotations. Boolean spinors define N Boolean statements in the set Ω^N so that one can in a natural manner assign a set with a Boolean spinor. In the real case this group is $SO(2N)$ and reduces to $SU(N)$ for Kähler manifolds. For pseudo-euclidian metric some non-compact variant of the tangent space group is involved. The selections of N mutually commuting generators are labelled by the flag-manifold $SO(2N)/SO(2)^N$ in real context and by the flag-manifold $U(N)/U(1)^N$ in the complex case. The selection of these generators defines a collection of N 2-dimensional linear subspaces of the tangent space.

Spinors are in general complex superpositions of spinor basis which can be taken as the product spinors. The quantum measurement of N spins representing the Cartan algebra of $SO(2N)$ ($SU(N)$) leads to a state representing a definite Boolean statement. This suggests that quantum jumps as moments of consciousness quite generally make universe classical, not only in geometric but also in logical sense. This is indeed what the state preparation process for the configuration space spinor field seems to do.

Quantum logic for finite-dimensional spinor fields

One can generalize the idea of the spinor logic also to the case of spinor fields. For a given choice of the local spinor basis (which is unique only modular local gauge rotation) spinor field assigns to each point of finite-dimensional space a quantum superposition of Boolean statements decomposing into product of N statements.

Also now one can ask whether it is possible to find a gauge in which each point corresponds to definite Boolean statement and is thus an eigen state of a maximal number of mutually commuting rotation generators Σ_{ij}. This is not trivial if one requires that Dirac equation is satisfied. In the case of flat space this is certainly true and constant spinors multiplied by functions which solve d'Alembert equation provide a global basis.

The solutions of Dirac equation in a curved finite-dimensional space do not usually possess a definite spin direction globally since spinor curvature means the presence of magnetic spin-flipping interaction and since there need not exist a global gauge transformation leading to an eigen state of the local Cartan algebra everywhere. What might happen is that the local gauge transformation becomes singular at some point: for instance, the direction of spin would be radial around given point and become ill defined at the point. This is much like the singularities for vector fields on sphere. The spinor field having this kind of singularity should vanish at singularity but the local gauge rotation rotating spin in same direction everywhere is necessarily ill-defined at the singularity.

In fact, this can be expressed using the language of category theory. The category in question corresponds to a presheaf which assigns to the points of the base space the fiber space of the spinor bundle. The presence of singularity means that there are no global section for this presheaf, that is a continuous choice of a non-vanishing spinor at each point of the base space. The so called Kochen-Specker theorem discussed in [20] is closely related to a completely analogous phenomenon involving non-existence of global sections and thus non-existence of a global truth value.

Thus in case of curved spaces is not necessarily possible to have spinor field basis representing globally Boolean statements and only the notion of locally Boolean logic makes sense. Indeed, one can select the basis to be eigen state of maximal set of mutually commuting rotation generators in single point of the compact space. Any such choice does.

Quantum logic and quantum topos defined by the prepared configuration space spinor fields

The prepared configuration space spinor fields occurring as initial and final states of quantum jumps are the natural candidates for defining quantum logic. The outcomes of the quantum jumps resulting in the state preparation process are maximally unentangled states and are as close to Boolean states as possible.

Configuration space spinors correspond to fermionic Fock states created by infinite number of fermionic (leptonic and quarklike) creation and annihilation operators. The spin degeneracy is replaced by the double-fold degeneracy associated with a given fermion mode: given state either contains fermion or not and these two states represent true and false now. If configuration space were flat, the Fock state basis with definite fermion and anti-fermion numbers in each mode would be in one-one correspondence with Boolean algebra.

Situation is however not so simple. Finite-dimensional curved space is replaced with the fiber degrees of freedom of the configuration space in which the metric is non-vanishing. The precise analogy with the finite-dimensional case suggests that if the curvature form of the configuration space spinor connection is nontrivial, it is impossible to diagonalize even the prepared maximally unentangled configuration space spinor fields Ψ_i in the entire fiber of the configuration space (quantum fluctuating degrees of freedom) for given values of the zero modes. Local singularities at which the spin quantum numbers of the diagonalized but vanishing configuration space spinor field become ill-defined are possible also now.

In the infinite-dimensional context the presence of the fermion-anti-fermion pairs in the state means that it does not represent a definite Boolean statement unless one defines a more general basis of configuration space spinors for which pairs are present in the states of the state basis: this generalization is indeed possible. The sigma matrices of the configuration space appearing in the spinor connection term of the Dirac operator of the configuration space indeed create fermion-fermion pairs. What is decisive, is not the absence of fermion-anti-fermion pairs, but the possibility that the spinor field basis cannot be reduced to eigen states of the local Cartan algebra in fiber degrees of freedom globally.

Also for bound states of fermions (say leptons and quarks) it is impossible to reduce the state to a definite Boolean statement even locally. This would suggest that fermionic logic does not reduce to a completely Boolean logic even in the case of the prepared states.

Thus configuration space spinor fields could have interpretation in terms of non-Boolean quantum logic possessing Boolean logics only as sub-logics and define what might be called quantum topos. Instead of Ω^N-valued maps the values for the maps are complex valued quantum superpositions of truth values in Ω^N.

An objection against the notion of quantum logic is that Boolean algebra operations AND and OR do not preserve fermion number so that quantum jump sequences leading from the product state defined by operands to the state representing the result of operation are therefore not possible. One manner to circumvent the objection is to consider the sub-algebra spanned by fermion and anti-fermion pairs for given mode so that fermion number conservation is not a problem. The objection can be also circumvented for pairs of space-time sheets with opposite time orientations and thus opposite signs of energies for particles. One can construct the algebra in question as pairs of many fermion states consisting of positive energy fermion and negative energy anti-fermion so that all states have vanishing fermion number and logical operations become possible. Pairs of MEs with opposite time orientations are excellent candidates for carries of these fermion-anti-fermion pairs.

Quantum classical correspondence and quantum logic

The intuitive idea is that the global Boolean statements correspond to sections of Z^2 bundle. Möbius band is a prototype example here. The failure of a global statement would reduce to the non-existence of global section so that true would transforms to false as one goes around full 2π rotation.

One can ask whether fermionic quantum realization of Boolean logic could have space-time counterpart in terms of Z_2 fiber bundle structure. This would give some hopes of having some connection between category theoretical and fermionic realizations of logic. The following argument stimulated by email discussion with Diego Lucio Rapoport suggests that this might be the case.

1. The hierarchy of Planck constants realized using the notion of generalized imbedding space involves only groups $Z_{n_a} \times Z_{n_b}$, $n_a, n_b \neq 2$ if one takes Jones inclusions as starting point. There is however no obvious reason for excluding the values $n_a = 2$ and $n_b = 2$ and the question concerns physical interpretation. Even if one allows only $n_i \geq 3$ one can ask for the physical interpretation for the factorization $Z_{2n} = Z_2 \times Z_n$. Could it perhaps relate to a space-time correlates for Boolean two-valuedness?

2. An important implication of fiber bundle structure is that the partonic 2-surfaces have $Z_{n_a} \times Z_{n_b} = Z_{n_a n_b}$ as a group of conformal symmetries. I have proposed that n_a or n_b is even for fermions so that Z_2 acts as a conformal symmetry of the partonic 2-surface. Both n_a and n_b would be odd for truly elementary bosons. Note that this hypothesis makes sense also for $n_i \geq 3$.

3. Z_2 conformal symmetry for fermions would imply that all partonic 2-surfaces associated with fermions are hyper-elliptic. As a consequence elementary particle vacuum functionals defined in modular degrees of freedom would vanish for fermions for genus $g > 2$ so that only three fermion families would be possible in accordance with experimental facts. Since gauge bosons and Higgs correspond to pairs of partonic 2-surfaces (the throats of the wormhole contact) one has 9 gauge boson states labelled by the pairs (g_1, g_2) which can be grouped to SU(3) singlet and octet. Singlet corresponds to ordinary gauge bosons.

   Super-canonical bosons are truly elementary bosons in the sense that they do not consist of fermion-antifermion pairs. For them both n_a and n_b should be odd if the correspondence is taken seriously and all genera would be possible. The super-conformal partners of these bosons have the quantum numbers of right handed neutrino. Since both spin directions are possible, one can ask whether Boolean Z_2 must be present also now. This need not be the case, ν_R generates only super-symmetries and does not define a family of fermionic oscillator operators. The electro-weak spin of ν_R is frozen and it does not couple at all to electro-weak intersections. Perhaps (only) odd values of n_i are possible in this case.

4. If fermionic Boolean logic has a space-time correlate, one can wonder whether the fermionic Z_2 conformal symmetry might correspond to a space-time correlate for the Boolean true-false dichotomy. If the partonic 2-surface contains points which are fixed points of Z_2 symmetry, there exists no everywhere non-vanishing sections. Furthermore, induced spinor fields should vanish at the fixed points of Z_2 symmetry since they correspond to singular orbifold points so that one could not actually have a situation in which true and false are true simultaneously. Global sections could however fail to exist since CP_2 spinor bundle is non-trivial.

7.5.3 Category theory and the modelling of aesthetic and ethical judgements

Consciousness theory should allow to model model the logics of ethics and aesthetics. Evolution (representable as p-adic evolution in TGD framework) is regarded as something positive and is a good candidate for defining universal ethics in TGD framework. Good deeds are such that they support this evolution occurring in statistical sense in any case. Moral provides a practical model for what good deeds are and moral right-wrong statements are analogous to logical statements. Often however the two-valued right-wrong logic seems to be too simplistic in case of moral statements. Same applies to aesthetic judgements. A possible application of the generalized logics defined by the inherent structure of categories relates to the understanding of the dilemmas associated with the moral and aesthetic rules.

As already found, quantum versions of sieves provide a formal generalization of Boolean truth values as a characteristic of a given category. Generalized moral rules could perhaps be seen as sieve valued statements about deeds. Deeds are either right or wrong in what might be called Boolean moral code. One can also consider Zen moral in which some deeds can be said to be right and wrong simultaneously. Some deeds could also be such that there simply exists no globally consistent moral rule: this would correspond to the nonexistence of what is called global section assigning to each object of the category consisting of the pairs formed by a moral agents and given deed) a sieve simultaneously.

7.6 Platonism, Constructivism, and Quantum Platonism

During years I have been trying to understand how Category Theory and Set Theory relate to quantum TGD inspired view about fundamentals of mathematics and the outcome section is added to this chapter several years after its first writing. I hope that reader does not experience too unpleasant discontinuity. I managed to clarify my thoughts about what these theories are by reading the article Structuralism, Category Theory and Philosophy of Mathematics by Richard Stefanik [27]. Blog discussions and email correspondence with Sampo Vesterinen have been very stimulating and inspired the attempt to represent TGD based vision about the unification of mathematics, physics, and consciousness theory in a more systematic manner.

Before continuing I want to summarize the basic ideas behind TGD vision. One cannot understand mathematics without understanding mathematical consciousness. Mathematical consciousness and its evolution must have direct quantum physical correlates and by quantum classical correspondence these correlates must appear also at space-time level. Quantum physics must allow to realize number as a conscious experience analogous to a sensory quale. In TGD based ontology there is no need to postulate physical world behind the quantum states as mathematical entities (theory is the reality). Hence number cannot be any physical object, but can be identified as a quantum state or its label and its number theoretical anatomy is revealed by the conscious experiences induced by the number theoretic variants of particle reactions. Mathematical systems and their axiomatics are dynamical evolving systems and physics is number theoretically universal selecting rationals and their extensions in a special role as numbers, which can can be regarded elements of several number fields simultaneously.

7.6.1 Platonism and structuralism

There are basically two philosophies of mathematics.

1. Platonism assumes that mathematical objects and structures have independent existence. Natural numbers would be the most fundamental objects of this kind. For instance, each natural number has its own number-theoretical anatomy decomposing into a product of prime numbers defining the elementary particles of Platonia. For quantum physicist this vision is attractive, and even more so if one accepts that elementary particles are labelled by primes (as I do)! The problematic aspects of this vision relate to the physical realization of the Platonia. Neither Minkowski space-time nor its curved variants understood in the sense of set theory have no room for Platonia and physical laws (as we know them) do not seem to allow the realization of all imaginable internally consistent mathematical structures.

2. Structuralist believes that the properties of natural numbers result from their relations to other natural numbers so that it is not possible to speak about number theoretical anatomy in the Platonic sense. Numbers as such are structureless and their relationships to other numbers provide them with their apparent structure. According to [27] structuralism is however not enough for the purposes of number theory: in combinatorics it is much more natural to use intensional definition for integers by providing them with inherent properties such as decomposition into

primes. I am not competent to take any strong attitudes on this statement but my physicist's intuition tells that numbers have number theoretic anatomy and that this anatomy can be only revealed by the morphisms or something more general which must have physical counterparts. I would like to regard numbers are analogous to bound states of elementary particles. Just as the decays of bound states reveal their inner structure, the generalizations of morphisms would reveal to the mathematician the inherent number theoretic anatomy of integers.

7.6.2 Structuralism

Set theory and category theory represent two basic variants of structuralism and before continuing I want to clarify to myself the basic ideas of structuralism: the reader can skip this section if it looks too boring.

Set theory

Structuralism has many variants. In set theory [28] the elements of set are treated as structureless points and sets with the same cardinality are equivalent. In number theory additional structure must be introduced. In the case of natural numbers one introduces the notion of successor and induction axiom and defines the basic arithmetic operations using these. Set theoretic realization is not unique. For instance, one can start from empty set Φ identified as 0, identify 1 as $\{\Phi\}$, 2 as $\{0, 1\}$ and so on. One can also identify 0 as Φ, 1 as $\{0\}$, 2 as $\{\{0\}\}$,.... For both physicist and consciousness theorist these formal definitions look rather weird.

The non-uniqueness of the identification of natural numbers as a set could be seen as a problem. The structuralist's approach is based on an extensional definition meaning that two objects are regarded as identical if one cannot find any property distinguishing them: object is a representative for the equivalence class of similar objects. This brings in mind gauge fixing to the mind of physicists.

Category theory

Category theory [29] represents a second form of structuralism. Category theorist does not worry about the ontological problems and dreams that all properties of objects could be reduced to the arrows and formally one could identify even objects as identity morphisms (looks like a trick to me). The great idea is that functors between categories respecting the structure defined by morphisms provide information about categories. Second basic concept is natural transformation which maps functors to functors in a structure preserving manner. Also functors define a category so that one can construct endless hierarchy of categories. This approach has enormous unifying power since functors and natural maps systemize the process of generalization. There is no doubt that category theory forms a huge piece of mathematics but I find difficult to believe that arrows can catch all of it.

The notion of category can be extended to that of n-category: in [18] I described a geometric realization of this hierarchy in which one defines 1-morphisms by parallel translations, 2-morphisms by parallel translations of parallel translations, and so on. In infinite-dimensional space this hierarchy would be infinite. Abstractions about abstractions about.., thoughts about thoughts about, statements about statements about..., is the basic idea behind this interpretation. Also the hierarchy of logics of various orders corresponds to this hierarchy. This encourages to see category theoretic thinking as being analogous to higher level self reflection which must be distinguished from the direct sensory experience.

In the case of natural numbers category theoretician would identify successor function as the arrow binding natural numbers to an infinitely long string with 0 as its end. If this approach would work, the properties of numbers would reflect the properties of the successor function.

7.6.3 The view about mathematics inspired by TGD and TGD inspired theory of consciousness

TGD based view might be called quantum Platonism. It is inspired by the requirement that both quantum states and quantum jumps between them are able to represent number theory and that all quantum notions have also space-time correlates so that Platonia should in some sense exist also at the level of space-time. Here I provide a brief summary of this view as it is now. The articles "TGD" [16] and "TGD inspired theory of consciousness" [17] provide an overview about TGD and TGD inspired theory of consciousness.

Physics is fixed from the uniqueness of infinite-D existence and number theoretic universality

1. The basic philosophy of quantum TGD relies on the geometrization of physics in terms of infinite-dimensional Kähler geometry of the "world of classical worlds" (configuration space), whose uniqueness is forced by the mere mathematical existence. Space-time dimension and imbedding space $H = M^4 \times CP_2$ are fixed among other things by this condition and allow interpretation in terms of classical number fields. Physical states correspond to configuration space spinor fields with configuration space spinors having interpretation as Fock states. Rather remarkably, configuration space Clifford algebra defines standard representation of so called hyper finite factor of II_1, perhaps the most fascinating von Neumann algebra.

2. Number theoretic universality states that all number fields are in a democratic position. This vision can be realized by requiring generalization of notions of imbedding space by gluing together real and p-adic variants of imbedding space along common algebraic numbers. All algebraic extensions of p-adic numbers are allowed. Real and p-adic space-time sheets intersect along common algebraics. The identification of the p-adic space-time sheets as correlates of cognition and intentionality explains why cognitive representations at space-time level are always discrete. Only space-time points belonging to an algebraic extension of rationals associated contribute to the data defining S-matrix. These points define what I call number theoretic braids. The interpretation in of algebraic discreteness terms of a physical realization of axiom of choice is highly suggestive. The axiom of choice would be dynamical and evolving quantum jump by quantum jump as the algebraic complexity of quantum states increases.

Holy trinity of existence

In TGD framework one would have 3-levelled ontology numbers should have representations at all these levels [17].

1. Subjective existence as a sequence of quantum jumps giving conscious sensory representations for numbers and various geometric structures would be the first level.

2. Quantum states would correspond to Platonia of mathematical ideas and mathematician- or if one is unwilling to use this practical illusion- conscious experiences about mathematic ideas, would be in quantum jumps. The quantum jumps between quantum states respecting the symmetries characterizing the mathematical structure would provide conscious information about the mathematical ideas not directly accessible to conscious experience. Mathematician would live in Plato's cave. There is no need to assume any independent physical reality behind quantum states as mathematical entities since quantum jumps between these states give rise to conscious experience. Theory-reality dualism disappears since the theory is reality or more poetically: painting is the landscape.

3. The third level of ontology would be represented by classical physics at the space-time level essential for quantum measurement theory. By quantum classical correspondence space-time physics would be like a written language providing symbolic representations for both quantum states and changes of them (by the failure of complete classical determinism of the fundamental variational principle). This would involve both real and p-adic space-time sheets corresponding to sensory and cognitive representations of mathematical concepts. This representation makes possible the feedback analogous to formulas written by mathematician crucial for the ability of becoming conscious about what one was conscious of and the dynamical character of this process allows to explain the self-referentiality of consciousness without paradox.

This ontology releases a deep Platonistic sigh of relief. Since there are no physical objects, there is no need to reduce mathematical notions to objects of the physical world. There are only quantum states identified as mathematical entities labelled naturally by integer valued quantum numbers; conscious experiences, which must represent sensations giving information about the number theoretical anatomy of a given quantum number; and space-time surfaces providing space-time correlates for quantum physics and therefore also for number theory and mathematical structures in general.

Factorization of integers as a direct sensory perception?

Both physicist and consciousness theorist would argue that the set theoretic construction of natural numbers could not be farther away from how we experience integers. Personally I feel that neither structuralist's approach nor Platonism as it is understood usually are enough. Mathematics is a conscious activity and this suggests that quantum theory of consciousness must be included if one wants to build more satisfactory view about fundamentals of mathematics.

Oliver Sack's book *The man who mistook his wife for a hat* [40] (see also [H3]) contains fascinating stories about those aspects of brain and consciousness which are more or less mysterious from the view point of neuroscience. Sacks tells in his book also a story about twins who were classified as idiots but had amazing number theoretical abilities. I feel that this story reveals something very important about the real character of mathematical consciousness.

The twins had absolutely no idea about mathematical concepts such as the notion of primeness but they could factorize huge numbers and tell whether they are primes. Their eyes rolled wildly during the process and suddenly their face started to glow of happiness and they reported a discovery of a factor. One could not avoid the feeling that they quite concretely saw the factorization process. The failure to detect the factorization served for them as the definition of primeness. For them the factorization was not a process based on some rules but a direct sensory perception.

The simplest explanation for the abilities of twins would in terms of a model of integers represented as string like structures consisting of identical basic units. This string can decay to strings. If string containing n units decaying into $m > 1$ identical pieces is not perceived, the conclusion is that a prime is in question. It could also be that decay to units smaller than 2 was forbidden in this dynamics. The necessary connection between written representations of numbers and representative strings is easy to build as associations.

This kind theory might help to understand marvellous feats of mathematicians like Ramanujan who represents a diametrical opposite of Groethendienck as a mathematician (when Groethendienck was asked to give an example about prime, he mentioned 57 which became known as Groethendienck prime!).

The lesson would be that one very fundamental representation of integers would be, not as objects, but conscious experiences. Primeness would be like the quale of redness. This of course does not exclude also other representations.

Experience of integers in TGD inspired quantum theory of consciousness

In quantum physics integers appear very naturally as quantum numbers. In quantal axiomatization or interpretation of mathematics same should hold true.

1. In TGD inspired theory of consciousness [17] quantum jump is identified as a moment of consciousness. There is actually an entire fractal hierarchy of quantum jumps consisting of quantum jumps and this correlates directly with the corresponding hierarchy of physical states and dark matter hierarchy. This means that the experience of integer should be reducible to a certain kind of quantum jump. The possible changes of state in the quantum jump would characterize the sensory representation of integer.

2. The quantum state as such does not give conscious information about the number theoretic anatomy of the integer labelling it: the change of the quantum state is required. The above geometric model translated to quantum case would suggest that integer represents a multiplicatively conserved quantum number. Decays of this this state into states labelled by integers n_i such that one has $n = \prod_i n_i$ would provide the fundamental conscious representation for the number theoretic anatomy of the integer. At the level of sensory perception based the space-time correlates a string-like bound state of basic particles representing n=1.

3. This picture is consistent with the Platonist view about integers represented as structured objects, now labels of quantum states. It would also conform with the view of category theorist in the sense that the arrows of category theorist replaced with quantum jumps are necessary to gain conscious information about the structure of the integer.

Infinite primes and arithmetic consciousness

Infinite primes [E3] were the first mathematical fruit of TGD inspired theory of consciousness and the inspiration for writing this posting came from the observation that the infinite primes at the lowest level of hierarchy provide a representation of algebraic numbers as Fock states of a super-symmetric arithmetic QFT so that it becomes possible to realize quantum jumps revealing the number theoretic anatomy of integers, rationals, and perhaps even that of algebraic numbers.

1. Infinite primes have a representation as Fock states of super-symmetric arithmetic QFT and at the lowest level of hierarchy they provide representations for primes, integers, rationals and algebraic numbers in the sense that at the lowest level of hierarchy of second quantizations the simplest infinite primes are naturally mapped to rationals whereas more complex infinite primes having interpretation as bound states can be mapped to algebraic numbers. Conscious experience of number can be assigned to the quantum jumps between these quantum states revealing information about the number theoretic anatomy of the number represented. It would be wrong to say that rationals only label these states: rather, these states represent rationals and since primes label the particles of these states.

2. More concretely, the conservation of number theoretic energy defined by the logarithm of the rational assignable with the Fock state implies that the allowed decays of the state to a product of infinite integers are such that the rational can decompose only into a product of rationals. These decays could provide for the above discussed fundamental realization of multiplicative aspects of arithmetic consciousness. Also additive aspects are represented since the exponents k in the powers p^k appearing in the decomposition are conserved so that only the partitions $k = \sum_i k_i$ are representable. Thus both product decompositions and partitions, the basic operations of number theorist, are represented.

3. The higher levels of the hierarchy represent a hierarchy of abstractions about abstractions bringing strongly in mind the hierarchy of n-categories and various similar constructions including n:th order logic. It also seems that the n+1:th level of hierarchy provides a quantum representation for the n:th level. Ordinary primes, integers, rationals, and algebraic numbers would be the lowest level, -the initial object- of the hierarchy representing nothing at low level. Higher levels could be reduced to them by the analog of category theoretic reductionism in the sense that there is arrow between n:th and n+1:th level representing the second quantization at this level. On can also say that these levels represent higher reflective level of mathematical consciousness and the fundamental sensory perception corresponds the lowest level.

4. Infinite primes have also space-time correlates. The decomposition of particle into partons can be interpreted as a infinite prime and this gives geometric representations of infinite primes and also rationals. The finite primes appearing in the decomposition of infinite prime correspond to bosonic or fermionic partonic 2-surfaces. Many-sheeted space-time provides a representation for the hierarchy of second quantizations: one physical prediction is that many particle bound state associated with space-time sheet behaves exactly like a boson or fermion. Nuclear string model is one concrete application of this idea: it replaces nucleon reductionism with reductionism occurs first to strings consisting of $A \leq 4$ nuclei and which in turn are strings consisting of nucleons. A further more speculative representation of infinite rationals as space-time surfaces is based on their mapping to rational functions.

Number theoretic Brahman=Atman identity

The notion of infinite primes leads to the notion of algebraic holography in which space-time points possess infinitely rich number-theoretic anatomy. This anatomy would be due to the existence of infinite number of real units defined as ratios of infinite integers which reduce to unit in the real sense and various p-adic senses. This anatomy is not visible in real physics but can contribute directly to mathematical consciousness [E3].

The anatomies of single space-time point could represent the entire world of classical worlds and quantum states of universe: the number theoretic anatomy is of course not visible in the structure of these these states. Therefore the basic building brick of mathematics - point- would become the Platonia able to represent all of the mathematics consistent with the laws of quantum physics. Space-time points would evolve, becoming more and more complex quantum jump by quantum jump. Configuration space and quantum states would be represented by the anatomies of space-time points. Some space-time points are more "civilized" than others so that space-time decomposes into "civilizations" at different levels of mathematical evolution.

Paths between space-time points represent processes analogous to parallel translations affecting the structure of the point and one can also define n-parallel translations up to $n = 4$ at level of space-time and $n = 8$ at level of imbedding space. At level of world of classical worlds whose points are representable as number theoretical anatomies arbitrary high values of n can be realized.

It is fair to say that the number theoretical anatomy of the space-time point makes it possible self-reference loop to close so that structured points are able to represent the physics of associated with with the structures constructed from structureless points. Hence one can speak about algebraic holography or number theoretic Brahman=Atman identity.

Finite measurement resolution, Jones inclusions, and number theoretic braids

In the history of physics and mathematics the realization of various limitations have been the royal road to a deeper understanding (Uncertainty Principle, Gödel's theorem). The precision of quantum measurement, sensory perception, and cognition are always finite. In standard quantum measurement theory this limitation is not taken into account but forms a corner stone of TGD based vision about quantum physics and of mathematics too as I want to argue in the following.

The finite resolutions has representation both at classical and quantum level.

1. At the level of quantum states finite resolution is represented in terms of Jones inclusions N subset M of hyper-finite factors of type II_1 (HFFs)[A9]. N represents measurement resolution in the sense that the states related by the action of N cannot be distinguished in the measurement considered. Complex rays are replaced by N rays. This brings in noncommutativity via quantum groups [C12]. Non-commutativity in TGD Universe would be therefore due to a finite measurement resolution rather than something exotic emerging in the Planck length scale. Same applies to p-adic physics: p-adic space-time sheets have literally infinite size in real topology!

2. At the space-time level discretization implied by the number theoretic universality could be seen as being due to the finite resolution with common algebraic points of real and p-adic variant of the partonic 3-surface chosen as representatives for regions of the surface. The solutions of modified Dirac equation are characterized by the prime in question so that the preferred prime makes itself visible at the level of quantum dynamics and characterizes the p-adic length scale fixing the values of coupling constants. Discretization could be also understood as effective non-commutativity of imbedding space points due to the finite resolution implying that second quantized spinor fields anticommute only at a discrete set of points rather than along stringy curve.

In this framework it is easy to imagine physical representations of number theoretical and other mathematical structures.

1. Every compact group corresponds to a hierarchy of Jones inclusions corresponding to various representations for the quantum variants of the group labelled by roots of unity. I would be surprised if non-compact groups would not allow similar representation since HFF can be regarded as infinite tensor power of n-dimensional complex matrix algebra for any value of n. Somewhat paradoxically, the finite measurement resolution would make possible to represent Lie group theory physically [A9].

2. There is a strong temptation to identify the Galois groups of algebraic numbers as the infinite permutation group S_∞ consisting of permutations of finite number of objects, whose projective representations give rise to an infinite braid group B_∞. The group algebras of these groups are HFFs besides the representation provided by the spinors of the world of classical worlds having physical identification as fermionic Fock states. Therefore physical states would provide a direct representation also for the more abstract features of number theory [E12].

3. Number theoretical braids crucial for the construction of S-matrix provide naturally representations for the Galois groups G associated with the algebraic extensions of rationals as diagonal

imbeddings $G \times G \times ....$ to the completion of S_∞ representable also as the action on the completion of spinors in the world of classical worlds so that the core of number theory would be represented physically [E12]. At the space-time level number theoretic braid having G as symmetries would represent the G. These representations are analogous to global gauge transformations. The elements of S_∞ are analogous to local gauge transformations having a natural identification as a universal number theoretical gauge symmetry group leaving physical states invariant.

Hierarchy of Planck constants and the generalization of imbedding space

Jones inclusions inspire a further generalization of the notion of imbedding space obtained by gluing together copies of the imbedding space H regarded as coverings $H \rightarrow H/G_a \times G_b$. In the simplest scenario $G_a \times G_b$ leaves invariant the choice of quantization axis and thus this hierarchy provides imbedding space correlate for the choice of quantization axes inducing these correlates also at space-time level and at the level of world of classical worlds [A9].

Dark matter hierarchy is identified in terms of different sectors of H glued together along common points of base spaces and thus forming a book like structure. For the simplest option elementary particles proper correspond to maximally quantum critical systems in the intersection of all pages. The field bodies of elementary particles are in the interiors of the pages of this "book".

One can assign to Jones inclusions quantum phase $q = exp(i2\pi/n)$ and the groups Z_n acts as exact symmetries both at level of M^4 and CP_2. In the case of M^4 this means that space-time sheets have exact Z_n rotational symmetry. This suggests that the algebraic numbers q^m could have geometric representation at the level of sensory perception as Z_n symmetric objects. We need not be conscious of this representation in the ordinary wake-up consciousness dominated by sensory perception of ordinary matter with $q = 1$. This would make possible the idea about transcendentals like π, which do not appear in any finite-dimensional extension of even p-adic numbers (p-adic numbers allow finite-dimensional extension by since e^p is ordinary p-adic number). Quantum jumps in which state suffers an action of the generating element of Z_n could also provide a sensory realization of these groups and numbers $exp(i2\pi/n)$.

Planck constant is identified as the ratio n_a/n_b of integers associated with M^4 and CP_2 degrees of freedom so that a representation of rationals emerge again. The so called ruler and compass rationals whose definition involves only a repeated square root operation applied on rationals are cognitively the simplest ones and should appear first in the evolution of mathematical consciousness. The successful [M3] quantum model for EEG is only one of the applications providing support for their preferred role. Other applications are to Bohr quantization of planetary orbits interpreted as being induced by the presence of macroscopically quantum coherent dark matter [D6].

7.6.4 Farey sequences, Riemann hypothesis, tangles, and TGD

Farey sequences allow an alternative formulation of Riemann Hypothesis and subsequent pairs in Farey sequence characterize so called rational 2-tangles. In TGD framework Farey sequences relate very closely to dark matter hierarchy, which inspires *"Platonia as the best possible world in the sense that cognitive representations are optimal"* as the basic variational principle of mathematics. This variational principle supports RH.

Possible TGD realizations of tangles, which are considerably more general objects than braids, are considered. One can assign to a given rational tangle a rational number a/b and the tangles labelled by a/b and c/d are equivalent if $ad - bc = \pm 1$ holds true. This means that the rationals in question are neighboring members of Farey sequence. Very light-hearted guesses about possible generalization of these invariants to the case of general N-tangles are made.

Farey sequences

Some basic facts about Farey sequences [32] demonstrate that they are very interesting also from TGD point of view.

1. Farey sequence F_N is defined as the set of rationals $0 \leq q = m/n \leq 1$ satisfying the conditions $n \leq N$ ordered in an increasing sequence.

2. Two subsequent terms a/b and c/d in F_N satisfy the condition $ad - bc = 1$ and thus define and element of the modular group $SL(2, Z)$.

3. The number $|F(N)|$ of terms in Farey sequence is given by

$$|F(N)| = |F(N-1)| + \phi(N-1) \ . \tag{7.6.1}$$

   Here $\phi(n)$ is Euler's totient function giving the number of divisors of n. For primes one has $\phi(p) = 1$ so that in the transition from p to $p+1$ the length of Farey sequence increases by one unit by the addition of $q = 1/(p+1)$ to the sequence.

   The members of Farey sequence F_N are in one-one correspondence with the set of quantum phases $q_n = exp(i2\pi/n)$, $0 \leq n \leq N$. This suggests a close connection with the hierarchy of Jones inclusions, quantum groups, and in TGD context with quantum measurement theory with finite measurement resolution and the hierarchy of Planck constants involving the generalization of the imbedding space. Also the recent TGD inspired ideas about the hierarchy of subgroups of the rational modular group with subgroups labelled by integers N and in direct correspondence with the hierarchy of quantum critical phases [C1] would naturally relate to the Farey sequence.

Riemann Hypothesis and Farey sequences

Farey sequences are used in two equivalent formulations of the Riemann hypothesis. Suppose the terms of F_N are $a_{n,N}$, $0 < n \leq |F_N|$. Define

$$d_{n,N} = a_{n,N} - \frac{n}{|F_N|} \ .$$

In other words, $d_{n,N}$ is the difference between the n:th term of the N:th Farey sequence, and the n:th member of a set of the same number of points, distributed evenly on the unit interval. Franel and Landau proved that both of the following statements

$$\sum_{n=1,\ldots,|F_N|} |d_{n,N}| = O(N^r) \ \text{for any} \ r > 1/2 \ ,$$
$$\sum_{n=1,\ldots,|F_N|} d_{n,N}^2 = O(N^r) \ \text{for any} \ r > 1 \ . \tag{7.6.1}$$

are equivalent with Riemann hypothesis.

One could say that RH would guarantee that the numbers of Farey sequence provide the best possible approximate representation for the evenly distributed rational numbers $n/|F_N|$.

Farey sequences and TGD

Farey sequences seem to relate very closely to TGD.

1. The rationals in the Farey sequence can be mapped to the roots of unity by the map $q \rightarrow exp(i2\pi q)$. The numbers $1/|F_N|$ are in turn mapped to the numbers $exp(i2\pi/|F_N|)$, which are also roots of unity. The statement would be that the algebraic phases defined by Farey sequence give the best possible approximate representation for the phases $exp(in2\pi/|F_N|)$ with evenly distributed phase angle.

2. In TGD framework the phase factors defined by F_N corresponds to the set of quantum phases corresponding to Jones inclusions labelled by $q = exp(i2\pi/n)$, $n \leq N$, and thus to the N lowest levels of dark matter hierarchy. There are actually two hierarchies corresponding to M^4 and CP_2 degrees of freedom and the Planck constant appearing in Schrödinger equation corresponds to the ratio n_a/n_b defining quantum phases in these degrees of freedom. $Z_{n_a \times n_b}$ appears as a conformal symmetry of "dark" partonic 2-surfaces and with very general assumptions this implies that there are only in TGD Universe [F1, C1].

3. The fusion of physics associated with various number fields to single coherent whole requires algebraic universality. In particular, the roots of unity, which are complex algebraic numbers, should define approximations to continuum of phase factors. At least the S-matrix associated with p-adic-to-real transitions and more generally $p_1 \rightarrow p_2$ transitions between states for which the partonic space-time sheets are p_1- resp. p_2-adic can involve only this kind of algebraic phases. One can also say that cognitive representations can involve only algebraic phases and algebraic numbers in general. For real-to-real transitions and real-to-padic transitions U-matrix might be non-algebraic or obtained by analytic continuation of algebraic U-matrix. S-matrix is by definition diagonal with respect to number field and similar continuation principle might apply also in this case.

4. The subgroups of the hierarchy of subgroups of the modular group with rational matrix elements are labelled by integer N and relate naturally to the hierarchy of Farey sequences. The hierarchy of quantum critical phases is labelled by integers N with quantum phase transitions occurring only between phases for which the smaller integer divides the larger one [C1].

Interpretation of RH in TGD framework

Number theoretic universality of physics suggests an interpretation for the Riemann hypothesis in TGD framework. RH would be equivalent to the statement that the Farey numbers provide best possible approximation to the set of rationals $k/|F_N|$ or to the statement that the roots of unity contained by F_N define the best possible approximation for the roots of unity defined as $exp(ik2\pi/|F_N|)$ with evenly spaced phase angles. The roots of unity allowed by the lowest N levels of the dark matter hierarchy allows the best possible approximate representation for algebraic phases represented exactly at $|F_N|$:th level of hierarchy.

A stronger statement would be that the Platonia, where RH holds true would be the best possible world in the sense that algebraic physics behind the cognitive representations would allow the best possible approximation hierarchy for the continuum physics (both for numbers in unit interval and for phases on unit circle). Platonia with RH would be cognitive paradise.

One could see this also from different view point. "Platonia as the cognitively best possible world" could be taken as the "axiom of all axioms": a kind of fundamental variational principle of mathematics. Among other things it would allow to conclude that RH is true: RH must hold true either as a theorem following from some axiomatics or as an axiom in itself.

Could rational N-tangles exist in some sense?

The article of Kauffman and Lambropoulou [33] about rational 2-tangles having commutative sum and product allowing to map them to rationals is very interesting from TGD point of view. The illustrations of the article are beautiful and make it easy to get the gist of various ideas. The theorem of the article states that equivalent rational tangles giving trivial tangle in the product correspond to subsequent Farey numbers a/b and c/d satisfying $ad - bc = \pm 1$ so that the pair defines element of the modular group SL(2,Z).

1. Rational 2-tangles

1. The basic observation is that 2-tangles are 2-tangles in both "s- and t-channels". Product and sum can be defined for all tangles but only in the case of 2-tangles the sum, which in this case reduces to product in t-channel obtained by putting tangles in series, gives 2-tangle. The so called rational tangles are 2-tangles constructible by using addition of $\pm[1]$ on left or right of tangle and multiplication by $\pm[1]$ on top or bottom. Product and sum are commutative for rational 2-tangles but the outcome is not a rational 2-tangle in the general case. One can also assign to rational 2-tangle its negative and inverse. One can map 2-tangle to a number which is rational for rational tangles. The tangles $[0]$, $[\infty]$, $\pm[1]$, $\pm 1/[1]$, $\pm[2]$, $\pm[1/2]$ define so called elementary rational 2-tangles.

2. In the general case the sum of $M-$ and $N-$tangles is $M + N - 2$-tangle and combines various $N-$tangles to a monoidal structure. Tensor product like operation giving $M + N$-tangle looks to me physically more natural than the sum.

3. The reason why general 2-tangles are non-commutative although 2-braids obviously commute is that 2-tangles can be regarded as sequences of $N-$tangles with 2-tangles appearing only as the initial and final state: N is actually even for intermediate states. Since $N > 2$-braid groups are non-commutative, non-commutativity results. It would be interesting to know whether braid group representations have been used to construct representations of $N-$tangles.

2. Does generalization to $N >> 2$ case exist?

One can wonder whether the notion of rational tangle and the basic result of the article about equivalence of tangles might somehow generalize to the $N > 2$ case.

1. Could the commutativity of tangle product allow to characterize the $N > 2$ generalizations of rational 2-tangles. The commutativity of product would be a space-time correlate for the commutativity of the S-matrices defining time like entanglement between the initial and final quantum states assignable to the N-tangle. For 2-tangles commutativity of the sum would have an analogous interpretation. Sum is not a very natural operation for N-tangles for $N >$ 2. Commutativity means that the representation matrices defined as products of braid group actions associated with the various intermediate states and acting in the same representation space commute. Only in very special cases one can expect commutativity for tangles since commutativity is lost already for braids.

2. The representations of 2-tangles should involve the subgroups of N-braid groups of intermediate braids identifiable as Galois groups of N:th order polynomials in the realization as number theoretic tangles. Could non-commutative 2-tangles be characterized by algebraic numbers in the extensions to which the Galois groups are associated? Could the non-commutativity reflect directly the non-commutativity of Galois groups involved? Quite generally one can ask whether the invariants should be expressible using algebraic numbers in the extensions of rationals associated with the intermediate braids.

3. Rational 2-tangles can be characterized by a rational number obtained by a projective identification $[a, b]^T \rightarrow a/b$ from a rational 2-spinor $[a, b]^T$ to which SL(2(N-1),Z) acts. Equivalence means that the columns $[a, b]^T$ and $[c, d]^T$ combine to form element of SL(2,Z) and thus defining a modular transformation. Could more general 2-tangles have a similar representation but in terms of algebraic integers?

4. Could N-tangles be characterized by $N - 1$ $2(N - 1)$-component projective column-spinors $[a_i^1, a_i^2, .., a_i^{2(N-1)}]^T$, $i = 1, ...N - 1$ so that only the ratios $a_i^k/a_i^{2(N-1)} \leq 1$ matter? Could equivalence for them mean that the $N - 1$ spinors combine to form $N - 1 + N - 1$ columns of $SL(2(N-1), Z)$ matrix. Could N-tangles quite generally correspond to collections of projective $N-1$ spinors having as components algebraic integers and could $ad-bc = \pm 1$ criterion generalize? Note that the modular group for surfaces of genus g is SL(2g,Z) so that $N-1$ would be analogous to g and $1 \leq N \geq 3$- braids would correspond to $g \leq 2$ Riemann surfaces.

5. Dark matter hierarchy leads naturally to a hierarchy of modular sub-groups of $SL(2, Q)$ labelled by N (the generator $\tau \rightarrow \tau + 2$ of modular group is replaced with $\tau \rightarrow \tau + 2/N$). What might be the role of these subgroups and corresponding subgroups of $SL(2(N - 1), Q)$. Could they arise in "anyonization" when one considers quantum group representations of 2-tangles with twist operation represented by an N:th root of unity instead of phase U satisfying $U^2 = 1$?

How tangles could be realized in TGD Universe?

The article of Kauffman and Lambropoulou stimulated the question in what senses N-tangles could be be realized in TGD Universe as fundamental structures.

1. Tangles as number theoretic braids?

The strands of number theoretical $N-$braids correspond to roots of N:th order polynomial and if one allows time evolutions of partonic 2-surface leading to the disappearance or appearance of real roots $N-$tangles become possible. This however means continuous evolution of roots so that the coefficients of polynomials defining the partonic 2-surface can be rational only in initial and final state but not in all intermediate "virtual" states.

2. Tangles as tangled partonic 2-surfaces?

Tangles could appear in TGD also in second manner.

1. Partonic 2-surfaces are sub-manifolds of a 3-D section of space-time surface. If partonic 2-surfaces have genus $g > 0$ the handles can become knotted and linked and one obtains besides ordinary knots and links more general knots and links in which circle is replaced by figure eight and its generalizations obtained by adding more circles (eyeglasses for $N-$eyed creatures).

2. Since these 2-surfaces are space-like, the resulting structures are indeed tangles rather than only braids. Tangles made of strands with fixed ends would result by allowing spherical partons elongate to long strands with fixed ends. DNA tangles would the basic example, and are discussed also in the article. DNA sequences to which I have speculatively assigned invisible (dark) braid structures might be seen in this context as space-like "written language representations" of genetic programs represented as number theoretic braids.

7.7 Quantum Quandaries

John Baez's [30] discusses in a physicist friendly manner the possible application of category theory to physics. The lessons obtained from the construction of topological quantum field theories (TQFTs) suggest that category theoretical thinking might be very useful in attempts to construct theories of quantum gravitation.

The point is that the Hilbert spaces associated with the initial and final state n-1-manifold of n-cobordism indeed form in a natural manner category. Morphisms of Hilb in turn are unitary or possibly more general maps between Hilbert spaces. TQFT itself is a functor assigning to a cobordism the counterpart of S-matrix between the Hilbert spaces associated with the initial and final n-1-manifold. The surprising result is that for $n \leq 4$ the S-matrix can be unitary S-matrix only if the cobordism is trivial. This should lead even string theorist to raise some worried questions.

In the hope of feeding some category theoretic thinking into my spine, I briefly summarize some of the category theoretical ideas discussed in the article and relate it to the TGD vision, and after that discuss the worried questions from TGD perspective. That space-time makes sense only relative to imbedding space would conform with category theoretic thinking.

7.7.1 The *-category of Hilbert spaces

Baez considers first the category of Hilbert spaces. Intuitively the definition of this category looks obvious: take linear spaces as objects in category Set, introduce inner product as additional structure and identify morphisms as maps preserving this inner product. In finite-D case the category with inner product is however identical to the linear category so that the inner product does not seem to be absolutely essential. Baez argues that in infinite-D case the morphisms need not be restricted to unitary transformations: one can consider also bounded linear operators as morphisms since they play key role in quantum theory (consider only observables as Hermitian operators). For hyper-finite factors of type II_1 inclusions define very important morphisms which are not unitary transformations but very similar to them. This challenges the belief about the fundamental role of unitarity and raises the question about how to weaken the unitarity condition without losing everything.

The existence of the inner product is essential only for the metric topology of the Hilbert space. Can one do without inner product as an inherent property of state space and reduce it to a morphism? One can indeed express inner product in terms of morphisms from complex numbers to Hilbert space and their conjugates. For any state Ψ of Hilbert space there is a unique morphisms T_Ψ from C to Hilbert space satisfying $T_\Psi(1) = \Psi$. If one assumes that these morphisms have conjugates T_Ψ^* mapping Hilbert space to C, inner products can be defined as morphisms $T_\Phi^* T_\Psi$. The Hermitian conjugates of operators can be defined with respect to this inner product so that one obtains *-category. Reader has probably realized that T_Ψ and its conjugate correspond to ket and bra in Dirac's formalism.

Note that in TGD framework based on hyper-finite factors of type II_1 (HFFs) the inclusions of complex rays might be replaced with inclusions of HFFs with included factor representing the finite measurement resolution. Note also the analogy of inner product with the representation of space-times as 4-surfaces of the imbedding space in TGD.

7.7.2 The monoidal *-category of Hilbert spaces and its counterpart at the level of nCob

One can give the category of Hilbert spaces a structure of monoid by introducing explicitly the tensor products of Hilbert spaces. The interpretation is obvious for physicist. Baez describes the details of this identification, which are far from trivial and in the theory of quantum groups very interesting things happen. A non-commutative quantum version of the tensor product implying braiding is possible and associativity condition leads to the celebrated Yang-Baxter equations: inclusions of HFFs lead to quantum groups [C12] too.

At the level of nCob the counterpart of the tensor product is disjoint union of n-1-manifolds. This unavoidably creates the feeling of cosmic loneliness. Am I really a disjoint 3-surface in emptiness which is not vacuum even in the geometric sense? Cannot be true!

This horrifying sensation disappears if n-1-manifolds are n-1-surfaces in some higher-dimensional imbedding space so that there would be at least something between them. I can emit a little baby manifold moving somewhere perhaps being received by some-one somewhere and I can receive radiation from some-one at some distance and in some direction as small baby manifolds making gentle tosses on my face!

This consoling feeling could be seen as one of the deep justifications for identifying fundamental objects as light-like partonic 3-surfaces in TGD framework. Their ends correspond to 2-D partonic surfaces at the boundaries of future or past directed light-cones (states of positive and negative energy respectively) and are indeed disjoint but not in the desperately existential sense as 3-geometries of General Relativity.

This disjointness has also positive aspect in TGD framework. One can identify the color degrees of freedom of partons as those associated with CP_2 degrees of freedom. For instance, SU(3) analogs for rotational states of rigid body become possible. 4-D space-time surfaces as preferred extremals of Kähler action connect the partonic 3-surfaces and bring in classical representation of correlations and thus of interactions. The representation as sub-manifolds makes it also possible to speak about positions of these sub-Universes and about distances between them. The habitants of TGD Universe are maximally free but not completely alone.

7.7.3 TQFT as a functor

The category theoretic formulation of TQFT relies on a very elegant and general idea. Quantum transition has as a space-time correlate an n-dimensional surface having initial final states as its n-1-dimensional ends. One assigns Hilbert spaces of states to the ends and S-matrix would be a unitary morphism between the ends. This is expressed in terms of the category theoretic language by introducing the category nCob with objects identified as n-1-manifolds and morphisms as cobordisms and *-category Hilb consisting of Hilbert spaces with inner product and morphisms which are bounded linear operators which do not however preserve the unitarity. Note that the morphisms of nCob cannot anymore be identified as maps between n-1-manifolds interpreted as sets with additional structure so that in this case category theory is more powerful than set theory.

TQFT is identified as a functor nCob $\rightarrow$ Hilb assigning to n-1-manifolds Hilbert spaces, and to cobordisms unitary S-matrices in the category Hilb. This looks nice but the surprise is that for $n \leq 4$ unitary S-matrix exists only if the cobordism is trivial so that topology changing transitions are not possible unless one gives up unitarity.

This raises several worried questions.

1. Does this result mean that in TQFT sense unitary S-matrix for topology changing transitions from a state containing n_i closed strings to a state containing $n_f \neq n_i$ strings does not exist? Could the situation be same also for more general non-topological stringy S-matrices? Could the non-converging perturbation series for S-matrix with finite individual terms matrix fail to no non-perturbative counterpart? Could it be that M-theory is doomed to remain a dream with no hope of being fulfilled?

2. Should one give up the unitarity condition and require that the theory predicts only the relative probabilities of transitions rather than absolute rates? What the proper generalization of the S-matrix could be?

3. What is the relevance of this result for quantum TGD?

7.7.4 The situation is in TGD framework

The result about the non-existence of unitary S-matrix for topology changing cobordisms allows new insights about the meaning of the departures of TGD from string models.

Cobordism cannot give interesting selection rules

When I started to work with TGD for more than 28 years ago, one of the first ideas was that one could identify the selection rules of quantum transitions as topological selection rules for cobordisms. Within week or two came the great disappointment: there were practically no selection rules. Could one revive this naive idea? Could the existence of unitary S-matrix force the topological selection rules after all? I am skeptic. If I have understood correctly the discussion of what happens in 4-D case [31] only the exotic diffeo-structures modify the situation in 4-D case.

Light-like 3-surfaces allow cobordism

In the physically interesting GRT like situation one would expect the cobordism to be mediated by a space-time surface possessing Lorentz signature. This brings in metric and temporal distance. This means complications since one must leave the pure TQFT context. Also the classical dynamics of quantum gravitation brings in strong selection rules related to the dynamics in metric degrees of freedom so that TQFT approach is not expected to be useful from the point of view of quantum gravity and certainly not the limit of a realistic theory of quantum gravitation.

In TGD framework situation is different. 4-D space-time sheets can have Euclidian signature of the induced metric so that Lorentz signature does not pose conditions. The counterparts of cobordisms correspond at fundamental level to light-like 3-surfaces, which are arbitrarily except for the light-likeness condition (the effective 2-dimensionality implies generalized conformal invariance and analogy with 3-D black-holes since 3-D vacuum Einstein equations are satisfied). Field equations defined by the Chern-Simons action imply that CP_2 projection is at most 2-D but this condition holds true only for the extremals and one has functional integral over all light-like 3-surfaces. The temporal distance between points along light-like 3-surface vanishes. The constraints from light-likeness bring in metric degrees of freedom but in a very gentle manner and just to make the theory physically interesting.

Feynmann cobordism as opposed to ordinary cobordism

In string model context the discouraging results from TQFT hold true in the category of nCob, which corresponds to trouser diagrams for closed strings or for their open string counterparts. In TGD framework these diagrams are replaced with a direct generalization of Feynman diagrams for which 3-D light-like partonic 3-surfaces meet along their 2-D ends at the vertices. In honor of Feynman one could perhaps speak of Feynman cobordisms. These surfaces are singular as 3-manifolds but vertices are nice 2-manifolds. I contrast to this, in string models diagrams are nice 2-manifolds but vertices are singular as 1-manifolds (say eye-glass type configurations for closed strings).

This picture gains a strong support for the interpretation of fermions as light-like throats associated with connected sums of CP_2 type extremals with space-time sheets with Minkowski signature and of bosons as pairs of light-like wormhole throats associated with CP_2 type extremal connecting two space-time sheets with Minkowski signature of induced metric. The space-time sheets have opposite time orientations so that also zero energy ontology emerges unavoidably. There is also consistency TGD based explanation of the family replication phenomenon in terms of genus of light-like partonic 2-surfaces.

One can wonder what the 4-D space-time sheets associated with the generalized Feynman diagrams could look like? One can try to gain some idea about this by trying to assign 2-D surfaces to ordinary Feynman diagrams having a subset of lines as boundaries. In the case of 2→2 reaction open string is pinched to a point at vertex. 1→2 vertex, and quite generally, vertices with odd number of lines, are impossible. The reason is that 1-D manifolds of finite size can have either 0 or 2 ends whereas in higher-D the number of boundary components is arbitrary. What one expects to happen in TGD context is that wormhole throats which are at distance characterized by CP_2 fuse together in the vertex so that some kind of pinches appear also now.

Zero energy ontology

Zero energy ontology gives rise to a second profound distinction between TGD and standard QFT. Physical states are identified as states with vanishing net quantum numbers, in particular energy. Everything is creatable from vacuum - and one could add- by intentional action so that zero energy ontology is profoundly Eastern. Positive *resp.* negative energy parts of states can be identified as states associated with 2-D partonic surfaces at the boundaries of future *resp.* past directed light-cones, whose tips correspond to the arguments of n-point functions. Each incoming/outgoing particle would define a mini-cosmology corresponding to not so big bang/crunch. If the time scale of perception is much shorter than time interval between positive and zero energy states, the ontology looks like the Western positive energy ontology. Bras and kets correspond naturally to the positive and negative energy states and phase conjugation for laser photons making them indeed something which seems to travel in opposite time direction is counterpart for bra-ket duality.

Finite temperature S-matrix defines genuine quantum state in zero energy ontology

In TGD framework one encounters two S-matrix like operators.

1. There is U-matrix between zero energy states. This is expected to be rather trivial but very important from the point of view of description of intentional actions as transitions transforming p-adic partonic 3-surfaces to their real counterparts.

2. The S-matrix like operator describing what happens in laboratory corresponds to the time-like entanglement coefficients between positive and negative energy parts of the state. Measurement of reaction rates would be a measurement of observables reducing time like entanglement and very much analogous to an ordinary quantum measurement reducing space-like entanglement. There is a finite measurement resolution described by inclusion of HFFs and this means that situation reduces effectively to a finite-dimensional one.

p-Adic thermodynamics strengthened with p-adic length scale hypothesis predicts particle masses with an amazing success. At first the thermodynamical approach seems to be in contradiction with the idea that elementary particles are quantal objects. Unitarity is however *not* necessary if one accepts that only relative probabilities for reductions to pairs of initial and final states interpreted as particle reactions can be measured.

The beneficial implications of unitarity are not lost if one replaces QFT with thermal QFT. Category theoretically this would mean that the time-like entanglement matrix associated with the product of cobordisms is a product of these matrices for the factors. The time parameter in S-matrix would be replaced with a complex time parameter with the imaginary part identified as inverse temperature. Hence the interpretation in terms of time evolution is not lost.

In the theory of hyper-finite factors of type III_1 the partition function for thermal equilibrium states and S-matrix can be neatly fused to a thermal S-matrix for zero energy states and one could introduce p-adic thermodynamics at the level of quantum states. It seems that this picture applies to HFFs by restriction. Therefore the loss of unitarity S-matrix might after all turn to a victory by more or less forcing both zero energy ontology and p-adic thermodynamics.

7.8 How to represent algebraic numbers as geometric objects?

I have found the blogs of mathematicians very interesting, in particular "Kea's blog" [34] has provided many stimuli in my attempts to gain some intuition about categories and their possible application to quantum TGD. Kea has generously explained what are the deep problems of category theoretic approach to mathematics and given references to articles: thanks to these references also this section saw the day light.

These blogs are also interesting because they allow to get some grasp about very different styles of thinking of a mathematician and physicist. For mathematician it is very important that the result is obtained by a strict use of axioms and deduction rules. Physicist is a cognitive opportunist: it does not matter how the result is obtained by moving along axiomatically allowed paths or not, and the new result is often more like a discovery of a new axiom and physicist is ever-grateful for Gödel for giving justification for what sometimes admittedly degenerates to a creative hand-waving. For physicist ideas form a kind of bio-sphere and the fate of the individual idea depends on its ability to survive, which is determined by its ability to become generalized, its consistency with other ideas, and ability to interact with other ideas to produce new ideas.

7.8.1 Can one define complex numbers as cardinalities of sets?

During few days before writing this we have had in Kea's blog a little bit of discussion inspired by the problem related to the categorification of basic number theoretical structures. I have learned that sum and product are natural operations for the objects of category. For instance, one can define sum as in terms of union of sets or direct sum of vector spaces and product as Cartesian product of sets and tensor product of vector spaces: rigs [35] are example of categories for which natural numbers define sum and product.

Subtraction and division are however problematic operations. Negative numbers and inverses of integers do not have a realization as a number of elements for any set or as dimension of vector space. The naive physicist inside me asks immediately: why not go from statics to dynamics and take operations (arrows with direction) as objects: couldn't this allow to define subtraction and division? Is the problem that the axiomatization of group theory requires something which purest categorification does not give? Or aren't the numbers representable in terms of operations of finite groups not enough? In any case cyclic groups would allow to realize roots of unity as operations (Z_2 would give -1).

One could also wonder why the algebraic numbers might not somehow result via the representations of permutation group of infinite number of elements containing all finite groups and thus Galois groups of algebraic extensions as subgroups? Why not take the elements of this group as objects of the basic category and continue by building group algebra and hyper-finite factors of type II_1 isomorphic to spinors of world of classical worlds, and so on.

After having written the first half of the section, I learned that something similar to the transition from statics to dynamics is actually carried out but by manner which is by many orders of magnitudes more refined than the proposal above and that I had never been able to imagine. The article *Objects of categories as complex numbers* of Marcelo Fiore and Tom Leinster [35] describes a fascinating idea summarized also by John Baez [36] about how one can assign to the objects of a category complex numbers as roots of a polynomial $Z = P(Z)$ defining an isomorphism of object. Z is the element of a category called rig, which differs from ring in that integers are replaced with natural numbers. One can replace Z with a complex number $|Z|$ defined as a root of polynomial. $|Z|$ is interpreted formally as the cardinality of the object. It is essential to have natural numbers and thus only product and sum are defined. This means a restriction: for instance, only complex algebraic numbers associated with polynomials having natural numbers as coefficients are obtained. Something is still missing.

Note that this correspondence assumes the existence of complex numbers and one cannot say that complex numbers are categorified. Maybe basic number fields must be left outside categorification. One can however require that all of them have a concrete set theoretic representation rather than only formal interpretation as cardinality so that one still encounters the problem how to represent algebraic complex number as a concrete cardinality of a set.

7.8.2 In what sense a set can have cardinality -1?

The discussion in Kea's blog led me to ask what the situation is in the case of p-adic numbers. Could it be possible to represent the negative and inverse of p-adic integer, and in fact any p-adic number, as a geometric object? In other words, does a set with -1 or $1/n$ or even $\sqrt{-1}$ elements exist? If this were in some sense true for all p-adic number fields, then all this wisdom combined together might provide something analogous to the adelic representation for the norm of a rational number as product of its p-adic norms. As will be found, alternative interpretations of complex algebraic numbers as p-adic numbers representing cardinalities of p-adic fractals emerge. The fractal defines the manner how one must do an infinite sum to get an infinite real number but finite p-adic number.

Of course, this representation might not help to define p-adics or reals categorically but might help to understand how p-adic cognitive representations defined as subsets for rational intersections of real and p-adic space-time sheets could represent p-adic number as the number of points of p-adic fractal having infinite number of points in real sense but finite in the p-adic sense. This would also give a fundamental cognitive role for p-adic fractals as cognitive representations of numbers.

How to construct a set with -1 elements?

The basic observation is that p-adic -1 has the representation

$$-1 = (p-1)/(1-p) = (p-1)(1+p+p^2+p^3....)$$

As a real number this number is infinite or -1 but as a p-adic number the series converges and has p-adic norm equal to 1. One can also map this number to a real number by canonical identification taking the powers of p to their inverses: one obtains p in this particular case. As a matter fact, any rational with p-adic norm equal to 1 has similar power series representation.

The idea would be to represent a given p-adic number as the infinite number of points (in real sense) of a p-adic fractal such that p-adic topology is natural for this fractal. This kind of fractals can be constructed in a simple manner: from this more below. This construction allows to represent any p-adic number as a fractal and code the arithmetic operations to geometric operations for these fractals.

These representations - interpreted as cognitive representations defined by intersections of real and p-adic space-time sheets - are in practice approximate if real space-time sheets are assumed to have a finite size: this is due to the finite p-adic cutoff implied by this assumption and the meaning a finite resolution. One can however say that the p-adic space-time itself could by its necessarily infinite size represent the *idea* of given p-adic number faithfully.

This representation applies also to the p-adic counterparts of algebraic numbers in case that they exist. For instance, roughly one half of p-adic numbers have square root as ordinary p-adic number and quite generally algebraic operations on p-adic numbers can give rise to p-adic numbers so that also these could have set theoretic representation. For $p\ mod\ 4 = 1$ also $\sqrt{(-1)}$ exists: for instance, for $p = 5$: $2^2 = 4 = -1\ mod\ 5$ guarantees this so that also imaginary unit and complex numbers would have a fractal representation. Also many transcendentals possess this kind of representation. For instance $exp(xp)$ exists as a p-adic number if x has p-adic norm not larger than 1: also $log(1+xp)$ does so.

Hence a quite impressive repertoire of p-adic counterparts of real numbers would have representation as a p-adic fractal for some values of p. Adelic vision would suggest that combining these representations one might be able to represent quite a many real numbers. In the case of π I do not find any obvious p-adic representation (for instance $sin(\pi/6) = 1/2$ does not help since the p-adic variant of the Taylor expansion of $\pi/6 = arcsin(1/2)$ does not converge p-adically for any value of p). It might be that there are very many transcendentals not allowing fractal representation for any value of p.

Conditions on the fractal representations of p-adic numbers

Consider now the construction of the fractal representations in terms of rational intersections of real real and p-adic space-time sheets. The question is what conditions are natural for this representation if it corresponds to a cognitive representation is realized in the rational intersection of real and p-adic space-time sheets obeying same algebraic equations.

1. Pinary cutoff is the analog of the decimal cutoff but is obtained by dropping away high positive rather than negative powers of p to get a finite real number: example of pinary cutoff is $-1 = (p-1)(1+p+p^2+...) \rightarrow (p-1)(1+p+p^2)$. This cutoff must reduce to a fractal cutoff meaning a finite resolution due to a finite size for the real space-time sheet. In the real sense the p-adic fractal cutoff means not forgetting details below some scale but cutting out all above some length scale. Physical analog would be forgetting all frequencies below some cutoff frequency in Fourier expansion.

   The motivation comes from the fact that TGD inspired consciousness assigns to a given biological body there is associated a field body or magnetic body containing dark matter with large $\hbar$ and quantum controlling the behavior of biological body and so strongly identifying with it so as to belief that this all ends up to a biological death. This field body has an onion like fractal structure and a size of at least order of light-life. Of course, also larger onion layers could be present and would represent those levels of cognitive consciousness not depending on the sensory input on biological body: some altered states of consciousness could relate to these levels. In any case, the larger the magnetic body, the better the numerical skills of the p-adic mathematician.

2. Lowest pinary digits of $x = x_0+x_1p+x_2p^2+...$, $x_n \leq p$ must have the most reliable representation since they are the most significant ones. The representation must be also highly redundant to guarantee reliability. This requires repetitions and periodicity. This is guaranteed if the representation is hologram like with segments of length p^n with digit x_n represented again and again in all segments of length p^m, $m > n$.

3. The TGD based physical constraint is that the representation must be realizable in terms of induced classical fields assignable to the field body hierarchy of an intelligent system interested in artistic expression of p-adic numbers using its own field body as instrument. As a matter, sensory and cognitive representations are realized at field body in TGD Universe and EEG is in a fundamental role in building this representation. By p-adic fractality fractal wavelets are the most natural candidate. The fundamental wavelet should represent the p different pinary digits and its scaled up variants would correspond to various powers of p so that the representation would reduce to a Fourier expansion of a classical field.

Concrete representation

Consider now a concrete candidate for a representation satisfying these constraints.

1. Consider a p-adic number

$$y = p^{n_0}x, \quad x = \sum x_n p^n \quad , \quad n \geq n_0 = 0 \quad .$$

   If one has a representation for a p-adic unit x the representation of is by a purely geometric fractal scaling of the representation by p^n. Hence one can restrict the consideration to p-adic units.

2. To construct the representation take a real line starting from origin and divide it into segments with lengths $1, p, p^2, ...$. In TGD framework this scalings come actually as powers of $p^{1/2}$ but this is just a technical detail.

3. It is natural to realize the representation in terms of periodic field patterns. One can use wavelets with fractal spectrum $p^n\lambda_0$ of "wavelet lengths", where λ_0 is the fundamental wavelength. Fundamental wavelet should have p different patterns correspond to the p values of pinary digit as its structures. Periodicity guarantees the hologram like character enabling to pick n:th digit by studying the field pattern in scale p^n anywhere inside the field body.

4. Periodicity guarantees also that the intersections of p-adic and real space-time sheets can represent the values of pinary digits. For instance, wavelets could be such that in a given p-adic scale the number of rational points in the intersection of the real and p-adic space-time sheet equals to x_n. This would give in the limit of an infinite pinary expansion a set theoretic realization of any p-adic number in which each pinary digit x_n corresponds to infinite copies of a set with x_n elements and fractal cutoff due to the finite size of real space-time sheet would bring in a finite precision. Note however that p-adic space-time sheet necessarily has an infinite size and it is only real world realization of the representation which has finite accuracy.

5. A concrete realization for this object would be as an infinite tree with $x_n + 1 \leq p$ branches in each node at level n ($x_n + 1$ is needed in order to avoid the splitting tree at $x_n = 0$). In 2-adic case -1 would be represented by an infinite pinary tree. Negative powers of p correspond to the of the tree extending to a finite depth in ground.

7.8.3 Generalization of the notion of rig by replacing naturals with p-adic integers

Previous considerations do not relate directly to category theoretical problem of assigning complex numbers to objects. It however turns out that p-adic approach allows to generalize the proposal of [35] by replacing natural numbers with p-adic integers in the definition of rig so that any algebraic complex number can define cardinality of an object of category allowing multiplication and sum and that these complex numbers can be replaced with p-adic numbers if they make sense as such so that previous arguments provide a concrete geometric representation of the cardinality. The road to the realization this simple generalization required a visit to the John Baez's Weekly Finds (Week 102) [36].

The outcome was the realization that the notion of rig used to categorify the subset of algebraic numbers obtained as roots of polynomials with *natural number* valued coefficients generalizes trivially by replacing natural numbers by *p-adic integers*. As a consequence one obtains beautiful p-adicization of the generating function F(x) of structure as a function which converges p-adically for any rational $x = q$ for which it has prime p as a positive power divisor.

Effectively this generalization means the replacement of natural numbers as coefficients of the polynomial defining the rig with all rationals, also negative, and *all* complex algebraic numbers find a category theoretical representation as "cardinalities". These cardinalities have a dual interpretation as p-adic integers which in general correspond to infinite real numbers but are mappable to real numbers by canonical identification and have a geometric representation as fractals.

Mapping of objects to complex numbers and the notion of rig

The idea of rig approach is to categorify the notion of cardinality in such a manner that one obtains a *subset* of algebraic complex numbers as cardinalities in the category-theoretical sense. One can assign to an object a polynomial with coefficients, which are *natural numbers* and the condition $Z = P(Z)$ says that $P(Z)$ acts as an isomorphism of the object. One can interpret the equation also in terms of complex numbers. Hence the object is mapped to a complex number Z defining a root of the polynomial interpreted as an ordinary polynomial: it does not matter which root is chosen. The complex number Z is interpreted as the "cardinality" of the object but I do not really understand the motivation for this. The deep further result is that also more general polynomial equations $R(|Z|) = Q(|Z|)$ satisfied by the generalized cardinality Z imply $R(Z) = Q(Z)$ as isomorphism.

I try to reproduce what looks the most essential in the explanation of John Baez and relate it to my own ideas but take this as my talk to myself and visit This Week's Finds [36], one of the many classics of Baez, to learn of this fascinating idea.

1. Baez considers first the ways of putting a given structure to n-element set. The set of these structures is denoted by F_n and the number of them by $|F_n|$. The generating function $|F|(x) = \sum_n |F_n| x^n$ packs all this information to a single function.

   For instance, if the structure is binary tree, this function is given by $T(x) = \sum_n C_{n-1} x^n$, where C_{n-1} are Catalan numbers and n¿0 holds true. One can show that T satisfies the formula

   $$T = X + T^2 \quad ,$$

   since any binary tree is either trivial or decomposes to a product of binary trees, where two trees emanate from the root. One can solve this second order polynomial equation and the power expansion gives the generating function.

2. The great insight is that one can also work directly with structures. For instance, by starting from the isomorphism $T = 1 + T^2$ applying to an object with cardinality 1 and substituting T^2 with $(1 + T^2)^2$ repeatedly, one can deduce the amazing formula $T^7(1) = T(1)$ mentioned by Kea, and this identity can be interpreted as an isomorphism of binary trees.

3. This result can be generalized using the notion of rig category [35]. In rig category one can add and multiply but negatives are not defined as in the case of ring. The lack of subtraction and division is still the problem and as I suggested in previous posting p-adic integers might resolve the problem.

Whenever Z is object of a rig category, one can equip it with an isomorphism $Z = P(Z)$ where $P(Z)$ is polynomial with *natural numbers* as coefficients and one can assign to object "cardinality" as any root of the equation $Z = P(Z)$. Note that set with n elements corresponds to $P(|Z|) = n$. Thus subset of algebraic complex numbers receive formal identification as cardinalities of sets. Furthermore, if the cardinality satisfies another equation $Q(|Z|) = R(|Z|)$ such that neither polynomial is constant, then one can construct an isomorphism $Q(Z) = R(Z)$. Isomorphisms correspond to equations!

4. This is indeed nice that there is something which is not so beautiful as it could be: why should we restrict ourselves to *natural numbers* as coefficients of $P(Z)$? Could it be possible to replace them with integers to obtain *all complex algebraic numbers* as cardinalities? Could it be possible to replace natural numbers by p-adic integers?

p-Adic rigs and Golden Object as p-adic fractal

The notions of generating function and rig generalize to the p-adic context.

1. The generating function $F(x)$ defining isomorphism Z in the rig formulation converges p-adically for any p-adic number containing p as a factor so that the idea that all structures have p-adic counterparts is natural. In the real context the generating function typically diverges and must be defined by analytic continuation. Hence one might even argue that p-adic numbers are more natural in the description of structures assignable to finite sets than reals.

2. For rig one considers only polynomials $P(Z)$ (Z corresponds to the generating function F) with coefficients which are natural numbers. Any p-adic integer can be however interpreted as a non-negative integer: natural number if it is finite and "super-natural" number if it is infinite. Hence can generalize the notion of rig by replacing natural numbers by p-adic integers. The rig formalism would thus generalize to arbitrary polynomials with integer valued coefficients so that all complex algebraic numbers could appear as cardinalities of category theoretical objects. Even rational coefficients are allowed. This is highly natural number theoretically.

3. For instance, in the case of binary trees the solutions to the isomorphism condition $T = p + T^2$ giving $T = [1 \pm (1-4p)^{1/2}]/2$ and T would be complex number $[p \pm (1-4p)^{1/2}]/2$. $T(p)$ can be interpreted also as a p-adic number by performing power expansion of square root in case that the p-adic square root exists: this super-natural number can be mapped to a real number by the canonical identification and one obtains also the set theoretic representations of the category theoretical object $T(p)$ as a p-adic fractal. This interpretation of cardinality is much more natural than the purely formal interpretation as a complex number. This argument applies completely generally. The case $x = 1$ discussed by Baez gives $T = [1 \pm (-3)^{1/2}]/2$ allows p-adic representation if $-3 == p - 3$ is square mod p. This is the case for $p = 7$ for instance.

4. John Baez [36] poses also the question about the category theoretic realization of "Golden Object", his big dream. In this case one would have $Z = G = -1 + G^2 = P(Z)$. The polynomial on the right hand side does not conform with the notion of rig since -1 is not a natural number. If one allows p-adic rigs, $x = -1$ can be interpreted as a p-adic integer $(p-1)(1+p+...)$, positive and infinite and "super-natural", actually largest possible p-adic integer in a well defined sense.

   A further condition is that Golden Mean converges as a p-adic number: this requires that $\sqrt{5}$ must exist as a p-adic number: $(5 = 1+4)^{1/2}$ certainly converges as power series for $p = 2$ so that Golden Object exists 2-adically. By using [37] of Euler, one finds that 5 is square mod p only if p is square mod 5. To decide whether given p is Golden it is enough to look whether p mod 5 is 1 or 4. For instance, $p = 11, 19, 29, 31$ ($=M_5$) are Golden. Mersennes M_k, $k = 3, 7, 127$ and Fermat primes are not Golden. One representation of Golden Object as p-adic fractal is the p-adic series expansion of $[1/2 \pm 5^{1/2}]/2$ representable geometrically as a binary tree such that there are $0 \leq x_n + 1 \leq p$ branches at each node at height n if n:th p-adic coefficient is x_n. The "cognitive" p-adic representation in terms of wavelet spectrum of classical fields is discussed in the previous posting.

5. It would be interesting to know how quantum dimensions of quantum groups assignable to Jones inclusions [C6, A9, C12] relate to the generalized cardinalities. The root of unity property of

quantum phase ($q^{n+1} = q$) suggests $Q = Q^{n+1} = P(Q)$ as the relevant isomorphism. For Jones inclusions the cardinality $q = exp(i2\pi/n)$ would not be however equal to quantum dimension $D(n) = 4cos^2(\pi/n)$.

Is there a connection with infinite integers?

Infinite primes [E3] correspond to Fock states of a super-symmetric arithmetic quantum field theory and there is entire infinite hierarchy of them corresponding to repeated second quantization. Also infinite primes and rationals make sense. Besides free Fock states spectrum contains at each level also what might be identified as bound states. All these states can be mapped to polynomials. Since the roots of polynomials represent complex algebraic numbers and as they seem to characterize objects of categories, there are reasons to expect that infinite rationals might allow also interpretation in terms of say rig categories or their generalization. Also the possibility to identify space-time coordinate as isomorphism of a category might be highly interesting concerning the interpretation of quantum classical correspondence.

7.9 Gerbes and TGD

The notion of gerbes has gained much attention during last years in theoretical physics and there is an abundant gerbe-related literature in hep-th archives. Personally I learned about gerbes from the excellent article of Jouko Mickelson [38] (Jouko was my opponent in PhD dissertation for more than two decades ago: so the time flows!).

I have already applied the notion of bundle gerbe in TGD framework in the construction of the Dirac determinant which I have proposed to define the Kähler function for the configuration space of 3-surfaces (see the chapter "Configuration Space Spinor Structure"). The insights provided by the general results about bundle gerbes discussed in [38] led, not only to a justification for the hypothesis that Dirac determinant exists for the modified Dirac action, but also to an elegant solution of the conceptual problems related to the construction of Dirac determinant in the presence of chiral symmetry. Furthermore, on basis of the special properties of the modified Dirac operator there are good reasons to hope that the determinant exists even without zeta function regularization. The construction also leads to the conclusion that the space-time sheets serving as causal determinants must be geodesic sub-manifolds (presumably light like boundary components or "elementary particle horizons"). Quantum gravitational holography is realized since the exponent of Kähler function is expressible as a Dirac determinant determined by the local data at causal determinants and there would be no need to find absolute minima of Kähler action explicitly.

In the sequel the emergence of 2-gerbes at the space-time level in TGD framework is discussed and shown to lead to a geometric interpretation of the somewhat mysterious cocycle conditions for a wide class of gerbes generated via the $\wedge d$ products of connections associated with 0-gerbes. The resulting conjecture is that gerbes form a graded-commutative Grassmman algebra like structure generated by -1- and 0-gerbes. 2-gerbes provide also a beautiful topological characterization of space-time sheets as structures carrying Chern-Simons charges at boundary components and the 2-gerbe variant of Bohm-Aharonov effect occurs for perhaps the most interesting asymptotic solutions of field equations especially relevant for anyonics systems, quantum Hall effect, and living matter [E9].

7.9.1 What gerbes roughly are?

Very roughly and differential geometrically, gerbes can be regarded as a generalization of connection. Instead of connection 1-form (0-gerbe) one considers a connection $n+1$-form defining n-gerbe. The curvature of n-gerbe is closed $n+2$-form and its integral defines an analog of magnetic charge. The notion of holonomy generalizes: instead of integrating n-gerbe connection over curve one integrates its connection form over n+1-dimensional closed surface and can transform it to the analog of magnetic flux.

There are some puzzling features associated with gerbes. Ordinary $U(1)$-bundles are defined in terms of open sets U_α with gauge transformations $g_{\alpha\beta} = g_{\beta\alpha}^{-1}$ defined in $U_\alpha \cap U_\beta$ relating the connection forms in the patch U_β to that in patch U_α. The 3-cocycle condition

$$g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha} = 1 \tag{7.9.1}$$

makes it possible to glue the patches to a bundle structure.

In the case of 1-gerbes the transition functions are replaced with the transition functions $g_{\alpha\beta\gamma} = g^{-1}_{\gamma\beta\alpha}$ defined in triple intersections $U_\alpha \cap U_\beta \cap U_\gamma$ and 3-cocycle must be replaced with 4-cocycle:

$$g_{\alpha\beta\gamma}g_{\beta\gamma\delta}g_{\gamma\delta\alpha}g_{\delta\alpha\beta} = 1 \ . \tag{7.9.2}$$

The generalizations of these conditions to n-gerbes is obvious.

In the case of 2-intersections one can build a bundle structure naturally but in the case of 3-intersections this is not possible. Hence the geometric interpretation of the higher gerbes is far from obvious. One possible interpretation of non-trivial 1-gerbe is as an obstruction for lifting projective bundles with fiber space CP_n to vector bundles with fiber space C^{n+1} [38]. This involves the lifting of the holomorphic transition functions g_α defined in the projective linear group $PGL(n+1,C)$ to $GL(n+1,C)$. When the 3-cocycle condition for the lifted transition functions $\overline{g}_{\alpha\beta}$ fails it can be replaced with 4-cocycle and one obtains 1-gerbe.

7.9.2 How do 2-gerbes emerge in TGD?

Gerbes seem to be interesting also from the point of view of TGD, and TGD approach allows a geometric interpretation of the cocycle conditions for a rather wide class of gerbes.

Recall that the Kähler form J of CP_2 defines a non-trivial magnetically charged and self-dual $U(1)$-connection A. The Chern-Simons form $\omega = A\wedge J = A\wedge dA$ having CP_2 Abelian instanton density $J\wedge J$ as its curvature form and can thus be regarded as a 3-connection form of a 2-gerbe. This 2-gerbe is induced by 0-gerbe.

The coordinate patches U_α are same as for $U(1)$ connection. In the transition between patches A and ω transform as

$$\begin{aligned} A &\rightarrow A + d\phi \ , \\ \omega &\rightarrow \omega + dA_2 \ , \\ A_2 &= \phi\wedge J \ . \end{aligned} \tag{7.9.0}$$

The transformation formula is induced by the transformation formula for $U(1)$ bundle. Somewhat mysteriously, there is no need to define anything in the intersections of U_α in the recent case.

The connection form of the 2-gerbe can be regarded as a second $\wedge d$ power of Kähler connection:

$$A_3 \equiv A\wedge dA \ . \tag{7.9.1}$$

The generalization of this observation allows to develop a different view about n-gerbes generated as $\wedge d$ products of 0-gerbes.

The hierarchy of gerbes generated by 0-gerbes

Consider a collection of $U(1)$ connections $A^{i)}$. They generate entire hierarchy of gerbe-connections via the $\wedge d$ product

$$A_3 = A^{1)}\wedge dA^{2)} \tag{7.9.2}$$

defining 2-gerbe having a closed curvature 4-form

$$F_4 = dA^{1)}\wedge dA^{2)} \ . \tag{7.9.3}$$

$\wedge d$ product is commutative apart from a gauge transformation and the curvature forms of $A^{1)} \wedge dA^{2)}$ and $A^{2)} \wedge dA^{1)}$ are the same.

Quite generally, the connections A_m of $m-1$ gerbe and A_n of $n-1$-gerbe define $m+n+1$ connection form and the closed curvature form of $m+n$-gerbe as

$$\begin{aligned} A_{m+n+1} &= A_m^{1)} \wedge dA_n^{2)} \ , \\ F_{m+n+2} &= dA_m^{1)} \wedge dA_n^{2)} \ . \end{aligned} \tag{7.9.3}$$

The sequence of gerbes extends up to $n = D-2$, where D is the dimension of the underlying manifold. These gerbes are not the most general ones since one starts from 0-gerbes. One can of course start from $n > 0$-gerbes too.

The generalization of the $\wedge d$ product to the non-Abelian situation is not obvious. The problems stem from the that the Lie-algebra valued connection forms $A^{1)}$ and $A^{2)}$ appearing in the covariant version $D = d + A$ do not commute.

7.9.3 How to understand the replacement of 3-cycles with n-cycles?

If n-gerbes are generated from 0-gerbes it is possible to understand how the intersections of the open sets emerge. Consider the product of 0-gerbes as the simplest possible case. The crucial observation is that the coverings U_α for $A^{1)}$ and V_β for $A^{2)}$ need not be same (for CP_2 this was the case). One can form a new covering consisting of sets $U_\alpha \cap V_{\alpha_1}$. Just by increasing the index range one can replace V with U and one has covering by $U_\alpha \cap U_{\alpha_1} \equiv U_{\alpha\alpha_1}$.

The transition functions are defined in the intersections $U_{\alpha\alpha_1} \cap U_{\beta\beta_1} \equiv U_{\alpha\alpha_1\beta\beta_1}$ and cocycle conditions must be formulated using instead of intersections $U_{\alpha\beta\gamma}$ the intersections $U_{\alpha\alpha_1\beta\beta_1\gamma\gamma_1}$. Hence the transition functions can be written as $g_{\alpha\alpha_1\beta\beta_1}$ and the 3-cocycle are replaced with 5-cocycle conditions since the minimal co-cycle corresponds to a sequence of 6 steps instead of 4:

$$U_{\alpha\alpha_1\beta\beta_1} \rightarrow U_{\alpha_1\beta\beta_1\gamma} \rightarrow U_{\beta\beta_1\gamma\gamma_1} \rightarrow U_{\beta_1\gamma\gamma_1\alpha} \rightarrow U_{\gamma\gamma_1\alpha\alpha_1} \ .$$

The emergence of higher co-cycles is thus forced by the modification of the bundle covering necessary when gerbe is formed as a product of lower gerbes. The conjecture is that any even gerbe is expressible as a product of 0-gerbes.

An interesting application of the product structure is at the level of configuration space of 3-surfaces ("world of classical worlds"). The Kähler form of the configuration space defines a connection 1-form and this generates infinite hierarchy of connection $2n+1$-forms associated with $2n$-gerbes.

7.9.4 Gerbes as graded-commutative algebra: can one express all gerbes as products of -1 and 0-gerbes?

If one starts from, say 1-gerbes, the previous argument providing a geometric understanding of gerbes is not applicable as such. One might however hope that it is possible to represent the connection 2-form of any 1-gerbe as a $\wedge d$ product of a connection 0-form ϕ of "-1"-gerbe and connection 1-form A of 0-gerbe:

$$A_2 = \phi dA \equiv A \wedge d\phi \ ,$$

with different coverings for ϕ and A. The interpretation as an obstruction for the modification of the underlying bundle structure is consistent with this interpretation.

The notion of -1-gerbe is not well-defined unless one can define the notion of -1 form precisely. The simplest possibility that 0-form transforms trivially in the change of patch is not consistent. One could identify contravariant n-tensors as $-n$-forms and d for them as divergence and d^2 as the antisymmetrized double divergence giving zero. ϕ would change in a gauge transformation by a divergence of a vector field. The integral of a divergence over closed M vanishes identically so that if the integral of ϕ over M is non-vanishing it corresponds to a non-trivial 0-connection. This interpretation of course requires the introduction of metric.

The requirement that the minimal intersections of the patches for 1-gerbes are of form $U_{\alpha\beta\gamma}$ would be achieved if the intersections patches can be restricted to the intersections $U_{\alpha\beta\gamma}$ defined by $U_\alpha \cap V_\gamma$

and $U_\beta \cap V_\gamma$ (instead of $U_\beta \cap V_\delta$), where the patches V_γ would be most naturally associated with -1-gerbe. It is not clear why one could make this restriction. The general conjecture is that any gerbe decomposes into a multiple $\wedge d$ product of -1 and 0-gerbes just like integers decompose into primes. The $\wedge d$ product of two odd gerbes is anti-commutative so that there is also an analogy with the decomposition of the physical state into fermions and bosons, and gerbes for a graded-commutative super-algebra generalizing the Grassmann algebra of manifold to a Grassmann algebra of gerbe structures for manifold.

7.9.5 The physical interpretation of 2-gerbes in TGD framework

2-gerbes could provide some insight to how to characterize the topological structure of the many-sheeted space-time.

1. The cohomology group H^4 is obviously crucial in characterizing 2-gerbe. In TGD framework many-sheetedness means that different space-time sheets with induced metric having Minkowski signature are separated by elementary particle horizons which are light like 3-surfaces at which the induced metric becomes degenerate. Also the time orientation of the space-time sheet can change at these surfaces since the determinant of the induced metric vanishes.

   This justifies the term elementary particle horizon and also the idea that one should treat different space-time sheets as generating independent direct summands in the homology group of the space-time surface: as if the space-time sheets not connected by join along boundaries bonds were disjoint. Thus the homology group H^4 and 2-gerbes defining instanton numbers would become important topological characteristics of the many-sheeted space-time.

2. The asymptotic behavior of the general solutions of field equations can be classified by the dimension D of the CP_2 projection of the space-time sheet. For $D = 4$ the instanton density defining the curvature form of 2-gerbe is non-vanishing and instanton number defines a topological charge. Also the values of the Chern-Simons invariants associated with the boundary components of the space-time sheet define topological quantum numbers characterizing the space-time sheet and their sum equals to the instanton charge. CP_2 type extremals represent a basic example of this kind of situation. From the physical view point $D = 4$ asymptotic solutions correspond to what might be regarded chaotic phase for the flow lines of the Kähler magnetic field. Kähler current vanishes so that empty space Maxwell's equations are satisfied.

3. For $D = 3$ situation is more subtle when boundaries are present so that the higher-dimensional analog of Aharonov-Bohm effect becomes possible. In this case instanton density vanishes but the Chern-Simons invariants associated with the boundary components can be non-vanishing. Their sum obviously vanishes. The space-time sheet can be said to be a neutral C-S multipole. Separate space-time sheets can become connected by join along boundaries bonds in a quantum jump replacing a space-time surface with a new one. This means that the cohomology group H^4 as well as instanton charges and C-S charges of the system change.

Concerning the asymptotic dynamics of the Kähler magnetic field, $D = 3$ phase corresponds to an extremely complex but highly organized phase serving as an excellent candidate for the modelling of living matter. Both the TGD based description of anyons and quantum Hall effect and the model for topological quantum computation based on the braiding of magnetic flux tubes rely heavily on the properties $D = 3$ phase [E9].

The non-vanishing of the C-S form implies that the flow lines of the Kähler magnetic are highly entangled and have as an analog mixing hydrodynamical flow. In particular, one cannot define non-trivial order parameters, say phase factors, which would be constant along the lines. The interpretation in terms of broken super-conductivity suggests itself. Kähler current can be non-vanishing so that there is no counterpart for this phase at the level of Maxwell's equations.

7.10 Appendix: Category theory and construction of S-matrix

The construction of configuration space geometry, spinor structure and of S-matrix involve difficult technical and conceptual problems and category theory might be of help here. As already found, the application of category theory to the construction of configuration space geometry allows to understand how the arrow of psychological time emerges.

The construction of the S-matrix involves several difficult conceptual and technical problems in which category theory might help. The incoming states of the theory are what might be called free states and are constructed as products of the configuration space spinor fields. One can effectively regard them as being defined in the Cartesian power of the configuration space divided by an appropriate permutation group. Interacting states in turn are defined in the configuration space.

Cartesian power of the configuration space of 3-surfaces is however in geometrical sense more or less identical with the configuration space since the disjoint union of N 3-surfaces is itself a 3-surface in configuration space. Actually it differs from configuration space itself only in that the 3-surfaces of many particle state can intersect each other and if one allows this, one has paradoxical self-referential identification $CH = \overline{CH^2}/S_2 = ... = \overline{CH^N}/S_N...$, where over-line signifies that intersecting 3-surfaces have been dropped from the product.

Note that arbitrarily small deformation can remove the intersections between 3-surfaces and four-dimensional general coordinate invariance allows always to use non-intersecting representatives. In case of the spinor structure of the Cartesian power this identification means that the tensor powers SCH^N of the configuration space spinor structure are in some sense identical with the spinor structure SCH of the configuration space. Certainly the oscillator operators of the tensor factors must be assumed to be mutually anti-commuting.

The identities $CH = \overline{CH^2}/S_2 = ..$ and corresponding identities $SCH = SCH^2 = ...$ for the space SCH of configuration space spinor fields might imply very deep constraints on S-matrix. What comes into mind are counterparts for the Schwinger-Dyson equations of perturbative quantum field theory providing defining equations for the n-point functions of the theory [27]. The isomorphism between SCH^2 and SCH is actually what is needed to calculate the S-matrix elements. Category theory might help to understand at a general level what these self-referential and somewhat paradoxical looking identities really imply and perhaps even develop TGD counterparts of Schwinger-Dyson equations.

There is also the issue of bound states. The interacting states contain also bound states not belonging to the space of free states and category theory might help also here. It would seem that the state space must be constructed by taking into account also the bound states as additional 'free' states in the decomposition of states to product states.

A category naturally involved with the construction of the S-matrix (or U-matrix) is the space of the absolute minima $X^4(X^3)$ of the Kähler action which might be called interacting category. The canonical transformations acting as isometries of the configuration space geometry act naturally as the morphisms of this category. The group $Diff^4$ of general coordinate transformations in turn acts as gauge symmetries.

S-matrix relates free and interacting states and is induced by the classical interactions induced by the absolute minimization of Kähler action. S-matrix elements are essentially Glebch-Gordan coefficients relating the states in the tensor power of the interacting supercanonical representation with the interacting supercanonical representation itself. More concretely, N-particle free states can be seen as configuration space spinor fields in CH^N obtained as tensor products of ordinary CH spinor fields. Free states correspond classically to the unions of space-time surfaces associated with the 3-surfaces representing incoming particles whereas interacting states correspond classically to the space-time surfaces associated with the unions of the 3-surfaces defining incoming states. These two states define what might be called free and interacting categories with canonical transformations acting as morphisms.

The classical interaction is represented by a functor $S : \overline{CH^N}/S_N \to CH$ mapping the classical free many particle states, that is objects of the product category defined by $\overline{CH^N}/S_N$ to the interacting category CH. This functor assigns to the union $\cup_i X^4(X_i^3)$ of the absolute minima $X^4(X_i^3)$ of Kähler action associated with the incoming, free states X_i^3 the absolute minimum $X^4(\cup X_i^3)$ associated with the union of three-surfaces representing the outgoing interacting state. At quantum level this functor maps the state space SCH^N associated with $\cup_i X^4(X_i^3)$ to SCH in a unitary manner. An important constraint on S-matrix is that it acts effectively as a flow in zero modes correlating the quantum numbers in fiber degrees of freedom in one-to-one manner with the values of zero modes so that quantum jump $U\Psi_i \to \Psi_0...$ gives rise to a quantum measurement.

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Chapter 8

Riemann Hypothesis and Physics

8.1 Introduction

Riemann hypothesis states that the nontrivial zeros of Riemann Zeta function lie on the axis $x = 1/2$. Since Riemann zeta function allows interpretation as a thermodynamical partition function for a quantum field theoretical system consisting of bosons labelled by primes, it is interesting to look Riemann hypothesis from the perspective of physics. Quantum TGD and also TGD inspired theory of consciousness provide additional view points to the hypothesis and suggests sharpening of Riemann hypothesis, detailed strategies of proof of the sharpened hypothesis, and heuristic arguments for why the hypothesis is true.

The idea that the evolution of cognition involves the increase of the dimensions of finite-dimensional extensions of p-adic numbers associated with p-adic space-time sheets emerges naturally in TGD inspired theory of consciousness. A further input that led to a connection with Riemann Zeta was the work of Hardmuth Mueller [48] suggesting strongly that e and its $p-1$ powers at least should belong to the extensions of p-adics. The basic objects in Mueller's approach are so called logarithmic waves $exp(iklog(u))$ which should exist for $u = n$ for a suitable choice of the scaling momenta k.

Logarithmic waves appear also as the basic building blocks (the terms $n^s = exp(log(n)(Re[s] + iIm[s]))$ in Riemann Zeta. This inspires naturally the hypothesis that also Riemann Zeta function is universal in the sense that it is defined at is zeros $s = 1/2 + iy$ not only for complex numbers but also for all p-adic number fields provided that an appropriate finite-dimensional extensions involving also transcendentals are allowed. This allows in turn to algebraically continue Zeta to any number field. The zeros of Riemann zeta are determined by number theoretical quantization and are thus universal and should appear in the physics of critical systems. The hypothesis $log(p) = \frac{q_1(p)exp[q_2(p)]}{\pi}$ explains the length scale hierarchies based on powers of e, primes p and Golden Mean.

Mueller's logarithmic waves lead also to an elegant concretization of the Hilbert Polya conjecture and to a sharpened form of Riemann hypothesis: the phases q^{-iy} for the zeros of Riemann Zeta belong to a finite-dimensional extension of R_p for any value of primes q and p and any zero $1/2 + iy$ of ζ. The question whether the imaginary parts of the Riemann Zeta are linearly independent (as assumed in the previous work) or not is of crucial physical significance. Linear independence implies that the spectrum of the super-canonical weights is essentially an infinite-dimensional lattice. Otherwise a more complex structure results. The numerical evidence supporting the translational invariance of the correlations for the spectrum of zeros together with p-adic considerations leads to the working hypothesis that for any prime p one can express the spectrum of zeros as the product of a subset of Pythagorean phases and of a fixed subset U of roots of unity. The spectrum of zeros could be expressed as a union over the translates of the same basic spectrum defined by the roots of unity translated by the phase angles associated with a subset of Pythagorean phases: this is consistent with what the spectral correlations strongly suggest. That decompositions defined by different primes p yield the same spectrum would mean a powerful number theoretical symmetry realizing p-adicities at the level of the spectrum of Zeta.

A second strategy is based on, what I call, Universality Principle. The function, that I refer to as $\hat{\zeta}$, is defined by the product formula for ζ and exists in the infinite-dimensional algebraic extension Q_∞ of rationals containing all roots of primes. $\hat{\zeta}$ is defined for all values of s for which the partition functions $1/(1-p^{-z})$ appearing in the product formula have value in Q_∞. Universality Principle states that $|\hat{\zeta}|^2$, defined as the product of the p-adic norms of $|\hat{\zeta}|^2$ by reversing the order of producting in the adelic formula, equals to $|\zeta|^2$ and, being an infinite dimensional vector in Q_∞, vanishes only if it contains a rational factor which vanishes. This factor is present only provided an infinite number of partition functions appearing in the product formula of $\hat{\zeta}$ have rational valued norm squared: this locates the plausible candidates for the zeros on the lines $Re[s] = n/2$.

Universality Principle implies the following stronger variant about sharpened form of the Riemann hypothesis: the real part of the phase p^{-iy} is rational for an infinite number of primes for zeros of ζ. Universality Principle, even if proven, does not however yield a proof of the Riemann hypothesis. The failure of the Riemann hypothesis becomes however extremely implausible. An important outcome of this approach is the realization that super-conformal invariance is a natural symmetry associated with ζ (not surprisingly, since the symmetry group of complex analysis is in question!).

Super-conformal invariance inspires a strategy for proving the Riemann hypothesis. The vanishing of the Riemann Zeta reduces to an orthogonality condition for the eigenfunctions of a non-Hermitian operator D^+ having the zeros of Riemann Zeta as its eigenvalues. The construction of D^+ is inspired by the conviction that Riemann Zeta is associated with a physical system allowing super-conformal transformations as its symmetries and second quantization in terms of the representations of the super-conformal algebra. The eigenfunctions of D^+ are analogous to coherent states of a harmonic oscillator and in general they are not orthogonal to each other. The states orthogonal to a vacuum state (having a negative norm squared) correspond to the zeros of Riemann Zeta. The physical states having a positive norm squared correspond to the zeros of Riemann Zeta at the critical line. Riemann hypothesis follows both from the hermiticity and positive definiteness of the metric in the space of states corresponding to the zeros of ζ. Also conformal symmetry in appropriate sense implies Riemann hypothesis and after one year from the discovery of the basic idea it became clear that one can actually construct a rigorous twenty line long analytic proof for the Riemann hypothesis using a standard argument from Lie group theory.

These approaches concretize the vision about TGD based physics as a generalized number theory. Two new realizations of the super-conformal algebra result and the second realization has direct application to the modelling of $1/f$ noise. The zeros of ζ code for the states of an arithmetic quantum field theory coded also by infinite primes: also the hierarchical structure of the many-sheeted space-time is coded. Even some basic quantum numbers of particles of TGD Universe might have number theoretical representation.

8.2 General vision

Quantum TGD has inspired several strategies of proof of the Riemann hypothesis. The first strategy is based on the modification of Hilbert Polya hypothesis by requiring that the physical system in question has super-conformal transformations as its symmetries. Second strategy is based on considerations based on TGD inspired quantum theory of cognition and a generalization of the number concept inspired by it. Together with some physical inputs one ends up to a hypothesis that Riemann Zeta is well defined in all number fields near its zeros provided finite-dimensional extensions of p-adic numbers are allowed. This hypothesis generalizes the earlier hypothesis assuming that the extensions are trivial or at most algebraic. Third strategy is based on, what I call, Universality Principle.

There are also strong physical motivations to say something explicit about the spectrum of zeros and here p-adicization program inspires the hypothesis the numbers q^{iy}, q prime, belong to a finite algebraic extension of p-adic number field R_p for every prime p. The findings about the correlations of the spectrum of zeros inspire very concrete hypothesis about the spectrum of zeros as a union of translates of the same basic spectrum and this hypothesis is supported by the physical identification of the zeros of Zeta as super-canonical conformal weights.

8.2.1 Generalization of the number concept and Riemann hypothesis

The hypothesis about p-adic physics as physics of cognition leads to a generalization of the notion of number obtained by gluing reals and various p-adic number fields together along rational numbers common to all of them. This structure is visualizable as a book like structure with pages represented by the number fields and the rim of the book represented by rationals. Even this structure can be generalized by allowing all finite-dimensional extensions of p-adic numbers including also those containing transcendental numbers and performing similar identification. Kind of fractal book might serve as a visualization of this structure.

In TGD inspired theory of consciousness intentions are assumed to correspond to quantum jumps involving the transformation of p-adic space-time sheets to real ones. An intuitive expectation is p-adic and real space-time sheets to each other must have a maximum number of common rational points. The building of idealized model for this transformation leads to the problem of defining functions having Taylor series with rational coefficients and continuable to both real and p-adic functions from a subset of rational numbers (or points of space-time sheet with rational coordinates). In this manner one ends up with the hypothesis that p-adic space-time sheets correspond to finite-dimensional extensions of p-adic numbers, which can involve also transcendental numbers such as e. This leads to a series of number theoretic conjectures.

The idea that the evolution of cognition involves the increase of the dimensions of finite-dimensional extensions of p-adic numbers associated with p-adic space-time sheets emerges naturally in TGD inspired theory of consciousness. A further input that led to a connection with Riemann Zeta was the work of Hardmuth Mueller [48] suggesting strongly that e and its $p-1$ powers at least should belong to extensions of p-adics. The basic objects in Mueller's approach are so called logarithmic waves $exp(iklog(u))$ which should exist for $u=n$ for a suitable choice of the scaling momenta k.

Logarithmic waves appear also as the basic building blocks (the terms $n^s = exp(log(n)(Re[s] + iIm[s]))$ in Riemann Zeta. This inspires naturally the hypothesis that also Riemann Zeta function is universal in the sense that it is defined at is zeros $s = 1/2 + iy$ not only for complex numbers but also for all p-adic number fields provided that an appropriate finite-dimensional extensions involving also transcendentals are allowed. This allows in turn to algebraically continue Zeta to any number field. The zeros of Riemann zeta are determined by number theoretical quantization and are thus universal and should appear in the physics of critical systems. A hierarchy of number theoretical conjectures stating that a finite number of iterated logarithms about transcendentals appearing in the extension forms a closed system under the operation of taking logarithms. Mueller's logarithmic waves lead also to an elegant concretization of the Hilbert Polya conjecture and to a sharpened form of Riemann hypothesis: the complex numbers p^{-iy} for the zeros of Riemann Zeta belong to a finite-dimensional extension of R_p for any value of p and any zero $1/2 + iy$ of ζ.

8.2.2 Modified form of Hilbert-Polya hypothesis

Super-conformal invariance inspires a strategy for proving (not a proof of, as was the first over-optimistic belief) the Riemann hypothesis. The vanishing of Riemann Zeta reduces to an orthogonality condition for the eigenfunctions of a non-Hermitian operator D^+ having the zeros of Riemann Zeta as its eigenvalues. The construction of D^+ is inspired by the conviction that Riemann Zeta is associated with a physical system allowing super-conformal transformations as its symmetries and second quantization in terms of the representations of super-conformal algebra. The eigenfunctions of D^+ are analogous to the so called coherent states and in general not orthogonal to each other. The states orthogonal to a vacuum state (having a negative norm squared) correspond to the zeros of Riemann Zeta. The physical states having a positive norm squared correspond to the zeros of Riemann Zeta at the critical line and possibly those having $Re[s] > 1/2$.

A possible proof of the Riemann hypothesis by reductio ad absurdum results if one assumes that the states corresponding to zeros of ζ span a space with a hermitian metric. Riemann hypothesis follows both from the hermiticity and positive definiteness of the metric in the space of states corresponding to the zeros of ζ. Also conformal invariance in appropriate sense implies Riemann hypothesis. Indeed, a rather rigorous proof of Riemann hypothesis results from the observation that certain generator of conformal algebra permutes the two zeros located symmetrically with respect to the critical line. If the action of this generator exponentiates, Riemann hypothesis follows since exponentiation would imply the existence of infinite number of zeros along a line parallel to $Re[s]$-axis. One can formulate this argument rigorously using first order differential equation, and if one forgets all the preceiding refined philosophical arguments, one can prove Riemann hypothesis using twenty line long analytic argument! Perhaps Ramajunan could have made this!

As already noticed, the state space metric can be made positive definite provided Riemann hypothesis holds true. Thus the system in question might quite well serve as a concrete physical model for quantum critical systems possessing super-conformal invariance as both dynamical and gauge symmetry.

8.2.3 Universality Principle

The function, what I call $\hat{\zeta}$, is defined by the product formula for ζ and exists in the infinite-dimensional algebraic extension of rationals containing all roots of primes. $\hat{\zeta}$ is defined for all values of s for which the partition functions $1/(1-p^{-s})$ appearing in the product formula have value in the algebraic extension. Universality Principle states that $|\hat{\zeta}|^2$, defined as the product of the p-adic norms of $|\hat{\zeta}|^2$ by reversing the order of producting in the adelic formula, equals to $|\zeta|^2$ and, being an infinite dimensional vector in the algebraic extension of the rationals, vanishes only if it contains a rational factor which vanishes. This factor is present only provided an infinite number of partition functions appearing in the product formula of $\hat{\zeta}$ have rational valued norm squared: this locates the plausible candidates for the zeros on the lines $Re[s]=n/2$.

Universality Principle generalizes the original sharpened form of the Riemann hypothesis: the real parts of the phases p^{-iy} are rational. Universality Principle, even if proven, does not however yield a proof of the Riemann hypothesis. The failure of Riemann hypothesis becomes however extremely implausible and one could consider the possibility of regarding Riemann Hypothesis as an axiom. An important outcome of this approach is the realization that super-conformal invariance is a natural symmetry associated with Riemann Zeta (not surprisingly, since the symmetry group of complex analysis is in question!).

8.2.4 Physics, Zetas, and Riemann Zeta

Although the original naive speculations are probably not correct, the work with Riemann Zeta led to several new mathematical concepts and rather concrete ideas about how physics in TGD Universe might reduce to generalized number theory.

Do M- and U-matrices exist in all number fields simultaneously?

TGD predicts two kinds of fundamental matrices [C1, C2]. S-matrix of particle physics is replaced with M-matrix defining time-like entanglement coefficients between positive and negative energy parts of zero energy states (all conserved quantum numbers vanish for these states so that they are creatable from vacuum). M-matrix equals to the product of a square root of density matrix and unitary matrix and cannot have elements between different number fields. U-matrix characterizes the unitary process associated with quantum jump between zero energy states. Therefore U can have elements also between different number fields and should be number theoretically universal. U-matrix would describe quantum jumps describing a transformation of intention to action for instance, or transformation of zero energy state to pure cognition.

One must consider the possibility that M-matrix can be constructed independently in all number fields. On the other hand, the assumption M-matrix is continuable from a matrix whose elements are algebraic numbers is however very attractive (ordinary S-matrix has 3-momenta of particles as continuous indices). One must of course be cautious in order to avoid the situation in which the theory effectively reduces to that in the field of algebraic numbers. To achieve this pit-hole one must understand how real and p-adic physics differ from each other. p-Adic variants of light-like 3-surfaces can obey same algebraic equations as their real counterparts. Real 4-D space-time sheets serving as classical correlates of classical degrees of freedom in quantum measurement theory however obey genuine field equations and it is not at all whether their solutions allow an algebraic continuation to the p-adic context. Since it is not possible to measure cognition, one might argue that p-adic space-time sheets are not needed at all.

Both U- and S-matrices could exist in a well-defined sense simultaneously in all number fields provided finite-dimensional extensions of p-adic numbers are allowed. It is also natural to expect that the structure of the these matrices reflects the evolution of cognition as a gradual increase of the p-adic prime characterizing the space-time sheet and of the dimension of the algebraic extension involved. These matrices should have a hierarchical decomposition into increasingly complex S- and U-matrices using direct sum and direct product. One might even hope of identifying universal elementary S and U-matrices serving as basic building blocks in this construction so that a number-theoretical bootstrap might make sense.

Do conformal weights of the generators of super-canonical algebra correspond to zeros of some zeta function?

For long time the zeros of Riemann Zeta remained excellent candidates for the conformal weights labelling the generators of super-canonical algebra [B2, B3, A6]. The basic motivation was that the radial conformal weights have very naturally real part which equals to -1/2 as does also the negative of the real part of complex zeros of Riemann Zeta. Also other conformal weights are possible but not so natural.

1. Why Riemann Zeta does not work

The following observations have however changed the situation.

1. The almost defining property of zeta functions is that their complex zeros reside at the critical line. There exists a lot of zeta functions [E3] so that the spectrum of super-canonical conformal weights allows to consider also other zetas.

2. The zeta functions analogous to the basic building blocks of Riemann Zeta labelled by prime p are especially natural from the point of view of p-adic length scale hypothesis and they have automatically the nice algebraic properties required by the number theoretic universality whereas in the case of Riemann Zeta they must be conjectured.

3. The generalized eigenvalues of the modified Dirac operator define in a very natural manner zeta functions coding geometric information about partonic 2-surfaces whereas Riemann Zeta has no obvious interpretation of this kind.

These findings do not of course exclude Riemann zeta or zetas analogous to it. For instance, one can assign Riemann Zeta to the purely bosonic infinite primes very naturally. The spectrum of the scaling generator L_0 consists of non-negative integers and the positive part of spectrum defines a zeta function of form $\sum_{n>0} g(n)n^{-s}$, which might be relevant for quantum TGD. I do not known about the zeros of this zeta function.

A further natural speculation was that the zeros of polyzetas $\zeta(z_1, ..., z_K)$ label the super-canonical conformal weights of K-particle bound states. The vanishing of loop corrections could be understood as being due to the fact that they are proportional to polyzetas having super-canonical conformal weights as arguments. This speculation was inspired by the fact that polyzetas with integer arguments emerge in loop corrections of quantum field theories.

2. Zeta functions assignable to the modified Dirac operator

In the case of the modified Dirac operator and super-canonical conformal weights Riemann Zeta is naturally replaced by a zeta function determined by purely physical considerations (detailed argument can be found in [A6, C1]).

1. The determinant of the modified Dirac operator D gives rise to the vacuum functional of TGD and the conjecture is that it reduces to a product of exponents of Kähler function and Chern-Simons action. The construction assigns to a given 3-D light-like surface X_l^3 a 4-D space-time sheet conjectured to be a preferred extremal of Kähler action [A6].

2. The generalized eigenvalue λ of D is actually a scalar field depending on the coordinates of partonic 2-surface X^2 (and light-like 3-surface X_l^3). λ codes purely geometric information about the light-like 3-surface, and Higgs vacuum expectation is naturally proportional to λ.

3. The minima of the modulus of the holomorphic function λ in X^2 give rise to what I call number theoretic braids. Dirac determinant is product of the eigenvalues at the minima of $|\lambda|$ interpreted as a function X_l^3.

4. One can assign to the values of λ at the points of the number theoretic braid also zeta function, call it ζ. ζ codes geometric information about 3-surface and super-canonical conformal weights correspond naturally to its zeros. ζ is sum over a finite number of terms only, and if it is rational function of a suitable coordinate, it has all the required number theoretic properties whereas in the case of Riemann Zeta these properties require strong number theoretic conjectures.

The notion of polyzeta might generalize in a natural manner to a dynamical polyzeta. Suppose that one has a collection X_i^2 of partonic 2-surfaces assignable to a connected space-like 3-surface defined by the intersection $X^3 = X^4 \cap \delta M_+^4 \times CP_2$. In this kind of situation one might hope that the notion of polyzeta generalizes and can be defined in terms of the generalized eigenvalues of the modified Dirac operator assigned with various partonic 2-surfaces X_i^2. If X^3 is connected, the polyzeta cannot be a mere product of independent zetas associated with X_i^2 obtained by assigning separate space-time sheets to the light-like orbits of X_i^2. Even if it reduces to a product, the eigenvalues assignable to X_i^2 are correlated by the constraint that the minimization of λ_i is consistent with the condition $X_i^2 \subset X^3$. This polyzeta would naturally characterize the bound state character of the resulting state.

8.2.5 General number theoretical ideas inspired by the number theoretic vision about cognition and intentionality

The following two ideas serve as guide lines in the attempt to relate cognition, intentionality and number theory to each other so that number theory would allow to construct a more detailed view about the realization of intentionality and cognition. As a matter fact, the general ideas about intention and cognition in turn generate very general number theoretical conjectures.

1. Real and p-adic number fields form a book like structure with pages represented by number fields glued together along rationals forming the rim of the book. For the extensions of p-adic numbers further common points result and the book becomes fractal if all possible extensions are allowed. This picture generalizes to the level of the imbedding space and allows to see space-time surfaces as consisting of real and p-adic space-time sheets belonging to various extensions of these numbers. This generalized view about numbers gives hopes about an unambigious definition of what some number, say e, appearing in an extension of p-adic numbers really means.

2. The first new idea is roughly that the discovery of notion of any algebraic or transcendental number x (such as Φ or e) involves a quantum jump in which there is generated a p-adic space-time sheet for which the existing finite-dimensional extension of p-adic numbers is replaced by a finite-dimensional extension involving also x. Also some higher powers of the number are involved. For instance, for e $p-1$ powers are necessarily needed (e^p exists p-adically).

3. The p-adic-to-real transition serving as a correlate for the transformation of intention to action is most probable if the number of common rational valued points for the p-adic and real space-time sheet is high. The requirement of real and p-adic continuity and even smoothness however forces upper and lower p-adic length scale cutoffs so that common points are in certain length scale range.

4. The points of M_+^4 with integer valued Minkowski coordinates using CP_2 length related fundamental length scale as a basic unit is a good guess for the subset of M_+^4 defining the rational points of the M_+^4 involved. CP_2 coordinates as functions of M_+^4 coordinates should be rational or belong to some finite-dimensional extension of p-adics. Of course, also rational points of M_+^4 are possible, and the evolution of cognition should correspond to the increase of the algebraic dimension of the extension.

5. A very powerful hypothesis is that the p-adic and real functions have the same analytic form besides coinciding at the chosen rational points defining the p-adic pseudo constant involved. Since the pseudo constant defines the corresponding real function in rational points, there are indeed good hopes that the transformation of p-adic intention to real action is possible. This assumption favors functions which allow at some point (most naturally origin) a Taylor series with rational valued Taylor coefficients.

Is e an exceptional transcendental?

Neper number is obviously the simplest one and only the powers e^k, $k = 1, ..., p-1$ of e are needed to define p-adic counterpart of e^x for $x = n$. In case of trigonometric functions deriving from e^{ix}, also e^i and its $p-1$ powers must belong to the extension.

An interesting question is whether e is a number theoretically exceptional transcendental or whether it could be easy to find also other transcendentals defining finite-dimensional extensions of p-adic numbers.

1. Consider functions $f(x)$, which are analytic functions with rational Taylor coefficients, when expanded around origin for $x > 0$. The values of $f(n)$, $n = 1, ..., p-1$ should belong to an extension, which should be finite-dimensional.

2. The expansion of these functions to Taylor series generalizes to the p-adic context if also the higher derivatives of f at $x = n$ belong to the extension. This is achieved if the higher derivatives are expressible in terms of the lower derivatives using rational coefficients and rational functions or functions, which are defined at integer points (such as exponential and logarithm) by construction. A differential equation of some finite order involving only rational functions with rational coefficients must therefore be satisfied (e^x satisfying the differential equation $df/dx = f$ is the optimal case in this sense). The higher derivatives could also reduce to rational functions at some step ($log(x)$ satisfying the differential equation $df/dx = 1/x$).

3. The differential equation allows to develop $f(x)$ in power series, say in origin

   $$f(x) = \sum f_n \frac{x^n}{n!}$$

   such that f_{n+m} is expressible as a rational function of the m lower derivatives and is therefore a rational number.

   The series converges when the p-adic norm of x satisfies $|x|_p \leq p^k$ for some k. For definiteness one can assume $k = 1$. For $x = 1, ..., p-1$ the series does not converge in this case, and one can introduce and extension containing the values $f(k)$ and hope that a finite-dimensional extension results.

Finite-dimensionality requires that the values are related to each other algebraically although they need not be algebraic numbers. This means symmetry. In the case of exponent function this relationship is exceptionally simple. The algebraic relationship reflects the fact that exponential map represents translation and exponent function is an eigen function of a translation operator. The necessary presence of symmetry might mean that the situation reduces always to either exponential action. Also the phase factors $exp(iq\pi)$ could be interpreted in terms of exponential symmetry. Hence the reason for the exceptional role of exponent function reduces to group theory.

Also other extensions than those defined by roots of e are possible. Any polynomial has n roots and for transcendental coefficients the roots define a finite-dimensional extension of rationals. It would seem that one could allow the coefficients of the polynomial to be functions in an extension of rationals by powers of a root of e and algebraic numbers so that one would obtain infinite hierarchy of transcendental extensions.

Some no-go theorems

Elementary functions like exp(x), log(1+x), cos(x), sin(x), are obviously favored by the previous considerations, in particular by the requirement of the form invariance of the function in p-adic-to-real transition. They indeed have p-adic Taylor expansion which converges for $|x|_p < 1$. The definition at integer valued points for which $x \ mod \ p = n$, $n = 0, 1, ..., p-1$, requires the introduction of an extension of p-adic numbers. The natural first guess is that this extension is finite-dimensional. Of course, this is just a hypothesis to be discussed and motivated by the idea that p-adic extensions reflect our own finite intelligence.

1. Can powers of $log(p)$ define a finite-dimensional extension of p-adics?

The number theoretical entropy associated with any p-adic prime for which the ordinary logarithm $log(p_n)$ is replaced by the logarithm of the p-adic norm of p_n, is proportional to a $log(p)$-factor. As already noticed, if bit is used as unit, then only the rationality of $log(p)/log(2)$ would be needed and $log(p)$ need not correspond to a finite-dimensional extension of p-adics. Unfortunately, also this conjecture turns out to be false.

The first observation is that $log(1+x)$, $x = O(p)$ exists as an ordinary p-adic number and the logarithm of $log(m)$, $m < p$ such that the powers of m span the numbers $1, ..., p-1$ besides $log(p)$ need be introduced to the extension in order that logarithm of any integer and in fact of any rational number exists p-adically. The problem is however that the powers of $log(m)$ and $log(p)$ might generate an infinite-dimensional extension of p-adic numbers.

First some no-go theorems inspired by wishful conjectures (professional number theorists must regard me as an idiot!).

1. $log(p) = q/t$, where t is a fixed transcendental number, say π, cannot hold true. The reason is that the rationality of $log(p_1)/log(p_2) = q_1/q_2 = r/s$ implies that $p_1^s = p_2^r$ in contradiction with the prime number property of p_1 and p_2. This excludes also the rationality of $log(q_1)/log(q_2)$. It is however possible to have *single* rational q for which say $\pi/log(q)$ is rational.

2. $log(q)$, q prime, cannot correspond to a finite dimensional extension of R_p in the sense that a finite power of $log(q)$ would be a rational number. Assume that this is the case, i.e. $(log(q))^{m_{p,q}} = x_{p,q}$, where $x_{p,q}$ is an ordinary p-adic number in R_p, and assume that e belongs to extension. For definiteness let us assume $|x_{p,q}| < 1$ and write

$$q = exp(log(q)) = \sum_n log(q)^n/n! = \sum_{k=0}^{m-1} c_k log(q)^k \quad , \quad c_k = \sum_n \frac{x_{p,q}^n}{(k+nm_{p,q})!} \quad .$$

The righthand side gives m terms corresponding to the m powers of $log(q)$ and only the lowest term can be non-vanishing and equals to q. The convergence of series requires that $x_{p,q}$ has p-adic norm smaller than one. This however implies that lowest order term has p-adic norm equal to one. For $q = p$ this leads to contradiction since one would have $p = 1 + O(p)$. For $|x_{p,q}|_p \geq 1$ the argument fails since the expansion does not make sense. For $q = exp(p^k log(q))$, k sufficiently large, the expansion exists and in this case one as $q^{p^k} = 1 + O(p)$, which for $q = p$ gives a contradiction.

3. One might hope that $log(p)$ belongs to an extension containing e or its root, or in the most general case root of a polynomial with coefficients which belongs to an extension of rationals by e and algebraic numbers. For instance, the ansatz $log(p) = e^{q_1(p)} q_2(p)$ with $q_2(p_1) \neq q_2(p_2)$ for all pairs of primes, would guarantee that logarithms belong to a finite-dimensional extension. There are no problems with the prime property as is clear from the expression

$$p_1 = p_2^{[exp(q_1(p_1)-q_1(p_2)] \times \frac{q_2(p_1)}{q_2(p_2)}} \quad .$$

From the assumption it follows that the exponent cannot reduce to a rational number.

Unfortunately the ansatz does not work! One can write

$$p_1 = exp\left(e^{q_1(p_1)} q_2(p_1)\right)$$

and for those primes p_2 whose positive power divides $q_2(p_1)$, one can expand the exponential in a converging power series in powers of a root of e, and one obtains that ordinary p-adic number is expressible as a non-trivial combination of powers of a root of e.

4. Obviously one must give up hopes for obtaining a finite-dimensional extension for the logarithms. Also the hope that $log(p)/log(2)$ is always rational guaranteing that p-adic entropy would be always rational multiple of bit must be given up. There could however exist single rational for which $log(q)/pi$ is rational. In fact, the rather speculative considerations related to Kähler couplings strength inspire the question whether the number $log[(2^{127}-1) \times 2 \times 3 \times 5 \times 7 \times 11 \times 13 \times 17 \times 19 \times 23]/\pi$ could be rational [C6]. If this conjecture were true it would fix completely the p-adic evolution of Kähler coupling strength.

3. π cannot belong to a finite-dimensional extension of p-adic numbers

A simple argument excludes the possibility that π could belong to some finite-dimensional extension $\pi = \sum c_n e_n$. If this is the case one can write $exp(ip^k\pi) = -1$ as a converging Taylor expansion in powers of p for high enough value of k, and the coefficients of all e_n expect $e_0 = 1$ must vanish. Since the terms in this series come in powers of p it is highly implausible that they could sum up to zero. In fact, even the coefficient of $e_0 = 1$ has wrong sign. By considering more general numbers $exp(iq\pi)$ one obtains that the expansion in terms of e_i equals to the expression of phase in infinite number of different algebraic extensions. Thus it seems obvious that π cannot belong to a finite extension.

Does the integration of complex rational functions lead to rationals extended by a root of e and powers of π?

These cold showers suggest that the best one might hope is that the numbers like $log(p)$ and $log(\Phi)$ could be proportional to some power π with a coefficient which belongs to a finite extension of p-adic numbers containing e. This might make it possible to continue the theory to p-adic context and also make very strong predictions.

The elementary differential and integral calculus provides important hints for as how to proceed. Derivation takes rational functions to rational functions unlike integration since the integrals of $1/x$ and and $1/(1+x^2)$ give $log(x)$ and $arctan(x)$ leading outside the realm of rational numbers. One can go to complex plane and consider the integrals of complex rational functions with complex rational coefficients and here one encounters integrals over closed curves and between two points. The rational approach is to consider rational complex plane, and first restrict to Gaussian integers which allow primes.

i) The first observation is that residy calculus for rational functions gives always integrals which are of form $2\pi iq$, q a rational number.

ii) The integral $I = \int_a^b dz/z$, $a = m_1 + in_1$, $b = m_2 + in_2$ in turn gives

$$I = log(a/b) = \frac{1}{2}\left(log(m_2^2 + n_2^2) - log(m_1^2 + n_1^2)\right)$$

$$+i(arctan(n_2/m_2) - arctan(n_1/m_1)) \ .$$

1. The strongest hypothesis would be that logarithm and arctan are also rationally proportional to π so that all integrals of this kind lead to an infinite-dimensional transcendental extension of p-adic numbers containing π. The strong hypothesis cannot be correct. Consider arcus tangent as an example. $arctan(m/n) = r\pi/s$ would imply $tan(r\pi/s) = m/n$, and this cannot hold true since it would imply that s:th powers of Gaussian integer $n+im$ would give an ordinary integer. This would be also true for Gaussian primes and the decomposition of Gaussian integers as products of Gaussian primes would become non-unique. There is this kind of uniqueness but this is due the units $exp(i\pi/4)$ and its powers. Indeed, $arctan(1) = \pi/4$ and proportional to π.

2. One can overcome this difficulty by replacing the ansatz with

$$arctan(q) = e^{q_1(q)} q_2\pi$$

such that $q_1(q)$ is non-vanishing for $q \neq \pm 1 \pm i$ corresponding to the units of Gaussian primes. This ansatz is completely analogous to the ansatz for $log(p)$. The beauty of this ansatz would be that the imaginary parts for the integral of $1/(z-z_0)$ between complex rational points would be proportional to π irrespective of whether the integration is over a closed or open curve. The real parts of complex integrals in turn would be proportional to $1/\pi$ of $log(p) \propto 1/\pi$ ansatz holds true.

The requirement that complex integrals are powers of π could also mean quantization of topology in TGD framework. For instance, the conformal equivalence classes of Riemann surfaces of genus g are represented by period integrals of 1-forms defining elements of cohomology group H^1 over the circles representing the elements of homology group H_1. Restricting the cohomology to a rational cohomology, the periods with standard normalization would be quantized to complex rationals multiplied by a power of π. For surfaces characterized by a given power of π one might perhaps perform the p-adicization finite-dimensionally by suitable normalizations by powers of π.

Why should one have $p = q_1 exp(q_2)/\pi$?

There are good physical arguments suggesting that $log(p)$ should be proportional to $1/\pi$.

1. π appears naturally in the plane wave solutions of field equations $exp(in\pi u)$, $u = x/L)$. These phases are well defined in a finite-dimensional algebraic extension if x/L is rational. One can however consider also logarithmic plane waves

   $$exp(iku), \quad u = log(x/L) \ ,$$

   and ask under what conditions they are well defined and in particular, under what conditions the real/imaginary parts of these plane waves can have zeros at $u = e^n$ required by Mueller's hypothesis [48]. Mueller's hypothesis implies that $exp(ikn)$ has zeros so that $k = q\pi$ must hold true. Thus one obtains essentially ordinary plane waves.

   If one has $u = q_1 e^n$, q_1 rational, one obtains also the exponential $exp(iq\pi log(q_1))$. From the point of view of p-adicization program it would be very nice if also this exponent would exist p-adically. This is guaranteed if one has

   $$log(p) = \frac{q_1(p)exp\left[q_2(p)\right]}{\pi}$$

   for every prime p. One can write

   $$exp(iq\pi u) = exp\left[iqq_1(p)exp(q_2(p))\right] \ .$$

   The exponential exists for those primes p_1 for which the exponent is divisible by a positive power of p_1. This means quantization conditions favoring selected primes p_1 or alternatively scaling momenta q. An easy manner to satisfy these conditions is to assume that q is a multiple of a power of p.

2. Besides Mueller's hierarchy in powers of e there are also p-adic hierarchies and the hierarchies associated with Golden Mean and one can look whether these hierarchies are obtained for suitable logarithmic waves. For $u = x/L = mp^n$ the scaling wave reads

   $$exp(iku) = exp\left[iknlog(p)\right] exp\left[iklog(m)\right] \ .$$

   For $log(p) = q_1(p)exp\left[q_2(p)\right]/\pi$ the existence of nodes for the the first factor requires $k = q\pi^2 exp\left[-q_2(p)\right]$. The second factor exists only for $m = 1$ so that nodes are possible only at $u = p^n$.

   Note that $k = q\pi$ for e so that these length scale hierarchies are distinguishable number theoretically. This assumption implies that also the second exponential of product can exist in a finite-dimensional algebraic extension and can have even nodes. For the hierarchy defined by powers of Golden Mean the assumption $log(\Phi) = q_1 q exp(q_2)/\pi$ would lead to similar conclusions. Again one must leave door open for more general power of π.

p-Adicization of vacuum functional of TGD and infinite primes

A further input comes from TGD. The basic challenge is to continue the exponent $exp(K)$ of the Kähler function to p-adic number fields. K can be expressed as

$$K = \frac{S_K}{16\pi\alpha_K} \ ,$$

where α_K is so called Kähler coupling strength and $S_K = \int J_{\mu\nu}J^{\mu\nu}\sqrt{g}d^4x$ is Kähler action, which is essentially the Maxwell action for the induced Kähler form. The dream is that an algebraic continuation from the extensions of rational numbers defining finite extensions of p-adic numbers allows to define the theory in various number fields. The fulfillment of this dream requires that physically important quantities such as the exponent of Kähler function for CP_2 extremal and other fundamental extremals exist in a finite-dimensional extension of p-adic numbers.

1. What is the value of Kähler coupling strength?

The value of Kähler coupling strength is analogous to a critical temperature and can have only discrete values.

1. The discrete p-adic evolution of the Kähler coupling strength follows from the requirement that gravitational coupling constant is renormalization group invariant (see the chapter "Fusion of p-Adic and Real Variants of Quantum TGD to a More General Theory").

   When combined with the requirement that the exponent of CP_2 action is a power of prime, the argument would give

$$\frac{1}{\alpha_K(p)} = \frac{4}{\pi}log(K^2) \ \ , \ \ K^2 = \prod_{q=2,3,...23} q \times p$$

   with $\alpha_K(p = M_{127}) \simeq 136.5585$ and $\alpha/\alpha_K \simeq .9965$. Note that M_{127} corresponds to electron length scale. If the action is a rational fraction of CP_2 action, and the extension of p-adic numbers is by an appropriate root of p is enough to guarantee the existence of the Kähler function.

2. One can consider also an alternative ansatz based on the requirement that Kähler function is a rational number rather than a logarithm of a power of integer K^2. This requires an extension of p-adic numbers involving some root of e and a finite number of its powers. S_R must be rational valued using Kähler action $S_K(CP_2) = 2\pi^2$ of CP_2 type extremal as a basic unit. In fact, not only rational values of Kähler function but all values which differ from a rational value by a perturbation with a p-adic norm smaller than one and rationally proportional to a power of e or even its root exist p-adically in this case if they have small enough p-adic norm. The most general perturbation of the action is in the field defined by the extension of rationals defined by the root of e and algebraic numbers.

Since CP_2 action is rationally proportional to π^2, the exponent is rational if $4\pi\alpha_K$ satisfies the same condition. If the conjecture $log(p) = q_1(p)exp\left[q(p_2)\right]/\pi$ holds , then the earlier ansatz $1/\alpha_K(p) = (4/\pi)log(K^2)$ does not guarantee this, and $4/\pi$ must be replaced with a rational number $Q \simeq 4/\pi$. The presence of $log(K^2)$, K^2 product of primes, is well motivated also in this case because it gives the desired $1/\pi$ factor.

This gives for the Kähler function the expression

$$K = Q\left[q_1(p)exp\left[q_2(p)\right] + \sum_i q_1(q_i)exp\left[q_2(q_i)\right]\right]\frac{S}{S_{CP_2}} \ . \tag{8.2.1}$$

$exp(K)$ exists p-adically only provided that K has p-adic norm smaller than one. For given p this poses strong conditions unless one assumes that the condition $S/S_{CP_2} = p^n r$, r rational. In the case of many-particle state of CP_2 extremals this would mean that particle number is divisible by a power of p.

For single CP_2 extremal, the fact that p cannot divide $q_1(p)$ means that either Q contains a power of p or the sum of terms is proportional to a power of p. Obviously this condition is extremely strong and allows only very few primes. One might wander whether this could provide the first principle explanation for p-adic length scale hypothesis selecting primes $p \simeq 2^k$, k integer, and with prime power powers being preferred.

Since $k = 137$ (atomic length scale) and $k = 107$ (hadronic length scale) are the most important nearest p-adic neighbors of electron, one could make a free fall into number mysticism and try the replacement $4/\pi \rightarrow 137/107$. This would give $\alpha_K = 137.3237$ to be compared with $\alpha = 137.0360$: the deviation from α is .2 per cent (of course, α_K need not equal to α and the evolutions of these couplings are quite different). Thus it seems that $log(p) = q_1exp(q_2)/\pi$ hypothesis is supported also by the properties of Kähler action and might lead to an improved understanding of the origin of the mystery prime $k = 137$. Of course, one must be extremely cautious with the numerics. For instance, one could replace $137/107$ with the ratio of $137/log(M_{107}$ and in this case the M_{107} would become an "easy" prime.

2. Could infinite primes appear in the p-adicization of the exponent of Kähler action?

The difficulties related to the p-adic continuation of Kähler function to an arbitrary p-adic number field and the fact that infinities are every day life in quantum field theory bring in mind infinite primes discussed in the chapter "Quaternions, Octonions, and Infinite Primes".

Infinite primes are not divisible by any finite prime. The simplest infinite prime is of form $\Pi = 1+X$, $X = \prod_i p_i$, where product is over all finite primes. The factor $Y = X/(1+X)$ is in the real sense equivalent with 1. In p-adic sense it has norm $1/p$ for every prime. Thus one could multiply Kähler function by Y or its positive power in order to guarantee that the continuation to p-adic number fields exists for all primes. Of course, these states might differ physically in p-adic sense from the states having $Y = 1$. Thus it would seem that the physics of cognition could differentiate between states which are in real sense equivalent.

More general infinite primes are of form $\Pi = nX/m+n$, such that $m = \prod_i q_i$ and $n = \prod_i p_i^{n_i}$ have no common factors. The interpretation could be as a counterpart for a state of a super-symmetric theory containing fermion in each mode labelled by q_i and n_i bosons labelled in modes labelled by p_i. Also positive powers of the ratio $Y = X/\Pi$, Π some infinite prime, are possible as a multiplier of the Kähler function. In the real sense this ratio would correspond to the ratio m/n.

If this picture is correct, infinite primes would emerge naturally in the p-adicization of the theory. Since octonionic infinite primes could correspond to the states of a super-symmetric quantum field theory more or less equivalent with TGD, the presence of infinite primes could make it possible to code the quantum physical state to the vacuum functional via coupling constant renormalization.

One could also consider the possibility of defining functions like $exp(x)$ and $log(1+x)$ p-adically by replacing x with Yx without introducing the algebraic extension. The series would converge for all values of x also p-adically and would be in real sense equivalent with the function. This trick would apply to a very general class of Taylor series having rational coefficients. One could also say that p-adic physics allowing infinite primes would be very similar to real physics.

The fascination of infinite primes is that the ratios of infinite primes which are ordinary rational numbers in the real sense could code the particle number content of a super-symmetric arithmetic quantum field theory. For the octonic version of the theory natural in the TGD framework these states could represent the states of a real Universe. Universe would be an algebraic hologram in the sense that space-time points, something devoid of any structure in the standard view, could code for the quantum states of possible Universes!

The simplest manner to realize this scenario is to consider an extension of rational numbers by the multiplicative group of real units obtained from infinite primes and powers of X. Real number 1 would code everything in its structure! This group is generated as products of powers of $Y(m/n) = (m/n) \times [X/\Pi(m/n)]$ which is a unit in the real sense. Each $Y(m/n)$ would define a subgroup of units and the power of $Y(m/n)$ would code for the number of factors of a given integer with unit counted as a factor. This would give a hierarchy of integers with their p-adic norms coming as powers of p with the prime factors of m and n forming an exception and being reflected in p-adic physics of cognition, Universe would "feel" its real or imagined state with its every point, be it a point of space-time surface, of imbedding space, or of configuration space.

8.2.6 How to understand Riemann hypothesis

The considerations of the preceding subsection led to the requirement that the logarithmic waves $e^{iKlog(u)}$ exist in all number fields for $u = n$ (and thus for any rational value of u) implying number theoretical quantization of the scaling momenta K. Since the logarithmic waves appear also in Riemann Zeta as the basic building blocks, there is an interesting connection with Riemann hypothesis, which states that all non-trivial zeros of $\zeta(z) = \sum_n 1/n^z$ lie at the line $Re(z) = 1/2$.

I have applied two basic strategies in my attempts to understand Riemann hypothesis. Both approaches rely heavily on conformal invariance but being realized in a different manner. The universality of the scaling momentum spectrum implied by the number theoretical quantization allows to understand the relationship between these approaches.

1. First approach

In this approach (see the preprint in [16] in Los Alamos archives and the article published in Acta Mathematica Universitatis Comeniae [17]) one constructs a simple conformally invariant dynamical system for which the vanishing of Riemann Zeta at the critical line states that the coherent quantum states, which are eigen states of a generalized annihilation operator, are orthogonal to a vacuum state possessing a negative norm. This condition implies that the eigenvalues are given by the nontrivial zeros of ζ. Riemann hypothesis reduces to conformal invariance and the outcome is an analytic reductio ad absurdum argument proving Riemann hypothesis with the standards of rigor applied in theoretical physics.

2. Second approach

The basic idea is that Riemann Zeta is in some sense defined for all number fields. The basic question is what "some" could mean. Since Riemann Zeta decomposes into a product of harmonic oscillator partition functions $Z_p(z) = 1/(1-p^z)$ associated with primes p the natural guess is that $p^{1/2+iy}$ exists p-adically for the zeros of Zeta. The first guess was that for every prime p (and hence every integer n) and every zero of Zeta p^{iy} might define complex rational number (Pythagorean phase) or perhaps a complex algebraic number.

The transcendental considerations that one should try to generalize this idea: for every p and y appearing in the zero of Zeta the number p^{iy} belongs to a finite-dimensional extension of rationals involving also rational roots of e. This would imply that also the quantities n^{iy} make sense for all number fields and one can develop Zeta into a p-adic power series. Riemann Zeta would be defined for any number field in the set linearly spanned by the integer multiples of the zeros y of Zeta and it is easy to get convinced that this set is dense at the Y-axis. Zeta would therefore be defined at least in the set $X \times Y$ where X is some subset of real axis depending on the extension used.

If $log(p) = q_1 exp(q_2)/\pi$ holds true, then $y = q(y)\pi$ should hold true for the zeros of ζ. In this case one would have

$$p^{iy} = exp\left[iq(y)q_1(p)exp\left(q_2(p)\right)\right] \quad .$$

This quantity exists p-adically if the exponent has p-adic norm smaller than one. $q_1(p)$ is divisible by finite number of primes p_1 so that p^{iy} does not exist in a finite-dimensional extension of R_{p_1} unless $q(y)$ is proportional to a positive power of p_1. Also in this case the multiplication of y by the units defined by infinite primes (to be discussed later) would save the day and would be completely invisible operation in real context.

3. Logarithmic plane waves and Hilbert-Polya conjecture

Logarithmic plane waves allow also a fresh insight on how to physically understand Riemann hypothesis and the Hilbert-Polya conjecture stating that the imaginary parts of the zeros of Riemann Zeta correspond to the eigenvalues of some Hamiltonian in some Hilbert space.

1. At the critical line $Re(z) = 1/2$ (z=x+iy) the numbers $n^{-z} = n^{-1/2-iy}$ appearing in the definition of the Riemann Zeta allow an interpretation as logarithmic plane waves $\Psi_y(v) = e^{iylog(v)}v^{-1/2}$ with the scaling momentum $K = 1/2 - iy$ estimated at integer valued points $v = n$. Riemann hypothesis would follow from two facts. First, logarithmic plane waves form a complete basis equivalent with the ordinary plane wave basis from which sub-basis is selected by number theoretical quantization. Secondly, for all other powers v^k other than $v^{-1/2}$ in the denominator the norm diverges due to the contributions coming from either short ($k < -1/2$) or long distances ($k > -1/2$).

2. Obviously the logarithmic plane waves provide a concrete blood and flesh realization for the conjecture of Hilbert and Polya and the eigenvalues of the Hamiltonian correspond to the universal scaling momenta. Note that Hilbert-Polya realization is based on mutually orthogonal plane waves whereas the Approach 1 relies on coherent states orthogonal to the negative norm vacuum state. That eigenvalue spectra coincide follows from the universality of the number theoretical quantization conditions. The universality of the number theoretical quantization predicts that the zeros should appear in the scaling eigenvalue spectrum of any physical system obeying conformal invariance. Also the Hamiltonian generating by definition an infinitesimal time translation could act as an infinitesimal scaling.

3. The vanishing of the Riemann Zeta could code the conditions stating that the extensions involved are finite-dimensional: it would be interesting to understand this aspect more clearly.

Connection with the conjecture of Berry and Keating

The idea that the imaginary parts y for the zeros of Riemann zeta function correspond to eigenvalues of some Hermitian operator H is not new. Berry and Keating [25] however proposed quite recently that the Hamilton in question is super-symmetric and given by

$$H = xp - \frac{i}{2} \ . \tag{8.2.2}$$

Here the momentum operator p is defined as $p = -id/dx$ and x has non-negative real values.

H can be indeed expressed as a square $H = Q^2$ of a Hermitian super symmetry generator Q:

$$\begin{aligned}
Q &= \sqrt{i}\,[ix\sigma_1 + p\sigma_2] + \sqrt{\frac{i}{2}}\sigma_3 \ , \\
\sigma_1 &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \ , \\
\sigma_2 &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \ , \\
\sigma_3 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ .
\end{aligned} \tag{8.2.0}$$

By a direct calculation one finds that the following relationship holds true:

$$Q^2 = \begin{pmatrix} xp + \frac{i}{2} & 0 \\ 0 & xp - \frac{i}{2} \end{pmatrix} \ .$$

The eigen spinors of Q can be written as

$$\psi = \begin{pmatrix} u \\ v \end{pmatrix} = x^{-iy} \begin{pmatrix} x^{1/2} \\ \sqrt{\frac{y}{i}}x^{-1/2} \end{pmatrix} \ .$$

The eigenvalues of Q are $q = \sqrt{y}$. For $y \geq 0$ the eigenvalues are real so that Q is Hermitian when inner product is defined appropriately. Obviously y is eigenvalue of Hamiltonian.

Orthogonality requirement for the solutions of the Dirac equation requires that the inner product reduces to the inner product for plane waves $exp(iu)$, $u = log(x)$. This is achieved if inner product for spinors $\psi_i = (u_i, v_i)$ is defined as

$$\langle \psi_1 | \psi_2 \rangle = \int_0^\infty \frac{dx}{x} \left[\overline{u}_1 v_2 + \overline{v}_1 u_2 \right] \ . \tag{8.2.-1}$$

In the basis formed by solutions of Dirac equation this inner product is indeed positive definite as one finds by a direct calculation.

The actual spectrum assumed to give the zeros of the Riemann Zeta function however remains open without additional hypothesis. An attractive hypothesis motivated by previous considerations is that the sharpened form of Riemann hypothesis stating that n^{iy} exists for any number field provided finite-dimensional extensions are allowed for the zeros of Riemann zeta function, holds true. This implies that x^{iy} satisfies the same condition for any rational value of x. $x^{\pm 1/2}$ in turn belongs to the infinite-dimensional algebraic extension Q_C^∞ of complex rationals, when x is rational. Therefore the solutions of Dirac equation, being of form $x^{iy}x^{\pm 1/2}$, exist for all number fields for rational values of argument x.

Connection with arithmetic quantum field theory and quantization of time

There is also a very interesting connection with arithmetic quantum field theory and sharpened form of Riemann hypothesis. The Hamiltonian for a bosonic/fermionic arithmetic quantum field theory is given by

$$H = \sum_p log(p) a_p^\dagger a_p \ . \tag{8.2.0}$$

where $a_p^\dagger$ and a_p satisfy standard bosonic/fermionic anti-commutation relations

$$\{a_{p_1}^\dagger, a_{p_2}\}_\pm = \delta(p_1, p_2) \ . \tag{8.2.1}$$

Here $\pm$ refers to anti-commutator/commutator. The sum of Hamiltonians defines super-symmetric arithmetic QFT. The states of the bosonic QFT are in one-one correspondence with non-negative integers and the decomposition of a non-negative integer to powers or prime corresponds to the decomposition of state to many boson states corresponding to various modes p. Analogous statement holds true for fermionic QFT.

The matrix element for the time development operator $U(t) \equiv exp(iHt)$ between states $|m\rangle$ and $|n\rangle$ can be written as

$$\langle m|U(t)|n\rangle = \delta(m, n) n^{it} \ . \tag{8.2.2}$$

Same form holds true both in bosonic and fermionic QFT:s. These matrix elements are defined for all number fields allowing finite-dimensional extensions if this holds true for n^{it} so that the allowed values of t corresponds to zeros of Riemann Zeta. Similar statement holds in the case of fermionic QFT. One can say that the durations for the time evolutions are quantized in a well defined sense and allowed values of time coordinate correspond to the zeros of Riemann zeta function!

The result is very interesting from the point of view of quantum TGD since it would mean that $U(t)$ allows for the preferred values of the time parameter p-adicization ($p \ mod \ 4 = 3$) obtained by mapping the diagonal phases to their p-adic counterparts by phase preserving canonical identification. For phases this map means only the re-interpretation of the rational phase factor as a complexified p-adic number. For these quantized values of the time parameter time evolution operator of the arithmetic quantum field theory makes sense in all p-adic number fields besides complex numbers.

In the case of Berry's super-symmetric Hamiltonian the assumption that p^{iy} exists in all number fields with finite extensions allowed and the requirement that same holds true for the time evolution operator implies that allowed time durations for time evolution are given by $t = log(n)$. This means that there is nice duality between Berry's theory and arithmetic QFT. The allowed time durations (energies) in Berry's theory correspond to energies (allowed time durations) in arithmetic QFT.

8.2.7 Stronger variants for the sharpened form of the Riemann hypothesis

The previous form of the sharpened form of Riemann hypothesis was preceded by conjectures, which were much stronger. The strongest variant of the sharpening is that the phases p^{iy} are complex rational numbers for all primes and for all zeros ζ. A weaker form assumes that these phases belong to the square root allowing infinite-dimensional extension of rationals. Although these conjectures are probably unrealistic, they deserve a brief discussion.

Could the phases p^{iy} exist as complex rationals for the zeros of ζ?

The set $z = n/2 + iy$, $n > 0$ such that p^{-iy} is Pythagorean phase, is the set in which both real Riemann zeta function and the p-adic counterparts of Z_p exist for $p \ mod \ 4 = 3$. They exists also for $p \ mod \ 4 = 1$, if one *defines* $exp(ix) \equiv cos(x) + \sqrt{-1} sin(x)$: $\sqrt{-1}$ would be ordinary p-adic number for $p \ mod \ 4 = 1$. One could also allow phase factors in square root allowing algebraic extension of p-adics.

What is important that $x = 1/2$ is the smallest value of x for which the p-adic counterpart of $Z_B(p, x_p)$ exists. Already Riemann showed that the nontrivial zeros of Riemann Zeta function lie symmetrically around the line $x = 1/2$ in the interval $0 \leq x \leq 1$.

If one assumes that the zeros of Riemann zeta belong to the set at which the p-adic counterparts of Riemann zeta are defined, Riemann hypothesis follows in sharpened form.

1. Sharpened form of Riemann hypothesis does not necessarily exclude zeros with $x = 0$ or $x = 1$ as zeros of Riemann zeta unless they are explicitly excluded. It is however known that the lines $x = 0$ and $x = 1$ do not contains zeros of Riemann Zeta so that sharpened form implies also Riemann hypothesis.

2. The sharpening of the Riemann hypothesis following from p-adic considerations implies that the phases p^{iy} exist as rational complex phases for all values of $p \ mod \ 4 = 3$ when y corresponds to a zero of Riemann Zeta. Obviously the rational phases p^{iy} form a group with respect to multiplication isomorphic with the group of integers in case that y does not vanish. The same is also true for the phases corresponding to integers continuing only powers of primes $p \ mod \ 4 = 3$ phase factor.

3. A stronger form of sharpened hypothesis is that all primes p and all integers are allowed. This would mean that each zero of the Riemann Zeta would generate naturally group isomorphic with the group of integers. Pythagorean phases form a group and should contain this group as a subgroup. It might be that very simple number theoretic considerations exclude this possibility. If not, one would have infinite number of conditions on each zero of Riemann function and much sharper form of Riemann hypothesis which could fix the zeros of Riemann zeta completely:

   The zeros of Riemann Zeta function lie on axis $x = 1/2$ and correspond to values of y such that the phase factor p^{iy} is rational complex number for all values of prime $p \ mod \ 4 = 3$ or perhaps even for all primes p.

   Of course, the proposed condition might be quite too strong. A milder condition is that $U_p(x_p)$ is rational for single value of p only: this would mean that the zeros of Riemann Zeta would correspond to Pythagorean angles labelled by primes. One can consider also the possibility that p^{iy} is rational for all y but for some primes only and that these preferred primes correspond to the p-adic primes characterizing the effective p-adic topologies realized in the physical world.

4. If this hypothesis is correct then each zero defines a subgroup of Pythagorean phases and also zeros have a natural group structure. Pythagorean phases contain an infinite number of subgroups generated by integer powers of phase. Each such subgroup has some number N of generators such that the subgroup is generated as products of these phases. From the fact that Pythagorean phases are in a one-one correspondence with rationals, it is obvious that there exists large number of subgroups of this kind. Every zero defines infinite number of Pythagorean phases and there are infinite number of zeros. The entire group generated by the phases is in one-one correspondence with the pairs (p, y).

5. If n^{iy} are rational numbers, there must exist imbedding map f: $(n, y) \rightarrow (r, s)$ from the set of phases n^{iy} to Pythagorean phases characterized by rationals $q = r/s$:

   $$(r, s) = (f_1(n, y), f_2(n, y)) \ .$$

   The multiplication of Pythagorean phases corresponds to certain map g

   $$\begin{aligned} &(r_1, s_1) \circ (r_2, s_2) = [g_1(r_1, s_1; r_2, s_2), \ g_2(r_1, s_1; r_2, s_2)] \\ &= (r_1 r_2 - s_1 s_2, r_1 s_2 + r_2 s_1) \equiv (r, s) \end{aligned}$$

   such that the values of r and s associated with the product can be calculated. Thus the product operation rise to functional equations giving constraints on the functional form of the map f.

   i) Multiplication of n^{iy_1} and n^{iy_2} gives rise to a condition

   $$f(n, y_1) \circ f(n, y_2) = f(n, y_1 + y_2) \ .$$

ii) Multiplication of n_1^{iy} and n_2^{iy} gives rise to a condition

$$f(n_1,y)\circ f(n_2,y) = f(n_1n_2,y) \ .$$

This variant of the sharpened form of the Riemann hypothesis has turned out to be un-necessarily strong. Universality Principle requires only that the real parts of the factors $p^{-x}p^{-iy}$ are rational numbers: this means that allowed phases correspond to triangles whose two sides have integer-valued length squared whereas the third side has integer-valued length.

Sharpened form of Riemann hypothesis and infinite-dimensional algebraic extension of rationals

The proposed variant for the sharpened form of Riemann hypothesis states that the zeros of Riemann zeta are on the line $x = 1/2$ and that p^{iy}, where p is prime, are complex rational (Pythagorean) phases for zeros. Furthermore, Riemann hypothesis is equivalent with the corresponding statement for the fermionic partition function Z_F. If the sharpened form of Riemann hypothesis holds true, the value of $Z_F(z)$ in the set of zeros $z = 1/2 + iy$ of Z_F can be interpreted as a complex (vanishing) image of certain function $Z_F^\infty(1/2 + iy)$ having values in the infinite-dimensional algebraic extension of rationals defined by adding the square roots of all primes to the set of rational numbers.

1. The general element q of the infinite-dimensional extension Q_C^∞ of complex rationals Q_C can be written as

$$\begin{aligned} q &= \sum_U q_U e_U \ , \\ e_U &= \prod_{i\in U}\sqrt{p}_i \ . \end{aligned} \quad (8.2.2)$$

   Here q_U are complex rational numbers, U runs over the subsets of primes and e_U are the units of the algebraic extension analogous to the imaginary unit. One can map the elements of Q_C^∞ to reals by interpreting the generating units $\sqrt{p}$ as real numbers. The real images $(e_U)_R$ of e_U are thus real numbers:

$$e_U \rightarrow [e_U]_R = \prod_i \sqrt{p_i} \ .$$

2. The value of $Z_F(z)$ at $z = 1/2 + iy$ can be written as

$$Z_F(z = 1/2 + iy) = \sum_U \left[\frac{1}{e_U}\right]_R \times (e_U^2)^{-iy} \ . \quad (8.2.3)$$

   Here $(e_U)_R$ means that e_U are interpreted as real numbers.

3. If one restricts the set of values of $z = 1/2 + iy$ to such values of y that p^{iy} is complex rational for every value of p, then the value of $Z_F(1/2 + iy)$ can be also interpreted as the real image of the value of a function $Z_F(Q_\infty|z = 1/2 + iy)$ restricted to the set of zeros of Riemann zeta and having values at Q_C^∞:

$$\begin{aligned} Z_F(1/2+iy) &= [Z_F(Q_\infty|1/2+iy)]_R \ , \\ Z_F(Q_\infty|1/2+iy) &\equiv \sum_U \frac{1}{e_U} \times (e_U^2)^{-iy} \ . \end{aligned} \quad (8.2.3)$$

Note that $Z_F(Q_\infty|z=1/2+iy)$ cannot vanish as element of Q_∞. One can also define the Q_C^∞ valued counterparts of the partition functions $Z_F(p,1/2+iy)$

$$\begin{aligned} Z_F(Q_\infty|1/2+iy) &= \prod_p Z_F(Q_\infty|p,z=1/2+iy) \ , \\ Z_F(Q_\infty|1/2+iy) &\equiv 1+p^{-1/2}p^{-iy} \ , \\ Z_F(p,1/2+iy) &= [Z_F(Q_\infty|p,1/2+iy)]_R \ . \end{aligned} \tag{8.2.2}$$

$Z_F(Q_\infty|1/2+iy)$ and $Z_F(Q_\infty|p,1/2+iy)$ belong to Q_C^∞ only provided p^{iy} is Pythagorean phase.

4. The requirement that p^{iy} is rational does not yet imply Riemann hypothesis. One can however strengthen this condition. The simplest condition is that the real image of $Z_F(Q_\infty|1/2+iy)$ is complex rational number for any value of Z_F. A stronger condition is that the complex images of the functions

$$\frac{Z_F^\infty}{\prod_{p\in U} Z_p^\infty}$$

are complex rational and U is finite set of primes. The complex counterparts of these functions are given by

$$\left[\frac{Z_F^\infty}{\prod_{p\in U} Z_p^\infty}\right]_R = \frac{Z_F}{\prod_{p\in U} Z_F(p,..)} \ . \tag{8.2.3}$$

Obviously these conditions can be true only provided that $Z_F(1/2+iy)$ vanishes identically for allowed values of y. This implies that sharpened form of Riemann hypothesis is true. "Physically" this means that the fermionic partition function restricted to any subset of integers not divisible by some finite set of primes, has real counterpart which is complex rational valued.

8.2.8 Are the imaginary parts of the zeros of Zeta linearly independent of not?

Concerning the structure of the weight space of super-canonical algebra the crucial question is whether the imaginary parts of the zeros of Zeta are linearly independent or not. If they are independent, the space of conformal weights is infinite-dimensional lattice. Otherwise points of this lattice must be identified. The model of the scalar propagator identified as a suitable partition function in the super-canonical algebra for which the generators have zeros of Riemann Zeta as conformal weights demonstrates that the assumption of linear independence leads to physically unrealistic results and the the propagator does not exist mathematically for the entire super-canonical algebra. Also the findings about the distribution of zeros of Zeta favor a hypothesis about the structure of zeros implying a linear dependence.

Imaginary parts of non-trivial zeros as additive counterparts of primes?

The natural looking (and probably wrong) working hypothesis is that the imaginary parts y_i of the nontrivial zeros $z_i = 1/2 + y_i$, $y_i > 0$, of Riemann Zeta are linearly independent. This would mean that y_i define play the role of primes but with respect to addition instead of multiplication. If there exists no relationship of form $y_i = n2\pi + y_j$, the exponents e^{iy_i} define a multiplicative representation of the additive group, and these factors satisfy the defining condition for primeness in the conventional sense. The inverses e^{-iy_i} are analogous to the inverses of ordinary primes, and the products of the phases are analogous to rational numbers.

There would exist an algebra homomorphism from $\{y_i\}$ to ordinary primes ordered in the obvious manner and defined as the map as $y_i \leftrightarrow p_i$. The beauty of this identification would be that the hierarchies of p-adic cutoffs identifiable in terms of the p-adic length scale hierarchy and y-cutoffs identifiable in terms p-adic phase resolution (the higher the p-adic phase resolution, the higher-dimensional extension of p-adic numbers is needed) would be closely related. The identification would allow to see Riemann Zeta as a function relating two kinds of primes to each other.

A rather general assumption is that the phases p^{iy_i} are expressible as products of roots of unity and Pythagorean phases:

$$\begin{aligned}
p^{iy} &= e^{i\phi_P(p,y)} \times e^{i\phi(p,y)} \ , \\
e^{i\phi_P(p,y)} &= \frac{r^2 - s^2 + i2rs}{r^2+s^2} \ , \ r = r(p,y) \ , \ s = s(p,y) \ , \\
e^{i\phi(p,y)} &= e^{i\frac{2\pi m}{n}} \ , \ m = m(p,y) \ , \ n = n(p,y) \ .
\end{aligned} \tag{8.2.2}$$

If the Pythagorean phases associated with two different zeros of zeta are different a linear independence over integers follows as a consequence.

Pythagorean phases form a multiplicative group having "prime" phases, which are in one-one correspondence with the squares of Gaussian primes, as its generators and Gaussian primes which are in many-to-one correspondence with primes $p_1 \ mod \ 4 = 1$. If p^{iy} is a product of algebraic phase and Pythagorean phase for any prime p, one should be able to decompose any zero y into two parts $y = y_1(p) + y_P(p)$ such that one has

$$log(p)y_1(p) = \frac{m2\pi}{n} \ , \ log(p)y_P(p) = \Phi_P = arctan\left[\frac{2rs}{r^2+s^2}\right] \ . \tag{8.2.3}$$

Note that the decomposition is not unique without additional conditions. The integers appearing in the formula of course depend on p.

Does the space of zeros factorize to a direct sum of multiples Pythagorean prime phase angles and algebraic phase angles?

As already noticed, the linear independence of the y_i follows if the Pythagorean prime phases associated with different zeros are different. The reverse of this implication holds also true. Suppose that there are two zeros $log(p)y_{1i} = \Phi_{P_1} + q_{1i}2\pi$, $i = a, b$ and two zeros $log(p)y_{2i} = \Phi_{P_2} + q_{2i}2\pi$, $i = a, b$, where q_{ij} are rational numbers. Then the linear combinations $n_1y_{1a} + n_2y_{2a}$ and $n_1y_{1b} + n_2y_{2b}$ represent same zeros if one has $n_1/n_2 = (q_{2a} - q_{2b})/(q_{1b} - q_{1a})$.

One can of course consider the possibility that linear independence holds true only in the weaker sense that one cannot express any zero of zeta as a linear combination of other zeros. For instance, this guarantees that the super-canonical algebra generated by generators labelled by the zeros has indeed these generates as a minimal set of generating elements.

For instance, one can imagine the possibility that for any prime p a given Pythagorean phase angle $log(p)y_{P_k}$ corresponds to a set of zeros by adding to $\Phi_{P_k} = log(p)y_{P_k}$ rational multiples $q_{k,i}2\pi$ of 2π, where $Q_p(k) = \{q_{k,i}|i = 1, 2, ..\}$ is a subset of rationals so that one obtains subset $\{\Phi_{P_k} + q_{k,i}2\pi|q_{k,i} \in Q_p(k)\}$. Note that the definition of y_P involves an integer multiple of 2π which must be chosen judiciously: for instance, if y_P is taken to be minimal possible (that is in the range $(0, \pi/2)$, one obviously ends up with a contradiction. The same is true if $q_{k,i} < 1$ is assumed. Needless to say, the existence of this kind of decomposition for every prime p is extremely strong number theoretic condition.

The facts that Pythagorean phases are linearly independent and not expressible as a rational multiple of 2π imply that no zero is expressible as a linear combination of other zeros whereas the linear independence fails in a more general sense as already found. An especially interesting situation results if the set $Q_p(k)$ for given p does not depend on the Pythagorean phase so that one can write $Q_p(k) = Q_p$. In this case the set of zeros of Zeta would be obtained as a union of translates of the set Q_p by a subset of Pythagorean phase angles and approximate translational invariance realized in a statistical sense would result. Note that the Pythagorean phases need not correspond to Pythagorean prime phases: what is needed is that a multiple of the same prime phase appears only once.

An attractive interpretation for the existence of this decomposition to Pythagorean and algebraic phases factors for every prime is in terms of the p-adic length scale evolution. The possibility to express the zeros of Zeta in an infinite number of manners labelled by primes could be seen as a number theoretic realization of the renormalization group symmetry of quantum field theories. Primes p define kind of length scale resolution and in each length scale resolution the decomposition of the phases makes sense. This assumption implies the following relationship between the phases associated with y:

$$\frac{\left[\Phi_{P(p_1)} + q(p_1)2\pi\right]}{log(p_1)} = \frac{\left[\Phi_{P(p_2)} + q(p_2)2\pi\right]}{log(p_2)} \quad . \tag{8.2.4}$$

In accordance with earlier number theoretical speculations, assume that $log(p_2)/log(p_1) \equiv Q(p_2, p_1)$ is rational. This condition allows to deduce how the phases p_1^{iy} transform in $p_1 \to p_2$ transformation. Let $p_1^{iy} = U_{P,p_1,y}U_{q,p_1,y}$ be the representation of p_1^{iy} as a product of Pythagorean and algebraic phases. Using the previous equation, one can write

$$p_2^{iy} = U_{P,p_2,y}U_{q,p_2,y} = U_{P,p_1,y}^{Q(p_2,p_1)}U_{q,p_1,y}^{Q(p_2,p_1)} \quad . \tag{8.2.5}$$

This means that the phases are mapped to rational powers of phases. In the case of Pythagorean phases this means that Pythagorean phase becomes a product of some Pythagorean and an algebraic phase whereas algebraic phases are mapped to algebraic phases. The requirement that the set of phases p_2^{iy} is same as the set of phases p_1^{iy} implies that the rational power $U_{P,p_1,y}^{Q(p_2,p_1)}$ is proportional to some Pythagorean phase U_{P,p_1,y_1} times algebraic phase U_q such that the the product of $U_qU_{q,p_1,y}^{Q(p_2,p_1)}$ gives an allowed algebraic phase. The map $U_{P,p_1,y} \to U_{P,p_1,y_1}$ from Pythagorean phases to Pythagorean phases induced in this manner must be one-to one must be the map between algebraic phases. Thus it seems that in principle the hypothesis might make sense.

The basic question is why the phases q^{iy} should exist p-adically in some finite-dimensional extension of R_p for every p. Obviously some function coding for the zeros of Zeta should exist p-adically. The factors $G_q = 1/(1 - q^{-iy-1/2})$ of the product representation of Zeta obviously exist if this assumption is made for every prime p but the product is not expected to converge p-adically.

Also the logarithmic derivative of Zeta codes for the zeros and can be written as

$$\frac{\zeta'}{\zeta} = -\sum_q log(q)\frac{q^{-1/2-iy}}{1 - q^{-1/2-iy}} \quad . \tag{8.2.6}$$

As such this function does not exist p-adically but dividing by $log(p)$ one obtains

$$\frac{1}{log(p)}\frac{\zeta'}{\zeta} = -\sum_q Q(q,p)\frac{q^{-1/2-iy}}{1 - q^{-1/2-iy}} \quad . \tag{8.2.7}$$

This function exists if the the p-adic norms rational numbers $Q(q, p)$ approach to zero for $q \to \infty$: $|Q(q, p)|_p \to 0$ for $q \to \infty$. The p-adic existence of the logarithmic derivative would thus give hopes of universal coding for the zeros of Zeta and also give strong constraints to the behavior of the factors $Q(q, p)$. The simplest guess would be $Q(q, p) \propto p^q$ for $q \to \infty$.

Correlation functions for the spectrum of zeros favors the factorization of the space of zeros

The idea that the imaginary parts of the zeros of Zeta are linearly independent is a very attractive but must be tested against what is known about the distribution of the zeros of Zeta.

There exists numerical evidence for the linear independence of y_i as well as for the hypothesis that the zeros correspond to a union of translates of a basic set Q_1 by subset of Pythagorean phase angles. Lu and Sridhar have studied the correlation among the zeros of ζ [32]. They consider the correlation functions for the fluctuating part of the spectral function of zeros smoothed out from a sum of delta functions to a sum of Lorentzian peaks. The correlation function between two zeros with a constant distance $K_2 - K_1 + s$ with the first zero in the interval $[K_1, K_1 + \Delta]$ and second zero in the interval $[K_2, K_2 + \Delta]$ is studied. The choice $K_1 = K_2$ assigns a correlation function for single interval at K_1 as a function of distance s between the zeros.

1. The first interesting finding, made already by Berry and Keating, is that the peaks for the negative values of the correlation function correspond to the lowest zeros of Riemann Zeta (only those contained in the interval Δ can appear as minima of correlation function). This phenomenon observed already by Berry and Keating is known as resurgence. That the anti-correlation is maximal when the distance of two zeros corresponds to a low lying zero of zeta can be understood if linear combinations of the zeros of Zeta are the least probable candidates for zeros. Stating it differently, large zeros tend to avoid the points which represent linear combinations of the smaller zeros.

2. Direct numerical support the hypothesis that the correlation function is approximately translationally invariant, which means that it depends on $K_2 - K_1 + s$ only. Correlation function is also independent of the width of the spectral window Δ. In the special $K_1 = K_2$ the finding means that correlation function does not depend at all on the position K_1 of the window and depends only on the variable s. Prophecy means that the correlation function between the interval $[K, K+\Delta]$ and its mirror image $[-K-\Delta, -K]$ is the correlation function for the interval $[2K+\Delta]$ and depends only on the variable $2K+s$ allowing to allows to deduce information about the distribution of zeros outside the range $[-K, K]$. This property obviously follows from the proposed hypothesis implying that the spectral function is a sum of translates of a basic distribution by a subset of Pythagorean prime phase angles.

This hypothesis is consistent with the properties of the the smoothed out spectral density for the zeros given by

$$\langle \rho(k) \rangle = \frac{1}{2\pi} log(\frac{k}{2\pi}) \ . \tag{8.2.8}$$

This implies that the smoothed out number of zeros y smaller than Y is given by

$$N(Y) = \frac{Y}{2\pi}(log(\frac{Y}{2\pi}) - 1) \ . \tag{8.2.9}$$

$N(Y)$ increases faster than linearly, which is consistent with the assumption that the distribution of zeros with positive imaginary part is sum over translates of a single spectral function ρ_{Q_0} for the rational multiples $q_i X_p$, $X_p = 2\pi/log(p)$, $q_i \in Q_p$, for every prime p.

If the smoothed out spectral function for $q_i \in Q_p$ is constant:

$$\rho_{Q_p} = \frac{1}{K_p 2\pi} \ , \ K_p > 0 \ , \tag{8.2.10}$$

the number $N_P(Y, p)$ of Pythagorean prime phases increases as

$$N_P(Y|p) = K_p(log(\frac{Y}{2\pi}) - 1) \ , \tag{8.2.11}$$

so that the smoothed out spectral function associated with $N_P(Y|p)$ is given by the function

$$\rho_P(k|p) = \frac{K_p}{k} \tag{8.2.12}$$

for sufficiently large values of k. Therefore the distances between subsequent zeros could quite well correspond to the same Pythagorean phase for a given p and thus should allow to deduce information about the spectral function ρ_{Q_0}. A convenient parametrization of K_p is as $K = K_{p,0}/4\pi^2$ since the points of Q_p are of form $q_i 2\pi = (n(q_i) + q_1(q_i))2\pi$, $q_1 < 1$, and $n(q_i)$ must in the average sense form an evenly spaced subset of reals.

Physical considerations favor the linear dependence of the zeros

The numerical evidence is at best suggestive and one can always argue that by an arbitrary small deformation of the linearly dependent zeros one obtains linearly independent zeros. This would however require that each zero of form $y_{P_i} + q2\pi$, $q \in Q_p$ is very near to a zero $\Phi_{P_{k(i,q)}} + q_{k(i,q)}2\pi$. In other words, the union of the translates of Q_p by a subset of Pythagorean phases would approximate the zeros in one-one correspondence with a larger subset of Pythagorean phases (given prime phase appears only once). This should hold for every prime and this seems rather implausible.

On the other hand, the linear dependence between zeros has deep physical implications for the basic quantum TGD, and as the following arguments demonstrate, is physically highly desirable. The precise arguments are developed later and here only the skeleton of the argument is given.

1. The zeros label the generating elements of the super-canonical algebra and the failure of the linear independence means that the weight system is not just the infinite-dimensional lattice spanned by the zeros but can be regarded as a kind of bundle like structure such that the linear combinations $log(p)y_b = \sum_{i=1}^{N} n_i \Phi_{P_{k_i}}$ form N-dimensional lattice, and the fiber at a given point of this lattice consists of the points $log(p)y_f = \sum_i n_i q_i 2\pi$. The set of these points is the lattice $n_1 Q_p \times n_2 Q_p \times ...$ divided by the equivalence defined by $y_{f,1} = y_{f,2}$ and for given values of n_i a discrete analog of the one-dimensional space of parallel hyper-planes of an N-dimensional defined by the equation $\sum_{i=1}^{N} n_i x^i = y$ space parameterized by the values of y. What is essential that the space of the planes is different for each point $y_b = \sum_{i=1}^{N} n_i y_{P_{k_i}}$.

2. The calculation of the scalar propagator as a partition function for the super-canonical algebra assuming linear independence gives without any restrictions to the super-canonical weights an infinite number of delta-function resonances of form $\delta(p^2 - m_n^2)$, and at the limit when all zeros of the Riemann Zeta are included in the sub-algebra of super-canonical algebra the set of delta function resonances defines a dense set on real axis. If only the super-canonical conformal weights generated by the positive zeros of Zeta are included, delta function resonances become ordinary poles of form $1/(p^2 - m_k^2)$. The resonances are infinitely narrow and form also now a dense set of real axis.

3. This result, which can be claimed to be non-physical, can be avoided if the the zeros are not linearly independent. Although the partition function cannot be calculated explicitly in this case, one can expect that the linear independence gives a reasonable first approximation and that the failure of the approximation is due to the multiple counting caused by the neglect of the fact that the planes of the fiber space can contain several equivalent points. If the zeros are linearly dependent, resonances get a finite width and singularities are avoided for real values of the masses and there are good hopes that the partition function is well-defined for the entire super-canonical algebra.

4. A further argument favoring the proposed form of zeros relates to the two hierarchies strongly suggested by quantum TGD. The first hierarchy corresponds to ordinary primes labelling p-adic length scales and corresponds to length scale resolution. The second hierarchy corresponds to a hierarchy of algebraic extensions of p-adic numbers and there is strong feeling that this hierarchy should correspond to the hierarchy of Beraha numbers $B_n = 4cos^2(\pi/n)$ associated with the phases $exp(i2\pi/n)$. The phases $exp(i\pi/p)$ or their non-trivial powers, for p prime, are even more interesting because of the structure of finite field $G(p,1)$.

   One could consider the possibility that the rationals $q \in Q_p$ for any p can be ordered by their size in such a manner that this ordering corresponds to the ordering of primes with respect to size. Obviously the condition $Q_p = Q_1$ must hold true. This would imply that the products of the powers of the phases $exp(iq2\pi)$ for the lowest N values of q_i would give the Beraha phases corresponding to square free integers having corresponding primes p_i, $i = 1, ..., N$, as factors. All Beraha phases are obtained if the phases $exp(i2\pi/p^n)$, $n = 1, 2, ..$ or their non-trivial powers, are also present. If this waves the case the full p-adic length scale hierarchy with powers of p would correspond to the hierarchy of Beraha phases. This would mean that the addition of new super-canonical conformal weights of increasing size to the sub-algebra of the super-canonical algebra would mean the increase of the dimension of the extension of p-adic numbers needed to represent the resulting phases p-adically as well as an increasing phase resolution.

5. With the assumptions about the structure of zeros of Zeta, the hierarchies defined by the subset y_{P_i} of multiples of Pythagorean prime phase angles and algebraic phases would neatly factorize and the latter would correspond to the p-adic length scale hierarchy. Pythagorean phases correspond to phases of the squares of Gaussian integers $r+is$ and the squares of Gaussian primes define naturally Pythagorean primes. The norm squared of the Gaussian prime is obviously prime: $r^2+s^2=p_1$, and satisfies $p_1\ mod\ 4=1$. Hence there is a natural correspondence between Pythagorean prime phases and primes $p\ mod\ 4=1$. One can wonder whether also Pythagorean prime phase angles could be mapped to a subset of primes such that that size ordering for y_{P_i} would correspond to the size ordering for the subset of primes. As already noticed, the primeness property is actually an un-necessary strong requirement for Pythagorean phases. Needless to emphasize, these speculative assumptions would pose very strong constraints on the spectrum of zeros and are certainly testable numerically.

The notion of dual Zeta

These considerations lead to the idea that Riemann Zeta has a dual for which the role of multiplicative primes is taken by the additive primes. This function, call it $\zeta_d(u)$ should either vanish or diverge at points $u=p$. The partition functions for super-canonical conformal weights discussed in the chapter "Equivalence of Loop Diagrams with Tree Diagrams and Cancellation of Infinities in Quantum TGD" define analogs of Riemann Zeta involving analog of restriction of summation to integers which are products of even and odd integers and these functions indeed are singular at powers $u=p^{kx}$, $x=2\pi k/y$, $k=1,2,...$, where the transcendental values x do not depend on p. That the singularities do not occur for rational values of u is physically very satisfactory since this would mean that the scattering rates could become infinite.

The precise dual ζ_d of ζ would be the function

$$\zeta_d(u) \quad = \sum_{\sum n(y)y, y\in Y} u^{i\sum n(y)y} = \prod_{y>0,y\in Y} \frac{1}{1-u^{iy}} \quad , \tag{8.2.13}$$

where the summation is over all possible formal linear combinations of positive imaginary parts y of zeros or subset of them with non-negative coefficients $n(y)$. In the case that the zeros of Riemann Zeta are linearly independent, the set Y corresponds to all zeros. If the zeros are of the form $y=y_{P_i}+q2\pi$, $q\in Q_0$, one can restrict the consideration to a subset Y of zeros obtains by selecting only single value of $q\in Q_0$ for each y_{P_i}. The simplest option is that q is same for all values of y_{P_i}.

The interpretation as a product of bosonic partition functions defined by the zeros of ζ or subset of them, obviously makes sense, and the form of the partition function is the same as that of Riemann Zeta in the product representation. By writing $u=\rho exp(i\phi)$, $\phi\geq 0$ one finds that all terms in the product converge if the term corresponding to the smallest value $y_{min}\simeq 14.124725$ of y converges. This gives the condition $\phi > 1/y_{min}\sim 2\pi/14$. One can however extract arbitrary number of the lowest terms in the product as a separate well-defined factor and obtain a convergence above arbitrarily small $\phi_{min}=\epsilon>0$. Thus the product is well-defined arbitrary near to real axis above it.

The limit $\phi\rightarrow 2\pi$ is well-defined and at $z=\rho e^{i2\pi}$, $\rho>0$ the product can be written as

$$\zeta_d(\rho e^{i2\pi}) \quad = \prod_{y\in Y} \frac{1}{1-\rho^{-2\pi y}\rho^{iy}} \quad . \tag{8.2.14}$$

This expression converges to a finite result at the real axis and pole is not possible. This expression is not consistent with the requirement that $u\rightarrow 1/u$ induces a complex conjugation of ζ_d at the real axis.

The conjecture is that the limit $\phi\rightarrow 0_+$ limit of ζ_d vanishes or diverges for $u=p^{\pm 1}$. Also now the powers of $u_p=p^{kx}$ define poles of the individual factors in the product at real axis. For $u=p$ one can write

$$\zeta_d(p)\overline{\zeta}_d(p) \quad = \prod_{y>0,y\in Y} \frac{1}{4sin^2\left[\frac{\phi(p,y)+\phi_P(y)}{2}\right]} \quad . \tag{8.2.15}$$

Here U refers to the subset of zeros of Zeta. This expansion diverges for $sin^2[(\phi(p,y)+\phi_P(y))/2] < 1/4$ for sufficiently many values of y. An interesting possibility inspired by the connection with braid groups and Beraha numbers $B_n = 4cos^2(\pi/n)$ is that the numbers $4cos^2\,[\phi(p,y)]$ are Beraha numbers so that one would have $\phi(p,y) = \pi/n(p,y)$, $n(p,y) \geq 3$. For $n(p,y) \geq 3$ and $\phi_P(y) = 0$, all factors in the product would be larger than or equal to one so that the product would diverge. The vanishing would be thus due the Pythagorean phases. Of course, these arguments cannot be however taken completely seriously since the product expansion does not converge at the real axis.

Also the zeros $z_i = 1/2 + iy_i$, $y_i > 0$, are generators of an Abelian algebra with integers $n/2 + \sum_i n_i y_i$, $\sum n_i = n > 0$. The corresponding zeta function is

$$\zeta_d(u) = \prod_y \frac{1}{1-u^{-1/2-iy}} \ . \tag{8.2.16}$$

This function has even nearer resemblance to the ordinary ζ. Interestingly, the product $\prod_d \zeta_d(p)$ satisfies the identity

$$\prod_p \zeta_d(p) = \prod_y \zeta(1/2+y) \ , \tag{8.2.17}$$

if one exchanges freely the order of producting. The fact that all factors on the right hand side vanish would suggest that also $\zeta_d(p)$ vanishes for all values of p.

8.2.9 Why the zeros of Zeta should correspond to number theoretically allowed values of conformal weights?

The following argument provides support for the belief that the conformal weights $s = 1/2 + iy$ for which $p^{1/2+iy}$ exist in a finite-dimensional extension of rationals for all values of prime p, indeed correspond to the non-trivial zeros of Zeta.

1. The basic idea of the number theoretical approach is that the conformal weights $1/2 + iy$ are such that the radial waves $r^{-1/2-iy}$ exist for all rational (and thus for integer) values of r in some finite-dimensional extension of rationals. The logarithms $log(n)$ of integers can be interpreted as quantum numbers of a system defined by an arithmetic quantum field theory and Zeta function $\zeta = \sum_n n^{-iy-1/2}$ with $s = 1/2 + iy$ interpreted as an inverse temperature, defines the partition function of this system.

2. On the other hand, so called Selberg's Zeta function characterizes the eigen values of the Laplacian in 2-dimensional quantum billiard systems defined in the fundamental domain of some hyperbolic subgroup G of $SL(2,Z)$ acting in the hyperbolic plane $SL(2,R)/SO(2)$ [33]. The fundamental domain is analogous to a box containing the particle. At quantum level the boundary conditions are satisfied by summing over all the G translates of $SL(2,R)$ invariant Green function with respect to the second argument. Physically this is analogous to putting to all copies of the fundamental domain an image charge. The confinement to the fundamental domain selects from the continuous energy spectrum a discrete sub-spectrum. Selberg's Zeta (its logarithmic derivative) has the allowed energy eigen values as its zeros (poles). Furthermore, the energy eigen values of Laplacian are of form $E = -l(l+1)$, where $l = -1/2-iy$ is identifiable as the counterpart of conformal weight and has the same form as the zeros of Zeta. y has discrete spectrum of values characterized by the choice of G. The density of the energy eigenvalues is amazingly similar to that of Zeta.

3. On basis of above resemblances one can argue that Riemann Zeta (its logarithmic derivative) characterizes the purely number theoretical spectrum as its zeros (poles). If this is the case, the zeros of Zeta would coincide with the number theoretically allowed conformal weights $1/2 + iy$.

The p-adically existing conformal weights are zeros of Zeta for 1-dimensional systems allowing discrete scaling invariance

The obvious question is whether one could reduce number theory to symmetry. The following considerations suggests that $D \geq 2$-dimensional spaces do not allow a system having zeros of Zeta as its spectrum.

1. The density of states of the Selberg Zeta function differs in some aspects from that of Zeta so that Riemann Zeta probably has no interpretation as a Selberg Zeta function of a number theoretical system. For instance, the average density of states with respect to y grows linearly rather than logarithmically although the fluctuating part of the density of states is formally very similar to that of Zeta.

2. Lobatchevski space (the hyperboloid of the 4-dimensional future light cone) has $SL(2,C)$ as its isometry group. The energy spectrum of Laplacian in this case is of the form $E = -l(l+2) = 1 + y^2$ with $l = -1 - iy$ and thus different from the spectrum of 2-dimensional case and of Riemann Zeta. Due to the higher dimension of the system the mean density of states grows even faster than in the 2-dimensional case so that there seems to be no hope of getting the density of states of Riemann Zeta.

Only one-dimensional systems give hopes of the required logarithmically varying mean density of states. The simplest candidate one can imagine is a system with discrete scaling invariance.

1. Instead of Laplacian, and in complete accordance with the view that conformal invariance is the key to the understanding of Riemann Zeta, one can consider the scaling operator $L_0 = xd/dx$ acting at the half line R_+ so that the Green functions defined by the equation

$$(L_0 + z)G(x, x_1) = (xd/dx + z)G(x, x_1) = \delta(\frac{x}{x_1} - 1) \qquad (8.2.18)$$

become the object of interest. The solution can be written as

$$G(x, x_1|z) = (\frac{x}{x_1})^z \times \theta(\frac{x}{x_1} - 1) \ . \qquad (8.2.19)$$

Here $\theta(x)$ denotes the step function. The requirement that the integrals

$$\int \overline{G}(x, x_1|z_1)G(x, x_1|z_2)dx$$

reduce to the inner products of ordinary plane waves when $ln(x/y)$ is taken as an integration variable forces the condition $z = 1/2 + iy$. In fact, this might be seen as the physicist's "proof" of the Riemann hypothesis.

2. Following the construction of the automorphic Green functions in the hyperbolic plane described in [33], the next step is to form a sum over the $x-$ scaling transforms of $G(x, x_1|z)$ by summing over the integer scaled values nx of x to form a well defined Green function in the fundamental domain associated with the semigroup of integer scalings. Any interval $[n, 2n]$ forms a fundamental domain. This gives

$$\begin{aligned} G_I(x, x_1|\frac{1}{2} + iy) &= \sum_n G(nx, x_1|\frac{1}{2} + iy) = \sum_n (\frac{nx}{x_1})^{\frac{1}{2}+iy} \\ &= \zeta(\frac{1}{2} + iy) \times (\frac{x}{x_1})^{\frac{1}{2}+iy} \ . \end{aligned} \qquad (8.2.18)$$

The resulting Green function is proportional to Riemann Zeta at the critical line and vanishes for the zeros of Zeta. Note that the logarithmic derivative of ζ divided by $log(p)$ exists in a finite-dimensional extension of R_p for $x = n/2+i\sum_k m_k y_k$ if the basic number theoretical requirements on the phases p^{iy} defined by the zeros of Zeta are satisfied: in particular $log(p_1)/log(p)$ must have R_p norm which approaches zero for larger values of p_1. Hence the logarithmic derivative of Zeta could codes the number theoretical physics universally.

3. In the usual approach [33] the integral of G_I over the fundamental domain would give the density of states $d(E)$. In the recent case the integration over the fundamental domain $[1,2]$ gives just ζ function

$$\int_1^2 G_I(x,x|-\frac{1}{2}+iy)dx \quad = \quad \sum_n n^{-\frac{1}{2}-iy} = \zeta(\frac{1}{2}+iy) \ . \tag{8.2.19}$$

The interpretation as a density of states is obviously not possible. The proof for the Riemann hypothesis to be discussed later allows to interpret the vanishing of Riemann Zeta as as orthogonality of physical states labelled by zeros of Zeta with a tachyonic vacuum state with a vanishing conformal weight. The vanishing of Green function could also now have an interpretation stating that the physical states labelled by non-trivial zeros are orthogonal to the scaling invariant tachyonic vacuum.

4. Quite generally, the imaginary part of the logarithmic derivative of any real function $f(E)$ for which energy eigenvalues E_n correspond to zeros of unit multiplicity, defines the density of states as a sum over delta functions. $G(y) = \zeta(1/2+iy)$ is real at the critical line as is also its logarithmic derivative apart from delta function singularities of the imaginary part at the zeros of Zeta so that its logarithmic derivative indeed gives the density of zeros of Zeta:

$$d(y) \quad = \quad \frac{1}{\pi}Im\left[i\frac{dlog\left[\zeta(\frac{1}{2}+iy)\right]}{dy}\right] = \sum_n \delta(y-y_n) \ . \tag{8.2.20}$$

This ultra simple model realizes the idea that the logarithmic derivative of Green function naturally associated with a system invariant under the semi-group of integer scalings codes as its poles the zeros of Zeta. The p-adic existence of the Green function in turn is equivalent with the requirement that the spectrum corresponds to the zeros of Zeta.

Realization of discrete scaling invariance as discrete 2-dimensional Lorentz invariance

Both the role of the hyperbolic groups and the fact that in quantum TGD zeros of Zeta label representations of Lorentz group, encourage to think that the 1-dimensional hyperbolic subspace $t^2-x^2 = constant$ of 2-dimensional Minkowski space having Lorentz group $SO(1,1)$ as its symmetries realizes the above described system physically. The counterpart of the hyperbolic subgroup G of $SL(2,R)$ would the semigroup of Lorentz transformations defining integer scalings of the second light like coordinate:

$$u \equiv t+z \rightarrow nu \ , \quad v \equiv t-z \rightarrow \frac{1}{n}v \ .$$

This semigroup corresponds to the diagonal semi-subgroup of $SL(2,Q)$ consisting of matrices $diag(\lambda, 1/lambda) = diag(n, 1/n)$. The reduction to semigroup is natural by the presence of the p-adic length scale cutoff unavoidable in p-adicization.

Taking $u = t+z$ as the coordinate of the hyperboloid, the situation reduces to that already considered. Infinitesimal Lorentz boost acts as a scaling operator and its eigenvalues correspond to the zeros of Zeta by number theoretic existence requirements. The matrices $diag(p, 1/p)$, p prime, are completely analogous to the group elements g_0 defining primitive periodic orbits in the higher-dimensional case so that prime numbers are naturally realized as discrete Lorentz transformations. Prime Lorentz transformations and their inverses generate rational Lorentz group. The length of the primitive periodic orbit corresponds to the scaling parameter $log(p)$ defining the scaling by p as an exponentiated scaling transformation $u \rightarrow exp(log(p))u = pu$.

8.3 Universality Principle and Riemann hypothesis

The basic definition of $\zeta(s=x+iy)$ based on the product formula does not converge for $Re[s] \leq 1$. One can however define 'universal' ζ, call it $\hat{\zeta}$, as the product of the partition functions $Z_{p_1}(s) = 1/(1-p^{-s})$, in the subset of complex plane, where the factors Z_{p_i} are complex algebraic numbers. The idea is to regard the value of $\hat{\zeta}$ as an element of an infinite-dimensional algebraic extension of the rationals containing all roots of primes. $\hat{\zeta}$ can be regarded as a vector with infinite number of components and is completely well defined despite the fact that the product expansion does not converge as an ordinary complex number unless one somehow specifies how the 'producting' is done.

In case that the factors $|Z_{p_1}|^2$ of the partition functions $Z_{p_1} = 1/(1-p^{-z})$ are complex rationals, one can rewrite the product formula by applying adelic formula to the norm squared $|Z_{p_1}|^2$ appearing in the product formula. The basic hypothesis is that the product of the p-adic norms of the complex norm squared of the function $\hat{\zeta}$ defined by the product formula obtained by changing the order of producting gives the norm squared of the analytically continued ζ in the region $(Re[s] < 1\ ,\ Im[s] \neq 0)$ at the points, where the factors $|Z_{p_1}|^2$ are algebraic numbers: $|\hat{\zeta}|^2 = \prod_p N_p(|\hat{\zeta}|^2) = |\zeta|^2$. A milder version of this hypothesis is that the product of the p-adic norms squared of $|\hat{\zeta}|^2$ converges to some function proportional to $|\zeta|^2$.

If this hypothesis is correct, the following vision giving good hopes about the proof of the Riemann hypothesis, suggests itself.

1. $|\hat{\zeta}|^2$ is a number in an infinite-dimensional algebraic extension of rationals and can vanish only if it contains a rational factor which vanishes. The vanishing of this factor is possible if it is a product of an infinite number of moduli squared $|Z_{p_1}(z)|^2$ having a rational value. For the values of y for which this is true on the line $Re[s] = n + 1/2$ correspond to the phases p_1^{-iy} having the following general form.

$$\begin{array}{l} p^{-iy} = U_1 U = \frac{(r_1+is_1\sqrt{k(p_1,y)})}{\sqrt{p_1}} \times \frac{(r+is\sqrt{k(p_1,y)})}{n_1} \quad , \\ r_1^2 + s_1^2 k(p_1,y) = p_1 \quad , \\ r^2 + s^2 k(p_1,y) = n_1^2 \quad . \end{array}$$

   $r_1^2 + s_1^2 k(p_1,y) = p_1$ condition is solved by $k(p_1,y) = \sqrt{p_1 - m^2}$, $m < \sqrt{p}$. $r^2 + s^2 k(p_1,y) = n_1^2$ condition is satisfied if U is a product of even powers of the phases of type U_1. Unless $k(p_1,y)$ is not square, the phases correspond to orthogonal triangles with one short side having integer valued length and the other sides having integer valued length squared.

2. If y defines rational value of $|Z_{p_1}(z)|^2$, also its integer multiples ny do the same. If the values of integers $k(p_1,y)$ do not depend on the value of y, the allowed values of y generate an additive group having integers as a coefficient ring. Even powers of the phases guaranteing the rationality of $|Z_{p_1}(z)|^2$ on the line $Re[s] = 1/2$, guarantee rationality on the lines $Re[s] = n$.

3. Especially important subset of these phases correspond to the choice $k_1 = 1$. These phases correspond to Gaussian primes having the form $G = r_1 + is_1$, $r_1^2 + s_1^2 = p_1$, $p_1\ mod\ 4 = 1$, and can compensate the irrationality of the $p_1^{-n-1/2}$ factor only in this case. The products of the squares of Gaussian primes define Pythagorean triangles and the corresponding phases are rational. Rather interestingly, the linear superpositions $y = n_1 y_2 + n_2 y_2$ of only *two* Pythagorean values of y_i form a dense subset of reals. Eisenstein primes having the general form $r_1 + s_1 w$, $w = -1/2 \pm \sqrt{3/2}$, $r_1^2 + s_1^2 - r_1 s_1 = p_1$, $p_1\ mod\ 3 = 1$, are second, probably very important class of complex primes. They can compensate the irrationality of the $p_1^{-n-1/2}$ factor for $p_1\ mod\ 3 = 1$ (note that the 1/2 is not relevant for the phase). Also other phases are needed since for primes satisfying $p_1\ mod\ 4 = 3$ and $p_1\ mod\ 3 = 2$ simultaneously neither Gaussian nor Eisenstein primes can compensate the irrationality of the $p_1^{-1/2} p_1^{-iy}$ factor.

4. The lines on which the real parts for an infinite number of factors Z_{p_1} can be rational, correspond to the lines $Re[s] = n/2$. This in turns leads to the conclusion that the norm squared of $\hat{\zeta}$ can vanish only on the lines $Re[s] = n/2$. If the norm squared of the $\hat{\zeta}$ coincides with the norm squared of the analytically continued ζ, Riemann hypothesis follows since it is known that the lines $Re[s] = n/2\ ,\ n \neq 1$ do not contain zeros of ζ.

In the following this vision is developed in detail and it is shown that it survives the basic tests.

8.3.1 Detailed realization of the Universality Principle

Universality Principle states that ζ vanishes only if $|\hat{\zeta}|^2$ understood as a number in an infinite-dimensional algebraic extension of rationals vanishes and hence must contain a rational factor resulting from an infinite number of rational factors Z_{p_1}. This hypothesis alone makes Riemann hypothesis very plausible. In this section an attempt to reduce the Universality Principle to something more concrete is made. Adelic formula and the hypothesis that the norm of $|\hat{\zeta}|^2$ defined by the modified adelic formula equals to $|\zeta|^2$ are described and shown to imply Universality Principle if the modified adelic formula defines a norm in the infinite-dimensional algebraic extension of rationals. The conditions guaranteing the rationality and the reduction of the p-adic norm of $|Z_{p_1}|^2$ are derived, and the connection between Pythagorean phases and basic facts about Gaussian and Eisenstein primes are summarized.

Modified adelic formula and Universality Principle

Although the product representation of ζ does not converge absolutely for $Re[s] \leq 1$, one can consider the possibility that the convergence of the function $\hat{\zeta}$ defined by the product representation occurs in some exceptional points in some natural sense. The points at which the value of $\hat{\zeta}$ belongs to the infinite-dimensional algebraic extension of rationals are obviously excellent candidates for these points. $\hat{\zeta}$ identified as an element of this algebraic extension certainly exists mathematically as a vector with an infinite number of components. The convergence in the strong sense would mean that the interpretation of the algebraic numbers of the algebraic extension as real numbers in the expression of $\hat{\zeta}$ gives the analytically continued ζ somehow. In the weak sense the convergence would mean that the complex norm squared for $\hat{\zeta}$, if defined in a suitable sense, equals or is proportional, to the norm squared of the analytically continued ζ.

1. Modified Adelic formula and Universality Principle

The fact that the product formula for ζ at rational points converges only conditionally, suggests that one should be able to device a natural method of 'producting' giving rise to the norm squared of the analytically continued ζ. Adelic formula provides very attractive approach to this problem (the appearance of the norm squared instead of norm is motivated by the Adelic formula).

The adelic formula expresses the real norm of a rational number as a product of the inverses of the p-adic norms

$$\frac{1}{|x|_R} = \prod_p |x|_p \ . \tag{8.3.1}$$

This formula generalizes also to the norms of the complex rationals. How to generalize this formula to the infinite-dimensional algebraic extension of rationals? The simplest possibility is to write the complex norm squared as vector in the infinite-dimensional extension having rational coefficients and to apply adelic formula to each factor separately.

$$\begin{aligned} |x|_R &= \sum_k e_R^{k)} \prod_p |\frac{1}{x_k}|_p \ , \\ |x| &= \sum_k e^{k)} x_k \ . \end{aligned} \tag{8.3.1}$$

Here $e^{k)}$ denote the units of the infinite-dimensional algebraic extension (products of roots of primes and analogous to imaginary unit) and $e_R^{k)}$ denote the evaluations of these units identified as real numbers. The resulting norm is indeed equal to the real norm when the resulting number is interpreted as a real number.

In the case that the factors Z_{p_1} of ζ are complex rationals, one can write the real norm of the real ζ for $Re[s] > 1$ as a product

$$|\zeta(z)|^2 = \ = \prod_{p_1} \left[\prod_p N_p(|\frac{1}{Z_{p_1}(z)}|^2) \right] \equiv \prod_{p_1} \left[\prod_p N_p(|Z_{p_1}^{p)}(z)|^2) \right] \ . \tag{8.3.2}$$

Here $N_p(x)$ denotes the p-adic norm of number x. This formula explains why one must define the p-adic zeta as an arithmetic inverse of the real ζ. The generalization of this formula to the case that $\hat{\zeta}^2$ has values in the set of the complex rationals is straightforward.

The problem with this representation is that the product over primes p_1 does not converge in an absolute sense for $Re[s] \leq 1$. By a suitable rearrangement of a conditionally convergent product a convergence to any number can be achieved. This suggests that one could find some unique manner to rearrange the terms to a convergent expression converging to $|\zeta|^2$. A unique definition indeed suggests itself: the analytic continuation of ζ from the region $Re[s] > 1$ might be equivalent with the exchange of the order of 'producting' in the expression of ζ:

$$\begin{aligned} |\hat{\zeta}(z)|^2 &= \prod_p N_p(|\frac{1}{\zeta(z)}|^2 = \prod_p \left[\prod_{p_1} N_p(|\frac{1}{Z_{p_1}(z)}|) \right] \\ &= \prod_p N_p(|\frac{1}{\zeta}|^2) = \prod_p N_p(|\zeta^{p)}|)^2 \quad . \end{aligned} \tag{8.3.2}$$

The minimal working hypothesis is that $|\hat{\zeta}|^2$ defined as the product its p-adic norms equals to $|\zeta|^2$ at points, where its values are *rational*:

$$\prod_p N_p(|\hat{\zeta}|^2) = |\zeta|^2 \quad . \tag{8.3.3}$$

The generalization to the algebraic extension of rationals is straightforward since the p-adic norm squared is sum over the p-adic norms of the components of the algebraic extension with various units $e^{k)}$ of the algebraic extension multiplying them interpreted as real numbers $e_R^{k)}$

$$\begin{aligned} \prod_p N_p(|\hat{\zeta}|^2) &= \sum_k e_R^{k)} \prod_p N_p(\frac{1}{|\hat{\zeta}|_k^2}) = |\zeta|^2 \quad , \\ |\hat{\zeta}|^2 &= \sum_k e^{k)} |\zeta|_k^2 \quad . \end{aligned} \tag{8.3.3}$$

From this formula Universality Principle follows automatically. Since $|\hat{\zeta}|^2$ can be regarded as a vector having infinite number of components, the only manner to achieve the vanishing of $\prod_p N_p(|\hat{\zeta}|^2)$ is to require that it contains a vanishing rational factor. As will be found, the points at which infinite number of the factors of $|\hat{\zeta}|^2$ can be rational, very probably belong to the lines $Re(s) = n/2$. Thus the Universality Principle, and as it seems, also Riemann hypothesis, reduces to the statement that the modified Adelic formula defines a genuine norm which vanishes only when the vector is a null vector and is equal to $|\zeta|^2$. Of course, one could consider also the possibility that this norm is proportional to $|\zeta|^2$.

The conditions guaranteing the rationality of the factors $|Z_{p_1}|^2$

Universality Principle states that zeros of ζ correspond to zeros of $|\hat{\zeta}|^2$. This quantity, when well-defined, belongs to an infinite-dimensional real algebraic extension of rationals, and its vanishing is possible if it contains a vanishing rational factor which is product of an infinite number of factors Z_{p_1} which are rational. $|\hat{\zeta}|^2$ is the product of the factors

$$\frac{1}{Z_{p_1}(x+iy)Z_{p_1}(x-iy)} = 1 - 2p_1^{-x} Re[p_1^{iy}] + p_1^{-2x} \quad . \tag{8.3.4}$$

This expression equals to a rational number q, if one has

$$Re[p_1^{iy}] = \frac{qp_1^x - p_1^{-x}}{2} \quad . \tag{8.3.5}$$

In this case the integer multiples ny do not satisfy the rationality condition, to say nothing about the superpositions of different values of y. It is also implausible that this condition would hold true for an infinite number of primes p_1 required by the vanishing of a rational factor of $\hat{\zeta}$.

An alternative manner to achieve rationality is by requiring that the two terms are separately rational. p_1^{-2x} factor is rational only if one has $x = n/2$. To achieve rationality $Re[p_1^{iy}]$ should contain a factor compensating the irrationality of the $p_1^{-n/2}$ factor somehow. On the lines $Re[s] = x = n/2$ one has

$$\frac{1}{Z_{p_1}(n/2+iy)Z_{p_1}(n/2-iy)} = 1 - 2p_1^{-n/2}Re[p_1^{iy}] + p_1^{-n} \ .$$

It is of crucial importance that the moduli squared depend on the real part of p_1^{iy} only. If this is rational, rationality is achieved for even values of n.

On the lines $Re[s] = n + 1/2$ rationality is achieved provided that p_1^{iy} factors contain the phase factor $(r_1 + is_1\sqrt{k})/\sqrt{p_1}$ compensating the $p_1^{-1/2}$ factor and multiplying a factor which of the same type:

$$\begin{aligned} p_1^{iy} &= U_1 U = \frac{(r_1+is_1\sqrt{k})}{\sqrt{p_1}} \times \frac{(r+is\sqrt{k})^2}{r^2+s^2k} \ , \\ r_1^2 + s_1^2 k_1 &= p_1 \ . \end{aligned} \quad (8.3.5)$$

The latter equation is satisfied if one has

$$k = \sqrt{p_1 - m^2} \ , \ 0 < m < \sqrt{p} \ . \quad (8.3.6)$$

On the lines $Re[s] = n$ one must have

$$p_1^{iy} = \frac{(r+is\sqrt{k})^2}{r^2+s^2k} \ . \quad (8.3.7)$$

The overall conclusions are following.

1. The vanishing of $|\hat{\zeta}|^2$ requires only the rationality of the *real parts* of Z_{p_1} for infinite number of values of p_1. The basic ansatz allows rationality only on the lines $Re[s] = n/2$ and my subjective feeling is that it is extremely implausible that exceptional ansatz gives rise to the rationality of an infinite number of $|Z_{p_1}|^2$ factors. That this is really the case might turn out to be difficult part in attempts to prove Riemann hypothesis even if one has proved the identity $\prod_p N_p(|\hat{\zeta}|^2) = |\zeta|^2$ and that this product defines a norm.

2. Rationality requirement allows p_1^{-iy} to consist of the products of the phases of very general algebraic numbers $r + is\sqrt{k}$. The products of these numbers are always of same form and their norm squared is $r^2 + s^2k$. Geometrically these numbers correspond to orthogonal triangles with one or two sides having integer valued length and remaining side having integer valued length squared.

3. For given value of y all integer multiples ny of y provide a solution of the rationality conditions. It is not necessary to require that the algebraic extensions $r + is\sqrt{k(p_1, y_i)}$ associated with y_1 and y_2 satisfying the condition, are same for given value of p_1: that is, one can have

$$k(p_1, y_1) \neq k(p_1, y_2) \ .$$

For $k(p_1, y_1) = k(p_1, y_2)$ also the linear combinations $m_1 y_1 + n_1 y_2$ satisfy rationality conditions. For the minimal solution to the rationality conditions, only multiples of each y solve the rationality conditions. For the maximal solution all solutions y_i correspond to the same algebraic extension for given p_1 and unrestricted linear superposition of the y_i holds true.

4. For $p\ mod\ 4=1$ rational phase factors p_1^{-iy} defined by the powers of the Gaussian primes provide the minimal manner to achieve rationality such that unrestricted superposition of solutions holds true. For $p_1\ mod\ 4=3$ and $p_1\ mod\ 3=1$ the minimal manner to achieve compensation is by using Eisenstein primes. For the primes $p_1\ mod\ 4=3$ and $p_1\ mod\ 3=1$ one cannot compensate $\sqrt{p_1}$ factor using Gaussian or Eisenstein primes and a more general algebraic extension of integers is necessary. For given prime p_1 there is finite number of possible algebraic extensions.

The conditions guaranteing the reduction of the p-adic norm

The term p_1^{-iy} appearing in the factors $1/Z_{p_1}$ is inversely proportional to integers and thus have p-adic norm which is larger than one for the primes appearing as factors of the integer n_1. Some mechanism guaranteing the reduction of the p-adic norm must be at work and this mechanism gives strong conditions on the allowed phases p_1^{iy}.

The condition guaranteing the reduction is very general. What is required is the reduction of the p-adic norm

$$|X\overline{X}|_p \ , \quad X=1-Up_1^{iy} \ , \quad U=(\epsilon p_1)^{-n/2} \ . \tag{8.3.8}$$

Here one has $\epsilon=1$ for even values of n whereas for for odd values of n one has $\epsilon=\pm 1$ depending on whether the square root exists or not p-adically: the sole purpose of this factor is to take care that the p-adic counterpart of U is an ordinary p-adic number.

By writing

$$p_1^{-iy} \equiv cos(\phi)+isin(\phi) \ ,$$

one obtains

$$|X\overline{X}|_p = |1-2Ucos(\phi)+U^2|_p \ .$$

Not surprisingly, the vanishing of the norm modulo p implies in modulo p accuracy

$$U=cos(\phi)-\sqrt{-1}sin(\phi) \ .$$

Since U must be real, the only possible manner to satisfy the condition is to require that

$$sin(\phi)=0\ \ mod\ \ p \ , \quad cos(\phi)=1\ \ mod\ \ p \ . \tag{8.3.9}$$

Clearly, ϕ must correspond to angle 0 or π in modulo p accuracy. What this condition says is that partition functions Z_{p_1} are real in order p. This is very natural condition on the line $Re[s]=1/2$ where the ζ is indeed real.

The condition $cos^2(\phi)=1\ mod\ p$ implies

$$p_1^n\ mod\ p=1 \ . \tag{8.3.10}$$

p_1 can be always written as a power $p_1=a^k$ of a primitive root a satisfying $a^{p-1}=1$ modulo p such that k divides $p-1$. Thus $p_1^n\ mod\ p=1$ holds true only only if $n\ mod\ (p-1)/k=0$ is satisfied.

The conditions guaranteing modulo p reality of Z_{p_1} for prime p dividing the denominator of p_1^{-iy}, when written explicitly, give

$$\begin{array}{lll} Re[s]=n\ : & r^2-s^2k=r^2+s^2k \ , & \frac{2rs}{r^2+s^2k}=0 \ , \\ Re[s]=n+\frac{1}{2}\ : & (r^2-s^2k)r_1-2rss_1k=r^2+s^2k \ , & \frac{2rsr_1+(r^2-s^2k)s_1}{r^2+s^2k}=0 \ . \end{array} \tag{8.3.10}$$

In the case of Gaussian primes ($k=1$) also second option is possible since the multiplication with $\pm i$ yields new rational phase factor: this option corresponds simply the exchange of r^2-s^2 and $2rs$ factors in the formula above.

Rather general solution to the conditions can be written rather immediately. In both cases the conditions

$$s \ mod \ p^2 = 0 \ , \quad r \ mod \ p = 0 \tag{8.3.11}$$

are satisfied. Note that $s \ mod \ p^2 = 0$ is necessary since $r^2+s^2k \ mod \ p = 0$ holds true. Besides this the conditions

$$\begin{array}{ll} r_1^2 + s_1^2 k \ mod \ p = 1 & \text{for } Re[s] = n \ , \\ s_1 \ mod \ p = 0 \ \& \ r_1 \ mod \ p = 1 & \text{for } Re[s] = n + \frac{1}{2} \ , \end{array} \tag{8.3.12}$$

are satisfied.

If p_1^{-iy} is inversely proportional to integer containing as factors powers of a prime p larger than p_1, the reduction of the norm cannot occur for $Re[s]=1/2$ but is possible for sufficiently large values of $Re[s]=n/2$. For $p_1=2$ and $p_1=3$ factors the reduction of the norm is certainly not possible on the line $Re[s]=1/2$ since the condition $2p+1\leq p_1$ cannot be satisfied for any prime in these cases. The reduction of the p-adic norm of the ζ suggests strongly that the condition $2p_i+1\leq p_1$ is satisfied for large primes p_1 and odd primes p_i. The condition is satisfied always for $p_i=2$ and $p_1\geq 3$. If it is satisfied completely generally, the phase factors associated with Z_3 must be of the general form

$$3^{-iy} = \frac{(\pm 1 \pm \sqrt{2}i)}{\sqrt{3}} \times \frac{(r(y)+i\sqrt{2}s(y))^2}{r^2(y)+2s^2(y)} \ , \quad r^2(y)+2s^2(y) = 3^k \text{ or } 2\times 3^k \ .$$

This condition and similar conditions associated with larger primes give very strong constraints on the zeros.

The general conclusions are following.

1. The reduction of the p-adic norm and the related modulo p reality of Z_{p_1} is the p-adic counterpart for the reality of ζ on the critical line which suggests that it might occur completely generally. It requires that $p_1^n \ mod \ p = 1$ holds true for all primes appearing as factors of the denominator n_1 of the rational part of the phase p_1^{-iy}.

2. If the denominator of p_1^{-iy} is square-free integer, the p-adic norm of Z_{p_1} is never larger than unity except possibly in the diagonal case $p=p_1$.

3. In the diagonal case the norm grows like p_1^{n+1} for $Re[s]=n+1/2$ and p_1^n for $Re[s]=n$. This conforms with the fact that ζ has no zeros for $Re[s]\geq 1$ but has zeros for $Re[s]=-2n$.

4. If rational points of ζ obey linear superposition, then the rational points on the lines $Re[s]=n$ contain an even number of y_i:s needed to achieve the rationality of $Re[p^{-iy}]$. Hence the denominator tends to have larger p-adic norm than it can have on the line $Re[s]=1/2$. This means that the line $Re[s]=1/2$ is optimal as far as zeros of $|\hat{\zeta}|^2$ are considered. It can however happen that in the product $p_1^{iy_1}p_1^{iy_2}$ complex conjugates of factor phases can compensate each other so that the p-adic norm of $p_1^{i(y_1+y_2)}$ is not always larger than the norms of the factors. In particular, the factors $(r_1+is_1\sqrt{k})/\sqrt{p_1}$ could cancel in the product $p_1^{iy_1}p_1^{-iy_2}$ This mechanism could imply the emergence small values of ζ for $y_{ij}=y_i-y_j$ on the line $Re[s]=1$ required by the inner product property of the Hermitian form defined by the super-conformal model for the zeros of ζ.

Gaussian primes and Eisenstein primes

The general manner to satisfy the rationality requirement is to assume that the phases p_1^{iy} correspond to orthogonal triangles with one or two sides with an integer valued length and one side with integer valued length squared. A rather general and mathematically highly interesting manner to realize the rationality of the the phases $p_1^{-n/2}p_1^{iy}$ is by choosing the phases to be products of Gaussian or Eisenstein primes [20].

Gaussian primes consist of complex integers $e_i \in \{\pm 1, \pm i+\}$, ordinary primes $p\ mod\ 4 = 3$ multiplied by the units e_i to give four different primes, and complex Gaussian primes $r \pm is$ multiplied by the units e_i to give 8 primes with the same modulus squared equal to prime $p\ mod\ 4 = 1$. Every prime $p\ mod\ 4 = 1$ gives rise to 8 non-degenerate Gaussian primes. Pythagorean phases correspond to the phases of the squares of complex Gaussian integers $m + in$ expressible as products of even powers of Gaussian primes $G_p = r + is$:

$$G_p = r + is \ , \quad G\overline{G} = r^2 + s^2 = p \ , \quad p \text{ prime } \& \ p \ \ mod \ \ 4 = 1 \ . \tag{8.3.13}$$

The general expression of a Pythagorean phase expressible as a product of even number of Gaussian primes is

$$U = \frac{r^2 - s^2 + i2rs}{r^2 + s^2} \ . \tag{8.3.14}$$

By multiplying this expression by a Gaussian prime i, one obtains second type of Pythagorean phase

$$U = \frac{2rs + i(r^2 - s^2)}{r^2 + s^2} \ . \tag{8.3.15}$$

Gaussian primes allow to achieve rationality of $p_1^{-n+1/2}p^{-iy}$ factor for $p_1\ mod\ 4 = 1$. The generality of the mechanism suggests that Gaussian primes should be very important. For $Re[s] \neq n/2$ it is not possible to achieve complex rationality with any decomposition of p_1^{iy} to Gaussian primes.

Besides Gaussian primes also so called Eisenstein primes are known to exist [20] and the fact that only the rationality of the real parts of $1/Z_{p_1}$ factors is necessary for the rationality of $|Z_{p_1}|^2$ means that they are also possible. Note however that now the multiplication the phase by $\pm i$ makes the real part of the phase irrational, and is thus not allowed. Thus only four-fold degeneracy is present now for ζ.

Whereas Gaussian primes rely on modulo 4 arithmetics for primes, Eisenstein primes rely on modulo 3 arithmetics. Let $w = exp(i\phi)$, $\phi = \pm 2\pi/3$, denote a nontrivial third root of unity. The number 1-w and its associates obtained by multiplying this number by ± 1 and $\pm i$; the rational primes $p\ mod\ 3 = 2$ and its associates; and the factors $r + sw$ of primes $p\ mod\ 3 = 1$ together with their associates, are Eisenstein primes. One can write Eisenstein prime in the form

$$w = r - \frac{s}{2} + is\frac{\sqrt{3}}{2} \ . \tag{8.3.16}$$

What might be called Eisenstein triangles correspond to the products of powers of the squares of Eisenstein primes and have integer-valued long side. The sides of the orthogonal triangle associated with a square of Eisenstein prime E_p have lengths

$$(r^2 - rs - \frac{3s^2}{2} \ , \ s\frac{\sqrt{3}}{2} \ , \ p = r^2 + s^2 - rs) \ .$$

Eisenstein primes clearly span the ring of the complex integers having the general form $z = (r + i\sqrt{3}s)/2$, r and s integers.

One can use Eisenstein prime E_p to achieve the replacement of the $p_1^{-1/2}$-factor with $1/p_1$-factor in the partition functions Z_{p_1} the same effect for $p_1\ mod\ 4 = 1$ and $p_1\ mod\ 3 = 1$ with the net result that $i\sqrt{3}$ term appears. This trick does not work for $p_1\ mod\ 4 = 3$ and $p_1\ mod\ 3 = 2$. Note that the presence of *both* Gaussian and Eisenstein primes in the same factor Z_{p_1} is not allowed since in this case also the real part of Z_{p_1} would contain $\sqrt{3}$. This suggests that quite generally $p\ mod\ 4 = 1$ *resp.* $p\ mod\ 4 = 3 \wedge p\ mod\ 3 = 1$ parts of $\hat{\zeta}$ could correspond to Gaussian *resp.* Eisenstein primes.

For the factors Z_{p_1} satisfying $p_1\ mod\ 4 = 3$ & $p_1\ mod\ 3 = 2$ simultaneously, neither Gaussian nor Eisenstein primes can affect the rationalization of $p^{-n+1/2-iy}$ factor, and in this case more general algebraic extension of complex numbers is necessary as already found.

The algebraic extensions of rational numbers allow the notion of algebraic integer and prime quite generally [26]. In the general case however the decomposition of an algebraic integer into primes is not unique. In case of complex extensions of form $r+\sqrt{-d}s$ unique prime factorization is obtained only in nine cases corresponding to $d=1,2,3,7,11,19,46,67,163$ [26]. $\sqrt{-d}$ corresponds to a root of unity only for $d=1$ and $d=3$, which perhaps makes Gaussian and Eisenstein primes special.

8.3.2 Tests for the $|\hat{\zeta}|^2=|\zeta|^2$ hypothesis

The fact that the phases p_1^{iy} correspond to non-vanishing values of y, suggests that $|\hat{\zeta}|^2=|\zeta|^2$ equality holds on the real axis only in the sense of a limiting procedure $y\rightarrow 0$. If the the values of y giving rise to allowed phases obey linear superposition (that is $k_1(p_1,y)$ defining the algebraic extension does not depend on y), the allowed values of y form a dense set of the real axis, since arbitrarily small differences y_i-y_j are possible for the zeros of ζ. Hence the limiting procedure $y\rightarrow 0$ should be well-defined and give the expected answer if the basic hypothesis is correct.

What happens on the real axis?

The simplest test for the basic hypothesis is to look what happens on the real axis at the points $s=n$. Real ζ diverges at $s=1$ and $s=0$ and has trivial zeros are at the points $s=-2n$. The norm of $\hat{\zeta}$ is given by

$$|\hat{\zeta}(n)|_R \quad = \quad \prod_p\left[\prod_{p_1}|1-p_1^{-n}|_p\right] \quad . \tag{8.3.17}$$

For $n=0$ a straightforward substitution to the formula implies that $|\hat{\zeta}(0)|$ vanishes. For $n>0$ one has

$$|\hat{\zeta}(n)|_R \quad = \quad \prod_p\left[\prod_{p_1}|\frac{p_1^n-1}{p_1^n}|_p\right]=\prod_p p^n\left[\prod_k \prod_{p_1^n \; mod \; p^k=1} p^{-k}\right] \quad . \tag{8.3.18}$$

Since the number of primes p_1 satisfying the condition $p_1^n \; mod \; p^k=1$ is infinite, the norm vanishes for all values $n>0$. For $s=-n<0$ one has,

$$|\hat{\zeta}(n)|_R \quad = \quad \prod_p\left[\prod_{p_1}|1-p_1^{n}|_p\right] \quad . \tag{8.3.19}$$

and also this product vanishes always.

How to understand these results?

1. The results are consistent with the view that $|\zeta|_R$ on the real axis should be estimated by taking the limit $y\rightarrow 0$. Since the values of y in question involve necessarily differences of very large values of y, it is conceivable that the limiting procedure does not yield zero. That the limiting procedure can give zero for $Re[s]<0$ could be partially due to the fact that for $Re[s]=-n<0$ one has for the diagonal $p_1=p$ contribution $|Z_p(-n+iy)|_p=1$ whereas for $Re[s]=n>0$ one has $|Z_p(n+iy)|_p>1$ in general. Furthermore, for $Re[s]=-n$ only $p_1^n \; mod \; p^k=1$ condition leads to the reduction of the p-adic norm of $Z_{p_1\neq p}$ whereas for $Re[s]=-2n$ also $p_1^n \; mod \; p^k=-1$ condition has the same effect.

2. One cannot exclude the possibility that only the proportionality $|\hat{\zeta}|^2\propto|\zeta|^2$ holds true. For instance, in the super-conformal model predicting that the physical states of the model correspond to the zeros of ζ on the critical line, the Hermitian form defining the 'inner product' is proportional to the product of $sin(i\pi z)\Gamma(z)\zeta(z)$. This function vanishes for $Re[s]\notin\{0,1\}$ and the coefficient function of ζ is finite in the critical strip. For $s=0$ this function however has the value $-1/2$ and for $s=1$ the value is 1, whereas the naively evaluated value of $|\hat{\zeta}|$ vanishes identically at these points. Thus something else is necessarily involved.

3. It could also be that the product representation for the norm squared of $\hat{\zeta}$ as a product of its p-adic norms converges only in a restricted region. It would not be surprising if the negative values of y were excluded from the region of convergence for the representation of $|\hat{\zeta}|^2$ as a product of its p-adic norms. Concerning the proof of the Riemann hypothesis, the minimal requirement is that the region $[1/2 \leq Re[s] \leq 1\ ,\ y \neq 0]$ is included in the region of convergence.

One might think that $|\zeta|^2 = |\hat{\zeta}|^2$ hypothesis is testable simply by comparing the norm squared of the real zeta with the product of the p-adic norms of $|\hat{\zeta}|^2$. The problems are that the value for the product of p-adic norms is extremely sensitive to numerical errors since the p-adic norm of Pythagorean triangles phases fluctuates wildly as a function of the phase angle, and that one does not actually know what the values of p_1^{iy} actually are. One testable prediction, also following from the super-conformal model of the Riemann Zeta, is that the superpositions of the zeros are probably small values or minima of $|\zeta|_R$ on the lines $Re[s] = n/2$. More precisely, it is the function $G(1 + iy_{12})$ which should have values smaller than one if the metric defined by G is Hermitian. One could also try to understand whether the the norm of $\hat{\zeta}$ allows a continuation to a continuous function of the complex argument identifiable as a modulus of an analytic function.

Can the imaginary part of $\hat{\zeta}$ vanish on the critical line?

Riemann Zeta is real on the critical line $Re[s] = 1/2$. A natural question is whether also $\hat{\zeta}$ has a vanishing imaginary part on this line. This is certainly not necessary since $\hat{\zeta}$ has values in the infinite-dimensional algebraic extension of rationals. It would be however highly desirable if this condition would hold true.

One cannot formulate the vanishing condition for the imaginary part in terms of the norm squared of any quantity defined by using the generalization of the adelic formula. The vanishing of the imaginary part of $\hat{\zeta}$ is however consistent with the Universality Principle. One can see this by expanding the factors $Z_{p_1} = 1/(1 - p_1^{-1/2-iy})$ to a geometric series in powers of the irrational imaginary part of $p_1^{-1/2-iy}$. Each odd term in this series is proportional to $\sqrt{k(p_1, y)}$. One can combine the product of all these geometric series with the same value $k(p_1, y) = k$ to a sum of a rational part and an irrational part proportional to $\sqrt{k}$. If the irrational parts vanish separately for all allowed values of k, the imaginary part of $\hat{\zeta}$ indeed vanishes. This requires that the same value of $k(p_1, y) = k$ is associated with an infinite number of factors Z_{p_1}.

What is interesting is that the terms appearing in the sum over primes p_1 with the same value of k are proportional to $1/p_1^n$, $n \geq 1$: $n = 1$ terms are on the borderline at which the absolute convergence fails. If the number of primes p_1 with the same value of k is sufficiently small, also the sum over $n = 1$ terms with given k converges. The allowed values of k are given by $k = \sqrt{p_1 - m^2}$, $m \leq \sqrt{p_1}$ and the simplest hypothesis is that each value of k appears with same probability so that for a given prime p_1 the probability for a $k(p_1, y) = k$ is $P(k) \sim 1/\sqrt{p_1}$. This would suggest that the lowest terms in the sum defining the imaginary part behaves as $1/p_1^{3/2}$ so that convergence is indeed achieved. Note that convergence requirement does not support the special role of Gaussian or Eisenstein primes in the set of algebraic numbers appearing in the expansion of $\hat{\zeta}$.

The general algebraic properties of $\hat{\zeta}$ must be consistent with the vanishing of $Im[\zeta]$ on the critical line. The reality of ζ on the critical line follows from the symmetry with respect to the critical line reducing on the critical line to the condition $\zeta(s) = \zeta(1 - s)$ implying the reality of $\zeta(s)\zeta(1 - s)$. This condition makes sense also for $\hat{\zeta}$. In general case, one has

$$\hat{\zeta}(s)\hat{\zeta}(1-s) = \prod_{p_1} Z_{p_1}(x+iy)Z_{p_1}(1-x-iy) = \prod_{p_1} \frac{1}{\left[1 - p_1^{-x}p_1^{-iy} - p_1^{-1+x}p_1^{iy} + \frac{1}{p_1}\right]} \ .$$

Due to the presence of p^{-x} terms, the moduli squared for these factors are complex irrational numbers.

On the line $Re[s] = 1/2$, the product representation for this function reduces to the product of *real* factors

$$\frac{1}{Z_{p_1}(1/2+iy)Z_{p_1}(1/2-iy)} = 1 - p_1^{-1/2}(p_1^{iy} + p_1^{-iy}) + \frac{1}{p_1} \tag{8.3.20}$$

in the algebraic extension of rationals. Thus the reality and rationality of the function $\hat{\zeta}(s)\hat{\zeta}(1-s)$ on the critical line corresponds in a very transparent manner the reality of ζ on the critical line. Note also that the modulo p reality of the factors Z_{p_1} implied by the reduction of the p-adic norm can be regarded as the p-adic counterpart for the reality of ζ on the critical line.

What about non-algebraic zeros of ζ?

In principle real ζ could also have non-algebraic zeros. The following argument however demonstrates that they do not pose a problem. If Universality Principles holds true, and if the norm squared of $\hat{\zeta}$ defined as a product of its p-adic norms indeed equals to the norm squared of the real ζ in the set of of complex plane in which the factors $1/(1-p^{-s})$ are algebraic numbers, one obtains strict bounds for the norm of the real ζ excluding the zeros in the dense set inside the critical strip. The continuity of the real ζ in turn implies that it cannot vanish except on the critical line.

8.4 Riemann hypothesis and super-conformal invariance

Hilbert and Polya [22] conjectured a long time ago that the non-trivial zeroes of Riemann Zeta function could have spectral interpretation in terms of the eigenvalues of a suitable self-adjoint differential operator H such that the eigenvalues of this operator correspond to the imaginary parts of the non-trivial zeros $z = x + iy$ of ζ. One can however consider a variant of this hypothesis stating that the eigenvalue spectrum of a non-hermitian operator D^+ contains the non-trivial zeros of ζ. The eigen states in question are eigen states of an annihilation operator type operator D^+ and analogous to the so called coherent states encountered in quantum physics [27]. In particular, the eigenfunctions are in general non-orthogonal and this is a quintessential element of the the proposed strategy of proof.

In the following an explicit operator having as its eigenvalues the non-trivial zeros of ζ is constructed.

1. The construction relies crucially on the interpretation of the vanishing of ζ as an orthogonality condition in a hermitian metric which is is a priori more general than Hilbert space inner product.

2. Second basic element is the scaling invariance motivated by the belief that ζ is associated with a physical system which has super-conformal transformations [28] as its symmetries.

The core elements of the construction are following.

1. All complex numbers are candidates for the eigenvalues of D^+ (formal hermitian conjugate of D) and genuine eigenvalues are selected by the requirement that the condition $D^\dagger = D^+$ holds true in the set of the genuine eigenfunctions. This condition is equivalent with the hermiticity of the metric defined by a function proportional to ζ.

2. The eigenvalues turn out to consist of $z = 0$ and the non-trivial zeros of ζ and only the eigenfunctions corresponding to the zeros with $Re[s] = 1/2$ define a subspace possessing a hermitian metric. The vanishing of ζ tells that the 'physical' positive norm eigenfunctions (in general *not* orthogonal to each other), are orthogonal to the 'un-physical' negative norm eigenfunction associated with the eigenvalue $z = 0$.

The proof of the Riemann hypothesis by reductio ad absurdum results if one assumes that the space $\mathcal{V}$ spanned by the states corresponding to the zeros of ζ inside the critical strip has a hermitian induced metric. Riemann hypothesis follows also from the requirement that the induced metric in the spaces subspaces $\mathcal{V}_s$ of $\mathcal{V}$ spanned by the states Ψ_s and $\Psi_{1-\overline{s}}$ does not possess negative eigenvalues: this condition is equivalent with the positive definiteness of the metric in $\mathcal{V}$. Conformal invariance in the sense of gauge invariance allows only the states belonging to $\mathcal{V}$. Riemann hypothesis follows also from a restricted form of a dynamical conformal invariance in $\mathcal{V}$. This allows the reduction of the proof to a standard analytic argument used in Lie-group theory.

8.4.1 Modified form of the Hilbert-Polya conjecture

One can modify the Hilbert-Polya conjecture by assuming scaling invariance and giving up the hermiticity of the Hilbert-Polya operator. This means introduction of the non-hermitian operators D^+ and D which are hermitian conjugates of each other such that D^+ has the nontrivial zeros of ζ as its complex eigenvalues

$$D^+\Psi = z\Psi. \tag{8.4.1}$$

The counterparts of the so called coherent states [27] are in question and the eigenfunctions of D^+ are not expected to be orthogonal in general. The following construction is based on the idea that D^+ also allows the eigenvalue $z = 0$ and that the vanishing of ζ at z expresses the orthogonality of the states with eigenvalue $z = x + iy \neq 0$ and the state with eigenvalue $z = 0$ which turns out to have a negative norm.

The trial

$$\begin{aligned} D &= L_0 + V, \quad D^+ = -L_0 + V \\ L_0 &= t\tfrac{d}{dt}, \qquad V = \tfrac{dlog(F)}{d(log(t))} = t\tfrac{dF}{dt}\tfrac{1}{F} \end{aligned} \tag{8.4.2}$$

is motivated by the requirement of invariance with respect to scalings $t \rightarrow \lambda t$ and $F \rightarrow \lambda F$. The range of variation for the variable t consists of non-negative real numbers $t \geq 0$. The scaling invariance implying conformal invariance (Virasoro generator L_0 represents scaling which plays a fundamental role in the super-conformal theories [28]) is motivated by the belief that ζ codes for the physics of a quantum critical system having, not only super-symmetries [25], but also super-conformal transformations as its basic symmetries.

8.4.2 Formal solution of the eigenvalue equation for operator D^+

One can formally solve the eigenvalue equation

$$D^+\Psi_z = \left[-t\frac{d}{dt} + t\frac{dF}{dt}\frac{1}{F}\right]\Psi_z = z\Psi_z. \tag{8.4.3}$$

for D^+ by factoring the eigenfunction to a product:

$$\Psi_z = f_z F. \tag{8.4.4}$$

The substitution into the eigenvalue equation gives

$$L_0 f_z = t\frac{d}{dt}f_z = -zf_z \tag{8.4.5}$$

allowing as its solution the functions

$$f_z(t) = t^z. \tag{8.4.6}$$

These functions are nothing but eigenfunctions of the scaling operator L_0 of the super-conformal algebra analogous to the eigen states of a translation operator. A priori all complex numbers z are candidates for the eigenvalues of D^+ and one must select the genuine eigenvalues by applying the requirement $D^\dagger = D^+$ in the space spanned by the genuine eigenfunctions.

It must be emphasized that Ψ_z is *not* an eigenfunction of D. Indeed, one has

$$D\Psi_z = -D^+\Psi_z + 2V\Psi_z = z\Psi_z + 2V\Psi_z. \tag{8.4.7}$$

This is in accordance with the analogy with the coherent states which are eigen states of annihilation operator but not those of creation operator.

8.4.3 $D^+ = D^\dagger$ condition and hermitian form

The requirement that D^+ is indeed the hermitian conjugate of D implies that the hermitian form satisfies

$$\langle f|D^+g\rangle = \langle Df|g\rangle . \tag{8.4.8}$$

This condition implies

$$\langle \Psi_{z_1}|D^+\Psi_{z_2}\rangle = \langle D\Psi_{z_1}|\Psi_{z_2}\rangle . \tag{8.4.9}$$

The first (not quite correct) guess is that the hermitian form is defined as an integral of the product $\overline{\Psi}_{z_1}\Psi_{z_2}$ of the eigenfunctions of the operator D over the non-negative real axis using a suitable integration measure. The hermitian form can be defined by continuing the integrand from the non-negative real axis to the entire complex t-plane and noticing that it has a cut along the non-negative real axis. This suggests the definition of the hermitian form, not as a mere integral over the non-negative real axis, but as a contour integral along curve C defined so that it encloses the non-negative real axis, that is C

1. traverses the non-negative real axis along the line $Im[t] = 0_-$ from $t = \infty + i0_-$ to $t = 0_+ + i0_-$,
2. encircles the origin around a small circle from $t = 0_+ + i0_-$ to $t = 0_+ + i0_+$,
3. traverses the non-negative real axis along the line $Im[t] = 0_+$ from $t = 0_+ + i0_+$ to $t = \infty + i0_+$.

   Here $0_\pm$ signifies taking the limit $x = \pm\epsilon$, $\epsilon > 0$, $\epsilon \rightarrow 0$.

C is the correct choice if the integrand defining the inner product approaches zero sufficiently fast at the limit $Re[t] \rightarrow \infty$. Otherwise one must assume that the integration contour continues along the circle S_R of radius $R \rightarrow \infty$ back to $t = \infty + i0_-$ to form a closed contour. It however turns out that this is not necessary. One can deform the integration contour rather freely: the only constraint is that the deformed integration contour does not cross over any cut or pole associated with the analytic continuation of the integrand from the non-negative real axis to the entire complex plane.

Scaling invariance dictates the form of the integration measure appearing in the hermitian form uniquely to be dt/t. The hermitian form thus obtained also makes possible to satisfy the crucial $D^+ = D^\dagger$ condition. The hermitian form is thus defined as

$$\langle \Psi_{z_1}|\Psi_{z_2}\rangle = -\frac{K(z_{12})}{2\pi i}\int_C \overline{\Psi_{z_1}}\Psi_{z_2}\frac{dt}{t} . \tag{8.4.10}$$

$K(z_{12})$ is real from the hermiticity requirement and the behavior as a function of $z_{12} = z_1 + \overline{z}_2$ by the requirement that the resulting Hermitian form defines a positive definite inner product. The value of $K(1)$ can can be fixed by requiring that the states corresponding to the zeros of ζ at the critical line have unit norm: with this choice the vacuum state corresponding to $z = 0$ has negative norm. Physical intuition suggests that $K(z_{12})$ is responsible for the Gaussian overlaps of the coherent states and this suggests the behavior

$$K(z_{12}) = exp(-\alpha|z_{12}|^2) , \tag{8.4.11}$$

for which overlaps between states at critical line are

proportional to $exp(-\alpha(y_1 - y_2)^2)$ so that for $\alpha > 0$ Schwartz inequalities are certainly satisfied for large values of $|y_{12}|$. Small values of y_{12} are dangerous in this respect but since the matrix elements of the metric decrease for small values of y_{12} even for $K(z_{12}) = 1$, it is possible to satisfy Schwartz inequalities for sufficiently large value of α. It must be emphasized that the detailed behavior

of K is *not* crucial for the arguments relating to Riemann

hypothesis.

The possibility to deform the shape of C in wide limits realizes conformal invariance stating that the change of the shape of the integration contour induced by a conformal transformation, which is nonsingular inside the integration contour, leaves the value of the contour integral of an analytic function unchanged. This scaling invariant hermitian form is indeed a correct guess. By applying partial integration one can write

$$\langle \Psi_{z_1} | D^+ \Psi_{z_2} \rangle = \langle D\Psi_{z_1} | \Psi_{z_2} \rangle - \frac{K(z_{12})}{2\pi i} \int_C dt \frac{d}{dt} \left[\overline{\Psi}_{z_1}(t) \Psi_{z_2}(t) \right] . \tag{8.4.12}$$

The integral of a total differential comes from the operator $L_0 = td/dt$ and must vanish. For a non-closed integration contour C the boundary terms from the partial integration could spoil the $D^+ = D^\dagger$ condition unless the eigenfunctions vanish at the end points of the integration contour ($t = \infty + i0_\pm$).

The explicit expression of the hermitian form is given by

$$\begin{aligned} \langle \Psi_{z_1} | \Psi_{z_2} \rangle &= -\frac{K(z_{12})}{2\pi i} \int_C \frac{dt}{t} F^2(t) t^{z_{12}} , \\ z_{12} &= \overline{z}_1 + z_2 . \end{aligned} \tag{8.4.12}$$

It must be emphasized that it is $\overline{\Psi}_{z_1} \Psi_{z_2}$ rather than eigenfunctions which is continued from the non-negative real axis to the complex t-plane: therefore one indeed obtains an analytic function as a result.

An essential role in the argument claimed to prove the Riemann hypothesis is played by the crossing symmetry

$$\langle \Psi_{z_1} | \Psi_{z_2} \rangle = \langle \Psi_0 | \Psi_{\overline{z}_1 + z_2} \rangle \tag{8.4.13}$$

of the hermitian form. This symmetry is analogous to the crossing symmetry of particle physics stating that the S-matrix is symmetric with respect to the replacement of the particles in the initial state with their antiparticles in the final state or vice versa [27].

The hermiticity of the hermitian form implies

$$\langle \Psi_{z_1} | \Psi_{z_2} \rangle = \overline{\langle \Psi_{z_2} | \Psi_{z_1} \rangle} . \tag{8.4.14}$$

This condition, which is *not* trivially satisfied, in fact determines the eigenvalue spectrum.

8.4.4 How to choose the function F?

The remaining task is to choose the function F in such a manner that the orthogonality conditions for the solutions Ψ_0 and Ψ_z reduce to the condition that ζ or some function proportional to ζ vanishes at the point $-z$. The definition of ζ based on analytical continuation performed by Riemann suggests how to proceed. Recall that the expression of ζ converging in the region $Re[s] > 1$ following from the basic definition of ζ and elementary properties of Γ function [29] reads as

$$\Gamma(s)\zeta(s) = \int_0^\infty \frac{dt}{t} \frac{exp(-t)}{[1 - exp(-t)]} t^s . \tag{8.4.15}$$

One can analytically continue this expression to a function defined in the entire complex plane by noticing that the integrand is discontinuous along the cut extending from $t = 0$ to $t = \infty$. Following Riemann it is however more convenient to consider the discontinuity for a function obtained by multiplying the integrand with the factor

$$(-1)^s \equiv exp(-i\pi s) .$$

The discontinuity $Disc(f) \equiv f(t) - f(texp(i2\pi))$ of the resulting function is given by

$$Disc\left[\frac{exp(-t)}{[1-exp(-t)]}(-t)^{s-1}\right] = -2isin(\pi s)\frac{exp(-t)}{[1-exp(-t)]}t^{s-1}. \tag{8.4.16}$$

The discontinuity vanishes at the limit $t \to 0$ for $Re[s] > 1$. Hence one can define ζ by modifying the integration contour from the non-negative real axis to an integration contour C enclosing non-negative real axis defined in the previous section.

This amounts to writing the analytical continuation of $\zeta(s)$ in the form

$$-2i\Gamma(s)\zeta(s)sin(\pi s) = \int_C \frac{dt}{t}\frac{exp(-t)}{[1-exp(-t)]}(-t)^{s-1}. \tag{8.4.17}$$

This expression equals to $\zeta(s)$ for $Re[s] > 1$ and defines $\zeta(s)$ in the entire complex plane since the integral around the origin eliminates the singularity.

The crucial observation is that the integrand on the righthand side of Eq. 8.4.17 has precisely the same general form as that appearing in the hermitian form defined in Eq. 8.4.12 defined using the same integration contour C. The integration measure is dt/t, the factor t^s is of the same form as the factor $t^{\overline{z}_1+z_2}$ appearing in the hermitian form, and the function $F^2(t)$ is given by

$$F^2(t) = \frac{exp(-t)}{1-exp(-t)}.$$

Therefore one can make the identification

$$F(t) = \left[\frac{exp(-t)}{1-exp(-t)}\right]^{1/2}. \tag{8.4.18}$$

Note that the argument of the square root is non-negative on the non-negative real axis and that $F(t)$ decays exponentially on the non-negative real axis and has $1/\sqrt{t}$ type singularity at origin. From this it follows that the eigenfunctions $\Psi_z(t)$ approach zero exponentially at the limit $Re[t] \to \infty$ so that one can use the non-closed integration contour C.

With this assumption, the hermitian form reduces to the expression

$$\begin{aligned}
\langle \Psi_{z_1}|\Psi_{z_2}\rangle &= -\frac{K(z_{12})}{2\pi i}\int_C \frac{dt}{t}\frac{exp(-t)}{[1-exp(-t]}(-t)^{z_{12}} \\
&= \frac{K(z_{12})}{\pi}sin(\pi z_{12})\Gamma(z_{12})\zeta(z_{12}).
\end{aligned} \tag{8.4.17}$$

Recall that the definition $z_{12} = \overline{z}_1 + z_2$ is adopted. Thus the orthogonality of the eigenfunctions is equivalent to the vanishing of $\zeta(z_{12})$ if $K(z_{12})$ is positive definite.

8.4.5 Study of the hermiticity condition

In order to derive information about the spectrum one must explicitly study what the statement that $D^\dagger$ is hermitian conjugate of D means. The defining equation is just the generalization of the equation

$$A^\dagger_{mn} = \overline{A}_{nm}. \tag{8.4.18}$$

defining the notion of hermiticity for matrices. Now indices m and n correspond to the eigenfunctions Ψ_{z_i}, and one obtains

$$\langle \Psi_{z_1}|D^+\Psi_{z_2}\rangle = z_2\langle \Psi_{z_1}|\Psi_{z_2}\rangle = \overline{\langle \Psi_{z_2}|D\Psi_{z_1}\rangle} = \overline{\langle D^+\Psi_{z_2}|\Psi_{z_1}\rangle} = z_2\overline{\langle \Psi_{z_2}|\Psi_{z_1}\rangle}.$$

Thus one has

$$\begin{aligned} G(z_{12}) &= \overline{G(z_{21})} = \overline{G(\overline{z}_{12})} \\ G(z_{12}) &\equiv \langle \Psi_{z_1} | \Psi_{z_2} \rangle . \end{aligned} \tag{8.4.18}$$

The condition states that the hermitian form defined by the contour integral is indeed hermitian. This is *not* trivially true. Hermiticity condition obviously determines the spectrum of the eigenvalues of D^+.

To see the implications of the hermiticity condition, one must study the behavior of the function $G(z_{12})$ under complex conjugation of both the argument and the value of the function itself. To achieve this one must write the integral

$$G(z_{12}) = -\frac{K(z_{12})}{2\pi i} \int_C \frac{dt}{t} \frac{exp(-t)}{[1 - exp(-t)]} (-t)^{z_{12}}$$

in a form from which one can easily deduce the behavior of this function under complex conjugation. To achieve this, one must perform the change $t \rightarrow u = log(exp(-i\pi)t)$ of the integration variable giving

$$G(z_{12}) = -\frac{K(z_{12})}{2\pi i} \int_D du \frac{exp(-exp(u))}{[1 - exp(-(exp(u)))]} exp(z_{12}u). \tag{8.4.18}$$

Here D denotes the image of the integration contour C under $t \rightarrow u = log(-t)$. D is a fork-like contour which

1. traverses the line $Im[u] = i\pi$ from $u = \infty + i\pi$ to $u = -\infty + i\pi$,
2. continues from $-\infty + i\pi$ to $-\infty - i\pi$ along the imaginary u-axis (it is easy to see that the contribution from this part of the contour vanishes),
3. traverses the real u-axis from $u = -\infty - i\pi$ to $u = \infty - i\pi$.

The integrand differs on the line $Im[u] = \pm i\pi$ from that on the line $Im[u] = 0$ by the factor $exp(\mp i\pi z_{12})$ so that one can write $G(z_{12})$ as integral over real u-axis

$$G(z_{12}) = -\frac{K(z_{12})}{\pi} sin(\pi z_{12}) \int_{-\infty}^{\infty} du \frac{exp(-exp(u))}{[1 - exp(-(exp(u)))]} exp(z_{12}u). \tag{8.4.18}$$

From this form the effect of the transformation $G(z) \rightarrow \overline{G(\overline{z})}$ can be deduced. Since the integral is along the real u-axis, complex conjugation amounts only to the replacement $z_{21} \rightarrow z_{12}$, and one has

$$\begin{aligned} \overline{G(\overline{z}_{12})} &= -\frac{\overline{K(z_{21})}}{\pi} \times \overline{sin(\pi z_{21})} \int_{-\infty}^{\infty} du \frac{exp(-exp(u))}{[1 - exp(-(exp(u)))]} exp(z_{12}u) \\ &= \frac{\overline{K(z_{21})}}{K(z_{12})} \times \frac{\overline{sin(\pi z_{21})}}{sin(\pi z_{12})} G(z_{12}). \end{aligned} \tag{8.4.18}$$

Thus the hermiticity condition reduces to the condition

$$G(z_{12}) = \frac{\overline{K(z_{21})}}{K(z_{12})} \times \frac{\overline{sin(\pi z_{21})}}{sin(\pi z_{12})} \times G(z_{12}). \tag{8.4.19}$$

The reality of $K(z_{12})$ guarantees that the diagonal matrix elements of the metric are real.

For non-diagonal matrix elements there are two manners to satisfy the hermiticity condition.

1. The condition

$$G(z_{12}) = 0 \tag{8.4.20}$$

is the only manner to satisfy the hermiticity condition for $x_1+x_2 \neq n$, $y_1-y_2 \neq 0$. This implies the vanishing of ζ:

$$\zeta(z_{12}) = 0 \text{ for } 0 < x_1+x_2 < 1. \tag{8.4.21}$$

In particular, this condition must be true for $z_1 = 0$ and $z_2 = 1/2+iy$. Hence the physical states with the eigenvalue $z = 1/2+iy$ must correspond to the zeros of ζ.

2. For the non-diagonal matrix elements of the metric the condition

$$exp(i\pi(x_1+x_2)) = \pm 1 \tag{8.4.22}$$

guarantees the reality of $sin(\pi z_{12})$ factors. This requires

$$x_1+x_2 = n. \tag{8.4.23}$$

The highly non-trivial implication is that the the vacuum state Ψ_0 and the zeros of ζ at the critical line span a space having a hermitian. Note that for $x_1 = x_2 = n/2$, $n \neq 1$, the diagonal matrix elements of the metric vanish.

3. The metric is positive definite only if the function $K(z_{12})$ decays sufficiently fast: this is due to the exponential increase of the moduli of the matrix elements $G(1/2+iy_1, 1/2+iy_2)$ for $K(z_{12}) = 1$ and for large values of $|y_1-y_2|$ (basically due to the $sinh\left[\pi\left(y_1-y_2\right)\right]$-factor in the metric) implying the failure of the Schwartz inequality for $|y_1-y_2| \rightarrow \infty$. Unitarity, guaranteing probability interpretation in quantum theory, thus requires that the parameter α characterizing the Gaussian decay of $K(z_{12}) = exp(-\alpha|z_{12}|^2)$ is above some minimum value.

8.4.6 Various assumptions implying Riemann hypothesis

As found, the general strategy for proving the Riemann hypothesis, originally inspired by super-conformal invariance, leads to the construction of a set of eigen states for an operator D^+, which is effectively an annihilation operator acting in the space of complex-valued functions defined on the real half-line. Physically the states are analogous to coherent states and are not orthogonal to each other. The quantization of the eigenvalues for the operator D^+ follows from the requirement that the metric, which is defined by the integral defining the analytical continuation of ζ, and thus proportional to ζ ($\langle s_1, s_2\rangle \propto \zeta(\overline{s}_1+s_2)$), is hermitian in the space of the physical states.

The nontrivial zeros of ζ are known to belong to the critical strip defined by $0 < Re[s] < 1$. Indeed, the theorem of Hadamard and de la Vallee Poussin [30] states the non-vanishing of ζ on the line $Re[s] = 1$. If s is a zero of ζ inside the critical strip, then also $1-\overline{s}$ as well as $\overline{s}$ and $1-s$ are zeros. If Hilbert space inner product property is not required so that the eigenvalues of the metric tensor can be also negative in this subspace. There could be also un-physical zeros of ζ outside the critical line $Re[s] = 1/2$ but inside the critical strip $0 < Re[s] < 1$. The problem is to find whether the zeros outside the critical line are excluded, not only by the hermiticity but also by the positive definiteness of the metric necessary for the physical interpretation, and perhaps also by conformal invariance posed in some sense as a dynamical symmetry. This turns out to be the case.

Before continuing it is convenient to introduce some notations. Denote by $\mathcal{V}$ the subspace spanned by Ψ_s corresponding to the zeros s of ζ inside the critical strip, by $\mathcal{V}_{crit}$ the subspace corresponding to the zeros of ζ at the critical strip, and by $\mathcal{V}_s$ the space spanned by the states Ψ_s and $\Psi_{1-\overline{s}}$. The basic idea behind the following proposals is that the basic objects of study are the spaces $\mathcal{V}$, $\mathcal{V}_{crit}$ and $\mathcal{V}_s$.

How to restrict the metric to $\mathcal{V}$?

One should somehow restrict the metric defined in the space spanned by the states Ψ_s labelled by a continuous complex eigenvalue s to the space $\mathcal{V}$ inside the critical strip spanned by a basis labelled by discrete eigenvalues. Very naively, one could try to do this by simply putting all other components of the metric to zero so that the states outside $\mathcal{V}$ correspond to gauge degrees of freedom. This is consistent with the interpretation of $\mathcal{V}$ as a coset space formed by identifying states which differ from each other by the addition of a superposition of states which do not correspond to zeros of ζ.

An more elegant manner to realize the restriction of the metric to $\mathcal{V}$ is to Fourier expand states in the basis labelled by a complex number s and define the metric in $\mathcal{V}$ using double Fourier integral over the complex plane and Dirac delta function restricting the labels of both states to the set of zeros inside the critical strip:

$$\begin{aligned}
\langle\Psi^{1)}|\Psi^{2)}\rangle &= \int d\mu(s_1)\int d\mu(s_2)\overline{\Psi}^{1)}_{s_1}\Psi^{2)}_{s_2}G(s_2+\overline{s}_1)\delta(\zeta(s_1))\delta(\zeta(s_2)) \\
&= \sum_{\zeta(s_1)=0,\zeta(s_2)=0}\overline{\Psi}^{1)}_{s_1}\Psi^{2)}_{s_2}G(s_2+\overline{s}_1)\frac{1}{\sqrt{det(s_2)det(\overline{s}_1)}}, \\
d\mu(s) &= dsd\overline{s}, \qquad det(s)=\frac{\partial(Re\,[\zeta(s)]\,,Im\,[\zeta(s)])}{\partial(Re\,[s]\,,Im\,[s])}.
\end{aligned} \tag{8.4.21}$$

Here the integrations are over the critical strip. $det(s)$ is the Jacobian for the map $s\to\zeta(s)$ at s. The appearance of the determinants might be crucial for the absence of negative norm states. The result means that the metric $G_{\mathcal{V}}$ in $\mathcal{V}$ effectively reduces to a product

$$\begin{aligned}
G_{\mathcal{V}} &= \overline{D}GD, \\
D(s_i,s_j) &= D(s_i)\delta(s_i,s_j), \\
\overline{D}(s_i,s_j) &= D(\overline{s_i})\delta(s_i,s_j) \\
D(s) &= \frac{1}{\sqrt{det(s)}}.
\end{aligned} \tag{8.4.19}$$

In the sequel the metric G will be called reduced metric whereas $G_{\mathcal{V}}$ will be called the full metric. In fact, the symmetry $D(s)=D(\overline{s})$ holds true by the basic symmetries of ζ so that one has $D=\overline{D}$ and $G_{\mathcal{V}}=DGD$. This means that Schwartz inequalities for the eigen states of D^+ are not affected in the replacement of $G_{\mathcal{V}}$ with G. The two metrics can be in fact transformed to each other by a mere scaling of the eigen states and are in this sense equivalent.

Riemann hypothesis from the hermicity of the metric in $\mathcal{V}$

The mere requirement that the metric is hermitian in $\mathcal{V}$ implies the Riemann hypothesis. This can be seen in the simplest manner as follows. Besides the zeros at the critical line $Re[s]=1/2$ also the symmetrically related zeros inside critical strip have positive norm squared but they do not have hermitian inner products with the states at the critical line unless one assumes that the inner product vanishes. The assumption that the inner products between the states at critical line and outside it vanish, implies additional zeros of ζ and, by repeating the argument again and again, one can fill the entire critical interval $(0,1)$ with the zeros of ζ so that a reductio ad absurdum proof for the Riemann hypothesis results. Thus the metric gives for the states corresponding to the zeros of the Riemann Zeta at the critical line a special status as what might be called physical states.

It should be noticed that the states in $\mathcal{V}_s$ and $\mathcal{V}_{\overline{s}}$ have non-hermitian inner products for $Re[s]\neq 1/2$ unless these inner products vanish: for $Re[s]>1/2$ this however implies that ζ has a zero for $Re[s]>1$.

Riemann hypothesis from the requirement that the metric in $\mathcal{V}$ is positive definite

With a suitable choice of $K(z_{12})$ the metric is positive definite between states having $y_1\neq y_2$. For s and $1-\overline{s}$ one has $y_1=y_2$ implying $K(z_{12})=1$ in $\mathcal{V}_s$. Thus the positive definiteness of the metric in $\mathcal{V}$ reduces to that for the induced metric in the spaces $\mathcal{V}_s$. This requirement implies also Riemann hypothesis as following argument shows.

The explicit expression for the norm of a $Re[s] = 1/2$ state with respect to the full metric $G_{\mathcal{V}}^{ind}$ reads as

$$\begin{aligned} G_{\mathcal{V}}^{ind}(1/2+iy_n, 1/2+iy_n) &= D^2(1/2+iy)G^{ind}(1/2+iy_n, 1/2+iy_n), \\ G^{ind}(1/2+iy_n, 1/2+iy_n) &= -\frac{K(z_{12})}{\pi} sin(\pi)\Gamma(1)\zeta(1). \end{aligned} \quad (8.4.19)$$

Here G^{ind} is the metric in $\mathcal{V}_s$ induced from the reduced metric G. This expression involves formally a product of vanishing and infinite factors and the value of expression must be defined as a limit by taking in $Im[z_{12}]$ to zero. The requirement that the norm squared defined by G^{ind} equals to one fixes the value of $K(1)$:

$$K(1) = -\frac{\pi}{sin(\pi)\zeta(1)} = 1. \quad (8.4.20)$$

The components G^{ind} in $\mathcal{V}_s$ are given by

$$\begin{aligned} G^{ind}(s,s) &= -\frac{sin(2\pi Re[s])\Gamma(2Re[s])\zeta(2Re[s])}{\pi}, \\ G^{ind}(1-\overline{s}, 1-\overline{s}) &= -\frac{sin(2\pi(1-Re[s]))\Gamma(2-2Re[s])\zeta(2(1-[Re[s]))}{\pi}, \\ G^{ind}(s, 1-\overline{s}) &= G^{ind}(1-\overline{s}, s) = 1. \end{aligned} \quad (8.4.19)$$

The determinant of the metric $G_{\mathcal{V}}^{ind}$ induced from the full metric reduces to the product

$$Det(G_{\mathcal{V}}^{ind}) = D^2(s))D^2(1-\overline{s}) \times Det(G^{ind}). \quad (8.4.20)$$

Since the first factor is positive definite, it suffices to study the determinant of G^{ind}. At the limit $Re[s] = 1/2$ G^{ind} formally reduces to

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

This reflects the fact that the states Ψ_s and $\Psi_{1-\overline{s}}$ are identical. The actual metric is of course positive definite. For $Re[s] = 0$ the G^{ind} is of the form

$$\begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}.$$

The determinant of G^{ind} is negative so that the eigenvalues of both the full metric and reduced metric are of opposite sign. The eigenvalues for G^{ind} are given by $(-1 \pm \sqrt{5})/2$.

The determinant of G^{ind} in $\mathcal{V}_s$ as a function of $Re[s]$ is symmetric with respect to $Re[s] = 1/2$, equals to -1 at the end points $Re[s] = 0$ and $Re[s] = 1$, and vanishes at $Re[s] = 1/2$. Numerical calculation shows that the sign of the determinant of G^{ind} inside the interval $(0,1)$ is negative for $Re[s] \neq 1/2$. Thus the diagonalized form of the induced metric has the signature $(1,-1)$ except at the limit $Re[s] = 1/2$, when the signature formally reduces to $(1,0)$. Thus Riemann hypothesis follows if one can show that the metric induced to $\mathcal{V}_s$ does not allow physical states with a negative norm squared. This requirement is physically very natural. In fact, when the factor $K(z_{12})$ represents sufficiently rapidly vanishing Gaussian, this guarantees the metric to $\mathcal{V}_{crit}$ has only non-negative eigenvalues. Hence the positive-definiteness of the metric, natural if there is real quantum system behind the model, implies Riemann hypothesis.

Riemann hypothesis and conformal invariance

The basic strategy for proving Riemann hypothesis has been based on the attempt to reduce Riemann hypothesis to invariance under conformal algebra or some subalgebra of the conformal algebra in $\mathcal{V}$ or $\mathcal{V}_s$. That this kind of algebra should act as a gauge symmetry associated with ζ is very natural idea since conformal invariance is in a well-defined sense the basic symmetry group of complex analysis.

Consider now one particular strategy based on conformal invariance in the space of the eigen states of D^+.

1. Realization of conformal algebra as a spectrum generating algebra

The conformal generators are realized as operators

$$L_z = t^z D^+ \tag{8.4.21}$$

act in the eigen space of D^+ and obey the standard conformal algebra without central extension [28]. D^+ itself corresponds to the conformal generator L_0 acting as a scaling. Conformal generators obviously act as dynamical symmetries transforming eigen states of D^+ to each other. What is new is that now conformal weights z have all possible complex values unlike in the standard case in which only integer values are possible. The vacuum state Ψ_0 having negative norm squared is annihilated by the conformal algebra so that the states orthogonal to it (non-trivial zeros of ζ inside the critical strip) form naturally another subspace which should be conformally invariant in some sense. Conformal algebra could act as gauge algebra and some subalgebra of the conformal algebra could act as a dynamical symmetry.

2. Realization of conformal algebra as gauge symmetries

The definition of the metric in $\mathcal{V}$ involves in an essential manner the mapping $s \rightarrow \zeta(s)$. This suggests that one should define the gauge action of the conformal algebra as

$$\begin{aligned} \Psi_s &\rightarrow \Psi_{\zeta(s)} \rightarrow L_z\Psi_{\zeta(s)} = \zeta_s\Psi_{\zeta(s)+z} \\ &\rightarrow \zeta_s\Psi_{\zeta^{-1}(\zeta(s)+z)} . \end{aligned} \tag{8.4.21}$$

Clearly, the action involves a map of the conformal weight s to $\zeta(s)$, the action of the conformal algebra to $\zeta(s)$, and the mapping of the transformed conformal weight $z+\zeta(s)$ back to the complex plane by the inverse of ζ. The inverse image is in general non-unique but in case of $\mathcal{V}$ this does not matter since the action annihilates automatically all states in $\mathcal{V}$. Thus conformal algebra indeed acts as a gauge symmetry. This symmetry does not however force Riemann hypothesis.

3. Realization of conformal algebra as dynamical symmetries

One can also study the action of the conformal algebra or its suitable sub-algebra in $\mathcal{V}_s$ as a dynamical (as opposed to gauge) symmetry realized as

$$\Psi_s \rightarrow L_z\Psi_s = s\Psi_{s+z} . \tag{8.4.22}$$

The states Ψ_s and $\Psi_{1-\overline{s}}$ in $\mathcal{V}_s$ have non-vanishing norms and are obtained from each other by the conformal generators $L_{1-2Re[s]}$ and $L_{2Re[s]-1}$. For $Re[s] \neq 1/2$ the generators $L_{1-2Re[s]}$, $L_{2Re[s]-1}$, and L_0 generate $SL(2,R)$ algebra which is non-compact and generates infinite number of states from the states of $\mathcal{V}_s$. At the critical line this algebra reduces to the abelian algebra spanned by L_0. The requirement that the algebra naturally associated with $\mathcal{V}_s$ is a dynamical symmetry and thus generates only zeros of ζ leads to the conclusion that all points $s+n(1-2Re[s])$, n integer, must be zeros of ζ. Clearly, $Re[s] = 1/2$ is the only possibility so that Riemann hypothesis follows. In this case the dynamical symmetry indeed reduces to a gauge symmetry.

There is clearly a connection with the argument based on the requirement that the induced metric in $\mathcal{V}_s$ does not possess negative eigenvalues. Since $SL(2,R)$ algebra acts as the isometries of the induced metric for the zeros having $Re[s] \neq 1/2$, the signature of the induced metric must be $(1,-1)$.

4. Riemann hypothesis from the requirement that infinitesimal isometries exponentiate

One could even try to prove that the entire subalgebra of the conformal algebra spanned by the generators with conformal weights $n(1-2Re[s])$ acts as a symmetry generating new zeros of ζ so that corresponding states are annihilated by gauge conformal algebra. If this holds, $Re[s] = 1/2$ is the only possibility so that Riemann hypothesis follows. In this case the dynamical conformal symmetry indeed reduces to a gauge symmetry.

Since $L_{1-2Re[s]}$ acts as an infinitesimal isometry leaving the matrix element $\langle\Psi_0|\Psi_s\rangle = 0$ invariant, one can in spirit of Lie group theory argue that also the exponentiated transformations $exp(tL_{1-2Re[s]})$ have the same property for all values of t. The exponential action leaves Ψ_0 invariant and generates from Ψ_s a superposition of states with conformal weights $s+n(1-2Re[s])$, which all must be orthogonal to Ψ_0 since t is arbitrary. Since all zeros are inside the critical strip, $Re[s] = 1/2$ is the only possibility.

A more explicit formulation of this idea is based on a first order differential equation for the integral representation of ζ. One can write the matrix element of the metric using the analytical continuation of $\zeta(s)$:

$$\begin{aligned} G(s) &= -2i\Gamma(s)\zeta(s)sin(\pi s) = H(s,a)_{|a=0} , \\ H(s,a) &= \int_C \frac{dt}{t}\frac{exp(-t+a(-t)^{1-2x})}{[1-exp(-t)]}(-t)^{x+iy-1} . \end{aligned} \tag{8.4.22}$$

If $s = x+iy$ is zero of ζ then also $1-x+iy$ is zero of ζ and its is trivial to see that this means the both $H(x+iy,a)$ and its first derivative vanishes at $a=0$:

$$\begin{aligned} &H(s,a)_{|a=0} = 0 , \\ &\frac{d}{da}H(s,a)_{|a=0} = 0 . \end{aligned} \tag{8.4.22}$$

Suppose that $H(s,a)$ satisfies a differential equation of form

$$\frac{d}{da}H(x+iy,a) = I(x, H(x+iy,a)) , \tag{8.4.23}$$

where $I(x,H)$ is some function having no explicit dependence on a so that the differential equation defines an autonomous flow. If the initial conditions of Eq. 8.4.22 are satisfied, this differential equation implies that all derivatives of H vanish which in turn, as it is easy to see, implies that the points $s+m(1-2x)$ are zeros of ζ. This leaves only the possibility $x = 1/2$ so that Riemann hypothesis is proven. If I is function of also a, that is $I = I(a,x,H)$, this argument breaks down.

The following argument shows that the system is autonomous. One can solve a as function $a = a(x,H)$ from the Taylor series of H with respect to a by using implicit function theorem, substitute this series to the Taylor series of dH/da with respect to a, and by re-organizing the summation obtain a Taylor series with respect to H with coefficients which depend only on x so that one has $I = I(x,H)$.

5. Conclusions

To sum up, Riemann hypothesis follows from the requirement that the states in $\mathcal{V}$ can be assigned with a conformally invariant physical quantum system. This condition reduces to three mutually equivalent conditions: the metric induced to $\mathcal{V}$ is hermitian; positive definite; allows conformal symmetries as isometries. The hermiticity and positive definiteness properties reduce to the requirement that the dynamical conformal algebra naturally spanned by the states in $\mathcal{V}_s$ reduces to the abelian algebra defined by $L_0 = D^+$. If the infinitesimal isometries for the matrix elements $\langle\Psi_0|\Psi_s\rangle = 0$ generated by $L_{1-2Re[s]}$ can be exponentiated to isometries as Lie group theory based argument strongly suggests, then Riemann hypothesis follows.

8.4.7 Does the Hermitian form define inner product?

Before considering the question whether the Hermitian form defined by G or $G_{\mathcal{V}}$ defines a positive definite Hilbert space inner product, a couple of comments concerning the general properties of the Hermitian form G are in order.

1. The Hermitian form is proportional to the factor

$$sin(i\pi(y_2 - y_1)) \ ,$$

which vanishes for $y_1 = y_2$. For $y_1 = y_2$ and $x_1 + x_2 = 1$ ($x_1 + x_2 = 0$) the diverging factor $\zeta(1)$ ($\zeta(0)$) compensates the vanishing of this factor. Therefore the norms of the eigenfunctions Ψ_z with $z = 1/2+iy$ must be calculated explicitly from the defining integral. Since the contribution from the cut vanishes in this case, one obtains only an integral along a small circle around the origin. This gives the result

$$\langle\Psi_{z_1}|\Psi_{z_1}\rangle = K \ \text{ for } \ z_1 = \tfrac{1}{2} + iy \ \ , \ \ \langle\Psi_0|\Psi_0\rangle = -K \ \ . \tag{8.4.24}$$

Thus the norms of the eigenfunctions are finite. For $K = 1$ the norms of $z = 1/2 + iy$ eigenfunctions are equal to one. Ψ_0 has however negative norm -1 so that the Hermitian form in question is not a genuine inner product in the space containing Ψ_0.

2. For $x_1 = x_2 = 1/2$ and $y_1 \neq y_2$ the factor is non-vanishing and one has

$$\langle\Psi_{z_1}|\Psi_{z_2}\rangle = -\frac{1}{\pi i}\zeta(1 + i(y_2 - y_1))\Gamma(1 + i(y_2 - y_1))sinh(\pi(y_2 - y_1)) \ \ . \tag{8.4.24}$$

The nontrivial zeros of ζ are known to belong to the critical strip defined by $0 < Re[s] < 1$. Indeed, the theorem of Hadamard and de la Vallee Poussin [30] states the non-vanishing of ζ on the line $Re[s] = 1$. Since the non-trivial zeros of ζ are located symmetrically with respect to the line $Re[s] = 1/2$, this implies that the line $Re[s] = 0$ cannot contain zeros of ζ. This result implies that the states $\Psi_{z=1/2+y}$ are non-orthogonal unless $\Gamma(1 + i(y_2 - y_1))$ vanishes for some pair of eigenfunctions.

It is not at all obvious that the Hermitian form in question defines an inner product in the space spanned by the states Ψ_z, $z = 1/2+iy$ having real and positive norm. Besides Hermiticity, a necessary condition for this is that Schwartz inequality

$$|\langle\Psi_{z_1}|\Psi_{z_2}\rangle| \leq |\Psi_{z_1}||\Psi_{z_2}|$$

holds true. In case of eigen states of D^+ this condition is not affected by the determinant factors and one can apply it to the metric G. This gives

$$\frac{1}{\pi}|\zeta(1 + iy_{12})| \times |\Gamma(1 + iy_{12})| \times |sin(i\pi y_{12})| \leq 1 \ \ , \tag{8.4.25}$$

where the shorthand notation $y_{12} = y_2 - y_1$ has been used.

Numerical computation suggests that $\zeta(1 + iy_{12})$ varies in a finite range of values for large values of y_{12} and that $\Gamma(1 + iy)$ behaves essentially as $exp(-\pi y/2)$ asymptotically so that the left hand side increases faster than $exp(\pi y_{12}/2)$ so that Schwartz inequality fails for the eigen states. It took a considerable time do realize that the solution to this difficulty is trivial: the only thing that is needed is to multiply the metric with the factor $K(z_{12})$ introduced already earlier. $K(z_{12})$ is expected to behave like a sufficiently narrow Gaussian on basis of the intuition about the behavior of coherent states.

Possible problems are also caused by the small values of y_{12} for which one might have $|G(1+iy_{12})| > 1$ implying the failure of the Schwartz inequality

$$|\langle\Psi_{z_1}|\Psi_{z_2}\rangle| \ \ \leq \ \ |\Psi_{z_1}||\Psi_{z_2}| \tag{8.4.26}$$

characterizing positive definite metric. The direct calculation of $G(1+iy)$ at the limit $y \to 0$ by using $\zeta(1 + iy) \simeq 1/iy$ however gives

$$G(1) = 1 \ \ . \tag{8.4.27}$$

By a straightforward calculation one can also verify that $z = 1$ is a local maximum of $|G(z)|$. Note that the Jacobians do not affect the required inequality at all in case of eigen states.

It is easy to see that arbitrary small values of y_{12} are unavoidable. The estimate of Riemann for the number of the zeros of ζ in the interval $Im[s] \in [0,T]$ along the line $Re[s] = 1/2$ reads as

$$N(T) \simeq \frac{T}{2\pi}\left[log(\frac{T}{2\pi}) - 1\right] \ , \tag{8.4.28}$$

and allows to estimate the average density dN_T/dy of the zeros and to deduce an upper limit for the minimum distance y_{12}^{min} between two zeros in the interval T:

$$\begin{aligned} \frac{dN_T}{dy} &\simeq \frac{1}{2\pi}\left[log(\frac{T}{2\pi}) - 1\right] \ , \\ y_{12}^{min} &\leq \frac{1}{\frac{dN_T}{dy}} = \frac{2\pi}{\left[log(\frac{T}{2\pi}) - 1\right]} \rightarrow 0 \ \text{ for } \ T \rightarrow \infty \ . \end{aligned} \tag{8.4.28}$$

This implies that arbitrary small values of y_{12} are unavoidable.

8.4.8 Super-conformal symmetry

Before considering super-conformal symmetry it is good to summarize the basic results obtained hitherto.

1. Conformal invariance as a gauge symmetry is possible only in the space $\mathcal{V}$ spanned by the eigen states associated with the zeros of ζ.

2. The hermiticity of the metric in the space spanned by the eigen states associated with the zeros of ζ is possible only if the zeros are on the critical line.

3. The requirement that the algebra spanned by the generators $L_{2Re[s]-1}$, $L_{1-2Re[s]}$ act as a dynamical symmetry algebra generating new zeros of ζ, forces the zeros to be on the critical line: in this case the generators in question reduce to L_0 and the dynamical symmetry reduces to a gauge symmetry.

One can say that the relationship of the conformal invariance to Riemann hypothesis is understood. Although super-conformal invariance does not seem to bring in anything new in this respect, it is still interesting to look whether conformal symmetry could be generalized to super-conformal symmetry. Certainly the basic idea about the action as gauge symmetry remains the same as well as the manner how subalgebra of conformal algebra acts as a dynamical symmetry algebra.

In the following various approaches to the problem of finding a super-conformal generalization of the dynamical system associated with the Riemann Zeta are discussed.

Simplest variant of the super-conformal symmetry

One can indeed identify a conformal algebra naturally associated with the proposed dynamical system. Note first that the generators of the ordinary conformal algebra

$$L_z = \Psi_z D^+ \tag{8.4.29}$$

generate conformal algebra with commutation relations ($[A,B] \equiv AB - BA$)

$$[L_{z_1}, L_{z_2}] = (z_2 - z_1) L_{z_1+z_2} \ . \tag{8.4.30}$$

Fermionic generators G_z satisfy the following anti-commutation and commutation relations:

$$\{G_{z_1}, G_{z_2}\} = L_{z_1+z_2} \quad , \quad [L_{z_1}, G_{z_2}] = z_2 G_{z_1+z_2} \quad , \quad . \tag{8.4.30}$$

An explicit representation for the generators of the algebra extended to a super-algebra is obtained by introducing besides the bosonic coordinate t an anti-commuting coordinate θ. This means that the ordinary complex function algebra is replaced by the function algebra consisting of functions $f(t) + \theta g(t)$.

It is easy to verify that the generators defined as

$$L_z = t^z(D^+ + z\theta d_\theta) \quad , \quad G_z = \frac{1}{\sqrt{2}} t^z (d_\theta + \theta D^+) \quad . \tag{8.4.30}$$

satisfy the defining commutation and anti-commutation relations of the super conformal algebra. Notice that the definition of the operator $D^+ = L_0$ is not affected at all by the generalization and the eigenfunctions of D^+ come as doubly degenerate pairs consisting of a bosonic state Ψ_z and its fermionic partner $\Psi_z\theta$. Vacuum state however corresponds to the bosonic state since L_z and G_z do not annihilate the fermionic partner of the vacuum state.

The representation of this algebra as a gauge algebra is achieved in exactly the same manner as in the case of the ordinary conformal algebra. The gauge conditions for L_z are satisfied only by the bosonic eigen states so that actually nothing new seems to emerge from this generalization. The counterpart of the algebra generated by $L_{1-2Re[s]}$, $L_{2Re[s]-1}$ and L_0 is obtained by adding the generator G_0. Since any L_z commutes with G_0 the algebra closes. The requirement that this algebra acts as a symmetry in $\mathcal{V}$ implies Riemann hypothesis since the algebra reduces to that generated by L_0 and G_0 on the critical line. The super-symmetric variant of the theory is clearly somewhat disappointing exercise since it does not seem to bring anything genuinely new: even the space of the conformally invariant states remains the same.

Second quantized version of super-conformal symmetry

The following much more complex construction is essentially a construction of a second-quantized super-conformal quantum field theory for the super-symmetric system associated with D^+. It must be emphasized that this construction contains un-necessary complexities. In particular, the introduction of Kac Moody symmetry can be criticized since Kac Moody generators cannot annihilate physical states in the representation of the super-conformal symmetries as gauge symmetries in the space $\mathcal{V}$. It is however perhaps wise to keep also this option since it turn out to be of some value.

The extension of this algebra to super-conformal algebra requires the introduction of the fermionic generators G_z and $G_z^\dagger$. To avoid confusions it must be emphasized that following convention concerning Hermitian conjugation is adopted to make notation more fluent:

$$(O_w)^\dagger = O^\dagger_{\overline{w}} \quad . \tag{8.4.31}$$

Fermionic generators G_z and $G_z^\dagger$ satisfy the following anti-commutation and commutation relations:

$$\{G_{z_1}, G^\dagger_{z_2}\} = L_{z_1+z_2} \quad , \quad [L_{z_1}, G_{z_2}] = z_2 G_{z_1+z_2} \quad , \quad \left[L_{z_1}, G^\dagger_{z_2}\right] = -z_2 G^\dagger_{z_1+z_2} \quad . \tag{8.4.31}$$

This definition differs from that used in the standard approach [28] in that generators G_z and $G_z^\dagger$ are introduced separately. Usually one introduces only the the generators G_n and assumes Hermiticity condition $G_{-n} = G_n^\dagger$. The anti-commutation relations of G_z contain usually also central extension term. Now this term is not present as will be found.

Conformal algebras are accompanied by Kac Moody algebra which results as a central extension of the algebra of the local gauge transformations for some Lie group on circle or line [28]. In the standard approach Kac Moody generators are Hermitian in the sense that one has $T_{-n} = T_n^\dagger$ [28]. Now this condition is dropped and one introduces also the generators $T_z^\dagger$. In present case the counterparts for the generators $T_z^\dagger$ of the local gauge transformations act as translations $z_1 \to z_1 + z$ in the index space labelling eigenfunctions and geometrically correspond to the multiplication of Ψ_{z_1} with the function t^z

$$T_{z_1}^\dagger \Psi_{z_2} = t^{z_1}\Psi_{z_2} = \Psi_{z_1+z_2} \ . \tag{8.4.32}$$

These transformations correspond to the isometries of the Hermitian form defined by $G(z_{12})$ and are therefore natural symmetries at the level of the entire space of the eigenfunctions.

The commutation relations with the conformal generators follow from this definition and are given by

$$[L_{z_1}, T_{z_2}] = z_2 T_{z_1+z_2} \ , \quad \left[L_{z_1}, T_{z_2}^\dagger\right] = -z_2 T_{z_1+z_2}^\dagger \ , \tag{8.4.33}$$

The central extension making this commutative algebra to Kac-Moody algebra is proportional to the Hermitian metric

$$[T_{z_1}, T_{z_2}] = 0 \ , \quad \left[T_{z_1}^\dagger, T_{z_2}^\dagger\right] = 0 \ , \quad \left[T_{z_1}^\dagger, T_{z_2}\right] = (z_1 - z_2)G(z_1 + z_2) \ . \tag{8.4.34}$$

One could also consider the central extension $\left[T_{z_1}^\dagger, T_{z_2}\right] = G(z_1 + z_2)$, which is however not the standard Kac-Moody central extension.

One can extend Kac Moody algebra to a super Kac Moody algebra by adding the fermionic generators Q_z and $Q_z^\dagger$ obeying the anti-commutation relations ($\{A, B\} \equiv AB + BA$)

$$\{Q_{z_1}, Q_{z_2}\} = 0 \ , \quad \{Q_{z_1}^\dagger, Q_{z_2}^\dagger\} = 0 \ , \{Q_{z_1}, Q_{z_2}^\dagger\} = G(z_1 + z_2) \ . \tag{8.4.35}$$

Note that also Q_0 has a Hermitian conjugate $Q_0^\dagger$, and one has

$$\{Q_0, Q_0^\dagger\} \quad = \quad G(0) = -\frac{1}{2} \tag{8.4.36}$$

implying that also the fermionic counterpart of Ψ_0 has negative norm. One can identify the fermionic generators as the gamma matrices of the infinite-dimensional Hermitian space spanned by the eigenfunctions Ψ_z. By their very definition, the complexified gamma matrices $\Gamma_{\bar{z}_1}$ and Γ_{z_2} anti-commute to the Hermitian metric $\langle \Psi_{z_1} | \Psi_{z_2} \rangle = G(\overline{z}_1 + z_2)$.

The commutation relations of the conformal and Kac Moody generators with the fermionic generators are given by

$$\begin{array}{ll} [L_{z_1}, Q_{z_2}] = z_2 Q_{z_1+z_2} \ , & \left[L_{z_1}, Q_{z_2}^\dagger\right] = -z_2 Q_{z_1+z_2}^\dagger \ , \\ \left[T_{z_1}, Q_{z_2}^\dagger\right] = 0 \ , & [T_{z_1}, Q_{z_2}] = 0 \ . \end{array} \tag{8.4.37}$$

The non-vanishing commutation relations of T_z with G_z and non-vanishing anticomutation relations of Q_z with G_z are given by

$$\begin{array}{ll} \left[G_{z_1}, T_{z_2}^\dagger\right] = Q_{z_1+z_2} \ , & \left[G_{z_1}^\dagger, T_{z_2}\right] = -Q_{z_1+z_2}^\dagger \ , \\ \{G_{z_1}, Q_{z_2}^\dagger\} = T_{z_1+z_2} \ , & \{G_{z_1}^\dagger, Q_{z_2}\} = T_{z_1+z_2}^\dagger \ . \end{array} \tag{8.4.38}$$

Super-conformal generators clearly transform bosonic and fermionic Super Kac-Moody generators to each other.

The final step is to construct an explicit representation for the generators G_z and L_z in terms of the Super Kac Moody algebra generators as a generalization of the Sugawara representation [28]. To achieve this, one must introduce the inverse $G^{-1}(z_a z_b)$ of the metric tensor $G(z_a z_b) \equiv \langle \Psi_{z_a} | \Psi_{z_b} \rangle$, which geometrically corresponds to the contravariant form of the Hermitian metric defined by G.

Adopting these notations, one can write the generalization for the Sugawara representation of the super-conformal generators as

$$\begin{aligned} G_z &= \sum_{z_a} T_{z+z_a} G^{z_a z_b} Q^{\dagger}_{z_b} \ , \\ G^{\dagger}_z &= \sum_{z_a} T^{\dagger}_{z+z_a} G^{z_a z_b} Q_{z_b} \ . \end{aligned} \tag{8.4.38}$$

One can easily verify that the commutation and anti-commutation relations with the super Kac-Moody generators are indeed correct. The generators L_z are obtained as the anti-commutators of the generators G_z and $G^{\dagger}_z$. Due to the introduction of the generators T_z, $T^{\dagger}_z$ and G_z, $G^{\dagger}_z$, the anti-commutators $\{G_{z_1}, G^{\dagger}_{z_2}\}$ do not contain any central extension terms. The expressions for the anti-commutators however contains terms of form $T^{\dagger}TQ^{\dagger}Q$ whereas the generators in the usual Sugawara representation contain only bilinears of type $T^{\dagger}T$ and $Q^{\dagger}Q$. The inspiration for introducing the generators T_z,G_z and $T^{\dagger}_z$, $G^{\dagger}_z$ separately comes from the construction of the physical states as generalized super-conformal representations in quantum TGD [F2]. The proposed algebra differs from the standard super-conformal algebra [28] also in that the indices z are now complex numbers rather than half-integers or integers as in the case of the ordinary super-conformal algebras [28]. It must be emphasized that one could also consider the commutation relations $\left[T^{\dagger}_{z_1}, T_{z_2}\right] = iG(z_1 + z_2)$ and they might be more the physical choice since $z_2 - z_1$ is now a complex number unlike for ordinary super-conformal representations. It is not however clear how and whether one could construct the counterpart of the Sugawara representation in this case.

Imitating the standard procedure used in the construction of the representations of the super-conformal algebras [28], one can assume that the vacuum state is annihilated by *all* generators L_z irrespective of the value of z:

$$L_z|0\rangle = 0 \ , \quad G_z|0\rangle = 0 \ . \tag{8.4.39}$$

That all generators L_z annihilate the vacuum state follows from the representation $L_z = \Psi_z D_+$ because D_+ annihilates Ψ_0. If G_0 annihilates vacuum then also $G_z \propto [L_z, G_0]$ does the same.

The action of $T^{\dagger}_z$ on an eigenfunction is simply a multiplication by t^z: therefore one cannot require that T_z annihilates the vacuum state as is usually done [28]. The action of T_0 is multiplication by $t^0 = 1$ so that T^0 and $T^{\dagger}_0$ act as unit operators in the space of the physical states. In particular,

$$T_0|0\rangle = T^{\dagger}_0|0\rangle = |0\rangle \ . \tag{8.4.40}$$

This implies the condition

$$\left[T_0, T^{\dagger}_z\right] = izG(z) = 0 \tag{8.4.41}$$

in the space of the physical states so that physical states must correspond to the zeros of ζ and possibly to $z = 0$. Thus one can generate the physical states from vacuum by acting using operators $Q^{\dagger}_z$ and $T^{\dagger}_z$ with $\zeta(z) = 0$. If one requires that the physical states also have real and positive norm squared, only the zeros of ζ on the line $Re[s] = 1/2$ are allowed. Hence the requirement that a unitary representation of the super-conformal algebra is in question, forces Riemann hypothesis.

It is important to notice that $T^{\dagger}_z$ and $Q\dagger_z$ cannot annihilate the vacuum: this would lead to the condition $G(z_1 + z_2) = 0$ implying the vanishing of $\zeta(z_1 + z_2)$ for any pair $z_1 + z_2$. One can however assume that Q_z annihilates the vacuum state

$$Q_z|0\rangle = 0 \ . \tag{8.4.42}$$

The realization of these conditions in case of super-conformal algebra is achieved by mapping the eigen states Ψ_s to $\Psi_{\zeta(s)}$, acting to these states by the generators of the algebra and mapping the resulting state (which vanishes for zeros of ζ) back to a state proportional to $\Psi_{\zeta^{-1}(\zeta(s)+z)}$. It must be however emphasized that for Kac Moody generators not annihilating the vacuum state the action is not well-defined.

This inspires the hypothesis that only the generators with conformal weights $z = 1/2+iy$ generate physical states from vacuum realizable in the space of the eigenfunctions Ψ_z and their fermionic counterparts. This means that the action of the bosonic generators $T^\dagger_{1/2+iy}$ and fermionic generators $Q^\dagger_0$ and $Q^\dagger_{1/2+iy}$, as well as the action of the corresponding super-conformal generators $G^\dagger_{1/2+iy}$, generates bosonic and fermionic states with conformal weight $z = 1/2 + iy$ from the vacuum state:

$$|1/2 + iy\rangle_B \equiv T^\dagger_{1/2+iy}|0\rangle \ , \quad |1/2 + iy\rangle_F \equiv Q^\dagger_{1/2+iy}|0\rangle \ . \tag{8.4.43}$$

One can identify the states generated by the Kac Moody generators $T^\dagger_z$ from the vacuum as the eigenfunctions Ψ_z. The system as a whole represents a second quantized super-symmetric version of the bosonic system defined by the eigenvalue equation for D^+ obtained by assigning to each eigenfunction a fermionic counterpart and performing second quantization as a free quantum field theory.

It should be noticed that the ordinary Super Kac-Moody and super-conformal algebras with generators O_n labelled by integers $n > 0$ generate zero norm states from any state $|z\rangle$ with $Re[z] = 0$ or $Re[z] = 1/2$ $(G(n_1 + n_2) = 0)$. Thus ordinary super-conformal invariance holds true as gauge invariance. It is possible (although perhaps not absolutely necessary) to restrict the real parts of the conformal weights of the generators to be non-negative.

Is the proof of the Riemann hypothesis by reductio ad absurdum possible using second quantized super-conformal invariance?

Riemann hypothesis is proven if all eigenfunctions for which the Riemann Zeta function vanishes, correspond to the states having a real and positive norm squared. The expectation is that super-conformal invariance realized in some sense excludes all zeros of ζ except those on the line $Re[s] = 1/2$. The problem is to define precisely what one means with super-conformal invariance and one can generate large number of reduction ad absurdum type proofs depending on how super-conformal invariance is assumed to be realized. The following considerations are completely independent of the already described and more recent realization of the super-conformal gauge invariance by applying ζ and its inverse to the conformal weights of the eigen states. I have kept this material because I feel that it might be unwise to to throw it way yet.

The most conservative option is that super-conformal invariance is realized in the standard sense. The action of the ordinary super-conformal generators L_n, and G_n, $n \neq 0$ on the vacuum states $|0\rangle_{B/F}$ or on any state $|1/2+iy\rangle_{B/F}$ indeed creates zero norm states as is obvious from the vanishing of the factor $sin(\pi z_{12}) = sin(\pi(x_1 + x_2))$ associated with the inner inner products of these states. Thus the zeros of ζ define an infinite family of ground states for the representations of the ordinary super-conformal algebra. A generalization of this hypothesis is that the action of L_n and G_n, $n \neq 0$, on any state $|w\rangle_{B/F}$, $\zeta(w) = 0$, creates states which are orthogonal zero norm states. This implies $\zeta(n+2Re[w]) = 0$ for all values of $n \neq 0$ and, since the real axis contains zeros of ζ only at the points $Re[s] = -2n$, $n > 0$, leads to a reductio ad absurdum unless one has $Re[w] = 1/2$. Thus the proof of the Riemann hypothesis would reduce to showing that the action of the ordinary super-conformal algebra generates mutually orthogonal zero norm states from any state $|w\rangle_{B/F}$ with $\zeta(w) = 0$. The proof of this physically plausible hypothesis is not obvious.

One can imagine also other strategies. The minimal requirement is certainly that some subalgebra of the super-conformal algebra generates a space of states satisfying the Hermiticity condition. The quantity

$$\Delta(\overline{w}_1 + w_2) \quad \equiv \quad \langle w_1|w_2\rangle - \overline{\langle w_2|w_1\rangle} = G(\overline{w}_1 + w_2) - \overline{G(\overline{w}_2 + w_1)} \tag{8.4.44}$$

must define the conformal invariant in question since this quantity must vanish in the space of the physical states for which the metric is Hermitian. This requirement does not however imply anything nontrivial for the ordinary conformal algebra having generators L_n and G_n: for $Re[w] \neq 1/2$ the condition is indeed satisfied because $G(n+2Re[w])$ does *not* satisfy the Hermiticity condition for any value of n.

One can try to abstract some property of the states associated with the zeros of ζ on the line $Re[s] = 1/2$. The generators $L_{1/2-iy}$ and $G_{1/2-iy}$ generate zero norm states from the states $|1/2+iy\rangle_{B/F}$, when $1/2+iy$ corresponds to the zero of ζ on the line $Re[s] = 1/2$. One can try to generalize this observation so that it applies to an arbitrary state $|w\rangle_{B/F}$, $\zeta(w) = 0$. The generators $L_{1-\overline{w}}$ and $G_{1-\overline{w}}$ certainly generate zero norm states from the states $|w\rangle_{B/F}$. Also the Hermiticity condition holds true identically and does not have nontrivial implications. One can however consider alternative generalizations by assuming that

1. either the generators $L_{\overline{w}}$ and $G_{\overline{w}}$ or

2. $L_{1/2+iy}$ and $G_{1/2+iy}$ generate from the states $|w\rangle_{B/F}$, $\zeta(w) = 0$ states satisfying the Hermiticity condition.

These two hypothesis lead to two versions of a reductio ad absurdum argument. Suppose that w is a zero of ζ. This means that the inner product of the states $Q_0^\dagger|0\rangle$ and $Q_w^\dagger|0\rangle$ and thus also $\Delta(w)$ vanishes:

$$\langle 0|Q_0 Q_w^\dagger|0\rangle = 0 \ , \quad \Delta(w) = 0 \ . \tag{8.4.45}$$

1. By acting on this matrix element by the conformal algebra generator $L_{\overline{w}}$ (which acts like derivative operator on the arguments of the should-be Hermitian form), and using the fact that $L_{\overline{w}}$ annihilates the vacuum state, one obtains

$$\langle 0|Q_0 Q_{\overline{w}+w}^\dagger|0\rangle = G(w+\overline{w}) \ . \tag{8.4.46}$$

   The requirement $\Delta(w+\overline{w}) = 0$ implies the reality of $G(w+\overline{w})$ and thus the condition $Re[w] = 1/2$ leading to the Riemann hypothesis. Note that the argument implying the reality of $G(w+\overline{w})$ assumes only that L_w annihilates vacuum.

   If this line of approach is correct, the basic challenge would be to show on the basis of the super-conformal invariance alone that the condition $\zeta(w) = 0$ implies that the generators $L_{\overline{w}}$ and $G_{\overline{w}}$ generate new ground states satisfying the Hermiticity condition.

2. An alternative line of argument uses only the invariance under the generators $L_{1/2+iy}$ associated with the zeros of ζ, and thus certainly belonging to the conformal algebra associated with the physical states. By applying the generators $L_{1/2+iy_i}$ to the the matrix element $\langle 0|Q_0 Q_w^\dagger|0\rangle = 0$ and requiring that Hermiticity is respected, one can deduce that $G(w+1/2+iy_i)$ satisfies the Hermiticity condition. Hence the line $Re[s] = Re[w]+1/2$, and by the reflection symmetry also the line $Re[s] = 1/2 - Re[w]$, contain an infinite number of zeros of ζ if one has $Re[w] \neq 1/2$. By repeating this process once for the zeros on the line $Re[s] = 1/2 - Re[w]$, one finds that the lines $Re[s] = 1 - Re[w]$ and $Re[s] = Re[w]$ contain infinite number of the zeros of ζ of form $w_{ij} = w + i(y_i + y_j)$, where y_i and y_j are associated with the zeros of ζ on the line $Re[s] = 1/2$. By applying this two-step procedure repeatedly, one can fill the lines $Re[s] = Re[w]$,$1 - Re[w]$, $1/2 - Re[w]$, $1/2 + Re[w]$ with the zeros of ζ.

8.4.9 p-Adic version of the modified Hilbert-Polya hypothesis

Rather interestingly, the dynamical model generalizes in straightforward manner to the p-adic context. The first problem encountered in p-adicization of the results obtained thusfar relates to the definition of the p-adic eigenvalue problem. The functions t^{x+iy} do not exist p-adically unless one assumes that t is integer valued, p_1^{iy} defines Pythagorean phase and p_1^x exists for every prime. For arbitrary rational value of $x = m/n$ this requires that $p_1^{m/n}$ exists for every p_1 in the algebraic extension associated with R_p. These conditions also guarantee the existence of the p-adic Riemann Zeta.

The basic requirement is that orthogonality conditions lead to the vanishing of p-adic Riemann Zeta. This is achieved if one defines p-adic inner product simply as the sum

$$\langle \Psi_{z_1} | \Psi_{z_2} \rangle = \sum_n n^{z_{12}} = \zeta(z_{12}) \ , \quad z_{12} = x_1 + x_2 + i(y_2 - y_1) \ . \tag{8.4.47}$$

It is important to notice that p-adic Riemann Zeta is formally the inverse of the real Riemann Zeta: this is implied by the requirement that p-adic Riemann Zeta vanishes for $z = -2n$ and also suggested by adelic formula.

This definition means that in p-adic case the differential operator D is simply the formal differential operator $L_0 = td/dt$, that is free scaling operator without any interaction term and thus having as its eigenvalues exponents $x = x + iy$. $D^\dagger = -D$ obviously holds true. A possible interpretation is that conformal invariance is broken in real case by the emergence of the interaction potential $V(t)$ whereas in p-adic case this symmetry is unbroken. The study of p-adic Riemann Zeta indeed leads to the general view that infinite hierarchy of breakings of conformal symmetry occurs as p increases and destroys zeros of Riemann zeta so that at the limit $p \to \infty$ leaves only the zeros of Riemann Zeta located at line $x = 1/2$ remain.

What is fascinating is that for the representations of Super Virasoro only half-odd and integer eigenvalues of L_0 are possible in case that eigenvalues are real. Indeed, for Neveu-Schwartz type representations fermionic super-symmetry generators are labelled by half-odd integers. In quantum TGD these representations combine to form a larger algebra in which both conformal and super-conformal generators are labelled by half-integer valued conformal weight [B2, B3]. This would mean that $x = n/2$ are the only possible values of x and this would imply Riemann hypothesis since $x = 0$ and $x = 1$ are included by the previous considerations. The reason for half-odd integers is basically that the representations functions $z^{n/2}$ define representations of double-fold covering of Lorentz group acting as Möbius transformations of complex plane. This suggests that spin-statistics theorem allowing only single and double valued representation function is involved with Riemann hypothesis.

In p-adic case the requirement that probability density and thus also p-adic norm are *ordinary* p-adic numbers implies $x = n/2$. This does not however prove Riemann hypothesis unless all Ψ_z orthogonal to Ψ_0 belong to the state space. For a general rational value $x = m/n$ of x the values of the p-adic Riemann Zeta are in the algebraic extension and the number of vanishing conditions is much larger than the number of coordinate variables (x and y) so that with the rigour used by physicist one can conclude that the conditions are very probably not satisfied. If one could prove that irrational values of x do not belong to the spectrum of the operator D, one would be quite near to the proof of Riemann hypothesis if Local-Global principle is assumed. Super-conformal invariance might be the key for proving that only the values $x = n/2$ are possible.

8.5 Could local zeta functions take the role of Riemann Zeta in TGD framework?

The recent view about TGD leads to some conjectures about Riemann Zeta.

1. Non-trivial zeros should be algebraic numbers.
2. The building blocks in the product decomposition of ζ should be algebraic numbers for non-trivial zeros of zeta.
3. The values of zeta for their combinations with positive imaginary part with positive integer coefficients should be algebraic numbers.

These conjectures are motivated by the findings that Riemann Zeta seems to be associated with critical systems and by the fact that non-trivial zeros of zeta are analogous to complex conformal weights or perhaps more naturally, to complex square roots of real conformal weights [A9]. The necessity to make such a strong conjectures, in particular conjecture c), is an unsatisfactory feature of the theory and one could ask how to modify this picture. Also a clear physical interpretation of Riemann zeta is lacking.

It was also found that there are good reasons for expecting that the zetas in question should have only a finite number zeros. In the same section the self-referentiality hypothesis for ζ was proposed on basis of physical arguments. In this section (written before the emergence of self-referentiality hypothesis) the situation will be discussed from different view point.

8.5.1 Local zeta functions and Weil conjectures

Riemann Zeta is not the only zeta [26, 25]. There is entire zoo of zeta functions and the natural question is whether some other zeta sharing the basic properties of Riemann zeta having zeros at critical line could be more appropriate in TGD framework.

The so called local zeta functions analogous to the factors $\zeta_p(s) = 1/(1-p^{-s})$ of Riemann Zeta can be used to code algebraic data about say numbers about solutions of algebraic equations reduced to finite fields. The local zeta functions appearing in Weil's conjectures [27] associated with finite fields $G(p,k)$ and thus to single prime. The extensions $G(p,nk)$ of this finite field are considered. These local zeta functions code the number for the points of algebraic variety for given value of n. Weil's conjectures also state that if X is a mod p reduction of non-singular complex projective variety then the degree for the polynomial multiplying the product $\zeta(s) \times \zeta(s-1)$ equals to Betti number. Betti number is 2 times genus in 2-D case.

It has been proven that the zetas of Weil are associated with single prime p, they satisfy functional equation, their zeros are at critical lines, and rather remarkably, they are rational functions of p^{-s}. For instance, for elliptic curves zeros are at critical line [27].

The general form for the local zeta is $\zeta(s) = exp(G(s))$, where $G = \sum g_n p^{-ns}$, $g_n = N_n/n$, codes for the numbers N_n of points of algebraic variety for n^{th} extension of finite field F with nk elements assuming that F has $k = p^r$ elements. This transformation resembles the relationship $Z = exp(F)$ between partition function and free energy $Z = exp(F)$ in thermodynamics.

The exponential form is motivated by the possibility to factorize the zeta function into a product of zeta functions. Note also that in the situation when N_n approaches constant N_∞, the division of N_n by n gives essentially $1/(1 - N_\infty p^{-s})$ and one obtains the factor of Riemann Zeta at a shifted argument $s - log_p(N_\infty)$. The local zeta associated with Riemann Zeta corresponds to $N_n = 1$.

8.5.2 Local zeta functions and TGD

The local zetas are associated with single prime p, they satisfy functional equation, their zeros lie at the critical lines, and they are rational functions of p^{-s}. These features are highly desirable from the TGD point of view.

Why local zeta functions are natural in TGD framework?

In TGD framework modified Dirac equation assigns to a partonic 2-surface a p-adic prime p and inverse of the zeta defines local conformal weight. The intersection of the real and corresponding p-adic parton 2-surface is the set containing the points that one is interested in. Hence local zeta sharing the basic properties of Riemann zeta is highly desirable and natural. In particular, if the local zeta is a rational function then the inverse images of rational points of the geodesic sphere are algebraic numbers. Of course, one might consider a stronger constraint that the inverse image is rational. Note that one must still require that p^{-s} as well as s are algebraic numbers for the zeros of the local zeta (conditions 1) and 2) listed in the beginning) if one wants the number theoretical universality.

Since the modified Dirac operator assigns to a given partonic 2-surface a p-adic prime p, one can ask whether the inverse $\zeta_p^{-1}(z)$ of some kind of local zeta directly coding data about partonic 2-surface could define the generalized eigenvalues of the modified Dirac operator and radial super-symplectic conformal weights so that the conjectures about Riemann Zeta would not be needed at all.

The eigenvalues of the modified Dirac operator would in a holographic manner code for information about partonic 2-surface. This kind of algebraic geometric data are absolutely relevant for TGD since U-matrix and probably also S-matrix must be formulated in terms of the data related to the intersection of real and partonic 2-surfaces (number theoretic braids) obeying same algebraic equations and consisting of algebraic points in the appropriate algebraic extension of p-adic numbers. Note that the hierarchy of algebraic extensions of p-adic number fields would give rise to a hierarchy of zetas so that the algebraic extension used would directly reflect itself in the eigenvalue spectrum of the modified Dirac operator and super-symplectic conformal weights. This is highly desirable but not achieved if one uses Riemann Zeta.

One must of course leave open the possibility that for real-real transitions the inverse of the zeta defined as a product of the local zetas (very much analogous to Riemann Zeta defines the conformal weights. This kind of picture would conform with the idea about real physics as a kind of adele formed from p-adic physics.

Finite field hierarchy is not natural in TGD context

That local zeta functions are assigned with a hierarchy of finite field extensions do not look natural in TGD context. The reason is that these extensions are regarded as abstract extensions of $G(p,k)$ as opposed to a large number of algebraic extensions isomorphic with finite fields as abstract number fields and induced from the extensions of p-adic number fields. Sub-field property is clearly highly relevant in TGD framework just as the sub-manifold property is crucial for geometrizing also other interactions than gravitation in TGD framework.

The $O(p^n)$ hierarchy for the p-adic cutoffs would naturally replace the hierarchy of finite fields. This hierarchy is quite different from the hierarchy of finite fields since one expects that the number of solutions becomes constant at the limit of large n and also at the limit of large p so that powers in the function G coding for the numbers of solutions of algebraic equations as function of n should not increase but approach constant N_∞. The possibility to factorize $exp(G)$ to a product $exp(G_0)exp(G_\infty)$ would mean a reduction to a product of a rational function and factor(s) $\zeta_p(s) = 1/(1-p^{-s_1})$ associated with Riemann Zeta with argument s shifted to $s_1 = s - log_p(N_\infty)$.

What data local zetas could code?

The next question is what data the local zeta functions could code.

1. It is not at clear whether it is useful to code global data such as the numbers of points of partonic 2-surface modulo p^n. The notion of number theoretic braid occurring in the proposed approach to S-matrix suggests that the zeta at an algebraic point z of the geodesic sphere S^2 of CP_2 or of light-cone boundary should code purely local data such as the numbers N_n of points which project to z as function of p-adic cutoff p^n. In the generic case this number would be finite for non-vacuum extremals with 2-D S^2 projection. The n^{th} coefficient $g_n = N_n/n$ of the function G_p would code the number N_n of these points in the approximation $O(p^{n+1}) = 0$ for the algebraic equations defining the p-adic counterpart of the partonic 2-surface.

2. In a region of partonic 2-surface where the numbers N_n of these points remain constant, $\zeta(s)$ would have constant functional form and therefore the information in this discrete set of algebraic points would allow to deduce deduce information about the numbers N_n. Both the algebraic points and generalized eigenvalues would carry the algebraic information.

3. A rather fascinating self referentiality would result: the generalized eigen values of the modified Dirac operator expressible in terms of inverse of zeta would code data for a sequence of approximations for the p-adic variant of the partonic 2-surface. This would be natural since second quantized induced spinor fields are correlates for logical thought in TGD inspired theory of consciousness. Even more, the data would be given at points $\zeta(s)$, s a rational value of a super-symplectic conformal weight or a value of generalized eigenvalue of modified Dirac operator (which is essentially function $s = \zeta_p^{-1}(z)$ at geodesic sphere of CP_2 or of light-cone boundary).

8.5.3 Galois groups, Jones inclusions, and infinite primes

Langlands program [21, 22] is an attempt to unify mathematics using the idea that all zeta functions and corresponding theta functions could emerge as automorphic functions giving rise to finite-dimensional representations for Galois groups (Galois group is defined as a group of automorphisms of the extension of field F leaving invariant the elements of F). The basic example corresponds to rationals and their extensions. Finite fields $G(p,k)$ and their extensions $G(p,nk)$ represents another example. The largest extension of rationals corresponds to algebraic numbers (algebraically closed set). Although this non-Abelian group is huge and does not exist in the usual sense of the word its finite-dimensional representations in groups $GL(n,Z)$ make sense.

For instance, Edward Witten is working with the idea that geometric variant of Langlands duality could correspond to the dualities discovered in string model framework and be understood in terms of topological version of four-dimensional $N = 4$ super-symmetric YM theory [36]. In particular, Witten assigns surface operators to the 2-D surfaces of 4-D space-time. This brings unavoidably in mind partonic 2-surfaces and TGD as $N = 4$ super-conformal almost topological QFT.

This observation stimulates some ideas about the role of zeta functions in TGD if one takes the vision about physics as a generalized number theory seriously.

Galois groups, Jones inclusions, and quantum measurement theory

The Galois representations appearing in Langlands program could have a concrete physical/cognitive meaning.

1. The Galois groups associated with the extensions of rationals have a natural action on partonic 2-surfaces represented by algebraic equations. Their action would reduce to permutations of roots of the polynomial equations defining the points with a fixed projection to the above mentioned geodesic sphere S^2 of CP_2 or δM^4_+. This makes possible to define modes of induced spinor fields transforming under representations of Galois groups. Galois groups would also have a natural action on configuration space-spinor fields. One can also speak about configuration space spinors invariant under Galois group.

2. Galois groups could be assigned to Jones inclusions having an interpretation in terms of a finite measurement resolution in the sense that the discrete group defining the inclusion leaves invariant the operators generating excitations which are not detectable.

3. The physical interpretation of the finite resolution represented by Galois group would be based on the analogy with particle physics. The field extension K/F implies that the primes (more precisely, prime ideals) of F decompose into products of primes (prime ideals) of K. Physically this corresponds to the decomposition of particle into more elementary constituents, say hadrons into quarks in the improved resolution implied by the extension $F \rightarrow K$. The interpretation in terms of cognitive resolution would be that the primes associated with the higher extensions of rationals are not cognizable: in other words, the observed states are singlets under corresponding Galois groups: one has algebraic/cognitive counterpart of color confinement.

4. For instance, the system labelled by an ordinary p-adic prime could decompose to a system which is a composite of Gaussian primes. Interestingly, the biologically highly interesting p-adic length scale range 10 nm-5 μm contains as many as four Gaussian Mersennes ($M_k = (1+i)^k - 1$, $k = 151, 157, 163, 167$), which suggests that the emergence of living matter means an improved cognitive resolution.

Galois groups and infinite primes

In particular, the notion of infinite prime suggests a manner to realize the modular functions as representations of Galois groups. Infinite primes might also provide a new perspective to the concrete realization of Langlands program.

1. The discrete Galois groups associated with various extensions of rationals and involved with modular functions which are in one-one correspondence with zeta functions via Mellin transform defined as $\sum x_n n^{-s} \rightarrow \sum x_n z^n$ [28]. Various Galois groups would have a natural action in the space of infinite primes having interpretation as Fock states and more general bound states of an arithmetic quantum field theory.

2. The number theoretic anatomy of space-time points due to the possibility to define infinite number of number theoretically non-equivalent real units using infinite rationals [17] allows the imbedding space points themselves to code holographically various things. Galois groups would have a natural action in the space of real units and thus on the number theoretical anatomy of a point of imbedding space.

3. Since the repeated second quantization of the super-symmetric arithmetic quantum field theory defined by infinite primes gives rise to a huge space of quantum states, the conjecture that the number theoretic anatomy of imbedding space point allows to represent configuration space (the world of classical worlds associated with the light-cone of a given point of H) and configuration space spinor fields emerges naturally [17].

4. Since Galois groups G are associated with inclusions of number fields to their extensions, this inclusion could correspond at quantum level to a generalized Jones inclusion $\mathcal{N} \subset \mathcal{M}$ such that G acts as automorphisms of $\mathcal{M}$ and leaves invariant the elements of $\mathcal{N}$. This might be possible if one allows the replacement of complex numbers as coefficient fields of hyper-finite factors of type II_1 with various algebraic extensions of rationals. Quantum measurement theory with a finite measurement resolution defined by Jones inclusion $\mathcal{N} \subset \mathcal{M}$ [16] could thus have also a purely number theoretic meaning provided it is possible to define a non-trivial action of various Galois groups on configuration space spinor fields via the imbedding of the configuration space spinors to the space of infinite integers and rationals (analogous to the imbedding of space-time surface to imbedding space).

This picture allows to develop rather fascinating ideas about mathematical structures and their relationship to physical world. For instance, the functional form of a map between two sets the points of the domain and target rather than only its value could be coded in a holographic manner by using the number theoretic anatomy of the points. Modular functions giving rise to generalized zeta functions would emerge in especially natural manner in this framework. Configuration space spinor fields would allow a physical realization of the holographic representations of various maps as quantum states.

8.5.4 Connection between Hurwitz zetas, quantum groups, and hierarchy of Planck constants?

The action of modular group SL(2,Z) on Riemann zeta [43] is induced by its action on theta function [44]. The action of the generator $\tau \rightarrow -1/\tau$ on theta function is essential in providing the functional equation for Riemann Zeta. Usually the action of the generator $\tau \rightarrow \tau + 1$ on Zeta is not considered explicitly. The surprise was that the action of the generator $\tau \rightarrow \tau + 1$ on Riemann Zeta does not give back Riemann zeta but a more general function known as Hurwitz zeta $\zeta(s, z)$ for $z = 1/2$. One finds that Hurwitz zetas for certain rational values of argument define in a well defined sense representations of fractional modular group to which quantum group can be assigned naturally. This could allow to code the value of the quantum phase $q = exp(i2\pi/n)$ to the solution spectrum of the modified Dirac operator D. It has later turned out that there is very natural Zeta function associated with the generalized eigenvalue spectrum of the modified Dirac operator and since the number of various kinds of zeta functions is so immense, the hopes that this conjecture would hold true, are meager. Despite this it is worth to discuss Hurwitz zetas here: one of the reasons is that one end up with a very nice argument for why the number of observed fermion families is three.

Hurwitz zetas

Hurwitz zeta is obtained by replacing integers m with $m + z$ in the defining sum formula for Riemann Zeta:

$$\zeta(s, z) = \sum_m (m + z)^{-s} \quad . \tag{8.5.1}$$

Riemann zeta results for $z = n$.
Hurwitz zeta obeys the following functional equation for rational $z = m/n$ of the second argument [45]:

$$\zeta(1 - s, \frac{m}{n}) = \frac{2\Gamma(s)}{2\pi n}^s \sum_{k=1}^{n} cos(\frac{\pi s}{2} - \frac{2\pi k m}{n})\zeta(s, \frac{k}{n}) \quad . \tag{8.5.2}$$

The representation of Hurwitz zeta in terms of θ [45] is given by the equation

$$\int_0^\infty [\theta(z, it) - 1]\, t^{s/2} \frac{dt}{t} = \pi^{(1-s)/2}\Gamma(\frac{1-s}{2})\, [\zeta(1 - s, z) + \zeta(1 - s, 1 - z)] \quad . \tag{8.5.3}$$

By the periodicity of theta function this gives for $z = n$ Riemann zeta.

The action of $\tau \to \tau + 1$ transforms $\zeta(s, 0)$ to $\zeta(s, 1/2)$

The action of the transformations $\tau \to \tau + 1$ on the integral representation of Riemann Zeta [43] in terms of θ function [44]

$$\theta(z;\tau) - 1 = 2\sum_{n=1}^{\infty}[exp(i\pi\tau)]^{n^2} cos(2\pi nz) \tag{8.5.4}$$

is given by

$$\pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s) = \int_0^{\infty}[\theta(0;it) - 1]t^{s/2}\frac{dt}{t} \ . \tag{8.5.5}$$

Using the first formula one finds that the shift $\tau = it \to \tau + 1$ in the argument θ induces the shift $\theta(0;\tau) \to \theta(1/2;\tau)$. Hence the result is Hurwitz zeta $\zeta(s, 1/2)$. For $\tau \to \tau + 2$ one obtains Riemann Zeta.

Thus $\zeta(s, 0)$ and $\zeta(s, 1/2)$ behave like a doublet under modular transformations. Under the subgroup of modular group obtained by replacing $\tau \to \tau+1$ with $\tau \to \tau+2$ Riemann Zeta forms a singlet. The functional equation for Hurwitz zeta relates $\zeta(1-s, 1/2)$ to $\zeta(s, 1/2)$ and $\zeta(s, 1) = \zeta(s, 0)$ so that also now one obtains a doublet, which is not surprising since the functional equations directly reflects the modular transformation properties of theta functions. This doublet might be the proper object to study instead of singlet if one considers full modular invariance.

Hurwitz zetas form n-plets closed under the action of fractional modular group

The inspection of the functional equation for Hurwitz zeta given above demonstrates that $\zeta(s, m/n)$, $m = 0, 1, ..., n$, form in a well-defined sense an n-plet under fractional modular transformations obtained by using generators $\tau \to -1/\tau$ and $\tau \to \tau + 2/n$. The latter corresponds to the unimodular matrix $(a, b; c, d) = (1, 2/n; 0, 1)$. These matrices obviously form a group. Note that Riemann zeta is always one member of the multiplet containing n Hurwitz zetas.

These observations bring in mind fractionization of quantum numbers, quantum groups corresponding to the quantum phase $q = exp(i2\pi/n)$, and the inclusions for hyper-finite factors of type II_1 partially characterized by these quantum phases. Fractional modular group obtained using generator $\tau \to \tau + 2/n$ and Hurwitz zetas $\zeta(s, k/n)$ could very naturally relate to these and related structures.

Hurwitz zetas and TGD

These observations suggest a direct application to quantum TGD.

1. In TGD framework inclusions of HFFs of type II_1 are directly related to the hierarchy of Planck constants involving a generalization of the notion of imbedding space obtained by gluing together copies of 8-D $H = M^4 \times CP_2$ with a discrete bundle structure $H \to H/Z_{n_a} \times Z_{n_b}$ together along the 4-D intersections of the associated base spaces [A9]. A book like structure results and various levels of dark matter correspond to the pages of this book. One can say that elementary particles proper are maximally quantum critical and live in the 4-D intersection of these imbedding spaces whereas their "field bodies" reside at the pages of the Big Book. Note that analogous book like structures results when real and various p-adic variants of the imbedding space are glued together along common algebraic points.

2. The integers n_a and n_b give Planck constant as $\hbar/\hbar_0 = n_a/n_b$, whose most general value is a rational number. In Platonic spirit one can argue that number theoretically simple integers involving only powers of 2 and Fermat primes are favored physically. Phase transitions between different matters occur at the intersection.

3. The inclusions $\mathcal{N} \subset \mathcal{M}$ of HFFs relate also to quantum measurement theory with finite measurement resolution with $\mathcal{N}$ defining the measurement resolution so that N-rays replace complex rays in the projection postulate and quantum space $\mathcal{M}/\mathcal{N}$ having fractional dimension effectively replaces $\mathcal{M}$.

4. The basic hypothesis is that the inverses of zeta function or of more general variants of zeta coding information about the algebraic structure of the partonic 2-surface appear in the admittedly speculative fundamental formula for the generalized eigenvalues of modified Dirac operator D. This formula is consistent with the generalized eigenvalue equation for D but is not the only one that one can imagine.

5. The generalized eigen spectrum of D should code information both about the p-adic prime p characterizing particle and about quantum phases $q = exp(i2\pi/n)$ assignable to the particle in M^4 and CP_2 degrees of freedom. I understand how p-adic primes appear in the spectrum of D and therefore how coupling constant evolution emerges at the level of free field theory so that radiative corrections can vanish without the loss of coupling constant evolution [C4]. The problem has been to understand how the quantum phase characterizing the sector of the generalized imbedding space could make itself visible in these formulas and therefore in quantum dynamics at the level of free spinor fields. The replacement of Riemann zeta with an n-plet of Hurwitz zetas would resolve this problem.

6. Geometrically the fractional modular invariance would naturally relate to the fact that Riemann surface (partonic 2-surface) can be seen as an $n_a \times n_b$-fold covering of its projection to the base space of H: fractional modular transformations corresponding to n_a and n_b would relate points at different sheets of the covering of M^4 and CP_2. This suggests that the fractionization could be a completely general phenomenon happening also for more general zeta functions.

What about exceptional cases $n = 1$ and $n = 2$?

Also $n = 1$ and $n = 2$ are present in the hierarchy of Hurwitz zetas (singlet and doublet). They do not correspond to allowed Jones inclusion since one has $n > 2$ for them. What could this mean?

1. It would seem that the fractionization of modular group relates to Jones inclusions ($n > 2$) giving rise to fractional statistics. $n = 2$ corresponding to the full modular group Sl(2,Z) could relate to the very special role of 2-valued logic, to the degeneracy of $n = 2$ polygon in plane, to the very special role played by 2-component spinors playing exceptional role in Riemann geometry with spinor structure, and to the symplectic representation of HFFs of type II_1 as fermionic Fock space (spinors in the world of classical worlds). Note also that SU(2) defines the building block of compact non-commutative Lie groups and one can obtain Lie-algebra generators of Lie groups from n copies of SU(2) triplets and posing relations which distinguish the resulting algebra from a direct sum of SU(2) algebras.

2. Also n = 2-fold coverings $M^4 \rightarrow M^4/Z_2$ and $CP_2 \rightarrow CP_2/Z_2$ seem to make sense. One can argue that by quantum classical correspondence the spin half property of imbedding space spinors should have space-time correlate. Could $n = 2$ coverings allow to define the space-time correlates for particles having half odd integer spin or weak isospin? If so, bosons would correspond to $n = 1$ and fermions to $n = 2$. One could of course counter argue that induced spinor fields already represent fermions at space-time level and there is no need for the doubling of the representation.

   The trivial group Z_1 and Z_2 are exceptional since Z_1 does not define any quantization axis and Z_2 allows any quantization axis orthogonal to the line connecting two points. For $n \geq 3$ Z_n fixes the direction of quantization axis uniquely. This obviously correlates with $n \geq 3$ for Jones inclusions.

Dark elementary particle functionals

One might wonder what might be the dark counterparts of elementary particle vacuum functionals [F1]. Theta functions $\theta_{[a,b]}(z,\Omega)$ with characteristic $[a,b]$ for Riemann surface of genus g as functions of z and Teichmueller parameters Ω are the basic building blocks of modular invariant vacuum functionals defined in the finite-dimensional moduli space whose points characterize the conformal equivalence class of the induced metric of the partonic 2-surface. Obviously, kind of spinorial variants of theta functions are in question with $g + g$ spinor indices for genus g.

The recent case corresponds to $g = 1$ Riemann surface (torus) so that a and b are $g = 1$-component vectors having values 0 or 1/2 and Hurwitz zeta corresponds to $\theta_{[0,1/2]}$. The four Jacobi theta functions listed in Wikipedia [44] correspond to these thetas for torus. The values for a and b are 0 and 1 for them but this is a mere convention.

The extensions of modular group to fractional modular groups obtained by replacing integers with integers shifted by multiples of $1/n$ suggest the existence of new kind of q-theta functions with characteristics $[a,b]$ with a and b being g-component vectors having fractional values k/n, $k = 0, 1...n-1$. There exists also a definition of q-theta functions working for $0 \leq |q| < 1$ but not for roots of unity [46]. The q-theta functions assigned to roots of unity would be associated with Riemann surfaces with additional Z_n conformal symmetry but not with generic Riemann surfaces and obtained by simply replacing the value range of characteristics $[a,b]$ with the new value range in the defining formula

$$\Theta[a,b](z|\Omega) = \sum_n exp\left[i\pi(n+a)\cdot\Omega\cdot(n+a) + i2\pi(n+a)\cdot(z+b)\right] . \tag{8.5.5}$$

for theta functions. If Z_n conformal symmetry is relevant for the definition of fractional thetas it is probably so because it would make the generalized theta functions sections in a bundle with a finite fiber having Z_n action.

This hierarchy would correspond to the hierarchy of quantum groups for roots of unity and Jones inclusions and one could probably define also corresponding zeta function multiplets. These theta functions would be building blocks of the elementary particle vacuum functionals for dark variants of elementary particles invariant under fractional modular group. They would also define a hierarchy of fractal variants of number theoretic functions: it would be interesting to see what this means from the point of view of Langlands program [21] discussed also in TGD framework [E12] involving ordinary modular invariance in an essential manner.

This hierarchy would correspond to the hierarchy of quantum groups for roots of unity and Jones inclusions and one could probably define also corresponding zeta function multiplets. These theta functions would be building blocks of the elementary particle vacuum functionals for dark variants of elementary particles invariant under fractional modular group.

Dark matter hierarchy and hierarchy of quantum critical systems in modular degrees of freedom

Dark matter hierarchy corresponds to a hierarchy of conformal symmetries Z_n of partonic 2-surfaces with genus $g \geq 1$ such that factors of n define subgroups of conformal symmetries of Z_n. By the decomposition $Z_n = \prod_{p|n} Z_p$, where $p|n$ tells that p divides n, this hierarchy corresponds to an hierarchy of increasingly quantum critical systems in modular degrees of freedom. For a given prime p one has a sub-hierarchy Z_p, $Z_{p^2} = Z_p \times Z_p$, etc... such that the moduli at n+1:th level are contained by n:th level. In the similar manner the moduli of Z_n are sub-moduli for each prime factor of n. This mapping of integers to quantum critical systems conforms nicely with the general vision that biological evolution corresponds to the increase of quantum criticality as Planck constant increases.

The group of conformal symmetries could be also non-commutative discrete group having Z_n as a subgroup. This inspires a very short-lived conjecture that only the discrete subgroups of SU(2) allowed by Jones inclusions are possible as conformal symmetries of Riemann surfaces having $g \geq 1$. Besides Z_n one could have tedrahedral and icosahedral groups plus cyclic group Z_{2n} with reflection added but not Z_{2n+1} nor the symmetry group of cube. The conjecture is wrong. Consider the orbit of the subgroup of rotational group on standard sphere of E^3, put a handle at one of the orbits such that it is invariant under rotations around the axis going through the point, and apply the elements of subgroup. You obtain a Riemann surface having the subgroup as its isometries. Hence all discrete subgroups of SU(2) can act even as isometries for some value of g.

The number theoretically simple ruler-and-compass integers having as factors only first powers of Fermat primes and power of 2 would define a physically preferred sub-hierarchy of quantum criticality for which subsequent levels would correspond to powers of 2: a connection with p-adic length scale hypothesis suggests itself.

Spherical topology is exceptional since in this case the space of conformal moduli is trivial and conformal symmetries correspond to the entire $SL(2,C)$. This would suggest that only the fermions of lowest generation corresponding to the spherical topology are maximally quantum critical. This brings in mind Jones inclusions for which the defining subgroup equals to $SU(2)$ and Jones index equals to

$\mathcal{M}/\mathcal{N} = 4$. In this case all discrete subgroups of $SU(2)$ label the inclusions. These inclusions would correspond to fiber space $CP_2 \rightarrow CP_2/U(2)$ consisting of geodesic spheres of CP_2. In this case the discrete subgroup might correspond to a selection of a subgroup of $SU(2) \subset SU(3)$ acting non-trivially on the geodesic sphere. Cosmic strings $X^2 \times Y^2 \subset M^4 \times CP_2$ having geodesic spheres of CP_2 as their ends could correspond to this phase dominating the very early cosmology.

Fermions in TGD Universe allow only three families

What is nice that if fermions correspond to $n = 2$ dark matter with Z_2 conformal symmetry as strong quantum classical correspondence suggests, the number of ordinary fermion families is three without any further assumptions. To see this suppose that also the sectors corresponding to $M^4 \rightarrow M^4/Z_2$ and $CP_2 \rightarrow CP_2/Z_2$ coverings are possible. Z_2 conformal symmetry implies that partonic Riemann surfaces are hyper-elliptic. For genera $g > 2$ this means that some theta functions of $\theta_{[a,b]}$ appearing in the product of theta functions defining the vacuum functional vanish. Hence fermionic elementary particle vacuum functionals would vanish for $g > 2$ and only 3 fermion families would be possible for $n = 2$ dark matter.

This results can be strengthened. The existence of space-time correlate for the fermionic 2-valuedness suggests that fermions quite generally to even values of n, so that this result would hold for all fermions. Elementary bosons (actually exotic particles belonging to Kac-Moody type representations) would correspond to odd values of n, and could possess also higher families. There is a nice argument supporting this hypothesis. n-fold discretization provided by covering associated with H corresponds to discretization for angular momentum eigen states. Minimal discretization for $2j + 1$ states corresponds to $n = 2j + 1$. $j = 1/2$ requires $n = 2$ at least, $j = 1$ requires $n = 3$ at least, and so on. $n = 2j + 1$ allows spins $j \leq n - 1/2$. This spin-quantum phase connection at the level of space-time correlates has counterpart for the representations of quantum SU(2).

These rules would hold only for genuinely elementary particles corresponding to single partonic component and all bosonic particles of this kind are exotics (excitations in only "vibrational" degrees of freedom of partonic 2-surface with modular invariance eliminating quite a number of them): ordinary gauge bosons correspond to fermion pairs at throats of a wormhole contact and decompose to SU(3) singlet and octet, whose states are labelled by handle-number pairs (g_1, g_2): they define new kind of heavy bosons giving rise to neutral flavor changing currents (could they be visible in LHC?). Note that gravitons necessarily correspond to pairs of fermions or gauge bosons connected by flux tubes so that they are stringy objects in this sense.

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Chapter 9

Topological Quantum Computation in TGD Universe

9.1 Introduction

Quantum computation is perhaps one of the most rapidly evolving branches of theoretical physics. TGD inspired theory of consciousness has led to new insights about quantum computation and in this chapter I want to discuss these ideas in a more organized manner.

There are three mathematically equivalent approaches to quantum computation [21]: quantum Turing machines, quantum circuits, and topological quantum computation (TQC). In fact, the realization that TGD Universe seems to be ideal place to perform TQC [22, 32] served as the stimulus for writing this chapter.

Quite generally, quantum computation allows to solve problems which are NP hard, that is the time required to solve the problem increases exponentially with the number of variables using classical computer but only polynomially using quantum computer. The topological realization of the computer program using so called braids resulting when threads are weaved to 2-dimensional patterns is very robust so that de-coherence, which is the basic nuisance of quantum computation, ceases to be a problem. More precisely, the error probability is proportional to $exp(-\alpha l)$, where l is the length scale characterizing the distance between strands of the braid [32] .

9.1.1 Evolution of basic ideas of quantum computation

The notion of quantum computation goes back to Feynman [24] who demonstrated that some computational tasks boil down to problems of solving quantum evolution of some physical system, say electrons scattering from each other. Many of these computations are NP hard, which means that the number of computational steps required grows exponentially with the number of variables involved so that they become quickly unsolvable using ordinary computers. A quicker manner to do the computation is to make a physical experiment. A further bonus is that if you can solve one NP hard problem, you can solve many equivalent NP hard problems. What is new that quantum computation is not deterministic so that computation must be carried out several times and probability distribution for the outcomes allows to deduce the answer. Often however the situation is such that it is easy to check whether the outcome provides the sought for solution.

Years later David Deutch [25] transformed Feynman's ideas into a detailed theory of quantum computation demonstrating how to encode quantum computation in a quantum system and researchers started to develop applications. One of the key factors in the computer security is cryptography which relies on the fact that the factorization of large integers to primes is a NP hard problem. Peter Shor [26] discovered an algorithm, which allows to carry out the factorization in time, which is exponentially shorter than by using ordinary computers. A second example is problem of searching a particular from a set of N items, which requires time proportional to N classically but quantally only a time proportional to $\sqrt{N}$.

The key notion is quantum entanglement which allows to store information in the relationship between systems, qubits in the simplest situation. This means that information storage capacity increases exponentially as a function of number of qubits rather than only linearly. This explains why NP hard problems which require time increasing exponentially with the number of variables can be solved using quantum computers. It also means exponentially larger information storage capacity than possible classically.

Recall that there are three equivalent approaches to quantum computation: quantum Turing machine, quantum circuits, and topology based unitary modular functor approach. In quantum circuit approach the unitary time evolution defining the quantum computation is assumed to be decomposable to a product of more elementary operations defined by unitary operators associated with quantum gates. The number of different gates needed is surprisingly small: only 1-gates generating unitary transformations of single qubit, and a 2-gate representing a transformation which together with 1-gates is able to generate entanglement are needed to generate a dense subgroup of unitary group $U(2^n)$ in the case of n-qubit system. 2-gate could be conditional NOT (CNOT). The first 1-gate can induce a phase factor to the qubit 0 and do nothing for qubit 1. Second 1-gate could form orthogonal square roots of bits 1 and 0 as superposition of 1 and 0 with identical probabilities.

The formal definition of the quantum computation using quantum circuit is as a computation of the value of a Boolean function of n Boolean arguments, for instance the k:th bit of the largest prime factor of a given integer. The unitary operator U is constructed as a product of operators associated with the basic gates. It is said that the function coding the problem belongs to the class BQP (function is computable with a bounded error in polynomial time) if there exists a classical polynomial-time (in string length) algorithm for specifying the quantum circuit. The first qubit of the outgoing n-qubit is measured and the probability that the the value is 0 determines the value of the bit to be calculated. For instance, for $p(0) \geq 2/3$ the bit is 0 and for $p(0) \geq 1/3$ the bit is 1. The evaluation of the outcome is probabilistic and requires a repeat the computation sufficiently many times.

The basic problem of quantum computation is the extremely fragility of the physical qubit (say spin). The fragility can be avoided by mapping q-bits to logical qubits realized as highly entangled states of many qubits and quantum error-correcting codes and fault tolerant methods [27, 28, 29] rely on this.

The space W of the logical qubits is known as a code space. The sub-space W of physical states of space $Y = V \otimes V..... \otimes V$ is called k-code if the effect of any k-local operator (affecting only k tensor factors of Y linearly but leaving the remaining factors invariant) followed by an orthogonal projection to W is multiplication by scalar. This means that k-local operator modify the states only in directions orthogonal to W.

These spaces indeed exist and it can be shown that the quantum information coded in W is not affected by the errors operating in fewer than $k/2$ of the n particles. Note that $k = 3$ is enough to guarantee stability with respect to 1-local errors. In this manner it is possible to correct the errors by repeated quantum measurements and by a suitable choice of the sub-space eliminate the errors due to the local changes of qubits by just performing a projection of the state back to the subspace (quantum measurement).

If the the error magnitude is below so called accuracy threshold, arbitrary long quantum computations are reliable. The estimates for this constant vary between 10^{-5} and 10^{-3}. This is beyond current technologies. Error correction is based on the representation of qubit as a logical qubit defined as a state in a linear sub-space of the tensor product of several qubits.

Topological quantum computation [32] provides an alternative approach to minimize the errors caused by de-coherence. Conceptually the modular functor approach [33, 32] is considerably more abstract than quantum circuit approach. Unitary modular functor is the S-matrix of a topological quantum field theory. It defines a unitary evolution realizing the quantum computation in macroscopic topological ground states degrees of freedom. The nice feature of this approach is that the notion of physical qubit becomes redundant and the code space defined by the logical qubits can be represented in terms topological and thus non-local degrees of freedom which are stable against local perturbations as required.

9.1.2 Quantum computation and TGD

Concerning quantum computation [21] in general, TGD TGD inspired theory of consciousness provides several new insights.

Quantum jump as elementary particle of consciousness and cognition

Quantum jump is interpreted as a fundamental cognitive process leading from creative confusion via analysis to an experience of understanding, and involves TGD counterpart of the unitary process followed by state function reduction and state preparation. One can say that quantum jump is the elementary particle of consciousness and that selves consists of sequences of quantum jump just like hadrons, nuclei, atoms, molecules,... consist basically of elementary particles. Self loses its consciousness when it generates bound state entanglement with environment. The conscious experience of self is in a well-defined sense a statistical average over the quantum jump during which self exists. During macro-temporal quantum coherence during macro-temporal quantum coherence a sequence of quantum jumps integrates effectively to a single moment of consciousness and effectively defines single unitary time evolution followed by state function reduction and preparation. This means a fractal hierarchy of consciousness very closely related to the corresponding hierarchy for bound states of elementary particles and structure formed from them.

Negentropy Maximization Principle guarantees maximal entanglement

Negentropy Maximization Principle is the basic dynamical principle constraining what happens in state reduction and self measurement steps of state preparation. Each self measurement involves a decomposition of system into two parts. The decomposition is dictated by the requirement that the reduction of entanglement entropy in self measurement is maximal. Self measurement can lead to either unentangled state or to entangled state with density matrix which is proportional to unit matrix (density matrix is the observable measured). In the latter case maximally entangled state typically involved with quantum computers results as an outcome. Hence Nature itself would favor maximally entangling 2-gates. Note however that self measurement occurs only if it increases the entanglement negentropy.

Number theoretical information measures and extended rational entanglement as bound state entanglement

The emerging number theoretical notion of information allows to interpret the entanglement for which entanglement probabilities are rational (or belong to an extension of rational numbers defining a finite extension of p-adic numbers) as bound state entanglement with positive information content. Macro-temporal quantum coherence corresponds to a formation of bound entanglement stable against state function reduction and preparation processes.

Spin glass degeneracy, which is the basic characteristic of the variational principle defining space-time dynamics, implies a huge number of vacuum degrees of freedom, and is the key mechanism behind macro-temporal quantum coherence. Spin glass degrees of freedom are also ideal candidates qubit degrees of freedom. As a matter fact, p-adic length scale hierarchy suggests that qubit represents only the lowest level in the hierarchy of qupits defining p-dimensional state spaces, p prime.

Time mirror mechanism and negative energies

The new view about time, in particular the possibility of communications with and control of geometric past, suggests the possibility of circumventing the restrictions posed by time for quantum computation. Iteration based on initiation of quantum computation again and again in geometric past would make possible practically instantaneous information processing.

Space-time sheets with negative time orientation carry negative energies. Also the possibility of phase conjugation of fermions is strongly suggestive. It is also possible that anti-fermions possess negative energies in phases corresponding to macroscopic length scales. This would explain matter-antimatter asymmetry in elegant manner. Zero energy states would be ideal for quantum computation purposes and could be even created intentionally by first generating a p-adic surface representing the state and then transforming it to a real surface.

The most predictive and elegant cosmology assumes that the net quantum numbers of the Universe vanish so that quantum jumps would occur between different kinds of vacua. Crossing symmetry makes this option almost consistent with the idea about objective reality with definite conserved total quantum numbers but requires that quantum states of 3-dimensional quantum theory represent S-matrices of 2-dimensional quantum field theory. These quantum states are thus about something. The boundaries of space-time surface are most naturally light-like 3-surfaces space-time surface and are limiting cases of space-like 3-surface and time evolution of 2-surface. Hence they would act naturally as space-time correlates for the reflective level of consciousness.

9.1.3 TGD and the new physics associated with TQC

TGD predicts the new physics making possible to realized braids as entangled flux tubes and also provides a detailed model explaing basic facts about anyons.

Topologically quantized magnetic flux tube structures as braids

Quantum classical correspondence suggests that the absolute minimization of Kähler action corresponds to a space-time representation of second law and that the 4-surfaces approach asymptotically space-time representations of systems which do not dissipate anymore. The correlate for the absence of dissipation is the vanishing of Lorentz 4-force associated with the induced Kähler field. This condition can be regarded as a generalization of Beltrami condition for magnetic fields and leads to very explicit general solutions of field equations [D1].

The outcome is a general classification of solutions based on the dimension of CP_2 projection. The most unstable phase corresponds to $D = 2$-dimensional projection and is analogous to a ferromagnetic phase. $D = 4$ projection corresponds to chaotic demagnetized phase and $D = 3$ is the extremely complex but ordered phase at the boundary between chaos and order. This phase was identified as the phase responsible for the main characteristics of living systems [I4, I5]. It is also ideal for quantum computations since magnetic field lines form extremely complex linked and knotted structures.

The flux tube structures representing topologically quantized fields, which have $D = 3$ -dimensional CP_2 projection, are knotted, linked and braided, and carry an infinite number of conserved topological charges labelled by representations of color group. They seem to be tailor-made for defining the braid structure needed by TQC. The boundaries of the magnetic flux tubes correspond to light-like 3-surfaces with respect to the induced metric (being thus metrically 2-dimensional and allowing conformal invariance) and can be interpreted either as 3-surfaces or time-evolutions of 2-dimensional systems so that S-matrix of 2-D system can be coded into the quantum state of conformally invariant 3-D system.

Anyons in TGD

TGD suggests a many-sheeted model for anyons used in the modelling of quantum Hall effect [42, 44, 43]. Quantum-classical correspondence requires that dissipation has space-time correlates. Hence a periodic motion should create a permanent track in space-time. This kind of track would be naturally magnetic flux tube like structure surrounding the Bohr orbit of the charged particle in the magnetic field. Anyon would be electron plus its track.

The magnetic field inside magnetic flux tubes impels the anyons to the surface of the magnetic flux tube and a highly conductive state results. The partial fusion of the flux tubes along their boundaries makes possible delocalization of valence anyons localized at the boundaries of flux tubes and implies a dramatic increase of longitudinal conductivity. When magnetic field is gradually increased the radii of flux tubes and the increase of the net flux brings in new flux tubes. The competition of these effects leads to the emergence of quantum Hall plateaus and sudden increase of the longitudinal conductivity σ_{xx}.

The simplest model explains only the filling fractions $\nu = 1/m$, m odd. The filling fractions $\nu = N/m$, m odd, require a more complex model. The transition to chaos means that periodic orbits become gradually more and more non-periodic: closed orbits fail to close after the first turn and do so only after N 2π rotations. Tracks would become N-branched surfaces. In N-branched space-time the single-valued analytic two particle wave functions $(\xi_k - \xi_l)^m$ of Laughlin [43] correspond to multiple valued wave functions $(z_k - z_l)^{m/N}$ at its M^4_+ projection and give rise to a filling fraction $\nu = N/m$. The filling fraction $\nu = N/m$, m even, requires composite fermions [48]. Anyon tracks can indeed contain up to $2N$ electrons if both directions of spin are allowed so that a rich spectroscopy is predicted: in particular anyonic super-conductivity becomes possible by 2-fermion composites. The branching gives rise to Z_N-valued topological charge.

One might think that fractional charges could be only apparent and result from the multi-branched character as charges associated with a single branch. This does not seem to be the case. Rather, the fractional charges result from the additional contribution of the vacuum Kähler charge of the anyonic flux tube to the charge of anyon. For $D = 3$ Kähler charge is topologized in the sense that the charge density is proportional to the Chern-Simons. Also anyon spin could become genuinely fractional due to the vacuum contribution of the Kähler field to the spin. Besides electronic anyons also anyons associated with various ions are predicted and certain strange experimental findings about fractional Larmor frequencies of proton in water environment [60, 55] have an elegant explanation in terms of protonic anyons with $\nu = 3/5$. In this case however the magnetic field was weaker than the Earth's magnetic field so that the belief that anyons are possible only in systems carrying very strong magnetic fields would be wrong.

In TGD framework anyons as punctures of plane would be replaced by wormhole like tubes connecting different points of the boundary of the magnetic flux tube and are predicted to always appear as pairs as they indeed do. Detailed arguments demonstrate that TGD anyons are for $N = 4$ ($\nu = 4/m$) ideal for realizing the scenario of [32] for TQC.

The TGD inspired model of non-Abelian anyons is consistent with the model of anyons based on spontaneous symmetry breaking of a gauge symmetry G to a discrete sub-group H dynamically [49]. The breaking of electro-weak gauge symmetry for classical electro-weak gauge fields occurs at the space-time sheets associated with the magnetic flux tubes defining the strands of braid. Symmetry breaking implies that elements of holonomy group span H. This group is also a discrete subgroup of color group acting as isotropy group of the many-branched surface describing anyon track inside the magnetic flux tube. Thus the elements of the holonomy group are mapped to a elements of discrete subgroup of the isometry group leading from branch to another one but leaving many-branched surface invariant.

Witten-Chern-Simons action and light-like 3-surfaces

The magnetic field inside magnetic flux tube expels anyons at the boundary of the flux tube. In quantum TGD framework light-like 3-surfaces of space-time surface and future light cone are in key role since they define causal determinants for Kähler action. They also provide a universal manner to satisfy boundary conditions. Hence also the boundaries of magnetic flux tube structures could be light like surfaces with respect to the induced metric of space-time sheet and would be somewhat like black hole horizons. By their metric 2-dimensionality they allow conformal invariance and due the vanishing of the metric determinant the only coordinate invariant action is Chern-Simons action associated Kähler gauge potential or with the induced electro-weak gauge potentials.

The quantum states associated with the light-like boundaries would be naturally "self-reflective" states in the sense that they correspond to S-matrix elements of the Witten-Chern-Simons topological field theory. Modular functors could results as restriction of the S-matrix to ground state degrees of freedom and Chern-Simons topological quantum field theory is a promising candidate for defining the modular functors [36, 33].

Braid group B_n is isomorphic to the first homotopy group of the configuration space $C_n(R^2)$ of n particles. $C_n(R^2)$ is $((R^2)_n - D)/S_n$, where D is the singularity represented by the configurations in which the positions of 2 or more particles. and be regarded also as the configuration associated with plane with $n+1$ punctures with $n+1$:th particle regarded as inert. The infinite order of the braid group is solely due to the 2-dimensionality. Hence the dimension $D = 4$ for space-time is unique also in the sense that it makes possible TQC.

9.1.4 TGD and TQC

Many-sheeted space-time concept, the possibility of negative energies, and Negentropy Maximization Principle inspire rather concrete ideas about TQC. NMP gives good hopes that the laws of Nature could take care of building fine-tuned entanglement generating 2-gates whereas 1-gates could be reduced to 2-gates for logical qubits realized using physical qubits realized as Z^4 charges and not existing as free qubits.

Only 2-gates are needed

The entanglement of qubits is algebraic which corresponds in TGD Universe to bound state entanglement. Negentropy Maximization Principle implies that maximal entanglement results automatically in quantum jump. This might saves from the fine-tuning of the 2-gates. In particular, the maximally entangling Yang-Baxter R-matrix is consistent with NMP.

TGD suggests a rather detailed physical realization of the model of [32] for anyonic quantum computation. The findings about strong correlation between quantum entanglement and topological entanglement are apparently contradicted by the Temperley-Lie representations for braid groups using only single qubit. The resolution of the paradox is based on the observation that in TGD framework batches containing anyon Cooper pair (AA) and single anyon (instead of two anyons as in the model of [32]) allow to represent single qubit as a logical qubit, and that mixing gate and phase gate can be represented as swap operations s_1 and s_2. Hence also 1-gates are induced by the purely topological 2-gate action, and since NMP maximizes quantum entanglement, Nature itself would take care of the fine-tuning also in this case. The quantum group representation based on $q = exp(i2\pi/5)$ is the simplest representation satisfying various constraints and is also physically very attractive. [32, 33].

TGD makes possible zero energy TQC

TGD allows also negative energies: besides phase conjugate photons also phase conjugate fermions and anti-fermions are possible, and matter-antimatter asymmetry might be only apparent and due to the ground state for which fermion energies are positive and anti-fermion energies negative.

This would make in principle possible zero energy topological quantum computations. The least one could hope wold be the performance of TQC in doubles of positive and negative energy computations making possible error detection by comparison. The TGD based model for anyon computation however leads to expect that negative energies play much more important role.

The idea is that the quantum states of light-like 3-surfaces represent 2-dimensional time evolutions (in particular modular functors) and that braid operations correspond to zero energy states with initial state represented by positive energy anyons and final state represented by negative energy anyons. The simplest manner to realize braid operations is by putting positive *resp.* negative energy anyons near the boundary of tube T_1 *resp.* T_2. Opposite topological charges are at the ends of the magnetic threads connecting the positive energy anyons at T_1 with the negative energy anyons at T_2. The braiding for the threads would code the quantum gates physically.

Before continuing a humble confession is in order: I am not a professional in the area of quantum information science. Despite this, my hope is that the speculations below might serve as an inspiration for real professionals in the field and help them to realize that TGD Universe provides an ideal arena for quantum information processing, and that the new view about time, space-time, and information suggests a generalization of the existing paradigm to a much more powerful one.

9.2 Existing view about topological quantum computation

In the sequel the evolution of ideas related to topological quantum computation, dance metaphor, and the idea about realizing the computation using a system exhibiting so called non-Abelian Quantum Hall effect, are discussed.

9.2.1 Evolution of ideas about TQC

The history of the TQC paradigm is as old as that of QC and involves the contribution of several Fields Medalists. At 1987 to-be Fields Medalist Vaughan Jones [37] demonstrated that the von Neumann algebras encountered in quantum theory are related to the theory of knots and allow to distinguish between very complex knots. Vaughan also demonstrated that a given knot can be characterized in terms an array of bits. The knot is oriented by assigning an arrow to each of its points and projected to a plane. The bit sequence is determined by a sequence of bits defined by the self-intersections of the knot's projection to plane. The value of the bit in a given intersection changes when the orientation of either line changes or when the line on top of another is moved under it. Since the logic operations performed by the gates of computer can be coded to matrices consisting of 0s and 1s, this means that tying a know can encode the logic operations necessary for computation.

String theorist Edward Witten [36, 33], also a Fields Medalist, connected the work of Jones to quantum physics by showing that performing measurements to a system described by a 3-dimensional topological quantum field theory defined by non-Abelian Chern-Simons action is equivalent with performing the computation that a particular braid encodes. The braids are determined by linked word lines of the particles of the topological quantum field theory. What makes braids and quantum computation so special is that the coding of the braiding pattern to a bit sequence gives rise to a code, which corresponds to a code solving NP hard problem using classical computer.

1989 computer scientist Alexei Kitaev [40] demonstrated that Witten's topological quantum field theory could form a basis for a computer. Then Fields Medalist Michael Freedman entered the scene and in collaboration with Kitaev, Michael Larson and Zhenghan Wang developed a vision of how to build a topological quantum computer [33, 32] using system exhibiting so called non-Abelian quantum Hall effect [51].

The key notion is Z_4 valued topological charge which has values 1 and 3 for anyons and 0 and 2 for their Cooper pairs. For a system of $2n$ non-Abelian anyon pairs created from vacuum there are n-1 anyon qubits analogous to spin . The notion of physical qubit is not needed at all and logical qubit is coded to the topological charge of the anyon Cooper pair. The basic idea is to utilize entanglement between Z_4 valued topological charges to achieve quantum information storage stable against decoherence. The swap of neighboring strands of the braid is the topological correlate of a 2-gate which as such does not generate entanglement but can give rise to a transformation such as CNOT. When combined with 1-gates taking square root of qubit and relative phase, this 2-gate is able to generate $U(2^n)$.

The swap can be represented as the so called braid Yang-Baxter R-matrix characterizing also the deviation of quantum groups from ordinary groups [34]. Quite generally, all unitary Yang-Baxter R-matrices are entangling when combined with square root gate except for special values of parameters characterizing them and thus there is a rich repertoire of topologically realized quantum gates. Temperley-Lieb representation provides a 1-qubit representation for swaps in 3-braid system [34, 33]. The measurement of qubit reduces to the measurement of the topological charge of the anyon Cooper pair: in the case that it vanishes (qubit 0) the anyon Cooper pair can annihilate and this serves as the physical signature.

9.2.2 Topological quantum computation as quantum dance

Although topological quantum computation involves very abstract and technical mathematical thinking, it is possible to illustrate how it occurs by a very elegant metaphor. With tongue in cheek one could say that topological quantum computation occurs like a dance. Dancers form couples and in this dancing floor the partners can be also of same sex. Dancers can change their partners. If the partners are of the same sex, they define bit 1 and if they are of opposite sex they define bit 0.

To simplify things one can arrange dancers into a row or several rows such that neighboring partners along the row form a couple. The simplest situation corresponds to a single row of dancers able to make twists of 180 degrees permuting the dancers and able to change the partner to a new one any time. Dance corresponds to a pattern of tracks of dancers at the floor. This pattern can be lifted to a three-dimensional pattern introducing time as a third dimension. When one looks the tracks of a row of dancers in this 2+1-dimensional space-time, one finds that the tracks of the dancers form a complex weaved pattern known as braiding. The braid codes for the computation. The braiding consists of primitive swap operations in which two neighboring word lines twist around each other.

The values of the bits giving the result of the final state of the calculation can be detected since there is something very special which partners with opposite sex can do and do it sooner or later. Just by looking which pairs do it allows to deduce the values of the bits. The alert reader has of course guessed already now that the physical characterization for the sex is as a Z^4 valued topological charge, which is of opposite sign for the different sexes forming Cooper pairs, and that the thing that partners of opposite sex can do is to annihilate! All that is needed to look for those pairs which annihilate after the dance evening to detect the 0s in the row of bits. The coding of the sex to the sign of the topological charge implies also robustness.

It is however essential that the value of topological charge for a given particle in the final state is not completely definite (this is completely general feature of all quantum computations). One can tell only with certain probability that given couple in the final state is male-female or male-male or female-female and the probabilities in question code for the braid pattern in turn coding for quantum logic circuit. Hence one must consider an ensemble of braid calculations to deduce these probabilities.

The basic computational operation permuting the neighboring topological charges is topological so that the program represented by the braiding pattern is very stable against perturbations. The values of the topological charges are also stable. Hence the topological quantum computation is a very robust process and immune to quantum de-coherence even in the standard physics context.

9.2.3 Braids and gates

In order to understand better how braids define gates one must introduce some mathematical notions related to the braids.

Braid groups

Artin introduced the braid groups bearing his name as groups generated by the elements, which correspond to the cross section between neighboring strands of the braid. The definition of these groups is discussed in detail in [34]. For a braid having $n+1$ strands the Artin group B_{n1} has n generators s_i. The generators satisfy certain relations. Depending on whether the line coming from left is above the the line coming from right one has s_i or s_i^{-1}. The elements s_i and s_j commute for $i<j$ and $i>j+1$: $s_is_j = s_js_i$, which only says that two swaps which do not have common lines commute. For $i=j$ and $i=j+1$ commutativity is not assumed and this correspond to the situation in which the swaps act on common lines.

As already mentioned, Artin's braid group B_n is isomorphic with the homotopy group $\pi_1((R^2)^n/S_{n+1})$ of plane with $n+1$ punctures. B_n is infinite-dimensional because the conditions $s_i^2=1$ added to the defining relations in the case of permutation group S_n are not included. The infinite-dimensionality of homotopy groups reflects the very special topological role of 2-dimensional spaces.

One must consider also variants of braid groups encountered when all particles in question are not identical particles. The reason is that braid operation must be replaced by a 2π rotation of particle A around B when the particles are not identical.

1. Consider first the situation in which all particles are non-identical. The first homotopy group of $R^2)^n-D$, where D represents points configurations for which two or more points are identical is identical with the colored braid group B_n^c defined by $n+1$ punctures in plane such that $n+1$:th is passive (punctures are usually imagined to be located on line). Since particles are not identical the braid operation must be replaced by monodromy in which i:th particle makes 2π rotation around j:th particle. This group has generators

$$\gamma_{ij} = s_i ... s_{j-2}s_{j-1}^2 s_{j-2}... s_i^{-1} \quad , i<j \quad , \tag{9.2.1}$$

   and can be regarded as a subgroup of the braid group.

2. When several representatives of a given particle species are present the so called partially colored braid group B_n^{pc} is believed to describe the situation. For pairs of identical particles the generators are braid generators and for non-identical particles monodromies appear as generators. It will be found later that in case of anyon bound states, the ordinary braid group with the assumption that braid operation can lead to a temporary decay and recombination of anyons to a bound state, might be a more appropriate model for what happens in braiding.

3. When all particles are identical, one has the braid group B_n, which corresponds to the fundamental group of $C_n(R^2) = ((R^2)^n - D)/S_n$. Division by S_n expresses the identicality of particles.

Extended Artin's group

Artin's group can be extended by introducing any group G and forming its tensor power $G^{\otimes^n} = G \otimes ... \otimes G$ by assigning to every strand of the braid group G. The extended group is formed from elements of $g_1 \otimes g_2 .... \otimes g_n$ and s_i by posing additional relations $g_i s_j = s_j g_i$ for $i < j$ and $i > j+1$. The interpretation of these relations is completely analogous to the corresponding one for the Artin's group.

If G allows representation in some space V one can look for the representations of the extended Artin's group in the space $V^{\otimes^n}$. In particular, unitary representations are possible. The space in question can also represent physical states of for instance anyonic system and the element g_i associated with the lines of the braid can represent the unitary operators characterizing the time development of the strand between up to the moment when it experiences a swap operation represented by s_i after this operation g_i becomes $s_i g_i s_i^{-1}$.

Braids, Yang-Baxter relations, and quantum groups

Artin's braid groups can be related directly to the so called quantum groups and Yang-Baxter relations. Yang-Baxter relations follow from the relation $s_1 s_2 s_1 = s_2 s_1 s_2$ by noticing that these operations permute the lines 123 of the braid to the order 321. By assigning to a swap operation permuting i:th and j:th line group element R_{ij} when i:th line goes over the j:th line, and noticing that $R_{ij}i$ acts in the tensor product $V_i \otimes V_j$, one can write the relation for braids in a form

$$R_{32}R_{13}R_{12} = R_{12}R_{13}R_{23} \quad .$$

Braid Yang-Baxter relations are equivalent with the so called algebraic Yang-Baxter relations encountered in quantum group theory. Algebraic R can be written as $R_a = RS$, where S is the matrix representing swap operation as a mere permutation. For a suitable choice R_a provides the fundamental representations for the elements of the quantum group $SL(n)_q$ when V is n-dimensional.

The equations represent n^6 equations for n^4 unknowns and are highly over-determined so that solving the equations is a difficult challenge. Equations have symmetries which are obvious on basis of the topological interpretation. Scaling and automorphism induced by linear transformations of V act as symmetries, and the exchange of tensor factors in $V \otimes V$ and transposition are symmetries as also shift of all indices by a constant amount (using modulo N arithmetics).

Unitary R-matrices

Quite a lot is known about the general solutions of the Yang-Baxter equations and for $n = 2$ the general unitary solutions of equations is known [36]. All of these solutions are entangling and define thus universal 1-gates except for certain parameter values.

The first solution is

$$R = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & \cdot & \cdot & 1 \\ \cdot & 1 & -1 & \cdot \\ \cdot & 1 & 1 & \cdot \\ -1 & \cdot & \cdot & 1 \end{pmatrix} \tag{9.2.1}$$

and contains no free parameters (dots denote zeros). This R-matrix is strongly entangling. Note that the condition $R^8 = 1$ is satisfied. The defining relations for Artin's braid group allow also more general solutions obtained by multiplying R with an arbitrary phase factor. This would mean that $R^8 = 1$ constraint is not satisfied anymore. One can argue that over-all phase does not matter: on the other hand, the over all phase is visible in knot invariants defined by the trace of R.

The second and third solution come as families labelled four phases a, b, c and d:

$$R'(a,b,c,d) = \frac{1}{\sqrt{2}} \begin{pmatrix} a & \cdot & \cdot & \cdot \\ \cdot & & b & \cdot \\ \cdot & c & \cdot & \cdot \\ \cdot & \cdot & \cdot & d \end{pmatrix}$$

$$R''(a,b,c,d) = \frac{1}{\sqrt{2}} \begin{pmatrix} \cdot & \cdot & \cdot & a \\ \cdot & b & \cdot & \cdot \\ \cdot & \cdot & c & \cdot \\ d & \cdot & \cdot & \cdot \end{pmatrix} \tag{9.2.-1}$$

These matrices are not as such entangling. The products $U_1 \otimes U_2 R V_1 \otimes V_2$, where U_i and V_i are 2×2 unitary matrices, are however entangling matrices and thus act as universal gates for $ad - bc \neq 0$ guaranteeing that the state $a|11\rangle + b|10\rangle + |01\rangle + |00\rangle$ is entangled.

It deserves to be noticed that the swap matrix

$$S = R'(1,1,1,1) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & & 1 & \cdot \\ \cdot & 1 & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \end{pmatrix} \tag{9.2.-2}$$

permuting the qubits does not define universal gate. This is understandable since in this representation of braid group reduces it to permutation group and situation becomes completely classical.

One can write all solutions R of braid Yang-Baxter equation in the form $R = R_a$, where R_a is the solution of so called algebraic Yang-Baxter equation. The interpretation is that the swap matrix S represents the completely classical part of the swap operation since it acts as a mere permutation whereas R_a represents genuine quantum effects related to the swap operation.

In the article of Kauffman [34] its is demonstrated explicitly how to construct CNOT gate as a product MRN, where M and N are products of single particle gates. This article contains also a beautiful discussion about how the traces of the unitary matrices defined by the braids define knot invariants. For instance, the matrix R satisfies $R^8 = 1$ so that the invariants constructed using R as 2-gate cannot distinguish between knots containing n and $n + 8k$ sub-sequent swaps. Note however that the multiplication of R with a phase factor allows to get rid of the 8-periodicity.

Knots, links, braids, and quantum 2-gates

In [34] basic facts about knots, links, and their relation to braids are discussed. Knot diagrams are introduced, the so called Reidermeister moves and homeomorphisms of plane as isotopies of knots and links are discussed. Also the notion of braid closure producing knots or links is introduced together with the theorem of Markov stating that any knot and link corresponds to some (not unique) braid. Markov moves as braid deformations leaving corresponding knots and links invariant are discussed and it the immediate implication is that traces of the braid matrices define knot invariants. In particular, the traces of the unitary matrices defined by R-matrix define invariants having same value for the knots and links resulting in the braid closure.

In [34] the state preparation and quantum measurement allowing to deduce the absolute value of the trace of the unitary matrix associated with the braid defining the quantum computer is discussed as an example how quantum computations could occur in practice. The braid in question is product of the braid defining the invariant and trivial braid with same number n of strands. The incoming state is maximally entangled state formed $\sum_n |n\rangle \otimes |n\rangle$, where n runs over all possible bit sequences defined by the tensor product of n qubits. Quantum measurement performs a projection to this state and from the measurements it is possible to deduce the absolute value of the trace defining the knot invariant.

9.2.4 About quantum Hall effect and theories of quantum Hall effect

Using the dance metaphor for TQC, the system must be such that it is possible to distinguish between the different sexes of dancers. The proposal of [32] is that the system exhibiting so called non-Abelian Quantum Hall effect [44, 50] could make possible realization of the topological computation.

The most elegant models of quantum Hall effect are in terms of anyons regarded as singularities due to the symmetry breaking of gauge group G down to a finite sub-group H, which can be also non-Abelian. Concerning the description of the dynamics of topological degrees of freedom topological quantum field theories based on Chern-Simons action are the most promising approach.

Quantum Hall effect

Quantum Hall effect [42, 44] occurs in 2-dimensional systems, typically a slab carrying a longitudinal voltage V causing longitudinal current j. A magnetic field orthogonal to the slab generates a transversal current component j_T by Lorentz force. j_T is proportional to the voltage V along the slab and the dimensionless coefficient is known as transversal conductivity. Classically the coefficients is proportional ne/B, where n is 2-dimensional electron density and should have a continuous spectrum. The finding that came as surprise was that the change of the coefficient as a function of parameters like magnetic field strength and temperature occurred as discrete steps of same size. In integer quantum Hall effect the coefficient is quantized to $2\nu\alpha$, $\alpha = e^2/4\pi$, such that ν is integer.

Later came the finding that also smaller steps corresponding to the filling fraction $\nu = 1/3$ of the basic step were present and could be understood if the charge of electron would have been replaced with $\nu = 1/3$ of its ordinary value. Later also QH effect with wide large range of filling fractions of form $\nu = k/m$ was observed.

The model explaining the QH effect is based on pseudo particles known as anyons [49, 44]. According to the general argument of [42] anyons have fractional charge νe. Also the TGD based model for fractionization to be discussed later suggests that the anyon charge should be νe quite generally. The braid statistics of anyon is believed to be fractional so that anyons are neither bosons nor fermions. Non-fractional statistics is absolutely essential for the vacuum degeneracy used to represent logical qubits.

In the case of Abelian anyons the gauge potential corresponds to the vector potential of the divergence free velocity field or equivalently of incompressible anyon current. For non-Abelian anyons the field theory defined by Chern-Simons action is free field theory and in well-defined sense trivial although it defines knot invariants. For non-Abelian anyons situation would be different. They would carry non-Abelian gauge charges possibly related to a symmetry breaking to a discrete subgroup H of gauge group [49] each of them defining an incompressible hydrodynamical flow. Non-Abelian QH effect has not yet been convincingly demonstrated experimentally. According to [32] the anyons associated with the filling fraction $\nu = 5/2$ are a good candidate for non-Abelian anyons and in this case the charge of electron is reduced to $Q = 1/4$ rather than being $Q = \nu e$.

Non-Abelian anyons [51, 44] are always created in pairs since they carry a conserved topological charge. In the model of [32] this charge should have values in 4-element group Z_4 so that it is conserved only modulo 4 so that charges +2 and -2 are equivalent as are also charges 3 and -1. The state of n anyon pairs created from vacuum can be show to possess 2^{n-1}-dimensional vacuum degeneracy [50]: later a TGD based argument for why this is the case is constructed. When two anyons fuse the 2^{n-1}-dimensional state space decomposes to 2^{n-2}-dimensional tensor factors corresponding to anyon Cooper pairs with topological charges 2 and 0. The topological "spin" is ideal for representing logical qubits. Since free topological charges are not possible the notion of physical qubit does not make sense (note the analogy with quarks). The measurement of topological qubit reduces to a measurement of whether anyon Cooper pair has vanishing topological charge or not.

Quantum Hall effect as a spontaneous symmetry breaking down to a discrete subgroup of the gauge group

The system exhibiting quantum Hall effect is effectively 2-dimensional. Fractional statistics suggests that topological defects, anyons, allowing a description in terms of the representations of the homotopy group of $((R^2)^n - D)/S_n$. The gauge theory description would be in terms of spontaneous symmetry breaking of the gauge group G to a finite subgroup H by a Higgs mechanism [49, 44]. This would make all gauge degrees of freedom massive and leave only topological degrees of freedom. What is unexpected that also non-Abelian topological degrees of freedom are in principle possible. Quantum Hall effect is Abelian or non-Abelian depending on whether the group H has this property.

In the symmetry breaking $G \rightarrow H$ the non-Abelian gauge fluxes defined as non-integrable phase factors $Pexp(i \oint A_\mu dx^\mu)$ around large circles (surrounding singularities (so that field approaches a pure gauge configuration) are elements of the first homotopy group of G/H, which is H in the case that H is discrete group and G is simple. An idealized manner to model the situation [44] is to assume that the connection is pure gauge and defined by an H-valued function which is many-valued such that the values for different branches are related by a gauge transformation in H. In the general case a gauge transformation of a non-trivial gauge field by a multi-valued element of the gauge group would give rise to a similar situation.

One can characterize a given topological singularity magnetically by an element in conjugacy class C of H representing the transformation of H induced by a 2π rotation around singularity. The elements of C define states in given magnetic representation. Electrically the particles are characterized by an irreducible representations of the subgroup of $H_C \subset H$ which commutes with an arbitrarily chosen element of the conjugacy class C.

The action of $h(B)$ resulting on particle A when it makes a closed turn around B reduces in magnetic degrees of freedom to translation in conjugacy class combined with the action of element of H_C in electric degrees of freedom. Closed paths correspond to elements of the braid group $B_n(X^2)$ identifiable as the mapping class group of the punctured 2-surface X^2 and this means that symmetry breaking $G \rightarrow H$ defines a representation of the braid group. The construction of these representations is discussed in [44] and leads naturally via the group algebra of H to the so called quantum double $D(H)$ of H, which is a quasi-triangular Hopf algebra allowing non-trivial representations of braid group.

Anyons could be singularities of gauge fields, perhaps even non-Abelian gauge fields, and the latter ones could be modelled by these representations. In particular, braid operations could be represented using anyons.

Witten-Chern-Simons action and topological quantum field theories

The Wess-Zumino-Witten action used to model 2-dimensional critical systems consists of a 2-dimensional conformally invariant term for the chiral field having values in group G combined with 2+1-dimensional term defined as the integral of Chern-Simons 3-form over a 3-space containing 2-D space as its boundary. This term is purely topological and identifiable as winding number for the map from 3-dimensional space to G. The coefficient of this term is integer k in suitable normalization. k gives the value of central extension of the Kac-Moody algebra defined by the theory.

One can couple the chiral field $g(x)$ to gauge potential defined for some subgroup of G_1 of G. If the G_1 coincides with G, the chiral field can be gauged away by a suitable gauge transformation and the theory becomes purely topological Witten-Chern-Simons theory. Pure gauge field configuration represented either as flat gauge fields with non-trivial holonomy over homotopically non-trivial paths or as multi-valued gauge group elements however remain and the remaining degrees of freedom correspond to the topological degrees of freedom.

Witten-Chern-Simons theories are labelled by a positive integer k giving the value of central extension of the Kac-Moody algebra defined by the theory. The connection with Wess-Zumino-Witten theory come from the fact that the highest weight states associated with the representations of the Kac-Moody algebra of WZW theory are in one-one correspondence with the representations R_i possible for Wilson loops in the topological quantum field theory.

In the Abelian case case 2+1-dimensional Chern-Simons action density is essentially the inner product $A \wedge dA$ of the vector potential and magnetic field known as helicity density and the theory in question is a free field theory. In the non-Abelian case the action is defined by the 3-form

$$\frac{k}{4\pi} Tr \left(A \wedge (dA + \frac{2}{3} A \wedge A) \right)$$

and contains also interaction term so that the field theory defined by the exponential of the interaction term is non-trivial.

In topological quantum field theory the usual n-point correlation functions defined by the functional integral are replaced by the functional averages for $Diff^3$ invariant quantities defined in terms of non-integrable phase factors defined by ordered exponentials over closed loops. One can consider arbitrary number of loops which can be knotted, linked, and braided. These quantities define both knot and 3-manifold invariants (the functional integral for zero link in particular). The perturbative calculation of the quantum averages leads directly to the Gaussian linking numbers and infinite number of perturbative link and not invariants.

The experience gained from topological quantum field theories defined by Chern-Simons action has led to a very elegant and surprisingly simple category theoretical approach to the topological quantum field theory [35, 38] allowing to assign invariants to knots, links, braids, and tangles and also to 3-manifolds for which braids as morphisms are replaced with cobordisms. The so called modular Hopf algebras, in particular quantum groups $Sl(2)_q$ with q a root of unity, are in key role in this approach. Also the connection between links and 3-manifolds can be understood since closed, oriented, 3-manifolds can be constructed from each other by surgery based on links.

Witten's article [36] "Quantum Field Theory and the Jones Polynomial" is full of ingenious constructions, and for a physicist it is the easiest and certainly highly enjoyable manner to learn about knots and 3-manifolds. For these reasons a little bit more detailed sum up is perhaps in order.

1. Witten discusses first the quantization of Chern-Simons action at the weak coupling limit $k \rightarrow \infty$. First it is shown how the functional integration around flat connections defines a topological invariant for 3-manifolds in the case of a trivial Wilson loop. Next a canonical quantization is performed in the case $X^3 = \Sigma^2 \times R^1$: in the Coulomb gauge $A_3 = 0$ the action reduces to a sum of $n = dim(G)$ Abelian Chern-Simons actions with a non-linear constraint expressing the vanishing of the gauge field. The configuration space consists thus of flat non-Abelian connections, which are characterized by their holonomy groups and allows Kähler manifold structure.

2. Perhaps the most elegant quantal element of the approach is the decomposition of the 3-manifold to two pieces glued together along 2-manifold implying the decomposition of the functional integral to a product of functional integrals over the pieces. This together with the basic properties of Hilbert of complex numbers (to which the partition functions defined by the functional integrals over the two pieces belong) allows almost a miracle like deduction of the basic results about the behavior of 3-manifold and link invariants under a connected sum, and leads to the crucial skein relations allowing to calculate the invariants by decomposing the link step by step to a union of unknotted, unlinked Wilson loops, which can be calculated exactly for $SU(N)$. The decomposition by skein relations gives rise to a partition function like representation of invariants and allows to understand the connection between knot theory and statistical physics [41]. A direct relationship with conformal field theories and Wess-Zumino-Witten model emerges via Wilson loops associated with the highest weight representations for Kac Moody algebras.

3. A similar decomposition procedure applies also to the calculation of 3-manifold invariants using link surgery to transform 3-manifolds to each other, with 3-manifold invariants being defined as Wilson loops associated with the homology generators of these (solid) tori using representations R_i appearing as highest weight representations of the loop algebra of torus. Surgery operations are represented as mapping class group operations acting in the Hilbert space defined by the invariants for representations R_i for the original 3-manifold. The outcome is explicit formulas for the invariants of trivial knots and 3-manifold invariant of S^3 for $G = SU(N)$, in terms of which more complex invariants are expressible.

4. For $SU(N)$ the invariants are expressible as functions of the phase $q = exp(i2\pi/(k + N))$ associated with quantum groups. Note that for $SU(2)$ and $k = 3$, the invariants are expressible in terms of Golden Ratio. The central charge $k = 3$ is in a special position since it gives rise to $k + 1 = 4$-vertex representing naturally 2-gate physically. Witten-Chern-Simons theories define universal unitary modular functors characterizing quantum computations [33].

Chern-Simons action for anyons

In the case of quantum Hall effect the Chern-Simons action has been deduced from a model of electrons as a 2-dimensional incompressible fluid [43]. Incompressibility requires that the electron current has a vanishing divergence, which makes it analogous to a magnetic field. The expressibility of the current as a curl of a vector potential b, and a detailed study of the interaction Lagrangian leads to the identification of an Abelian Chern-Simons for b as a low energy effective action. This action is Abelian, whereas the anyonic realization of quantum computation would suggest a non-Abelian Chern-Simons action.

Non-Abelian Chern-Simons action could result in the symmetry breaking of a non-Abelian gauge group G, most naturally electro-weak gauge group, to a non-Abelian discrete subgroup H [49] so that states would be labelled by representations of H and anyons would be characterized magnetically H-valued non-Abelian magnetic fluxes each of them defining its own incompressible hydro-dynamical flow. As will be found, TGD predicts a non-Abelian Chern-Simons term associated with electroweak long range classical fields.

9.2.5 Topological quantum computation using braids and anyons

By the general mathematical results braids are able to code all quantum logic operations [41]. In particular, braids allow to realize any quantum circuit consisting of single particle gates acting on qubits and two particle gates acting on pairs of qubits. The coding of braid requires a classical computation which can be done in polynomial time. The coding requires that each dancer is able to remember its dancing history by coding it into its own state.

The general ideas are following.

1. The ground states of anyonic system characterize the logical qubits, One assumes non-Abelian anyons with Z_4 -valued topological charge so that a system of n anyon pairs created from vacuum allows 2^{n-1}-fold anyon degeneracy [50]. The system is decomposed into blocks containing one anyonic Cooper pair with $Q_T \in \{2, 0\}$ and two anyons with such topological charges that the net topological charge vanishes. One can say that the states $(0, 1-1)$ and $(0, -1, +1))$ represent logical qubit 0 whereas the states $(2, -1, -1)$ and $(2, +1, +1)$ represent logical qubit 1. This would suggest 2^2-fold degeneracy but actually the degeneracy is 2-fold.

   Free physical qubits are not possible and at least four particles are indeed necessarily in order to represent logical qubit. The reason is that the conservation of Z^4 charge would not allow mixing of qubits 1 and 0, in particular the Hadamard 1-gate generating square root of qubit would break the conservation of topological charge. The square root of qubit can be generated only if 2 units of topological charge is transferred between anyon and anyon Cooper pair. Thus qubits can be represented as entangled states of anyon Cooper pair and anyon and the fourth anyon is needed to achieve vanishing total topological charge in the batch.

2. In the initial state of the system the anyonic Cooper pairs have $Q_T = 0$ and the two anyons have opposite topological charges inside each block. The initial state codes no information unlike in ordinary computation but the information is represented by the braid. Of course, also more general configurations are possible. Anyons are assumed to evolve like free particles except during swap operations and their time evolution is described by single particle Hamiltonians.

   Free particle approximation fails when the anyons are too near to each other as during braid operations. The space of logical qubits is realized as k-code defined by the 2^{n-1} ground states, which are stable against local single particle perturbations for $k = 3$ Witten-Chern-Simons action. In the more general case the stability against n-particle perturbations with $n < [k/2]$ is achieved but the gates would become $[k/2]$-particle gates (for $k = 5$ this would give 6-particle vertices).

3. Anyonic system provides a unitary modular functor as the S-matrix associated with the anyon system whose time evolution is fixed by the pre-existing braid structure. What this means that the S-matrices associated with the braids can be multiplied and thus a unitary representation for the group formed by braids results. The vacuum degeneracy of anyon system makes this representation non-trivial. By the NP complexity of braids it is possible to code any quantum logic operation by a particular braid [39]. There exists a powerful approximation theorem allowing to achieve this coding classically in polynomial time [41]. From the properties of the R-matrices inducing gate operations it is indeed clear that two gates can be realized. The Hadamard 1-gate could be realized as 2-gate in the system formed by anyon Cooper pair and anyon.

4. In [32] the time evolution is regarded as a discrete sequence of modifications of single anyon Hamiltonians induced by swaps [38]. If the modifications define a closed loop in the space

of Hamiltonians the resulting unitary operators define a representation of braid group in a dense discrete sub-group of $U(2^n)$. The swap operation is 2-local operation acting like a 2-gate and induces quantum logical operation modifying also single particle Hamiltonians. What is important that this modification maps the space of the ground states to a new one and only if the modifications correspond to a closed loop the final state is in the same code space as the initial state. What time evolution does is to affect the topological charges of anyon Cooper pairs representing qubits inside the 4-anyon batches defined by the braids.

In quantum field theory the analog but not equivalent of this description would be following. Quite generally, a given particle in the final state has suffered a unitary transformation, which is an ordered product consisting of two kinds of unitary operators. Unitary single particle operators $U_n = Pexp(i\int_{t_n}^{t_{n+1}} H_0 dt)$ are analogs of operators describing single qubit gate and play the role of anyon propagators during no-swap periods. Two-particle unitary operators $U_{swap} = Pexp(i\int H_{swap}dt)$ are analogous to four-particle interactions and describe the effect of braid operations inducing entanglement of states having opposite values of topological charge but conserving the net topological charge of the anyon pair. This entanglement is completely analogous to spin entanglement. In particular, the braid operation mixes different states of the anyon. The unitary time development operator generating entangled state of anyons and defined by the braid structure represents the operation performed by the quantum circuit and the quantum measurement in the final state selects a particular final state.

5. Formally the computation halts with a measurement of the topological charge of the left-most anyon Cooper pair when the outcome is just single bit. If decay occurs with sufficiently high probability it is concluded that the value of the computed bit is 0, otherwise 1.

9.3 General implications of TGD for quantum computation

TGD based view about time and space-time could have rather dramatic implications for quantum computation in general and these implications deserve to be discussed briefly.

9.3.1 Time need not be a problem for quantum computations in TGD Universe

Communication with and control of the geometric past is the basic mechanism of intentional action, sensory perception, and long term memory in TGD inspired theory of consciousness. The possibility to send negative energy signals to the geometric past allows also instantaneous computations with respect to subjective time defined by a sequence of quantum jumps. The physicist of year 2100 can induce the quantum jump to turn on the quantum computer at 2050 to perform a simulation of field equations defined by the absolute minimization of Kähler action and lasting 50 geometric years, and if this is not enough iterate the process by sending the outcome of computation back to the past where it defines initial values of the next round of iteration. Time would cease to be a limiting factor to computation.

9.3.2 New view about information

The notion of information is very problematic even in the classical physics and in quantum realm this concept becomes even more enigmatic. TGD inspired theory consciousness has inspired number theoretic ideas about quantum information which are still developing. The standard definition of entanglement entropy relies on the Shannon's formula: $S = -\sum_k p_k log(p_k)$. This entropy is always non-negative and tells that the best one can achieve is entanglement with zero entropy.

The generalization of the notion of entanglement entropy to the p-adic context however led to realization that entanglement for which entanglement probabilities are rational or in an extension of rational numbers defining a finite extension of p-adics allows a hierarchy of entanglement entropies S_p labelled by primes. These entropies are defined as $S_p = -\sum_k p_k log(|p_k|_p)$, where $|p_k|_p$ denotes the p-adic norm of probability. S_p can be negative and in this case defines a genuine information measure. For given entanglement probabilities S_p has a minimum for some value p_0 of prime p, and S_{p_0} could be taken as a measure for the information carried by the entanglement in question whereas entanglement in real and p-adic continua would be entropic. The entanglement with negative entanglement entropy is identified as bound state entanglement.

Since quantum computers by definition apply states for which entanglement coefficients belong to a finite algebraic extension of rational numbers, the resulting states, if ideal, should be bound states. Also finite-dimensional extensions of p-adic numbers by transcendentals are possible. For instance, the extension by the $p-1$ first powers of e (e^p is ordinary p-adic number in R_p). As an extension of rationals this extension would be discrete but infinite-dimensional. Macro-temporal quantum coherence can be identified as being due to bound state formation in appropriate degrees of freedom and implying that state preparation and state function reduction effectively ceases to occur in these degrees of freedom.

Macro-temporal quantum coherence effectively binds a sequence of quantum jumps to single quantum jump so that the effective duration of unitary evolution is stretched from about 10^4 Planck times to arbitrary long time span. Also quantum computations can be regarded as this kind of extended moments of consciousness.

9.3.3 Number theoretic vision about quantum jump as a building block of conscious experience

The generalization of number concept resulting when reals and various p-adic number fields are fused to a book like structure obtaining by gluing them along rational numbers common to all these number fields leads to an extremely general view about what happens in quantum jump identified as basic building block of conscious experience. First of all, the unitary process U generates a formal superposition of states belonging to different number fields including their extensions. Negentropy Maximization Principle [H2] constrains the dynamics of state preparation and state function reduction following U so that the final state contains only rational or extended rational entanglement with positive information content. At the level of conscious experience this process can be interpreted as a cognitive process or analysis leading to a state containing only bound state entanglement serving as a correlate for the experience of understanding. Thus quantum information science and quantum theory of consciousness seem to meet each other.

In the standard approach to quantum computing entanglement is not bound state entanglement. If bound state entanglement is really the entanglement which is possible for quantum computer, the entanglement of qubits might not serve as a universal entanglement currency. That is, the reduction of the general two-particle entanglement to entanglement between N qubits might not be possible in TGD framework.

The conclusion that only bound state entanglement is possible in quantum computation in human time scales is however based on the somewhat questionable heuristic assumption that subjective time has the same universal rate, that is the average increment Δt of the geometric time in single quantum jump does not depend on the space-time sheet, and is of order CP_2 time about 10^4 Planck times. The conclusion could be circumvented if one assumes that Δt depends on the space-time sheet involved: for instance, instead of CP_2 time Δt could be of order p-adic time scale T_p for a space-time sheet labelled by p-adic prime p and increase like $\sqrt{p}$. In this case the unitary operator defining quantum computation would be simply that defining the unitary process U.

9.3.4 Dissipative quantum parallelism?

The new view about quantum jump implies that state function reduction and preparation process decomposes into a hierarchy of these processes occurring in various scales: dissipation would occur in quantum parallel manner with each p-adic scale defining one level in the hierarchy. At space-time level this would correspond to almost independent quantum dynamics at parallel space-time sheets labelled by p-adic primes. In particular, dissipative processes can occur in short scales while the dynamics in longer scales is non-dissipative. This would explain why the description of hadrons as dissipative systems consisting of quarks and gluons in short scales is consistent with the description of hadrons as genuine quantum systems in long scales. Dissipative quantum parallelism would also mean that thermodynamics at shorter length scales would stabilize the dynamics at longer length scales and in this manner favor scaled up quantum coherence.

NMR systems [21] might represent an example about dissipative quantum parallelism. Room temperature NMR (nuclear magnetic resonance) systems use highly redundant replicas of qubits which have very long coherence times. Quantum gates using radio frequency pulses to modify the spin evolution have been implemented, and even effective Hamiltonians have been synthesized. Quantum computations and dynamics of other quantum systems have been simulated and quantum error protocols have been realized. These successes are unexpected since the energy scale of cyclotron states is much below the thermal energy. This has raised fundamental questions about the power of quantum information processing in highly mixed states, and it might be that dissipative quantum parallelism is needed to explain the successes.

Magnetized systems could realize quite concretely the renormalization group philosophy in the sense that the magnetic fields due to the magnetization at the atomic space-time sheets could define a return flux along larger space-time sheets as magnetic flux quanta (by topological flux quantization) defining effective block spins serving as thermally stabilized qubits for a long length scale quantum parallel dynamics. For an external magnetic field $B \sim 10$ Tesla the magnetic length is $L \sim 10$ nm and corresponds to the p-adic length scale $L(k = 151)$. The induced magnetization is $M \sim n\mu^2 B/T$, where n is the density of nuclei and $\mu = ge/2m_p$ is the magnetic moment of nucleus. For solid matter density the magnetization is by a factor ~ 10 weaker than the Earth's magnetic field and corresponds to a magnetic length $L \sim 15$ μm: the p-adic length scale is around $L(171)$. For 10^{22} spins per block spin used for NMR simulations the size of block spin should be $\sim$ 1mm solid matter density so that single block spin would contain roughly 10^6 magnetization flux quanta containing 10^{16} spins each. The magnetization flux quanta serving as logical qubits could allow to circumvent the standard physics upper bound for scaling up of about 10 logical qubits [21].

9.3.5 Negative energies and quantum computation

In TGD universe space-times are 4-surfaces so that negative energies are possible due to the fact that the sign of energy depends on time orientation (energy momentum tensor is replaced by a collection of conserved momentum currents). This has several implications. Negative energy photons having phase conjugate photons as physical correlates of photons play a key role in TGD inspired theory of consciousness and living matter and there are also indications that magnetic flux tubes structures with negative energies are important.

Negative energies makes possible communications to the geometric past, and time mirror mechanism involving generation of negative energy photons is the key mechanism of intentional action and plays central role in the model for the functioning of bio-systems. In principle this could allow to circumvent the problems due to the time required by computation by initiating computation in the geometric past and iterating this process. The most elegant and predictive cosmology is that for which the net conserved quantities of the universe vanish due the natural boundary condition that nothing flows into the future light cone through its boundaries representing the moment of big bang.

Also topological quantum field theories describe systems for which conserved quantities associated with the isometries of space-time, such as energy and momentum, vanish. Hence the natural question is whether negative energies making possible zero energy states might also make possible also zero energy quantum computations.

Crossing symmetry and Eastern and Western views about what happens in scattering

The hypothesis that all physical states have vanishing net quantum numbers (Eastern view) forces to interpret the scattering events of particle physics as quantum jumps between different vacua. This interpretation is in a satisfactory consistency with the assumption about existence of objective reality characterized by a positive energy (Western view) if crossing symmetry holds so that configuration space spinor fields can be regarded as S-matrix elements between initial state defined by positive energy particles and negative energy state defined by negative energy particles. As a matter fact, the proposal for the S-matrix of TGD at elementary particle level relies on this idea: the amplitudes for the transition from vacuum to states having vanishing net quantum numbers with positive and negative energy states interpreted as incoming and outgoing states are assumed to be interpretable as S-matrix elements.

More generally, one could require that scattering between any pair of states with zero net energies and representing S-matrix allows interpretation as a scattering between positive energy states. This requirement is satisfied if their exists an entire self-reflective hierarchy of S-matrices in the sense that the S-matrix between states representing S-matrices S_1 and S_2 would be the tensor product $S_1 \otimes S_2$. At the observational level the experience the usual sequence of observations $|m_1\rangle \rightarrow |m_2\rangle ..... \rightarrow |m_n\rangle ...$ based on belief about objective reality with non-vanishing conserved net quantum numbers would correspond to a sequence $(|m_1 \rightarrow m_2\rangle \rightarrow |m_2 \rightarrow m_3\rangle ...$ between "self-reflective" zero energy states. These sequences are expected to be of special importance since the contribution of the unit matrix to S-matrix $S = 1+iT$ gives dominating contribution unless interactions are strong. This sequence would result in the approximation that $S_2 = 1 + iT_2$ in $S = S_1 \otimes S_2$ is diagonal. The fact that the scattering for macroscopic systems tends to be in forward direction would help to create the materialistic illusion about unique objective reality.

It should be possible to test whether the Eastern or Western view is correct by looking what happens strong interacting systems where the western view should fail. The Eastern view is consistent with the basic vision about quantum jumps between quantum histories having as a counterpart the change of the geometric past at space-time level.

Negative energy anti-fermions and matter-antimatter asymmetry

The assumption that space-time is 4-surface means that the sign of energy depends on time orientation so that negative energies are possible. Phase conjugate photons [52] are excellent candidates for negative energy photons propagating into geometric past.

Also the phase conjugate fermions make in principle sense and one can indeed perform Dirac quantization in four manners such that a) both fermions and anti-fermions have positive/negative energies, b) fermions (anti-fermions) have positive energies and anti-fermions (fermions) have negative energies. The corresponding ground state correspond to Dirac seas obtained by applying the product of a) all fermionic and anti-fermionic annihilation (creation) operators to vacuum, b) all fermionic creation (annihilation) operators and anti-fermionic annihilation (creation) operators to vacuum. The ground states of a) have infinite vacuum energy which is either negative or positive whereas the ground states of b) have vanishing vacuum energy. The case b) with positive fermionic and negative anti-fermionic energies could correspond to long length scales in which are matter-antisymmetric due to the effective absence of anti-fermions ("effective" meaning that no-one has tried to detect the negative energy anti-fermions). The case a) with positive energies could naturally correspond to the phase studied in elementary particle physics.

If gravitational and inertial masses have same magnitude and same sign, consistency with empirical facts requires that positive and negative energy matter must have been separated in cosmological length scales. Gravitational repulsion might be the mechanism causing this. Applying naively Newton's equations to a system of two bodies with energies $E_1 > 0$ and $-E_2 < 0$ and assuming only gravitational force, one finds that the sign of force for the motion in relative coordinates is determined by the sign of the reduced mass $-E_1E_2/(E_1 - E_2)$, which is negative for $E_1 > |E_2|$: positive masses would act repulsively on smaller negative masses. For $E_1 = -E_2$ the motion in the relative coordinate becomes free motion and both systems experience same acceleration which for E_1 corresponds to a repulsive force. The reader has probably already asked whether the observed acceleration of the cosmological expansion interpreted in terms of cosmological constant due to vacuum energy could actually correspond to a repulsive force between positive and negative energy matter.

It is possible to create pairs of positive energy fermions and negative energy fermions from vacuum. For instance, annihilation of photons and phase conjugate photons could create electron and negative energy positron pairs with a vanishing net energy. Magnetic flux tubes having positive and negative energies carrying fermions and negative energy positrons pairs of photons and their phase conjugates via fermion anti-fermion annihilation. The obvious idea is to perform zero energy topological quantum computations by using anyons of positive energy and anti-anyons of negative energy plus their Cooper pairs. This idea will be discussed later in more detail.

9.4 TGD based new physics related to topological quantum computation

The absolute minimization of Kähler action is the basic dynamical principle of space-time dynamics. For a long time it remained an open question whether the known solutions of field equations are building blocks of the absolute minima of Kähler action or represent only the simplest extremals one can imagine and perhaps devoid of any real significance. Quantum-classical correspondence meant a great progress in the understanding the solution spectrum of field equations. Among other things, this principle requires that the dissipative quantum dynamics leading to non-dissipating asymptotic self-organization patterns should have the vanishing of the Lorentz 4-force as space-time correlate. The absence of dissipation in the sense of vanishing of Lorentz 4-force is a natural correlate for the absence of dissipation in quantum computations. Furthermore, absolute minimization, if it is really a fundamental principle, should represent the second law of thermodynamics at space-time level. Of course, one cannot exclude the possibility that second law of thermodynamics at space-time level could replace absolute minimization as the basic principle.

The vanishing of Lorentz 4-force generalizes the so called Beltrami conditions [17, 19] stating the vanishing of Lorentz force for purely magnetic field configurations and these conditions reduce in many cases to topological conditions. The study of classical field equations predicts three phases corresponding to non-vacuum solutions of field equations possessing vanishing Lorentz force. The dimension D of CP_2 projection of the space-time sheet serves as classifier of the phases.

1. $D = 2$ phase is analogous to ferro-magnetic phase possible in low temperatures and relatively simple, $D = 4$ phase is in turn analogous to a chaotic de-magnetized high temperature phase.

2. $D = 3$ phase represents spin glass phase, kind of boundary region between order and chaos possible in a finite temperature range and is an ideal candidate for the field body serving as a template for living systems. $D = 3$ phase allows infinite number of conserved topological charges having interpretation as invariants describing the linking of the magnetic field lines. This phase is also the phase in which topological quantum computations are possible.

9.4.1 Topologically quantized generalized Beltrami fields and braiding

From the construction of the solutions of field equations in terms topologically quantized fields it is obvious that TGD Universe is tailor made for TQC.

$D = 3$ phase allows infinite number of topological charges characterizing the linking of magnetic field lines

When space-time sheet possesses a $D = 3$-dimensional CP_2 projection, one can assign to it a non-vanishing and conserved topological charge characterizing the linking of the magnetic field lines defined by Chern-Simons action density $A \wedge dA/4\pi$ for induced Kähler form. This charge can be seen as classical topological invariant of the linked structure formed by magnetic field lines. For $D = 2$ the topological charge densities vanish identically, for $D = 3$ they are in general non-vanishing and conserved, whereas for $D = 4$ they are not conserved. The transition to $D = 4$ phase can thus be used to erase quantum computer programs realized as braids. The 3-dimensional CP_2 projection provides an economical manner to represent the braided world line pattern of dancers and would be the space where the 3-dimensional quantum field theory would be defined.

The topological charge can also vanish for $D = 3$ space-time sheets. In Darboux coordinates for which Kähler gauge potential reads as $A = P_k dQ^k$, the surfaces of this kind result if one has $Q^2 = f(Q^1)$ implying $A = fdQ^1$, $f = P_1 + P_2\partial_{Q_1}Q^2$, which implies the condition $A \wedge dA = 0$. For these space-time sheets one can introduce Q^1 as a global coordinate along field lines of A and define the phase factor $exp(i\int A_\mu dx^\mu)$ as a wave function defined for the entire space-time sheet. This function could be interpreted as a phase of an order order parameter of super-conductor like state and there is a high temptation to assume that quantum coherence in this sense is lost for more general $D = 3$ solutions. Note however that in boundaries can still remain super-conducting and it seems that this occurs in the case of anyons.

Chern-Simons action is known as helicity in electrodynamics [20]. Helicity indeed describes the linking of magnetic flux lines as is easy to see by interpreting magnetic field as incompressible fluid flow having A as vector potential: $B = \nabla \times A$. One can write A using the inverse of $\nabla\times$ as $A = (1/\nabla\times)B$. The inverse is non-local operator expressible as

$$\frac{1}{\nabla\times}B(r) = \int dV' \frac{(r-r')}{|r-r'|^3} \times B(r') \ ,$$

as a little calculation shows. This allows to write $\int A \cdot B$ as

$$\int dV A\cdot B = \int dV dV' B(r)\cdot \left(\frac{(r-r')}{|r-r'|^3}\times B(r')\right) ,$$

which is completely analogous to the Gauss formula for linking number when linked curves are replaced by a distribution of linked curves and an average is taken.

For $D=3$ field equations imply that Kähler current is proportional to the helicity current by a factor which depends on CP_2 coordinates, which implies that the current is automatically divergence free and defines a conserved charge for $D=3$-dimensional CP_2 projection for which the instanton density vanishes identically. Kähler charge is not equal to the helicity defined by the inner product of magnetic field and vector potential but to a more general topological charge.

The number of conserved topological charges is infinite since the product of any function of CP_2 coordinates with the helicity current has vanishing divergence and defines a topological charge. A very natural function basis is provided by the scalar spherical harmonics of $SU(3)$ defining Hamiltonians of CP_2 canonical transformations and possessing well defined color quantum numbers. These functions define and infinite number of conserved charges which are also classical knot invariants in the sense that they are not affected at all when the 3-surface interpreted as a map from CP_2 projection to M^4_+ is deformed in M^4_+ degrees of freedom. Also canonical transformations induced by Hamiltonians in irreducible representations of color group affect these invariants via Poisson bracket action when the $U(1)$ gauge transformation induced by the canonical transformation corresponds to a single valued scalar function. These link invariants are additive in union whereas the quantum invariants defined by topological quantum field theories are multiplicative.

Also non-Abelian topological charges are well-defined. One can generalize the topological current associated with the Kähler form to a corresponding current associated with the induced electro-weak gauge fields whereas for classical color gauge fields the Chern-Simons form vanishes identically. Also in this case one can multiply the current by CP_2 color harmonics to obtain an infinite number of invariants in $D=3$ case. The only difference is that $A\wedge dA$ is replaced by $Tr(A\wedge(dA+2A\wedge A/3))$.

There is a strong temptation to assume that these conserved charges characterize colored quantum states of the conformally invariant quantum theory as a functional of the light-like 3-surface defining boundary of space-time sheet or elementary particle horizon surrounding wormhole contacts. They would be TGD analogs of the states of the topological quantum field theory defined by Chern-Simons action as highest weight states associated with corresponding Wess-Zumino-Witten theory. These charges could be interpreted as topological counterparts of the isometry charges of configuration space of 3-surfaces defined by the algebra of canonical transformations of CP_2.

The interpretation of these charges as contributions of light-like boundaries to configuration space Hamiltonians would be natural. The dynamics of the induced second quantized spinor fields relates to that of Kähler action by a super-symmetry, so that it should define super-symmetric counterparts of these knot invariants. The anti-commutators of these super charges would contribute to configuration space metric a part which would define a Kähler magnetic knot invariant. These Hamiltonians and their super-charge counterparts would be responsible for the topological sector of quantum TGD.

The color partial wave degeneracy of topological charges inspires the idea that also anyons could move in color partial waves identifiable in terms of "rigid body rotation" of the magnetic flux tube of anyon in CP_2 degrees of freedom. Their presence could explain non-Abelianity of Chern-Simons action and bring in new kind bits increasing the computational capacity of the topological quantum computer. The idea about the importance of macroscopic color is not new in TGD context. The fact that non-vanishing Kähler field is always accompanied by a classical color field (proportional to it) has motivated the proposal that colored excitations in macroscopic length scales are important in living matter and that colors as visual qualia correspond to increments of color quantum numbers in quantum phase transitions giving rise to visual sensations.

Knot theory, 3-manifold topology, and $D=3$ solutions of field equations

Topological quantum field theory (TQFT) [35] demonstrates a deep connection between links and 3-topology, and one might hope that this connection could be re-interpreted in terms of imbeddings of 3-manifolds to $H=M^4_+\times CP_2$ as surfaces having 3-dimensional CP_2 projection, call it X^3 in the sequel. $D=3$ suggests itself because in this case Chern-Simons action density for the induced Kähler field is generically non-vanishing and defines an infinite number of classical charges identifiable as Kähler magnetic canonical covariants invariant under $Diff(M^4_+)$. The field topology of Kähler magnetic field should be in a key role in the understanding of these invariants.

1. Could 3-D CP_2 projection of 3-surface provide a representation of 3-topology?

Witten-Chern-Simons theory for a given 3-manifold defines invariants which characterize both the topology of 3-manifold and the link. Why this is the case can be understood from the construction of 3-manifolds by drilling a tubular neighborhood of a link in S^3 and by gluing the tori back to get a new 3-manifolds. The links with some moves defining link equivalences are known to be in one-one correspondence with closed 3-manifolds and the axiomatic formulation of TQFT [35] as a modular functor clarifies this correspondence. The question is whether the CP_2 projection of the 3-surface could under some assumptions be represented by a link so that one could understand the connection between the links and topology of 3-manifolds.

In order to get some idea about what might happen consider the CP_2 projection X^3 of 3-surface. Assume that X^3 is obtained from S^3 represented as a 3-surface in CP_2 by removing from S^3 a tubular link consisting of linked and knotted solid tori $D^2 \times S^1$. Since the 3-surface is closed, it must have folds at the boundaries being thus representable as a two-valued map $S^3 \rightarrow M^4_+$ near the folds. Assume that this is the case everywhere. The two halves of the 3-surface corresponding to the two branches of the map would be glued together along the boundary of the tubular link by identification maps which are in the general case characterized by the mapping class group of 2-torus. The gluing maps are defined inside the overlapping coordinate batches containing the boundary $S^1 \times S^1$ and are maps between the pairs (Ψ_i, Φ_i), $i = 1, 2$ of the angular coordinates parameterizing the tori.

Define longitude as a representative for the $a + nb$ of the homology group of the 2-torus. The integer n defines so called framing and means that the longitude twists n times around torus. As a matter fact, TQFT requires bi-framing: at the level of Chern-Simons perturbation theory bi-framing is necessary in order to define self linking numbers. Define meridian as the generator of the homology group of the complement of solid torus in S^3. It is enough to glue the carved torus back in such a manner that meridian is mapped to longitude and longitude to minus meridian. This map corresponds to the $SL(2, C)$ element

$$\left(\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right) .$$

Also other identification maps defined by $SL(2, Z)$ matrices are possible but one can do using only this. Note that the two component $SL(2, Z)$ spinors defined as superpositions of the generators (a, b) of the homology group of torus are candidates for the topological correlates of spinors. In the gluing process the tori become knotted and linked when seen in the coordinates of the complement of the solid tori.

This construction would represent the link surgery of 3-manifolds in terms of CP_2 projections of 3-surfaces of H. Unfortunately this representation does not seem to be the only one. One can construct closed three-manifolds also by the so called Heegaard splitting. Remove from S^3 D_g, a solid sphere with g handles having boundary S_g, and glue the resulting surface with its oppositely oriented copy along boundaries. The gluing maps are classified by the mapping class group of S_g. Any closed orientable 3-manifold can be obtained by this kind of procedure for some value of g. Also this construction could be interpreted in terms of a fold at the boundary of the CP_2 projection for a 2-valued graph $S^3 \rightarrow M^4_+$. Whether link surgery representation and Heegaard splitting could be transformed to each other by say pinching D_g to separate tori is not clear to me.

When the graph $CP_2 \rightarrow M^4_+$ is at most 2-valued, the intricacies due to the imbedding of the 3-manifold are at minimum, and the link associated with the projection should give information about 3-topology and perhaps even characterize it. Also the classical topological charges associated with Kähler Chern-Simons action could give this kind of information.

2. Knotting and linking for 3-surfaces

The intricacies related to imbedding become important in small co-dimensions and it is of considerable interest to find what can happen in the case of 3-surfaces. For 1-dimensional links and knots the projection to a plane, the shadow of the knot, characterizes the link/knot and allows to deduce link and knot invariants purely combinatorially by gradually removing the intersection points and writing a contribution to the link invariant determined by the orientations of intersecting strands and by which of them is above the other. Thus also the generalization of knot and link diagrams is of interest.

Linking of m- and n-dimensional sub-manifolds of D-dimensional manifold H_D occurs when the condition $m+n=D-1$ holds true. The n-dimensional sub-manifold intersects $m+1$-dimensional surfaces having m-dimensional manifold as its boundary at discrete points, and it is usually not possible to remove these points by deforming the surfaces without intersections in some intermediate stage. The generalization of the link diagram results as a projection $D-1$-dimensional disk D^{D-1} of H_D.

3-surfaces link in dimension $D=7$ so that the linking of 3-surfaces occurs quite generally in time=constant section of the imbedding space. A link diagram would result as a projection to $E^2\times CP_2$, E^2 a 2-dimensional plane: putting CP_2 coordinates constant gives ordinary link diagram in E^2. For magnetic flux tubes the reduction to 2-dimensional linking by idealizing flux tubes with 1-dimensional strings makes sense.

Knotting occurs in codimension 2 that is for an n-manifold imbedded in $D=n+2$-dimensional manifold. Knotting can be understood as follows. Knotted surface spans locally $n+1$-dimensional 2-sided n+1-disk D^{n+1} (disk for ordinary knot). The portion of surface going through D^{n+1} can be idealized with a 1-dimensional thread going through it and by $n+2=D$ knotting is locally linking of this 1-dimensional thread with n-dimensional manifold. N-dimensional knots define n+1-dimensional knots by so called spinning. Take an n-knot with the topology of sphere S^n such that the knotted part is above $n+1$-plane of $n+2$-dimensional space R^{n+2} $(z\geq 0)$, cut off the part below plane $(z<0)$, introduce an additional dimension (t) and make a 2π rotation for the resulting knot in $z-t$ plane. The resulting manifold is a knotted S^{n+1}. The counterpart of the knot diagram would be a projection to $n+1$-dimensional sub-manifold, most naturally disk D^{n+1}, of the imbedding space.

3-surfaces could become knotted under some conditions. Vacuum extremals correspond to 4-surfaces $X^4\subset M^4_+\times Y^2$ whereas the four-surfaces $X^4\subset M^4_+\times S^2$, S^2 homologically non-trivial geodesic sphere, define their own "sub-theory". In both cases 3-surfaces in time=constant section of imbedding space can get knotted in the sense that un-knotting requires giving up the defining condition temporarily. The counterpart of the knot diagram is the projection to $E^2\times X^2$, $X^2=Y^2$ or S^2, where E^2 is plane of M^4_+. For constant values of CP_2 coordinates ordinary knot diagram would result. Reduction to ordinary knot diagrams would naturally occur for $D=2$ magnetic flux tubes. The knotting occurs also for 4-surfaces themselves in $M^4_+\times X^2$: knot diagram is now defined as projection to $E^3\times X^2$.

3. Could the magnetic field topology of 3-manifold be able to mimic other 3-topologies?

In $D=3$ case the topological charges associated with Kähler Chern-Simons term characterize the linking of the field lines of the Kähler gauge potential A. What $dA\wedge A\neq 0$ means that field lines are linked and it is not possible to define a coordinate varying along the field lines of A. This is impossible even locally since the $dA\wedge A\neq 0$ condition is equivalent with non existence of a scalar functions k and Φ such that $\nabla\Phi=kA$ guaranteing that Φ would be the sought for global coordinate.

One can idealize the situation a little bit and think of a field configuration for which magnetic flux is concentrated at one-dimensional closed lines. The vector potential would in this case be simply $A=\nabla(k\Psi+l\Phi)$, where Ψ is an angle coordinate around the singular line and Φ a coordinate along the singular circle. In this idealized situation the failure to have a global coordinate would be due to the singularities of otherwise global coordinates along one-dimensional linked and knotted circles. The reason is that the field lines of A and B rotate helically around the singular circle and the points (x,y,z) with constant values of x,y are on a helix which becomes singular at z-axis. Since the replacement of a field configuration with a non-singular field configuration but having same field line topology does not affect the global field line topology, one might hope of characterizing the field topology by its singularities along linked and knotted circles also in the general case.

Just similar linked and knotted circles are used to construct 3-manifolds in the link surgery which would suggest that the singularities of the field line topology of X^3 code the non-trivial 3-topology resulting when the singularities are removed by link surgery. Physically the longitude defining the framing $a+nb$ would correspond to the field line of A making an $n2\pi$ twist along the singular circle. Meridian would correspond to a circle in the plane of B. The bi-framing necessitated by TQFT would have a physical interpretation in terms of the helical field lines of A and B rotating around the singular circle. At the level of fields the gluing operation would mean a gauge transformation such that the meridians would become the field lines of the gauge transformed A and being non-helical could be continued to the the interior of the glued torus without singularities. Simple non-helical magnetic torus would be in question.

This means that the magnetic field patterns of a given 3-manifold could mimic the topologies of other 3-manifolds. The topological mimicry of this kind would be a very robust manner to represent information and might be directly relevant to TQC. For instance, the computation of topological invariants of 3-manifold Y^3 could be coded by the field pattern of X^3 representing the link surgery producing the 3-manifold from S^3, and the physical realization of TQC program could directly utilize the singularities of this field pattern. Topological magnetized flux tubes glued to the back-ground 3-surface along the singular field lines of A could provide the braiding.

This mimicry could also induce transitions to the new topology and relate directly to 3-manifold surgery performed by a physical system. This transition would quite concretely mean gluing of simple $D = 2$ magnetic flux tubes along their boundaries to the larger $D = 3$ space-time sheet from which similar flux tube has been cut away.

4. A connection with anyons?

There is also a possible connection with anyons. Anyons are thought to correspond to singularities of gauge fields resulting in a symmetry breaking of gauge group to a finite subgroup H and are associated with homotopically non-trivial loops of $C_n = ((R^2)^n - D)/S_n$ represented as elements of H. Could the singularities of gauge fields relate to the singularities of the link surgery so that the singularities would be more or less identifiable as anyons? Could N-branched anyons be identified in terms of framings $a + Nb$ associated with the gluing map? $D = 3$ solutions allow the so called contact structure [D1], which means a decomposition of the coordinates of CP_2 projection to a longitudinal coordinate s and a complex coordinate w. Could this decomposition generalize the notion of effective 2-dimensionality crucial for the notion of anyon?

5. What about Witten's quantal link invariants?

Witten's quantal link invariants define natural multiplicative factors of configuration space spinor fields identifiable as representations of two 2-dimensional topological evolution. In Witten's approach these invariants are defined as functional averages of non-integrable phase factors associated with a given link in a given 3-manifold. TGD does not allow any natural functional integral over gauge field configurations for a fixed 3-surface unless one is willing to introduce fictive non-Abelian gauge fields. Although this is not a problem as such, the representation of the invariants in terms of inherent properties of the 3-surface or corresponding 4-surfaces would be highly desirable.

Functional integral representation is not the only possibility. Quantum classical correspondence combined with topological field quantization implied by the absolute minimization of Kähler action generalizing Bohr rules to the field context gives hopes that the 3-surfaces themselves might be able to represent 3-manifold invariants classically. In $D = 3$ case the quantized exponents of Kähler-Chern-Simons action and $SU(2)_L$ Chern-Simons action could define 3-manifold invariants. These invariants would satisfy the obvious multiplicativity conditions and could correspond to the phase factors due to the framing dependence of Witten's invariants identifying the loops of surgery link as Wilson loops. These phase factors are powers of $U = exp(i2\pi c/24)$, where c is the central charge of the Virasoro representation defined by Kac Moody representation. One has $c = k \times dim(g)/(k + c_g/2)$, which gives $U = exp(i2\pi k/8(k + 2))$ for $SU(2)$. The dependence on k differs from what one might naively expect. For this reason, and also because the classical Wilson loops do not depend explicitly on k, the value of k appearing in Chern-Simons action should be fixed by the internal consistency and be a constant of Nature according to TGD. The guess is that k possesses the minimal value $k = 3$ allowing a universal modular functor for $SU(2)$ with $q = exp(i2\pi/5)$.

The loops associated with the topological singularities of the Kähler gauge potential (typically the center lines of helical field configurations) would in turn define natural Wilson loops, and since the holonomies around these loops are also topologically quantized, they could define invariants of 3-manifolds obtained by performing surgery around these lines. The behavior of the induced gauge fields should be universal near the singularities in the sense that the holonomies associated with the CP_2 projections of the singularities to CP_2 would be universal. This expectation is encouraged by the notion of quantum criticality in general and in particular, by the interpretation of $D = 3$ phase as a critical system analogous to spin glass.

The exponent of Chern-Simons action can explain only the phase factors due to the framing, which are usually regarded as an unavoidable nuisance. This might be however all that is needed. For the manifolds of type $X^2 \times S^1$ all link invariants are either equal to unity or vanish. Surgery would allow to build 3-manifold invariants from those of $S^2 \times S^1$. For instance, surgery gives the invariant $Z(S^3)$ in terms of $Z(S^2 \times S^1, R_i)$ and mapping class group action coded into the linking of the field lines.

Holonomies can be also seen as multi-valued $SU(2)_L$ gauge transformations and can be mapped to a multi-valued transformations in the $SU(2)$ subgroup of $SU(3)$ acting on 3-surface as a geometric transformations and making it multi-branched. This makes sense if the holonomies define a finite group so that the gauge transformation is finitely many-valued. This description might apply to the 3-manifold resulting in a surgery defined by the Wilson loops identifiable as branched covering of the initial manifold.

The construction makes also sense for the holonomies defined by the classical $SU(3)$ gauge fields defined by the projections of the isometry currents. Furthermore, the fact that any CP_2 Hamiltonian defines a conserved topological charge in $D = 3$ phase should have a deep significance. At the level of the configuration space geometry the finite-dimensional group defining Kac Moody algebra is replaced with the group of canonical transformations of CP_2. Perhaps one could extend the notion of Wilson loop for the algebra of canonical transformations of CP_2 so that the representations R_i of the gauge group would be replaced by matrix representations of the canonical algebra. That the trace of the identity matrix is infinite in this case need not be a problem since one can simply redefine the trace to have value one.

Braids as topologically quantized magnetic fields

$D = 3$ space-time sheets would define complex braiding structures with flux tubes possessing infinite number of topological charges characterizing the linking of field lines. The world lines of the quantum computing dancers could thus correspond to the flux tubes that can get knotted, linked, and braided. This idea conforms with the earlier idea that the various knotted and linked structures formed by linear bio-molecules define some kind of computer programs.

1. Boundaries of magnetic flux tubes as light-like 3-surfaces

Field equations for Kähler action are satisfied identically at boundaries if the boundaries of magnetic flux tubes (and space-time sheet in general) are light-like in the induced metric. In M^4_+ metric the flux tubes could look static structures. Light-likeness allows an interpretation of the boundary state either as a 3-dimensional quantum state or as a time-evolution of a 2-dimensional quantum state. This conforms with the idea that quantum computation is cognitive, self reflective process so that quantum state is about something rather than something. There would be no need to force particles to flow through the braid structure to build up time-like braid whereas for time-like boundaries of magnetic flux tubes a time-like braid results only if the topologically charged particles flow through the flux tubes with the same average velocity so that the length along flux tubes is mapped to time.

Using the terminology of consciousness theory, one could say that during quantum dance the dancers are in trance being entangled to a single macro-temporally coherent state which represents single collective consciousness, and wake up to individual dancers when the dance ends. Quantum classical correspondence suggests that the generation of bound state entanglement between dancers requires tangled join along boundaries bonds connecting the space-time sheets of anyons (braid of flux tubes again!): dancers share mental images whereas direct contact between magnetic flux tubes defining the braid is not necessary. The bound state entanglement between sub-systems of unentangled systems is made possible by the many-sheeted space-time. This kind of entanglement could be interpreted as entanglement not visible in scales of larger flux tubes so that the notion is natural in the philosophy based on the idea of length scale resolution.

2. How braids are generated?

The encoding of the program to a braid could be a mechanical process: a bundle of magnetic flux tubes with one end fixed would be gradually weaved to a braid by stretching and performing the needed elementary twists. The time to perform the braiding mechanically requires classical computer program and the time needed to carry out the braiding depends polynomially on the number of strands.

The process could also occur by a quantum jump generating the braided flux tubes in single flash and perhaps even intentionally in living systems (flux tubes with negative topological charge could have negative energy so that it would require no energy to generate the structure from vacuum). The interaction with environment could be used to select the desired braids. Also ensembles of braids might be imagined. Living matter might have discovered this mechanism and used int intentionally.

3. Topological quantization, many-sheetedness, and localization

Localization of modular functors is one of the key problems in topological quantum computation (see the article of Freedman [38]. For anyonic computation this would mean in the ideal case a decomposition of the system into batches containing 4 anyons each so that these anyon groups interact only during swap operations.

The role of topological quantization would be to select of a portion of the magnetic field defining the braid as a macroscopic structure. Topological field quantization realizes elegantly the requirement that single particle time evolutions between swaps involve no interaction with other anyons.

Also many-sheetedness is important. The (AA) pair and two anyons would correspond braids inside braids and as it turns out this gives more flexibility in construction of quantum computation since the 1-gates associated with logical qubits of 4-batch can belong to different representation of braid group than that associated with braiding of the batches.

9.4.2 Quantum Hall effect and fractional charges in TGD

In fractional QH effect anyons possess fractional electromagnetic charges. Also fractional spin is possible. TGD explains fractional charges as being due to multi-branched character of space-time sheets. Also the Z_n-valued topological charge associated with anyons has natural explanation.

Basic TGD inspired ideas about quantum Hall effect

Quantum Hall effect is observed in low temperature systems when the intensity of a strong magnetic field perpendicular to the current carrying slab is varied adiabatically. Classically quantum Hall effect can be understood as a generation of a transversal electric field, which exactly cancels the magnetic Lorentz force. This gives $E = -j \times B/ne$. The resulting current can be also understood as due to a drift velocity proportional to $E \times B$ generated in electric and magnetic fields orthogonal to each other and allowing to cancel Lorentz force. This picture leads to the classical expression for transversal Hall conductivity as $\sigma_{xy} = ne/B$. σ_{xy} should vary continuously as a function of the magnetic field and 2-dimensional electron density n.

In quantum Hall effect σ_{xy} is piece-wise constant and quantized with relative precision of about 10^{-10}. The second remarkable feature is that the longitudinal conductivity σ_{xx} is very high at plateaus: variations by 13 orders of magnitude are observed. The system is also very sensitive to small perturbations.

Consider now what these qualitative observations might mean in TGD context.

1. Sensitivity to small perturbations means criticality. TGD Universe is quantum critical and quantum criticality reduces to the spin glass degeneracy due to the enormous vacuum degeneracy of the theory. The $D = 2$ and $D = 3$ non-vacuum phases predicted by the generalized Beltrami ansatz are this in-stability might play important role in the effect.

2. The magnetic fields are genuinely classical fields in TGD framework, and for $D = 2$ proportional to induced Kähler magnetic field. The canonical symmetries of CP_2 act like $U(1)$ gauge transformations on the induced gauge field but are not gauge symmetries since canonical transformations change the shape of 3-surface and affect both classical gravitational fields and electro-weak and color gauge fields. Hence different gauges for classical Kähler field represent magnetic fields for which topological field quanta can have widely differing and physically non-equivalent shapes. For instance, tube like quanta act effectively as insulators whereas magnetic walls parallel to the slab act as conducting wires.

   Wall like flux tubes parallel to the slab perhaps formed by a partial fusion of magnetic flux tubes along their boundaries would give rise to high longitudinal conductivity. For disjoint flux tubes the motion would be around the flux tubes and the electrons would get stuck inside these tubes. By quantum criticality and by $D < 4$ property the magnetic flux tube structures are unstable against perturbations, in particular the variation of the magnetic field strength itself. The transitions from a plateau to a new one would correspond to the decay of the magnetic walls back to disjoint flux tubes followed by a generation of walls again so that conductivity is very high outside transition regions. The variation of any parameter, such as temperature, is expected to be able to cause similar effects implying dramatic changes in Hall conductivity.

The percolation model for the quantum Hall effect represents slab as a landscape with mountains and valleys and the varied external parameter, say B or free electron density, as the sea level. For the critical values of sea level narrow regions carrying so called edge states allow liquid to fill large regions appear and implies increase of conductivity. Obviously percolation model differs from the model based on criticality for which the landscape itself is highly fragile and a small perturbation can develop new valleys and mountains.

3. The effective 2-dimensionality implies that the solutions of Schrödinger equation of electron in external magnetic field are products of any analytic function with a Gaussian representing the ground state of a harmonic oscillator. Analyticity means that the kinetic energy is completely degenerate for these solutions. The Lauhglin ansatz for the state functions of electron in the external magnetic field is many-electron generalization of these solutions: the wave functions consists of products of terms of form $(z_i - z_j)^m$, m odd integer from Fermi statistics.

   The N-particle variant of Laughlin's ansatz allows to deduce that the system is incompressible. The key observation is that the probability density for the many-particle state has an interpretation as a Boltzmann factor for a fictive two-dimensional plasma in electric field created by constant charge density [42, 43]. The probability density is extremely sensitive to the changes of the positions of electrons giving rise to the constant electron density. The screening of charge in this fictive plasma implies the filling fraction $\nu = 1/m$, m odd integer and requires charge fractionization $e \to e/m$. The explanation of the filling fractions $\nu = N/m$ would require multi-valued wave functions $(z_i - z_j)^{N/m}$. In single-sheeted space-time this leads to problems. TGD suggests that these wave functions are single valued but defined on N-branched surface.

   The degeneracy with respect to kinetic energy brings in mind the spin glass degeneracy induced by the vacuum degeneracy of the Kähler action. The Dirac equation for the induced spinors is not ordinary Dirac equation but super-symmetrically related to the field equations associated with Kähler action. Also it allows vacuum degeneracy. One cannot exclude the possibility that also this aspect is involved at deeper level.

4. The fractionization of charge in quantum Hall effect challenges the idea that charged particles of the incompressible liquid are electrons and this leads to the notion of anyon. Quantum-classical correspondence inspires the idea that although dissipation is absent, it has left its signature as a track associated with electron. This track is magnetic flux tube surrounding the classical orbit of electron and electron is confined inside it. This reduces the dissipative effects and explains the increase of conductivity. The rule that there is single electron state per magnetic flux quantum follows if Bohr quantization is applied to the radii of the orbits. The fractional charge of anyon would result from a contribution of classical Kähler charge of anyon flux tube to the charge of the anyon. This charge is topologized in $D = 3$ phase.

Anyons as multi-branched flux tubes representing charged particle plus its track

Electrons (in fact, any charged particles) moving inside magnetic flux tubes move along circular paths classically. The solutions of the field equations with vanishing Lorentz 4-force correspond to asymptotic patterns for which dissipation has already done its job and is absent. Dissipation has however definite effects on the final state of the system, and one can argue that the periodic motion of the charged particle has created what might called its "track". The track would be realized as a circular or helical flux tube rotating around field lines of the magnetic field. The corresponding cyclotron states would be localized inside tracks. Simplest tracks are circular ones and correspond to absence of motion in the direction of the magnetic field. Anyons could be identified as systems formed as particles plus the tracks containing them.

1. Many-branched tracks and approach to chaos

When the system approaches chaos one expects the the periodic circular tracks become non-periodic. One however expects that this process occurs in steps so that the tracks are periodic in the sense that they close after N 2π rotations with the value of N increasing gradually. The requirement that Kähler energy stays finite suggests also this. A basic example of this kind of track is obtained when the phase angles Ψ and Φ of complex CP_2 coordinates (ξ^1, ξ^2) have finitely multi-valued dependence on the coordinate ϕ of cylindrical coordinates: $(\Psi, \Phi) = (m_1/N, m_2/N)\phi)$. The space-sheet would be many-branched and it would take N turns of 2π to get back to the point were one started. The phase factors behave as a phase of a spinning particle having effective fractional spin $1/N$. I have proposed this kind of mechanism as an explanation of so called hydrino atoms claimed to have the spectrum of hydrogen atom but with energies scaled up by N^2 [56, G2]. The first guess that N corresponds to m in $\nu = 1/m$ is wrong. Rather, N corresponds to N in $\nu = N/m$ which means many-valued Laughlin wave functions in single branched space-time.

Similar argument applies also in CP_2 degrees of freedom. Only the N-multiples of 2π rotations by CP_2 isometries corresponding to color hyper charge and color iso-spin would affect trivially the point of multi-branched surface. Since the contribution of Kähler charge to electromagnetic charge corresponds also to anomalous hyper-charge of spinor field in question, an additional geometric contribution to the anomalous hypercharge would mean anomalous electromagnetic charge.

It must be emphasized the fractionization of the isometry charges is only effective and results from the interpretation of isometries as space-time transformations rather than transformation rotating entire space-time sheet in imbedding space. Also classical charges are effectively fractionized in the sense that single branch gives in a symmetric situation a fraction of $1/n$ of the entire charge. Later it will be found that also a genuine fractionization occurs and is due to the classical topologized Kähler charge of the anyon track.

2. Modelling anyons in terms of gauge group and isometry group

Anyons can be modelled in terms of the gauge symmetry breaking $SU(2)_L \rightarrow H$, where H is discrete sub-group. The breaking of gauge symmetry results by the action of multi-valued gauge transformation $g(x)$ such that different branches of the multi-valued map are related by the action of H.

1. The standard description of anyons is based on spontaneous symmetry breaking of a gauge symmetry G to a discrete sub-group H dynamically [49]. The gauge field has suffered multi-valued gauge transformation such that the elements of H permute the different branches of $g(x)$. The puncture is characterized by the element of the H associated with the loop surrounding puncture. In the idealized situation that gauge field vanishes, the parallel translation of a particle around puncture affects the particle state, itself a representation of G, by the element of the homotopy $\pi_1(G/H) = H$ identifiable as non-Abelian magnetic charge. Thus holonomy group corresponds to homotopy group of G/H which in turn equals to H. This in turn implies that the infinite-dimensional braid group whose elements define holonomies in turn is represented in H.

2. In TGD framework the multi-valuedness of $g(x)$ corresponds to a many-branched character of 4-surface. This in turn induces a branching of both magnetic flux tube and anyon tracks describable in terms of $H \subset SU(2)_L$ acting as an isotropy group for the boundaries of the magnetic flux tubes. H can correspond only to a non-Abelian subgroup $SU(2)_L$ of the electro-weak gauge group for the induced (classical) electro-weak gauge fields since the Chern-Simons action associated with the classical color gauge fields vanishes identically. The electro-weak holonomy group would reduce to a discrete group H around loops defined by anyonic flux tubes surrounding magnetic field lines inside the magnetic flux tubes containing anyons. The reduction to H need to occur only at the boundaries of the space-time sheet where conducting anyons would reside: boundaries indeed correspond to asymptotia in well-defined sense. Electro-weak symmetry group can be regarded as a sub-group of color group of isometries in a well-defined sense so that H can be regarded also as a subgroup of color group acting as isotropies of the multi-branched surface at least in the in regions where gauge field vanishes.

3. For branched surfaces the points obtained by moving around the puncture correspond in a good approximation to some elements of $h \in H$ leading to a new branch but the 2-surface as a whole however remains invariant. The braid group of the punctured 2-surface would be also now represented as transformations of H. The simplest situation is obtained when H is a cyclic group Z_N of the $U(1)$ group of CP_2 geodesic in such a manner that 2π rotation around symmetry axis corresponds to the generating element $exp(i2\pi/N)$ of Z_N.

Dihedral group D_n having order $2n$ and acting as symmetries of n-polygon of the plane is especially interesting candidate for H. For $n = 2$ the group is Abelian group $Z_2 \times Z_2$ whereas for $n > 2$ D_n is a non-Abelian sub-group of the permutation group S_n. The cyclic group Z_4 crucial for TQC is a sub-group of D_4 acting as symmetries of square. D_4 has a 2-dimensional faithful representation. The numbers of elements for the conjugacy classes are 1,1,2,2,2. The sub-group commuting with a fixed element of a conjugacy class is D_4 for the 1-element conjugacy classes and cyclic group Z_4 for 2-element conjugacy classes. Hence 2-valued magnetic flux would be accompanied by Z_4 valued "electric charge" identifiable as a cyclic group permuting the branches.

3. Can one understand the increase in conductivity and filling fractions at plateaus?

Quantum Hall effect involves the increase of longitudinal conductivity by a factor of order 10^{13}[42]. The reduction of dissipation could be understood as being caused by the fact that anyonic electrons are closed inside the magnetic flux tubes representing their tracks so that their interactions with matter and thus also dissipation are reduced.

Laughlin's theory [43, 42] gives almost universal description of many aspects of quantum Hall effect and the question arises whether Laughlin's wave functions are defined on possibly multi-branched space-time sheet X^4 or at projection of X^4 to M^4_+. Since most theoreticians that I know still live in single sheeted space-time, one can start with the most conservative assumption that they are defined at the projection to M^4_+. The wave functions of one-electron state giving rise filling fraction $\nu = 1/m$ are constructed of $(z_i - z_j)^m$, where m is odd by Fermi statistics.

Also rational filling fractions of form $\nu = 1/m = N/n$ have been observed. These could relate to the presence of states whose projections to M^4 are multi-valued and which thus do not have any "classical" counterpart. For N-branched surface the single-valued wave functions $(\xi_i - \xi_j)^n$, n odd by Fermi statistics, correspond to apparently multi-valued wave functions $(z_i - z_j)^{n/N}$ at M^4 projection with fractional relative angular momenta $m = n/N$. The filling fraction would be $\nu = N/n$, n odd. All filling fractions reported in [42] have n odd with n varying in the range $1 - 7$. N has the values $1, 2, 3, 4, 5, 7, 9$. Also values $N = 12, 13$ for which $n = 5$ are reported [32].

The filling fractions $\nu = N/n = 5/2, 3/8, 3/10$ reported in [48] would require even values of n conflicting with Fermi statistics. Obviously Lauhglin's model fails in this case and the question is whether one these fractions could correspond to bosonic anyons, perhaps Cooper pairs of electrons inside track flux tubes. The Z_N valued charge associated with N-branched surfaces indeed allows the maximum $2N$ electrons per anyon. Bosonic anyons are indeed the building block of the TQC model of [32]. The anyon Cooper pairs could be this kind of states and their BE condensation would make possible genuine super-conductivity rather than only exceptionally high value of conductivity.

One can imagine also more complex multi-electron wave functions than those of Laughlin. The so called conformal blocks representing correlation functions of conformal quantum field theories are natural candidates for the wave functions [50] and they appear naturally as state functions of in topological quantum field theories. For instance, wave functions which are products of factors $(z^k - z^l)^2$ with the Pfaffian $Pf(A_{kl})$ of the matrix $A_{kl} = 1/(z_k - z_l)$ guaranteing anti-symmetrization have been used to explain even values of m [50].

4. N-branched space-time surfaces make possible Z_N valued topological charge

According to [50] that $2n$ non-Abelian anyon pairs with charge $1/4$ created from vacuum gives rise to a 2^{n-1}-fold degenerate ground state. It is also argued that filling fraction $5/2$ could correspond to this charge [32]. TGD suggests somewhat different interpretation. 4-fold branching implies automatically the Z_4-valued topological charge crucial for anyonic quantum computation. For 4-branched space-time surface the contribution of a single branch to electron's charge is indeed $1/4$ units but this has nothing to do with the actual charge fractionization. The value of ν is of form $\nu = /m$ and electromagnetic charge equals to $\nu = 4e/m$ in this kind of situation.

If anyons (electron plus flux tube representing its track) have Z_4 charges 1 and 3, their Cooper pairs have charges 0 and 2. The double-fold degeneracy for anyon's topological charge means that it possesses topological spin conserved modulo 4. In presence of $2n$ anyon pairs one would expect 2^n-fold degeneracy. The requirement that the net topological charge vanishes modulo 4 however fixes the topological charge of n:th pair so that 2^{n-1} fold degeneracy results.

A possible interpretation for Z_N-valued topological charge is as fractional angular momenta k/N

associated with the phases $exp(ik2\pi/N)$, $k = 0, 1, ..., n-1$ of particles in multi-branched surfaces. The projections of these wave functions to single-branched space-time would be many-valued. If electro-weak gauge group breaks down to a discrete subgroup H for magnetic flux tubes carrying anyonic "tracks", this symmetry breakdown could induce their multi-branched property in the sense rotation by 2π would correspond to H isometry leading to a different branch.

Topologization of Kähler charge as an explanation for charge fractionization

The argument based on what happens when one adds one anyon to the anyon system by utilizing Faraday's law [42] leads to the conclusion that anyon charge is fractional and given by νe. The anyonic flux tube along boundary of the flux tube corresponds to the left hand side in the Faraday's equation

$$\oint E \cdot dl = -\frac{d\Phi}{dt} \quad .$$

By expressing E in turns of current using transversal conductivity and integrating with respect to time, one obtains

$$Q = \nu e$$

for the charge associated with a single anyon. Hence the addition of the anyon means an addition of a fractional charge νe to the system. This argument should survive as such the 1-branched situation so that at least in this case the fractional charges should be real.

In N-branched case the closed loop $\oint E \cdot dl$ around magnetic flux tube corresponds to N-branched anyon and surrounds the magnetic flux tube N times. This would suggest so that net magnetic flux should be N times the one associated with single but unclosed 2π rotation. Hence the formula would seem to hold true as such also now for the total charge of the anyon and the conclusion is that charge fractionization is real and cannot be an effective effect due to fractionization of charge at single branch of anyon flux tube.

One of the basic differences between TGD and Maxwell's theory is the possibility of vacuum charges and this provides an explanation of the effect is in terms of vacuum Kähler charge. Kähler charge contributes $e/2$ to the charge of electron. Anyon flux tube can generate vacuum Kähler charge changing the net charge of the anyon. If the anyon charge equals to νe the conclusions are following.

1. The vacuum Kähler charge of the anyon track is $q = (\nu - 1)e$.

2. The dimension of the CP_2 projection of the anyon flux tube must be $D = 3$ since only in this case the topologization of anyon charge becomes possible so that the charge density is proportional to the Chern-Simons term $A \wedge dA/4\pi$. Anyon flux tubes cannot be super-conducting in the sense that non-integrable phase factor $exp(\int A \cdot dl)$ would define global order parameter. The boundaries of anyonic flux tubes could however remain potentially super-conducting and anyon Cooper pairs would be expelled there by Meissner effect. This gives super-conductivity in length scale of single flux tube. Conductivity and super-conductivity in long length scales requires that magnetic flux tubes are glued together along their boundaries partially.

3. By Bohr quantization anyon tracks can have $r_n = \sqrt{n} \times r_B$, $n \leq m$, where r_m corresponds to the radius of the magnetic flux tube carrying m flux quanta. Only the tracks with radius r_m contribute to boundary conductivity and super-conductivity giving $\nu = 1/m$ for singly branched surfaces.

   The states with $\nu = N/m$ cannot correspond to non-super-conducting anyonic tracks with radii r_n, $n < m$, n odd, since these cannot contribute to boundary conductivity. The many-branched character however allows an N-fold degeneracy corresponding to the fractional angular momentum states $exp(ik\phi/N)$, $k = 0, ..., N-1$ of electron inside anyon flux tubes of radius r_m. k is obviously a an excellent candidate for the Z_N-valued topological charge crucial for anyonic quantum computation. Z_4 is uniquely selected by the braid matrix R.

   Only part of the anyonic Fermi sea need to be filled so that filling fractions $\nu = k/m$, $k = 1, ..., N$ are possible. Charges νe are possible if each electron inside anyon track contributes $1/m$ units to the fractional vacuum Kähler charge. This is achieved if the radius of the anyonic flux tube grows as $\sqrt{k/m}$ when electrons are added. The anyon tracks containing several electrons give rise to composite fermions with fermion number up to $2N$ if both directions of electron spin are allowed.

4. Charge fractionization requires vacuum Kähler charge has rational values $Q_K = (\nu - 1)e$. The quantization indeed occurs for the helicity defined by Chern-Simons term $A \wedge dA/4\pi$. For compact 3-spaces without boundary the helicity can be interpreted as an integer valued invariant characterizing the linking of two disjoint closed curves defined by the magnetic field lines. This topological charge can be also related to the asymptotic Hopf invariant proposed by Arnold [18], which in non-compact case has a continuum of values. Vacuum Kähler current is obtained from the topological current $A \wedge dA/4\pi$ by multiplying it with a function of CP_2 coordinates completely fixed by the field equations. There are thus reasons to expect that vacuum Kähler charge and also the topological charges obtained by multiplying Chern Simons current by $SU(3)$ Hamiltonians are quantized for compact 3-surfaces but that the presence of boundaries replaces integers by rationals.

What happens in quantum Hall system when the strength of the external magnetic field is increased?

The proposed mechanism of anyonic conducitivity allows to understand what occurs in quantum Hall system when the intensity of the magnetic field is gradually increased.

1. Percolation picture encourages to think that magnetic flux tubes fuse partially along their boundaries in a transition to anyon conductivity so that the anyonic states localized at the boundaries of flux tubes become delocalized much like electrons in metals. Laughlin's states provide an idealized description for these states. Also anyons, whose tracks have Bohr radii r_m smaller than the radius r_B of the magnetic flux tube could be present but they would not participate in this localization. Clearly, the anyons at the boundaries of magnetic flux tubes are highly analogous to valence electrons in atomic physics.

2. As the intensity of the magnetic field B increases, the areas a of the flux tubes decreases as $a \propto 1/B$: this means that the existing contacts between neighboring flux tubes tend to be destroyed so that anyon conductivity is reduced. On the other hand, new magnetic flux tubes must emerge by the constancy of the average magnetic flux implying $dn/da \propto B$ for the average density of flux tubes. This increases the probability that the newly generated flux tubes can partially fuse with the existing flux tubes.

3. If the flux tubes are not completely free to move and change their shape by area preserving transformations, one can imagine that for certain value ranges of B the generation of new magnetic flux tubes is not favored since there is simply no room for the newcomers. The Fermi statistics of the anyonic electrons at the boundaries of flux tubes might relate to this non-hospitable behavior. At certain critical values of the magnetic field the sizes of flux tubes become however so small that the situation changes and the new flux tubes penetrate the system and via the partial fusion with the existing flux tubes increase dramatically the conductivity.

Also protonic anyons are possible

According to the TGD based model, any charged particle can form anyons and the strength of the magnetic field does not seem to be crucial for the occurrence of the effect and it could occur even in the Earth's magnetic field. The change of the cyclotron and Larmor frequencies of the charged particle in an external magnetic field to a value corresponding to the fractional charge provides a clear experimental signature for both the presence of anyons and for their the fractional charge.

Interestingly, water displays a strange scaling of proton's cyclotron frequency in an external magnetic field [60, 55]. In an alternating magnetic field of .1551 Gauss (Eearth's field has a nominal value of .58 Gauss) a strong absorption at frequency $f = 156$ Hz was observed. The frequency was halved when D_2O was used and varied linearly with the field strength. The resonance frequency however deviated from proton's Larmor frequency, which suggests that a protonic anyon is in question. The Larmor frequency would be in this case $f_L = r \times \nu eB/2m_p$, where $r = \mu_p/\mu_B = 2.2792743$ is the ratio of proton's actual magnetic moment to its value for a point like proton. The experimental data gives

$\nu = .6003 = 3/5$ with the accuracy of 5×10^{-4} so that 3-branched protonic anyons with $m = 5$ would be responsible for the effect.

If this interpretation is correct, entire p-adic hierarchy of anyonic NMR spectroscopies associated with various atomic nuclei would become possible. Bosonic anyon atoms and Cooper pairs of fermionic anyon atom could also form macroscopic quantum phases making possible super-conductivity very sensitive to the value of the average magnetic field and bio-systems and brain could utilize this feature.

9.4.3 Does the quantization of Planck constant transform integer quantum Hall effect to fractional quantum Hall effect?

The model for topological quantum computation inspired the idea that Planck constant might be dynamical and quantized. The work of Nottale [53] gave a strong boost to concrete development of the idea and it took year and half to end up with a proposal about how basic quantum TGD could allow quantization Planck constant associated with M^4 and CP_2 degrees of freedom such that the scaling factor of the metric in M^4 degrees of freedom corresponds to the scaling of $\hbar$ in CP_2 degrees of freedom and vice versa [A9]. The dynamical character of the scaling factors of M^4 and CP_2 metrics makes sense if space-time and imbedding space, and in fact the entire quantum TGD, emerge from a local version of an infinite-dimensional Clifford algebra existing only in dimension $D = 8$ [C6].

The predicted scaling factors of Planck constant correspond to the integers n defining the quantum phases $q = exp(i\pi/n)$ characterizing Jones inclusions. A more precise characterization of Jones inclusion is in terms of group $G_b \subset SU(2) \subset SU(3)$ in CP_2 degrees of freedom and $G_a \subset SL(2, C)$ in M^4 degrees of freedom. In quantum group phase space-time surfaces have exact symmetry such that to a given point of M^4 corresponds an entire G_b orbit of CP_2 points and vice versa. Thus space-time sheet becomes $N(G_a)$ fold covering of CP_2 and $N(G_b)$-fold covering of M^4. This allows an elegant topological interpretation for the fractionization of quantum numbers. The integer n corresponds to the order of maximal cyclic subgroup of G.

In the scaling $\hbar_0 \rightarrow n\hbar_0$ of M^4 Planck constant fine structure constant would scale as

$$\alpha = \frac{e^2}{4\pi\hbar c} \rightarrow \frac{\alpha}{n} \quad ,$$

and the formula for Hall conductance would transform to

$$\sigma_H \rightarrow \frac{\nu}{n}\alpha \quad .$$

Fractional quantum Hall effect would be integer quantum Hall effect but with scaled down α. The apparent fractional filling fraction $\nu = m/n$ would directly code the quantum phase $q = exp(i\pi/n)$ in the case that m obtains all possible values. A complete classification for possible phase transitions yielding fractional quantum Hall effect in terms of finite subgroups $G \subset SU(2) \subset SU(3)$ given by ADE diagrams would emerge (A_n, D_{2n}, E_6 and E_8 are possible). What would be also nice that CP_2 would make itself directly manifest at the level of condensed matter physics.

9.4.4 Why 2+1-dimensional conformally invariant Witten-Chern-Simons theory should work for anyons?

Wess-Zumino-Witten theories are 2-dimensional conformally invariant quantum field theories with dynamical variables in some group G. The action contains the usual 2-dimensional kinetic term for group variables allowing conformal group action as a dynamical symmetry plus winding number defined associated with the mapping of 3-surface to G which is $Diff^4$ invariant. The coefficient of this term is quantized to integer.

If one couples this theory to a gauge potential, the original chiral field can be transformed away and only a Chern-Simons term defined for the 3-manifold having the 2-dimensional space as boundary remains. Also the coefficient k of Chern-Simons term is quantized to integer. Chern-Simons-Witten action has close connection with Wess-Zumino-Witten theory. In particular, the states of the topological quantum field theory are in one-one correspondence with highest weights of the WZW action.

The appearance of 2+1-dimensional $Diff^3$ invariant action can be understood from the fundamentals of TGD.

1. Light-like 3-surfaces of both future light-cone M^4_+ and of space-time surface X^4 itself are in a key role in the construction of quantum TGD since they define causal determinants for Kähler action.

2. At the space-time level both the boundaries of X^4 and elementary particle horizons surrounding the orbits of wormhole contacts define light-like 3-surfaces. The field equations are satisfied identically at light-like boundaries. Of course, the projections of the the light-like surfaces of X^4 to Minkowski space need not look light-like at all, and even boundaries of magnetic flux tubes could be light-like.

   Light-like 3-surfaces are metrically 2-dimensional and allow a generalized conformal invariance crucial for the construction of quantum TGD. At the level of imbedding space conformal super-canonical invariance results. At the space-time level the outcome is conformal invariance highly analogous to the Kac Moody symmetry of super string models [B2, B3, E2]. In fact, there are good reasons to believe that the three-dimensional Chern-Simons action appears even in the construction of configuration space metric and give an additional contribution to the configuration space metric when the light-like boundaries of 3-surface have 3-dimensional CP_2 projection.

3. By the effective two-dimensionality the Wess-Zumino-Witten action containing Chern-Simons term is an excellent candidate for the quantum description of S-matrix associated with the light-like 3-surfaces since by the vanishing of the metric determinant one cannot define any general coordinate invariant 3-dimensional action other than Chern-Simons action. The boundaries of the braid formed by the magnetic flux tubes having light-like boundaries, perhaps having join along boundaries bonds between swapped flux tubes would define the 2+1-dimensional space-time associated with a braid, would define the arena of Witten-Chern-Simons theory describing anyons. This S-matrix can be interpreted also as characterizing either a 3-dimensional quantum state since light-like boundaries are limiting cases of space-like 3-surfaces.

4. Kähler action defines an Abelian Chern-Simons term and the induced electroweak gauge fields define a non-Abelian variant of this term. The Chern-Simons action associated with the classical color degrees of freedom vanishes as is easy to find. The classical color fields are identified as projections of Killing vector fields of color group: $A^c_\alpha = j^A_k \partial_\alpha s^k \tau_A = J_k^{\ r} \partial_r H^A \partial_\alpha s^k$. The classical color gauge field is proportional to the induced Kähler form: $F^c_{\alpha\beta} = H^A J_{\alpha\beta} \tau_A$. A little calculation shows that the instanton density vanishes by the identity $H_A H^A = 1$ (this identity is forced by the necessary color-singletness of the YM action density and is easy to check in the simpler case of S^2.

5. Since qubit realizes the fundamental representation of the quantum group $SU(2)_q$, $SU(2)$ is in a unique role concerning the construction of modular functors and quantum computation using Chern-Simons action. The quantum group corresponding to $q = exp(i2\pi/r)$, $r = 5$ is realized for the level $k = 3$ Chern-Simons action and satisfies the constraint $r = k + c_g$, where $c_g = 2$ is the so called dual Coxeter number of $SU(2)$ [33, 39, 32].

   The exponent non-Abelian $SU(2)_L \times U(1)$ Chern-Simons action combined with the corresponding action for Kähler form so that effective reduction to $SU(2)_L$ occurs, could appear as a multiplicative factor of the configuration space spinor fields defined in the configuration space of 3-surfaces. Since 3-dimensional quantum state would represent a 2-dimensional time evolution the role of these phase factor would be very analogous to the role of ordinary Chern-Simons action.

9.5 Topological quantum computation in TGD Universe

The general philosophy behind TQC inspires the dream that the existence of basic gates, in particular the maximally entangling 2-gate R, is guaranteed by the laws of Nature so that no fine tuning would be needed to build the gates. Negentropy Maximization Principle, originally developed in context of TGD inspired theory of consciousness, is a natural candidate for this kind of Law of Nature.

9.5.1 Concrete realization of quantum gates

The bold dream is that besides 2-gates also 1-gates are realized by the basic laws of Nature. The topological realization of the 3-braid representation in terms of Temperley-Lie algebra allows the reduction of 1-gates to 2-gates.

NMP and TQC

Quantum jump involves a cascade of self measurements in which the system under consideration can be though of as decomposing to two parts which are either un-entangled or possess rational or extended rational entanglement in the final state. The sub-system is selected by the requirement that entanglement negentropy gain is maximal in the measurement of the density matrix characterizing the entanglement of the sub-system with its complement.

In the case case that the density matrix before the self measurement decomposes into a direct sum of matrices of dimensions N_i , such that $N_i > 1$ holds true for some values of i, say i_0, the final state is a rationally entangled and thus a bound state. i_0 is fixed by the requirement that the number theoretic entropy for the final state maximally negative and equals to $klog(p)$, where p^k is the largest power of prime dividing N_{i_0}. This means that maximally entangled state results and the density matrix is proportional to a unit matrix as it is also for the entanglement produced by R. In case of R the density matrix is $1/2$ times 2-dimensional unit matrix so that bound state entanglement negentropy is 1 bit.

The question is what occurs if the density matrix contains a part for which entanglement probabilities are extended rational but not identical. In this case the entanglement negentropy is positive and one could argue that no self-measurement occurs for this state and it remains entangled. If so then the measurement of the density matrix would occur only when it increases entanglement negentropy. This looks the only sensible option since otherwise only bound state entanglement with identical entanglement probabilities would be possible. This question is relevant also because Temperley-Lieb representation using $(AA) - A - A$ system involves entanglement with entanglement probabilities which are not identical.

In the case that the 2-gate itself is not directly entangling as in case of R' and R'', NMP should select just the quantum history, that single particle gates at it guarantee maximum entanglement negentropy. Thus NMP would come in rescue and give hopes that various gates are realized by Nature.

Non-Abelian anyon systems are modelled in terms of punctures of plane and Chern-Simons action for the incompressible vector potential of hydrodynamical flow. It is interesting to find how these ideas relate to the TGD description.

Non-Abelian anyons reside at boundaries of magnetic flux tubes in TGD

In [32] anyons are modelled in terms of punctures of plane defined by the slab carrying Hall current. In TGD the punctures correspond naturally to magnetic flux tubes defining the braid. It is now however obvious under what conditions the braid containing the TGD counterpart of (AA)-A-A system can be described as a punctured disk if the flux tubes describing the tracks of valence anyons are very near to the boundaries of the magnetic flux tubes. Rather, the punctured disk is replaced with the closed boundary of the magnetic flux tube or of the structure formed by the partial fusion of several magnetic flux tubes. This microscopic description and is consistent with Laughlin's model only if it is understood as a long length scale description.

Non-Abelian charges require singularities and punctures but a two-surface which is boundary does not allow punctures. The punctures assigned with an anyon pair would become narrow wormhole threads traversing through the interior of the magnetic flux tube and connecting the punctures like wormholes connect two points of an apple. It is also possible that the threads connect the surfaces of two nearby magnetic flux tubes. The wormhole like character conforms with the fact that non-Abelian anyons appear always in pairs.

The case in which which the ends of the wormhole thread belong to different neighboring magnetic flux tubes, call them T_1 and T_2, is especially interesting as far as the model for TQC is considered. The state of $(AA) - A - A$ system before (after) the 3-braid operation would be identifiable as anyons near the surface of T_1 (T_2). If only sufficiently local operations are allowed, the braid group would be same as for anyons inside disk. This means consistency with the anyon model of [32] for TQC requiring that the dimension for the space of ground states is 2^{n-1} in a system consisting of n anyon pairs.

The possibility of negative energies allows inspires the idea that the anyons at T_2 have negative energies so that the anyon system would have a vanishing net energy. This would conform with the idea that the scattering from initial to final state is equivalent with the creation of zero energy state for which initial (final) state particles have positive (negative) energies, and with the fact that the boundaries of magnetic flux tubes are light-like systems for which 3-D quantum state is representation for a 2-D time evolution.

Since the correlation between anyons at the ends of the wormhole thread is purely topological, the most plausible option is that they behave as free anyons dynamically. Assuming 4-branched anyon surfaces, the charges of anyons would be of form $Q = \nu_A e$, $\nu_A = 4/m$, m odd.

Consider now the representation of 3-braid group. That the mapping class group for the 3-braid system should have a 2-dimensional representation is obvious from the fact that the group has same generators as the mapping class group for torus which is represented by as $SL(2,Z)$ matrices acting on the homology of torus having two generators a, b corresponding to the two non-contractible circles around torus. 3-braid group would be necessarily represented in Temperley-Lieb representation.

The character of the anyon bound state is important for braid representations.

1. If anyons form loosely bound states (AA), the electrons are at different tracks and the charge is additive in the process so that one has $Q_{AA} = 2Q_A = 8/m$, m odd, which is at odds with statistics. It might be that the naive rule of assigning fractional charge to the state does not hold true for loosely bound bosonic anyons. In this case $(AA) - A$ system with charge states $((1,-1),1)$ and $((1,1),-1)$ would be enough for realizing 1-gates in TQC. The braid operation s_2 of Temperley-Lieb representation represented $(A_1A_2) - A_3 \rightarrow (A_1A_3) - A_2$ would correspond to an exchange of the dance partner by a temporary decay of (A_1A_2) followed by a recombination to a quantum superposition of (A_1A_2) and (A_1A_3) and could be regarded as an ordinary braid operation rather than monodromy. The relative phase 1-gate would correspond to s_1 represented as braid operation for A_1 and A_2 inside (A_1A_2).

2. If anyons form tightly bound states (AA) in the sense that single anyonic flux tube carries two electrons, charge need not be additive so that bound states could have charges $Q = 4/2m_1$ so that the vacuum Kähler charge $Q_K = 4(1/m_1 - 2/m)$ would be created in the process. This would stabilize (AA) state and would mean that the braid operation $(A_1A_2) - A_3 \rightarrow (A_1A_3) - A_2$ cannot occur via a temporary decay to free anyons and it might be necessary to replace 3-braid group by a partially colored 3-braid group for $(AA) - A - A$ system which is sub-group of 3-braid group and has generators s_1^2 (two swaps for $(AA) - A$) and s_2 (swap for $A - A$) instead of s_1 and s_2. Also in this case a microscopic mechanism changing the value of (AA) Z^4 charge is needed and the situation might reduce to the case a) after all.

   The Temperley Lieb representation for this group is obtained by simply taking square of the generator inducing entanglement (s_2 rather than s_1 in the notation used!). The topological charge assignments for $(AA) - A - A$ system are $((1,-1),1,-1)$ and $((1,1),-1,-1)$. s_1^2 would correspond to the group element generating $(AA) - A$ entanglement and s_2 acting on $A - A$ pair would correspond to phase generating group element.

Braid representations and 4-branched anyon surfaces

Some comments about braid representations in relation to Z_N- valued topological charges are in order.

1. Yang-Baxter braid representation using the maximally entangling braid matrix R is especially attractive option. For anyonic computation with Z_4-valued topological charge R is the unique 2-gate conserving the net topological charge (note that the mixing of the $|1,1\rangle$ and $|-1,-1\rangle$ is allowed). On the other, R allows only the conservation of Z_4 value topological charge. This suggests that the the entanglement between logical qubits represented by $(AA) - A - A$ batches is is generated by R. The physical implication is that only $\nu = 4/n$ 4-branched anyons could be used for TQC.

2. In TGD framework the entangling braid representation inside batches responsible for 1-gates need not be the same since batches correspond to magnetic flux tubes. In standard physics con-

text it would be harder to defend this kind of assumption. As will be found 3-braid Temperley-Lieb representation is very natural for 1-gates. The implication is that the n-braid system with braids represented as 4-batches would have 2^n-dimensional space of logical qubits in fact identical with the space of realizable qubits.

3. Also n-braid Temperley-Lieb representations are possible and the explicit expressions of the braiding matrices for 6-braid case suggest that Z_4 topological charge is conserved also now [33]. In this case the dimension of the space of logical qubits is for highly favored value of quantum group parameter $q = exp(i\pi/5)$ given by the Fibonacci number $F(n)$ for n-braid case and behaves as Φ^{4n} asymptotically so that this option would be more effective. From $\Phi^4 = 1 + 3\Phi \simeq 8.03$ one can say that single 4-batch carries 3 bits of information instead of one. This is as it must be since topological charge is not conserved inside batches separately for this option.

4. $(AA) - A$ representation based on Z_4-valued topological charge is unique in that the space of logical qubits would be the space of topologically realizable qubits. Quantum superposition of logical qubits could could be represented $(AA) - A$ entangled state of form $a|2,-1\rangle + b|0,1\rangle$ generated by braid action. Relative phase could be generated by braid operation acting on the entangled state of anyons of (AA) Cooper pair. Since the superposition of logical cubits corresponds to an entangled state $a|2,-1\rangle + b|0,1\rangle$ for which coefficients are extended rational numbers, the number theoretic realization of the bound state property could pose severe conditions on possible relative phases.

9.5.2 Temperley-Lieb representations

The articles of Kaufmann [34] and Freedman [33, 38] provide enjoyable introduction to braid groups and to Tempeley-Lie representations. In the sequel Temperley-Lieb representations are discussed from TGD view point.

Temperley-Lieb representation for 3-braid group

In [34] it is explained how the so called Temperley-Lie algebra defined by 2×2-matrices I, U_1, U_2 satisfying the relations $U_1^2 = dU_1$, $U_2^2 = dU_2$, $U_1U_2U_1 = U_2$, $U_2U_1U_2 = U_1$ allows a unitary representation of Artin's braid group by unitary 2×2 matrices. The explicit representations of the matrices U_1 and U_2 (note that U_i/d acts as a projector) given by

$$\begin{aligned} U_1 &= \begin{pmatrix} d & 0 \\ 0 & 0 \end{pmatrix} \ , \\ U_2 &= \begin{pmatrix} \frac{1}{d} & \sqrt{1-\frac{1}{d^2}} \\ \sqrt{1-\frac{1}{d^2}} & d-\frac{1}{d} \end{pmatrix} \ . \end{aligned} \tag{9.5.0}$$

Note that the eigenvalues of U_i are d and 0. The representation of the elements s_1 and s_2 of the 3-braid group is given by

$$\begin{aligned} \Phi(s_1) &= AI + A^{-1}U_1 = \begin{pmatrix} -U^{-3} & 0 \\ 0 & U \end{pmatrix} \ , \\ \Phi(s_2) &= AI + A^{-1}U_2 = \begin{pmatrix} -\frac{U^3}{d} & \frac{U^{-1}}{\sqrt{1-(1/d)^2}} \\ \frac{U^{-1}}{\sqrt{1-(1/d)^2}} & \frac{U^{-5}}{d} \end{pmatrix} \ , \\ U &= exp(i\phi) \ . \end{aligned} \tag{9.5.-1}$$

Here the condition $d = -A^2 - A^{-2}$ is satisfied. For $A = exp(i\phi)$, with $|\phi| \leq \pi/6$ or $|\pi - \phi| \leq \pi/6$, the representation is unitary. The constraint comes from the requirement $d > 1$. From the basic representation it follows that the eigenvalues of $\Phi(s_i)$ are $-exp(-3i\phi)$ and $exp(i\phi)$.

Tihs 3-braid representation is a special case of a more general Temperley-Lieb-Jones representation discussed in [33] using notations $A = \sqrt{-1}exp(-i2\pi/4r)$, $s = A^2$, and $q = A^4$. In this case all eigenvalues of all representation matrices are -1 and $q = exp(-i2\pi/r)$. This representation results by multiplying Temperley-Lieb representation above with an over-all phase factor $exp(4i\phi)$ and by the replacement $A = exp(i\phi) \rightarrow \sqrt{-1}A$.

Constraints on the parameters of Temperley-Lieb representation

The basic mathematical requirement is that besides entangling 2-gate there is minimum set of 1-gates generating infinite sub-group of $U(2)$. Further conditions come from the requirement that a braid representation is in question. In the proposal of [32, 33] the 1-gates are realized using Temperley-Lieb 3-braid representation. It is found that there are strong constraints to the representation and that relative phase gate generating the phase $exp(i\phi) = exp(i2\pi/5)$ is the simplest solution to the constraints.

The motivation comes from the findings made already by Witten in his pioneering work related to the topological quantum field theories and one can find a good representation about what is involve din [35].

Topological quantum field theories can produce unitary modular functors when the $A = q^{1/4} = exp(i\phi)$ characterizing the quantum group multiplication is a root of unity so that the quantum enveloping algebra $U(Sl(2))_q$ defined as the quantum version of the enveloping algebra $U(Sl(2))$ is not homomorphic with $U(Sl(2))$ and theory does not trivialize. Besides this, q must satisfy some consistency conditions. First of all, $A^{4n} = 1$ must be satisfied for some value of n so that A is either a primitive l:th, $2l$:th of unity for l odd, or $4l$:th primitive root of unity.

This condition relates directly to the fact that the quantum integers $[n]_q = (A^{2n}-A^{-2n})/(A^2-A^{-2})$ vanish for $n \geq l$ so that the representations for a highest weight n larger than l are not irreducible. This implies that the theory simplifies dramatically since these representations can be truncated away but can cause also additional difficulties in the definition of link invariants. Indeed, as Witten found in his original construction, the topological field theories are unitary for $U(Sl(2))_q$ only for $A = exp(ik\pi/2l)$, k not dividing $2l$, and $A = exp(i\pi/l)$, l odd (no multiples are allowed) [35]. $n = 2l = 10$, which is the physically favored choice, corresponds to the relative phase $4\phi = 2\pi/5$.

Golden Mean and quantum computation

Temperley-Lieb representation based on $q = exp(i2\pi/5)$ is highly preferred physically.

1. One might hope that the Yang-Baxter representation based on maximally entangling braid matrix R might work. $R^8 = 1$ constraint is however not consistent with Temperley-Lieb representations. The reason is that $\Phi^8(s_1) = 1$ gives $\phi = \pi/4 > \pi/6$ so that unitarity constraint is not satisfied. $\phi = exp(i2\pi/16)$ corresponding $r = 4$ and to the matrix $\Phi(s_2) = \hat{R} = exp(i2\pi/16) \times R$ allows to satisfy the unitarity constraint. This would look like a very natural looking selection since $\Phi(s_2)$ would act as a Hadamard gate and NMP would imply identical entanglement probabilities if a bound state results in a quantum jump. Unfortunately, s_1 and s_2 do not generate a dense subgroup of $U(2)$ in this case as shown in [33].

2. $\phi = \pi/10$ corresponding to $r = 5$ and Golden Mean satisfies all constraints coming from quantum computation and knot theory. That is it spans a dense subgroup of $U(2)$, and allows the realization of modular functor defined by Witten-Chern-Simons $SU(2)$ action for $k = 3$, which is physically highly attractive since the condition

$$r = k + c_g(SU(2))$$

   connecting r, k and the dual Coxeter number $c_g(SU(N)) = n$ in WCS theories is satisfied for $SU(2)$ in this case for $r = 5$ and $k = 3$.

   $SU(2)$ would have interpretation as the left-handed electro-weak gauge group $SU(2)_L$ associated with classical electro-weak gauge fields. The symmetry breaking of $SU(2)_L$ down to a discrete subgroup of $SU(2)_L$ yielding anyons would relate naturally to this. The conservation of the topologized Kähler charge would correlate with the fact that there is no symmetry breaking in the classical $U(1)$ sector. $k = 3$ Chern-Simons theory is also known to share the same universality class as simple 4-body Hamiltonian [32] (larger values of k would correspond to $k + 1$-body Hamiltonians).

3. Number theoretical vision about intentional systems suggests that the preferred relative phases are algebraic numbers or more generally numbers which belong to a finite-dimensional extension of p-adic numbers. The idea about p-adic cognitive evolution as a gradual generation of increasingly complex algebraic extensions of rationals allows to see the extension containing Golden Mean $\Phi = (1+\sqrt{5})/2$ as one of the simplest extensions. The relative phase $exp(i4\phi) = exp(i2\pi/5)$ is expressible in an extension containing $\sqrt{\Phi}$ and Φ: one has $cos(4\phi) = (\Phi-1)/2$ and $sin(4\phi) = \sqrt{5\Phi}/2$.

The general number theoretical ideas about cognition support the view that Golden Mean is in a very special role in the number theoretical world order. This would be due to the fact that $log(\Phi)/\pi$ is a rational number. This hypothesis would explain scaling hierarchies based on powers of Golden Mean. One could argue that the geometry of the braid should reflect directly the value of the $A = exp(i2\phi)$. The angle increment per single DNA nucleotide is $\phi/2 = 2\pi/10$ for DNA double strand (note that q would be $exp(i\pi/10)$, which raises the question whether DNA might be a topological quantum computer.

Bratteli diagram for $n = 5$ case, Fibonacci numbers, and microtubuli

Finite-dimensional von Neumann algebras can be conveniently characterized in terms of Bratteli diagrams [41]. For instance, the diagram a) of the figure 9.5.2 at the end of the chapter represents the inclusion $N \subset M$, where $N = M_2(C)\otimes C$, $M = M_6(C)\otimes M_3(C)\otimes C$. The diagram expresses the imbeddings of elements $A\otimes x$ of $M_2(C)\otimes C$ to $M_6(C)$ as a tensor product $A_1\otimes A_2\otimes x$

$$\begin{aligned} A_1 &= \begin{pmatrix} A & . & . \\ . & A & . \\ . & . & A \end{pmatrix} , \\ A_2 &= \begin{pmatrix} A & . \\ . & x \end{pmatrix} . \end{aligned} \tag{9.5.-2}$$

Bratteli diagrams of infinite-dimensional von Neumann algebras are obtained as limiting cases of finite-dimensional ones.

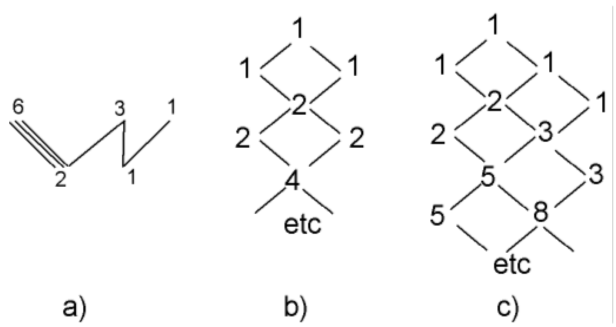

Figure 9.1: a) Illustration of Bratteli diagram. b) and c) give Bratteli diagrams for $n = 4$ and $n = 5$ Temperley Lieb algebras

2. Temperley Lieb algebras approximate II_1 factors

The hierarchy of inclusions of with $|M_{i+1} : M_i| = r$ defines a hierarchy of Temperley-Lieb algebras characterizable using Bratteli diagrams. The diagrams b) and c) of the figure 9.5.2 at the end of the chapter characterize the Bratteli diagrams for $n = 4$ and $n = 5$. For $n = 4$ the dimensions of algebras come in powers of 2 in accordance with the fact $r = 2$ is the dimension of the effective tensor factor of II_1.

For $n = 5$ and $B_m = \{1, e_1, ..., e_m\}$ the dimensions of the two tensor factors of the Temperley Lieb-representation are two subsequent Fibonacci numbers F_{m-1}, F_m ($F_{m+1} = F_m + F_{m-1}$, $F_1 = 1, F_2 = 1$) so that the dimension of the tensor product is $dim(B_m) = F_m F_{m-1}$. One has $dim(B_{m+1})/dim(B_m) = F_m/F_{m-2} \rightarrow \Phi^2 = 1 + \Phi$, the dimension of the effective tensor factor for the corresponding hierarchy of II_1 factors. Hence the two dimensional hierarchies "approximate" each other. In fact, this result holds completely generally.

The fact that r is approximated by an integer in braid representations is highly interesting from the point of view of TQC. For 3-braid representation the dimension of Temperley-Lieb representation is 2 for all values of n so that 3-braid representation defines single (topo)logical qubit as $(AA) - A - A$ realization indeed assumes. One could optimistically say that TGD based physics automatically realizes topological qubit in terms of 3-braid representation and the challenge is to understand the details of this realization.

2. Why Golden Mean should be favored?

The following argument suggests a physical reason for why just Golden Mean should be favored in the magnetic flux tube systems.

1. Arnold [18] has shown that if Lorentz 3-force satisfies the condition $F_B = q(\nabla \times B) \times B = q\nabla\Phi$, then the field lines of the magnetic field lie on $\Phi = constant$ tori. On the other hand, the vanishing of the Lorentz 4-forces for solutions of field equations representing asymptotic self-organized states, which are the "survivors" selected by dissipation, equates magnetic force with the negative of the electric force expressible as qE, $E = -\nabla\Phi + \partial_t A$, which is gradient if the vector potential does not depend on time. Since the vector potential depends on three CP_2 coordinates only for $D = 3$, this seems to be the case.

2. The celebrated Kolmogorov-Arnold-Moser (KAM) theorem is about the stability of systems, whose orbits are on invariant tori characterized by the frequencies associated with the n independent harmonic oscillator like degrees of freedom. The theorem states that the tori for which the frequency ratios are rational are highly unstable against perturbations: this is due to resonance effects. The more "irrational" the frequencies are, the higher the stability of the orbits is, and the most stable situation corresponds to frequencies whose ratio is Golden Mean. In quantum context the frequencies for wave motion on torus would correspond to multiples $\omega_i = n2\pi/L_i$, L_i the circumference of torus. This poor man's argument would suggest that the ratio of the circumferences of the most stable magnetic tori should be given by Golden Mean in the most stable situation: perhaps one might talk about Golden Tori!

3. Golden Mean and microtubuli

What makes this observation so interesting is that Fibonacci numbers appear repeatedly in the geometry of living matter. For instance, micro-tubuli, which are speculated to be systems performing quantum computation, represent in their structure the hierarchy Fibonacci numbers $5, 8, 13$, which brings in mind the tensor product representation $5 \otimes 8$ of B_5 (5 braid strands!) and leads to ask whether this Temperley-Lieb representation could be somehow realized using microtubular geometry.

According to the arguments of [32] the state of n anyons corresponds to 2^{n-1} topological degrees of freedom and code space corresponds to F_n-dimensional sub-space of this space. The two conformations of tubulin dimer define the standard candidate for qubit, and one could assume that the conformation correlates strongly with the underlying topological qubit. A sequence of 5 *resp.* 8 tubulin dimers would give 2^4 *resp.* 2^7-dimensional space with $F_5 = 5$- *resp.* $F_7 = 13$-dimensional code sub-space so that numbers come out nicely. The changes of tubulin dimer conformations would be induced by the braid groups B_4 and B_7. B_4 would be most naturally realized in terms of a unit of 5-dimers by regarding the 4 first tubulins as braided punctures and 5th tubulin as the passive puncture. B_7 would be realized in a similar manner using a unit of 8 tubulin dimers.

Flux tubes would connect the subsequent dimers along the helical 5-strand *resp.* 8-strand defined by the microtubule. Nearest neighbor swap for the flux tubes would induce the change of the tubulin conformation and induce also entanglement between neighboring conformations. A full 2π helical twist along microtubule would correspond to 13 basic steps and would define a natural TQC program module. In accordance with the interpretation of II_1 factor hierarchy, (magnetic or electric) flux tubes could be assumed to correspond to $r = 2$ II_1 factor and thus carry 2-dimensional representations of

$n = 5$ or $n = 4$ 3-braid group. These qubits could be realized as topological qubits using $(AA) - A$ system.

Topological entanglement as space-time correlate of quantum entanglement

Quantum-classical correspondence encourages to think that bound state formation is represented at the space-time level as a formation of join along boundaries bonds connecting the boundaries of 3-space sheets. In particular, the formation of entangled bound states would correspond to a topological entanglement for the join along boundaries bonds forming braids. The light-likeness of the boundaries of the bonds gives a further support for this identification. During macro-temporal quantum coherence a sequence of quantum jumps binds effectively to single quantum jump and subjective time effectively ceases to run. The light-likeness for the boundaries of bonds means that geometric time stops and is thus natural space-time correlate for the subjective experience during macro-temporal quantum coherence.

Also the work with TQC lends support for a a deep connection between quantum entanglement and topological entanglement in the sense that the knot invariants constructed using entangling 2-gate R can detect linking. Temperley-Lieb representation for 3-braids however suggests that topological entanglement allows also single qubit representations for with quantum entanglement plays no role. One can however wonder whether the entanglement might enter into the picture in some natural manner in the quantum computation of Temperley-Lieb representation. The idea is simple: perhaps the physics of $(AA) - A - A$ system forces single qubit representation through the simple fact that the state space reduces in 4-batch to single qubit by topological constraints.

For TQC the logical qubits correspond to entangled states of anyon Cooper pair (AA) and second anyon A so that the quantum superposition of qubits corresponds to an entangled state in general. Several arguments suggest that logical qubits would provide Temperley-Lieb representation in a natural manner.

1. The number of braids inside 4-anyon batch (or 3-anyon batch in case that (AA) can decay temporarily during braid operation) 3 so that by the universality this system allows to compute the unitary Temperley-Lieb braid representation. The space of logical qubits equals to the entire state space since the number of qubits represented by topological ground state degeneracy is 1 instead of the expected three since $2n$ anyon system gives rise to 2^{n-1}-fold vacuum degeneracy. The degeneracy is same even when two of the anyons fuse to anyon Cooper pair. Thus it would seem that the 3-braid system in question automatically produces 1-qubit representation of 3-braid group.

2. The braiding matrices $\Phi(s_1)$ and $\Phi(s_2)$ are different and only $\Phi(s_2)$ mixes qubit values. This can be interpreted as the presence of two inherently different braid operations such that only the second braiding operation can generate entanglement of states serving as building blocks of logical qubits. The description of anyons as 2-dimensional wormholes led to precisely this picture. The braid group reduces to braid group for one half of anyons since anyon and its partner at the end of wormhole are head and feet of single dancer, and the anyon pair (AA) forming bound state can change partner during swap operation with anyon A and this generates quantum entanglement. The swap for anyons inside (AA) can generate only relative phase.

3. The vanishing of the topological charge in a pairwise manner is the symmetry which reduces the dimension of the representation space to 2^{n-1} as already found. For $n = 4$ only single topological qubit results. The conservation and vanishing of the net topological charge inside each batch gives a constraint, which is satisfied by the maximally entangling R-matrix R so that it could take care of braiding between different 4-batches and one would have different braid representation for 4-batches and braids consisting of them. Topological quantization justifies this picture physically. Only phase generating *physical* 1-gates are allowed since Hadamard gate would break the conservation of topological charge whereas for *logical* 1-gates entanglement generating 2-gates can generate mixing without the breaking of the conservation of topological charges.

Summary

It deserves to summarize the key elements of the proposed model for which the localization (in the precise sense defined in [38]) made possible by topological field quantization and Z_4 valued topological charge are absolutely essential prerequisites.

1. $2n$-anyon system has 2^{n-1}-fold ground state degeneracy, which for $n = 2$ leaves only single logical qubit. In standard physics framework $(AA) - A - A$ is minimal option because the total homology charge of the system must vanish. In TGD $(AA) - A$ system is enough to represent 3-braid system if the braid operation between AA and A can be realized as an exchange of the dancing partner. This option makes sense because the anyons with opposite topological charges at the ends of wormhole threads can be negative energy anyons representing the final state of the braid operation. A pair of magnetic flux tubes is needed to realize single anyon-system containing braid.

2. Maximally entangling R-matrix realizes braid interactions between $(AA) - A$ systems realized as 3-braids inside larger braids and the space of logical qubits is equivalent with the space of realizable qubits. The topological charges are conserved separately for each $(AA) - A$ system. Also the more general realization based on n-braid representations of Temperley-Lieb algebra is formally possible but the different topological realization of braiding operations does not support this possibility.

3. Temperley-Lieb 3-braid representation for $(AA) - A - A$ system allows to realize also 1-gates as braid operations so that topology would allow to avoid the fine-tuning associated with 1-gates. Temperley-Lieb representation for $\phi = exp(i\pi/10)$ satisfies all basic constraints and provides representation of the modular functor expressible using $k = 3$ Witten-Chern-Simons action. Physically 1-gates are realizable using Φ_1 acting as phase gate for anyon pair inside (AA) and $\Phi(s_2)$ entangling (AA) and A by partner exchange. The existence of single qubit braid representations apparently conflicting with the identification of topological entanglement as a correlate of quantum entanglement has an explanation in terms of quantum computation under topological symmetries.

9.5.3 Zero energy topological quantum computations

As already described, TGD suggests a radical re-interpretation for matter antimatter asymmetry in long length scales. The asymmetry would be due to the fact that ground state for fermion system corresponds to infinite sea of negative energy fermions and positive energy anti-fermions so that fermions would have positive energies and anti-fermions negative energies.

The obvious implication is the possibility to interpret scattering between positive energy states as a creation of a zero energy state with outgoing particles represented as negative energy particles. The fact that the quantum states of 3-dimensional light-like boundaries of 3-surfaces represent evolutions of 2-dimensional quantum systems suggests a realization of topological quantum computations using physical boundary states consisting of positive energy anyons representing the initial state of anyon system and negative energy anyons representing the outcome of the braid operation.

The simplest scenario simply introduces negative energy charge conjugate of the $(AA) - A$ system so that no deviations from the proposed scenario are needed. Both calculation and its conjugate are performed. This picture is the only possible one if one assumes that given space-time sheet contains either positive or negative energy particles but not both and very natural if one assumes ordinary fermionic vacuum. The quantum computing system would could be generated without any energy costs and even intentionally by first generating the p-adic space-time sheets responsible for the magnetic flux tubes and anyons and then transformed to their real counterparts in quantum jump. This double degeneracy is analogous to that associated with DNA double strand and could be used for error correction purposes: if the calculation has been run correctly both anyon Cooper pairs and their charge conjugates should decay with the same probability.

Negative energies could have much deeper role in TQC. This option emerges naturally in the wormhole handle realization of TQC. The TGD realization of 1-gates in 3-braid Temperley-Lieb representation uses anyons of opposite topological charges at the opposite ends of threads connecting magnetic flux tube boundaries. Single 3-braid unit would correspond to positive energy electronic anyons at the first flux tube boundary and negative energy positronic anyons at the second flux tube boundary. The sequences of 1-gates represented as 3-braid operations would be coded by a sequence of 3-braids representing generators of 3-braid group along a pair of magnetic flux tubes. Of course, also n-braid operations could be coded in the similar manner in series. Hence TQC could be realized using only two magnetic flux tubes with n-braids connecting their boundaries in series.

Condensed matter physicist would probably argue that all this could be achieved by using electrons in strand and holes in the conjugate strand instead of negative energy positrons: this would require only established physics. One can however ask whether negative energy positrons could appear routinely in condensed matter physics. For instance, holes might in some circumstances be generated by a creation of an almost zero energy pair such that positron annihilates with a fermion below the Fermi surface. The signature for this would be a photon pair consisting of ordinary and phase conjugate photons.

The proposed interpretation of the S-matrix in the Universe having vanishing net quantum numbers encourages to think that the S-matrices of 2+1-dimensional field theories based on Witten-Chern-Simons action defined in the space of zero (net) energy states could define physical states for quantum TGD. Thus the 2+1-dimensional S-matrix could define quantum states of 4-dimensional theory having interpretation as states representing "self-reflective" level representing in itself the S-matrix of a lower-dimensional theory. The identification of the quantum state as S-matrix indeed makes sense for light-like surfaces which can be regarded as limiting cases of space-like 3-surfaces defining physical state and time-like surfaces defining a time evolution of the state of 2-dimensional system.

Time evolution would define also an evolution in topological degrees of freedom characterizing ground states. Quantum states associated with light-like (with respect to the induced metric of space-time sheet) 3-dimensional boundaries of say magnetic flux tubes would define quantum computations as modular functors. This conforms with quantum-classical correspondence since braids, the classical states, indeed define quantum computations.

The important implication would be that a configuration which looks static would code for the dynamic braiding. One could understand the quantum computation in this framework as signals propagating through the strands and being affected by the gate. Even at the limit when the signal propagates with light velocity along boundary of braid the situation looks static from outside. Time evolution as a state could be characterized as sequence of many-anyon states such that basic braid operations are realized as zero energy states with initial state realized using positive energy anyons and final state realized using negative energy energy anyons differing by the appropriate gate operation from the positive energy state.

In the case of n-braid system the state representing the S-matrix $S = S^1S^2....S^n$ associated with a concatenation of n elementary braid operations would look like

$$\begin{aligned} |S\rangle &= P_{k_1}S^1_{k_1k_2}P_{k_2}S^2_{k_2k_3}P_{k_3}S^3_{k_3k_4}... \ , \\ P_k &= |k,<\rangle|k,>\rangle \ . \end{aligned} \tag{9.5.-2}$$

Here S^k are S-matrices associated with gates representing simple braiding operations s_k for $n+1$ threads connecting the magnetic flux tubes. P_k represents a trivial transition $|k\rangle \rightarrow |k \rightarrow k\rangle$ as zero energy state $|k,>0\rangle|k,<\rangle$. The states P_k represent matrix elements of the identification map from positive energy Hilbert space to its negative energy dual.

What would happen can be visualized in two alternative manners.

1. For this option the braid maps occur always from flux tube 1 to flux tube 2. A braiding transition from 1 to 2 is represented by S^{k_1}; a trivial transition from 2 to 1 is represented by P_k; a braiding transition from 1 to 2 is represented by S^{k_2}, etc... In this case flux tube 1 contains positive energy anyons and flux tube 2 the negative energy anyons.

2. An alternative representation is the one in which P_k represents transition along the strand so that S^k *resp.* S^{k+1} corresponds to braiding transition from strand 1 to 2 *resp.* 2 to 1. In this case both flux tubes contain both positive and negative energy anyons.

9.6 Appendix: A generalization of the notion of imbedding space

In the following the recent view about structure of imbedding space forced by the quantization of Planck constant is described. This view has developed much before the original version of this chapter was written.

The original idea was that the proposed modification of the imbedding space could explain naturally phenomena like quantum Hall effect involving fractionization of quantum numbers like spin and charge. This does not however seem to be the case. $G_a \times G_b$ implies just the opposite if these quantum numbers are assigned with the symmetries of the imbedding space. For instance, quantization unit for orbital angular momentum becomes n_a where Z_{n_a} is the maximal cyclic subgroup of G_a.

One can however imagine of obtaining fractionization at the level of imbedding space for space-time sheets, which are analogous to multi-sheeted Riemann surfaces (say Riemann surfaces associated with $z^{1/n}$ since the rotation by 2π understood as a homotopy of M^4 lifted to the space-time sheet is a non-closed curve. Continuity requirement indeed allows fractionization of the orbital quantum numbers and color in this kind of situation.

9.6.1 Both covering spaces and factor spaces are possible

The observation above stimulates the question whether it might be possible in some sense to replace H or its factors by their multiple coverings.

1. This is certainly not possible for M^4, CP_2, or H since their fundamental groups are trivial. On the other hand, the fixing of quantization axes implies a selection of the sub-space $H_4 = M^2 \times S^2 \subset M^4 \times CP_2$, where S^2 is a geodesic sphere of CP_2. $\hat{M}^4 = M^4 \backslash M^2$ and $\hat{CP}_2 = CP_2 \backslash S^2$ have fundamental group Z since the codimension of the excluded sub-manifold is equal to two and homotopically the situation is like that for a punctured plane. The exclusion of these sub-manifolds defined by the choice of quantization axes could naturally give rise to the desired situation.

2. Zero energy ontology forces to modify this picture somewhat. In zero energy ontology causal diamonds (CDs) defined as the intersections of future and past directed light-cones are loci for zero energy states containing positive and negative energy parts of state at the two light-cone boundaries. The location of CD in M^4 is arbitrary but p-adic length scale hypothesis suggests that the temporal distances between tips of CD come as powers of 2 using CP_2 size as unit. Thus M^4 is replaces by CD and $\hat{M}^4$ is replaced with $\hat{CD}$ defined in obvious manner.

3. H_4 represents a straight cosmic string inside CD. Quantum field theory phase corresponds to Jones inclusions with Jones index $\mathcal{M} : \mathcal{N} < 4$. Stringy phase would by previous arguments correspond to $\mathcal{M} : \mathcal{N} = 4$. Also these Jones inclusions are labeled by finite subgroups of $SO(3)$ and thus by Z_n identified as a maximal Abelian subgroup.

   One can argue that cosmic strings are not allowed in QFT phase. This would encourage the replacement $\hat{CD} \times \hat{CP}_2$ implying that surfaces in $CD \times S^2$ and $(M^2 \cap CD) \times CP_2$ are not allowed. In particular, cosmic strings and CP_2 type extremals with M^4 projection in M^2 and thus light-like geodesic without zitterwebegung essential for massivation are forbidden. This brings in mind instability of Higgs=0 phase.

4. The covering spaces in question would correspond to the Cartesian products $\hat{CD}_{n_a} \times \hat{CP}_{2n_b}$ of the covering spaces of $\hat{CD}$ and $\hat{CP}_2$ by Z_{n_a} and Z_{n_b} with fundamental group is $Z_{n_a} \times Z_{n_b}$. One can also consider extension by replacing $M^2 \cap CD$ and S^2 with its orbit under G_a (say tedrahedral, octahedral, or icosahedral group). The resulting space will be denoted by $\hat{CD} \hat{\times} G_a$ *resp.* $\hat{CP}_2 \hat{\times} G_b$.

5. One expects the discrete subgroups of $SU(2)$ emerge naturally in this framework if one allows the action of these groups on the singular sub-manifolds $M^2 \cap CD$ or S^2. This would replace the singular manifold with a set of its rotated copies in the case that the subgroups have genuinely 3-dimensional action (the subgroups which corresponds to exceptional groups in the ADE correspondence). For instance, in the case of $M^2 \cap CD$ the quantization axes for angular momentum

would be replaced by the set of quantization axes going through the vertices of tedrahedron, octahedron, or icosahedron. This would bring non-commutative homotopy groups into the picture in a natural manner.

6. Also the orbifolds $\hat{CD}/G_a \times \hat{CP}_2/G_b$ can be allowed as also the spaces $\hat{CD}/G_a \times (\hat{CP}_2 \hat{\times} G_b)$ and $(\hat{CD}\hat{\times}G_a) \times \hat{CP}_2/G_b$. Hence the previous framework would generalize considerably by the allowance of both coset spaces and covering spaces.

There are several non-trivial questions related to the details of the gluing procedure and phase transition as motion of partonic 2-surface from one sector of the imbedding space to another one.

1. How the gluing of copies of imbedding space at $(M^2 \cap CD) \times CP_2$ takes place? It would seem that the covariant metric of M^4 factor proportional to $\hbar^2$ must be discontinuous at the singular manifold since only in this manner the idea about different scaling factor of M^4 metric can make sense. This is consistent with the identical vanishing of Chern-Simons action in $M^2 \times S^2$.

2. One might worry whether the phase transition changing Planck constant means an instantaneous change of the size of partonic 2-surface in CD degrees of freedom. This is not the case. Light-likeness in $(M^2 \cap CD) \times S^2$ makes sense only for surfaces $X^1 \times D^2 \subset (M^2 \cap CD) \times S^2$, where X^1 is light-like geodesic. The requirement that the partonic 2-surface X^2 moving from one sector of H to another one is light-like at $(M^2 \cap CD) \times S^2$ irrespective of the value of Planck constant requires that X^2 has single point of $(M^2 \cap CD)$ as M^2 projection. Hence no sudden change of the size X^2 occurs.

3. A natural question is whether the phase transition changing the value of Planck constant can occur purely classically or whether it is analogous to quantum tunneling. Classical non-vacuum extremals of Chern-Simons action have two-dimensional CP_2 projection to homologically non-trivial geodesic sphere S^2_I. The deformation of the entire S^2_I to homologically trivial geodesic sphere S^2_{II} is not possible so that only combinations of partonic 2-surfaces with vanishing total homology charge (Kähler magnetic charge) can in principle move from sector to another one, and this process involves fusion of these 2-surfaces such that CP_2 projection becomes single homologically trivial 2-surface. A piece of a non-trivial geodesic sphere S^2_I of CP_2 can be deformed to that of S^2_{II} using 2-dimensional homotopy flattening the piece of S^2 to curve. If this homotopy cannot be chosen to be light-like, the phase transitions changing Planck constant take place only via quantum tunnelling. Obviously the notions of light-like homotopies (cobordisms) and classical light-like homotopies (cobordisms) are very relevant for the understanding of phase transitions changing Planck constant.

9.6.2 Do factor spaces and coverings correspond to the two kinds of Jones inclusions?

What could be the interpretation of these two kinds of spaces?

1. Jones inclusions appear in two varieties corresponding to $\mathcal{M} : \mathcal{N} < 4$ and $\mathcal{M} : \mathcal{N} = 4$ and one can assign a hierarchy of subgroups of $SU(2)$ with both of them. In particular, their maximal Abelian subgroups Z_n label these inclusions. The interpretation of Z_n as invariance group is natural for $\mathcal{M} : \mathcal{N} < 4$ and it naturally corresponds to the coset spaces. For $\mathcal{M} : \mathcal{N} = 4$ the interpretation of Z_n has remained open. Obviously the interpretation of Z_n as the homology group defining covering would be natural.

2. $\mathcal{M} : \mathcal{N} = 4$ should correspond to the allowance of cosmic strings and other analogous objects. Does the introduction of the covering spaces bring in cosmic strings in some controlled manner? Formally the subgroup of $SU(2)$ defining the inclusion is $SU(2)$ would mean that states are $SU(2)$ singlets which is something non-physical. For covering spaces one would however obtain the degrees of freedom associated with the discrete fiber and the degrees of freedom in question would not disappear completely and would be characterized by the discrete subgroup of $SU(2)$.

   For anyons the non-trivial homotopy of plane brings in non-trivial connection with a flat curvature and the non-trivial dynamics of topological QFTs. Also now one might expect similar non-trivial contribution to appear in the spinor connection of $\hat{CD}\hat{\times}G_a$ and $\hat{CP}_2\hat{\times}G_b$. In conformal field theory models non-trivial monodromy would correspond to the presence of punctures in plane.

3. For factor spaces the unit for quantum numbers like orbital angular momentum is multiplied by n_a *resp.* n_b and for coverings it is divided by this number. These two kind of spaces are in a well defined sense obtained by multiplying and dividing the factors of $\hat{H}$ by G_a *resp.* G_b and multiplication and division are expected to relate to Jones inclusions with $\mathcal{M}:\mathcal{N}<4$ and $\mathcal{M}:\mathcal{N}=4$, which both are labeled by a subset of discrete subgroups of SU(2).

4. The discrete subgroups of $SU(2)$ with fixed quantization axes possess a well defined multiplication with product defined as the group generated by forming all possible products of group elements as elements of $SU(2)$. This product is commutative and all elements are idempotent and thus analogous to projectors. Trivial group G_1, two-element group G_2 consisting of reflection and identity, the cyclic groups Z_p, p prime, and tedrahedral, octahedral, and icosahedral groups are the generators of this algebra.

By commutativity one can regard this algebra as an 11-dimensional module having natural numbers as coefficients ("rig"). The trivial group G_1, two-element group G_2¡ generated by reflection, and tedrahedral, octahedral, and icosahedral groups define 5 generating elements for this algebra. The products of groups other than trivial group define 10 units for this algebra so that there are 11 units altogether. The groups Z_p generate a structure analogous to natural numbers acting as analog of coefficients of this structure. Clearly, one has effectively 11-dimensional commutative algebra in 1-1 correspondence with the 11-dimensional "half-lattice" N^{11} (N denotes natural numbers). Leaving away reflections, one obtains N^7. The projector representation suggests a connection with Jones inclusions. An interesting question concerns the possible Jones inclusions assignable to the subgroups containing infinitely manner elements. Reader has of course already asked whether dimensions 11, 7 and their difference 4 might relate somehow to the mathematical structures of M-theory with 7 compactified dimensions. One could introduce generalized configuration space spinor fields in the configuration space labelled by sectors of H with given quantization axes. By introducing Fourier transform in N^{11} one would formally obtain an infinite-component field in 11-D space.

The question how do the Planck constants associated with factors and coverings relate is far from trivial and I have considered several options.

1. If one assumes that $\hbar^2(X)$, $X=M^4$, CP_2 corresponds to the scaling of the covariant metric tensor g_{ij} and performs an over-all scaling of metric allowed by Weyl invariance of Kähler action by dividing metric with $\hbar^2(CP_2)$, one obtains $r^2\equiv\hbar^2/\hbar_0^2\hbar^2(M^4)/\hbar^2(CP_2)$. This puts M^4 and CP_2 in a very symmetric role and allows much more flexibility in the identification of symmetries associated with large Planck constant phases.

2. Algebraist would argue that Planck constant must define a homomorphism respecting multiplication and division (when possible) by G_i. This requires $r(X)=\hbar(X)\hbar_0=n$ for covering and $r(X)=1/n$ for factor space or vice versa. This gives two options.

3. Option I: $r(X)=n$ for covering and $r(X)=1/n$ for factor space gives $r\equiv\hbar/\hbar_0=r(M^4)/r(CP_2)$. This gives $r=n_a/n_b$ for $\hat{H}/G_a\times G_b$ option and $r=n_b/n_a$ for $\hat{H}\hat{times}(G_a\times G_b)$ option with obvious formulas for hybrid cases.

4. Option II: $r(X)=1/n$ for covering and $r(X)=n$ for factor space gives $r=r(CP_2)/r(M^4)$. This gives $r=n_b/n_a$ for $\hat{H}/G_a\times G_b$ option and $r=n_a/n_b$ for $\hat{H}\hat{times}(G_a\times G_b)$ option with obvious formulas for the hybrid cases.

5. At quantum level the fractionization would come from the modification of fermionic anti-commutation (bosonic commutation) relations involving $\hbar$ at the right hand side so that particle number becomes a multiple of $1/n$ or n. If one postulates that the total number states is invariant in the transition, the increase in the number of sheets is compensated by the increase of the fundamental phase space volume proportional to $\hbar$. This would give $r(X)\rightarrow r(X)/n$ for factor space and $r(X)\rightarrow nr(X)$ for the covering space to compensate the n-fold reduction/increase of states. This would favor Option II.

6. The second manner to distinguish between these two options is to apply the theory to concrete physical situations. Since G_a and G_b act as symmetries in CD and CP_2 degrees of freedom, one might of being able to distinguish between the two options if it is possible to distinguish between the action of G as symmetry of quantum states associated with covering and factor space. Also the quantization of the orbital spin quantum number at single particle level as multiples of n can be distinguished from that in multiples of $1/n$.

9.6.3 A simple model of fractional quantum Hall effect

The generalization of the imbedding space suggests that it could possible to understand fractional quantum Hall effect [45] at the level of basic quantum TGD. This section represents the first rough model of QHE constructed for a couple of years ago is discussed. Needless to emphasize, the model represents only the basic idea and involves ad hoc assumption about charge fractionization.

Recall that the formula for the quantized Hall conductance is given by

$$\begin{aligned} \sigma &= \nu \times \frac{e^2}{h} \ , \\ \nu &= \frac{n}{m} \ . \end{aligned} \tag{9.6.0}$$

Series of fractions in $\nu = 1/3, 2/5, 3/7, 4/9, 5/11, 6/13, 7/15..., 2/3, 3/5, 4/7, 5/9, 6/11, 7/13..., 5/3, 8/5, 11/7, 14/9...$ $1/5, 2/9, 3/13..., 2/7, 3/11..., 1/7....$ with odd denominator have been observed as are also $\nu = 1/2$ and $\nu = 5/2$ states with even denominator [45].

The model of Laughlin [43] cannot explain all aspects of FQHE. The best existing model proposed originally by Jain is based on composite fermions resulting as bound states of electron and even number of magnetic flux quanta [46]. Electrons remain integer charged but due to the effective magnetic field electrons appear to have fractional charges. Composite fermion picture predicts all the observed fractions and also their relative intensities and the order in which they appear as the quality of sample improves.

The generalization of the notion of imbedding space suggests the possibility to interpret these states in terms of fractionized charge, spin, and electron number. There are four combinations of covering and factors spaces of CP_2 and three of them can lead to the increase of Planck constant. Besides this there are two options for the formula of Planck constant so that which the very meager theoretical background one can make only guesses. On the following just for fun consideration option I is considered although the conservation of number of states in the phase transition changing $\hbar$ favors option II.

1. The easiest manner to understand the observed fractions is by assuming that both M^4 and CP_2 correspond to covering spaces so that both spin and electric charge and fermion number are fractionized. This means that e in electronic charge density is replaced with fractional charge. Quantized magnetic flux is proportional to e and the question is whether also here fractional charge appears. Assume that this does not occur.

2. With this assumption the expression for the Planck constant becomes for Option II as $r = \hbar/\hbar_0 = n_a/n_b$ and charge and spin units are equal to $1/n_b$ and $1/n_a$ respectively. This gives $\nu = nn_a/n_b$. The values $m = 2, 3, 5, 7, ..$ are observed. Planck constant can have arbitrarily large values. There are general arguments stating that also spin is fractionized in FQHE.

3. The appearance of $\nu = 5/2$ has been observed [47]. The fractionized charge is $e/4$ in this case. Since $n_i > 3$ holds true if coverings are correlates for Jones inclusions, this requires to $n_b = 4$ and $n_a = 10$. n_b predicting a correct fractionization of charge. The alternative option would be $n_b = 2$ that also Z_2 would appear as the fundamental group of the covering space. Filling fraction $1/2$ corresponds in the composite fermion model and also experimentally to the limit of zero magnetic field [46]. $n_b = 2$ is however inconsistent with the observed fractionization of electric charge and with the vision inspired by Jones inclusions.

4. A possible problematic aspect of the TGD based model is the experimental absence of even values of n_b except $n_b = 2$ (Laughlin's model predicts only odd values of n). A possible explanation is

that by some symmetry condition possibly related to fermionic statistics (as in Laughlin model) n_a/n_b must reduce to a rational with an odd denominator for $n_b > 2$. In other words, one has $n_a \propto 2^r$, where 2^r the largest power of 2 divisor of n_b.

5. Large values of n_a emerge as B increases. This can be understood from flux quantization. One has $e\int BdS = n\hbar(M^4) = nn_a\hbar_0$. By using actual fractional charge $e_F = e/n_b$ in the flux factor would give $e_F\int BdS = n(n_a/n_b)\hbar_0 = n\hbar$. The interpretation is that each of the n_a sheets contributes one unit to the flux for e. Note that the value of magnetic field in given sheet is not affected so that the build-up of multiple covering seems to keep magnetic field strength below critical value.

6. The understanding of the thermal stability is not trivial. The original FQHE was observed in 80 mK temperature corresponding roughly to a thermal energy of $T \sim 10^{-5}$ eV. For graphene the effect is observed at room temperature. Cyclotron energy for electron is (from $f_e = 6 \times 10^5$ Hz at $B = .2$ Gauss) of order thermal energy at room temperature in a magnetic field varying in the range 1-10 Tesla. This raises the question why the original FQHE requires so low temperature. The magnetic energy of a flux tube of length L is by flux quantization roughly $e^2B^2S \sim E_c(e)m_eL$ ($\hbar_0 = c = 1$) and exceeds cyclotron roughly by a factor L/L_e, L_e electron Compton length so that thermal stability of magnetic flux quanta is not the explanation. A possible explanation is that since FQHE involves several values of Planck constant, it is quantum critical phenomenon and is characterized by a critical temperature. The differences of the energies associated with the phase with ordinary Planck constant and phases with different Planck constant would characterize the transition temperature.

As already noticed, it is possible to imagine several other options and the identification of charge unit is rather ad hoc. Therefore this model can be taken only as a warm-up exercise. In [F12] Quantum Hall effect and charge fractionization are discussed in detail and one ends up with a rather detailed view about the delicacies of the Kähler structure of generalized imbedding space.

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Chapter 10

Langlands Program and TGD

10.1 Introduction

Langlands program [20, 21, 22, 23] is an attempt to unify number theory and representation theory of groups and as it seems all mathematics. About related topics I know frustratingly little at technical level. Zeta functions and theta functions [25, 26, 27, 28], and more generally modular forms [29] are the connecting notion appearing both in number theory and in the theory of automorphic representations of reductive Lie groups. The fact that zeta functions have a key role in TGD has been one of the reasons for my personal interest.

The vision about TGD as a generalized number theory [E1, E2, E3, C1, C2] gives good motivations to learn the basic ideas of Langlands program. I hasten to admit that I am just a novice with no hope becoming a master of the horrible technicalities involved. I just try to find whether the TGD framework could allow new physics inspired insights to Langlands program and whether the more abstract number theory relying heavily on the representations of Galois groups could have a direct physical counterpart in TGD Universe and help to develop TGD as a generalized number theory vision. After these apologies I however dare to raise my head a little bit and say aloud that mathematicians might get inspiration from physics inspired new insights.

The basic vision is that Langlands program could relate very closely to the unification of physics as proposed in TGD framework [16, 17, 18]. TGD can indeed be seen both as infinite-dimensional geometry, as a generalized number theory involving several generalizations of the number concept, and as an algebraic approach to physics relying on the unique properties of hyper finite factors of type II_1 so that unification of mathematics would obviously fit nicely into this framework. The fusion of real and various p-adic physics based on the generalization of the number concept, the notion of number theoretic braid, hyper-finite-factors of type II_1 andsub-factors, and the notion of infinite prime, inspired a new view about how to represent finite Galois groups and how to unify the number theoretic and geometric Langlands programs.

10.1.1 Langlands program very briefly

Langlands program [21] states that there exists a connection between number theory and automorphic representations of a very general class of Lie groups known as reductive groups (groups whose all representations are fully reducible). At the number theoretic side there are Galois groups characterizing extensions of number fields, say rationals or finite fields. Number theory involves also so called automorphic functions to which zeta functions carrying arithmetic information via their coefficients relate via so called Mellin transform $\sum_n a_n n^s \rightarrow \sum_n a_n z^n$ [28].

Automorphic functions, invariant under modular group $SL(2,Z)$ or subgroup $\Gamma_0(N) \subset SL(2,Z)$ consisting of matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \ , \ c \ mod \ N = 0 \ ,$$

emerge also via the representations of groups $GL(2,R)$. This generalizes also to higher dimensional groups $GL(n,R)$. The dream is that all number theoretic zeta functions could be understood in terms of representation theory of reductive groups. The highly non-trivial outcome would be possibility to deduce very intricate number theoretical information from the Taylor coefficients of these functions.

Langlands program relates also to Riemann hypothesis and its generalizations. For instance, the zeta functions associated with 1-dimensional algebraic curve on finite field F_q, $q = p^n$, code the numbers of solutions to the equations defining algebraic curve in extensions of F_q which form a hierarchy of finite fields F_{q^m} with $m = kn$ [27]: in this case Riemann hypothesis has been proven.

It must be emphasized that algebraic 1-dimensionality is responsible for the deep results related to the number theoretic Langlands program as far as 1-dimensional function fields on finite fields are considered [27, 22]. In fact, Langlands program is formulated only for algebraic extensions of 1-dimensional function fields.

One might also conjecture that Langlands duality for Lie groups reflects some deep duality on physical side. For instance, Edward Witten is working with the idea that geometric variant of Langlands duality could correspond to the dualities discovered in the framework of YM theories and string models. In particular, Witten proposes that electric-magnetic duality which indeed relates gauge group and its dual, provides a physical correlate for the Langlands duality for Lie groups and could be understood in terms of topological version of four-dimensional $N = 4$ super-symmetric YM theory [36]. Interestingly, Witten assigns surface operators to the 2-D surfaces of 4-D space-time. This brings unavoidably in mind partonic 2-surfaces and TGD as $N = 4$ super-conformal almost topological QFT. In this chapter it will be proposed that super-symmetry might correspond to the Langlands duality in TGD framework.

10.1.2 Questions

Before representing in more detail the TGD based ideas related to Langlands correspondence it is good to summarize the basic questions which Langlands program stimulates.

Could one give more concrete content to the notion of Galois group of algebraic closure of rationals?

The notion of Galois group for algebraic closure of rationals $Gal(\overline{Q}/Q)$ is immensely abstract and one can wonder how to make it more explicit? Langlands program adopts the philosophy that this group could be defined only via its representations. The so called automorphic representations constructed in terms of adeles. The motivation comes from the observation that the subset of adeles consisting of Cartesian product of invertible p-adic integers is a structure isomorphic with the maximal abelian subgroup of $Gal(\overline{Q}/Q)$ obtained by dividing $Gal(\overline{Q}/Q)$ with its commutator subgroup. Representations of finite abelian Galois groups are obtained as homomorphisms mapping infinite abelian Galois group to its finite factor group. In this approach the group $Gal(\overline{Q}/Q)$ remains rather abstract and adeles seem to define a mere auxiliary technical tool although it is clear that so called l-adic representations for Galois groups are are natural also in TGD framework.

This raises some questions.

1. Could one make $Gal(\overline{Q}/Q)$ more concrete? For instance, could one identify it as an infinite symmetric group S_∞ consisting of finite permutations of infinite number of objects? Could one imagine some universal polynomial of infinite degree or a universal rational function resulting as ratio of polynomials of infinite degree giving as its roots the closure of rationals?

2. S_∞ has only single normal subgroup consisting of even permutations and corresponding factor group is maximal abelian group. Therefore finite non-abelian Galois groups cannot be represented via homomorphisms to factor groups. Furthermore, S_{infty} has only infinite-dimensional non-abelian irreducible unitary representations as a simple argument to be discussed later shows.

   What is highly non-trivial is that the group algebras of S_∞ and closely related braid group B_∞ define hyper-finite factors of type II_1 (HFF). Could sub-factors characterized by finite groups G allow to realize the representations of finite Galois groups as automorphisms p HFF? The interpretation would be in terms of "spontaneous symmetry breaking" $Gal(\overline{Q}/Q) \rightarrow G$. Could it be possible to get rid of adeles in this manner?

3. Could one find a concrete physical realization for the action of S_∞? Could the permuted objects be identified as strands of braid so that a braiding of Galois group to infinite braid group

B_∞ would result? Could the outer automorphism action of Galois group on number theoretic braids defining the basic structure of quantum TGD allow to realize Galois groups physically as Galois groups of number theoretic braids associated with subset of algebraic points defined by the intersection of real and p-adic partonic 2-surface? The requirement that mathematics is able to represent itself physically would provide the reason for the fact that reality and various p-adicities intersect along subsets of rational and algebraic points only.

Could one understand the correspondences between the representations of finite Galois groups and reductive Lie groups?

Langlands correspondence involves a connection between the representations of finite-dimensional Galois groups and reductive Lie groups.

1. Could this correspondence result via an extension of the representations of finite groups in infinite dimensional Clifford algebra to those of reductive Lie groups identified for instance as groups defining sub-factors (any compact group can define a unique sub-factor)? If Galois groups and reductive groups indeed have a common representation space, it might be easier to understand Langlands correspondence.

2. Is there some deep difference between between general Langlands correspondence and that for $GL(2, F)$ and could this relate to the fact that subgroups of $SU(2)$ define sub-factors with quantized index $\mathcal{M} : \mathcal{N} \leq 4$.

3. McKay correspondence [52] relates finite subgroups of compact Lie groups to compact Lie group (say finite sub-groups of SU(2) to ADE type Lie-algebras or Kac-Moody algebras). TGD approach leads to a general heuristic explanation of this correspondence in terms of Jones inclusions and Connes tensor product. Could sub-factors allow to understand Langlands correspondence for general reductive Lie groups as both the fact that any compact Lie group can define a unique sub-factor and an argument inspired by McKay correspondence suggest.

Could one unify geometric and number theoretic Langlands programs?

There are two Langlands programs: algebraic [20, 22] and geometric [22, 23] one corresponding to ordinary number fields and function fields. The natural question is whether and how these approaches could be unified.

1. Could the discretization based on the notion of number theoretic braids induce the number theoretic Langlands from geometric Langlands so that the two programs could be unified by the generalization of the notion of number field obtained by gluing together reals with union of reals and various p-adic numbers fields and their extensions along common rationals and algebraics. Certainly the fusion of p-adics and reals to a generalized notion of number should be essential for the unification of mathematics.

2. Could the distinction between number fields and function fields correspond to two kinds of sub-factors corresponding to finite subgroups $G \subset SU(2)$ and SU(2) itself leaving invariant the elements of imbedded algebra? This would obviously generalize to imbeddings of Galois groups to arbitrary compact Lie group. Could gauge group algebras contra Kac Moody algebras be a possible physical interpretation for this. Could the two Langlands programs correspond to two kinds of ADE type hierarchies defined by Jones inclusions? Could minimal conformal field theories with finite number of primary fields correspond to algebraic Langlands and full string theory like conformal field theories with infinite number of primary fields to geometric Langlands? Could this difference correspond to sub-factors defined by disrete groups and Lie groups?

3. Could the notion of infinite rational [19] be involved with this unification? Infinite rationals are indeed mapped to elements of rational function fields (also algebraic extensions of them) so that their interpretation as quantum states of a repeatedly second quantized arithmetic supersymmetric quantum field theory might provide totally new mathematical insights.

Is it really necessary to replace groups $GL(n,F)$ with their adelic counterparts?

If the group of invertible adeles is not needed or allowed then a definite deviation from Langlands program is implied. It would seem that multiplicative adeles (ideles) are not favored by TGD view about the role of p-adic number fields. The l-adic representations of p-adic Galois groups corresponding to single p-adic prime l emerge however naturally in TGD framework.

1. The 2×2 Clifford algebra could be easily replaced with its adelic version. A generalization of Clifford algebra would be in question and very much analogous to $GL(2,A)$ in fact. The interpretation would be that real numbers are replaced with adeles also at the level of imbedding space and space-time. This interpretation does not conform with the TGD based view about the relationship between real and p-adic degrees of freedom. The physical picture is that H is 8-D but has different kind of local topologies and that spinors are in some sense universal and independent of number field.

2. Configuration space spinors define a hyper-finite factor of type II_1. It is not clear if this interpretation continues to make sense if configuration space spinors (fermionic Fock space) are replaced with adelic spinors. Note that this generalization would require the replacement of the group algebra of S_{infty} with its adelic counterpart.

10.2 Basic concepts and ideas related to the number theoretic Langlands program

The basic ideas of Langlands program are following.

1. $Gal(\overline{Q}/Q)$ is a poorly understood concept. The idea is to define this group via its representations and construct representations in terms of group $GL(2,A)$ and more generally $GL(n,A)$, where A refers to adeles. Also representations in any reductive group can be considered. The so called automorphic representations of these groups have a close relationship to the modular forms [29], which inspires the conjecture that n-dimensional representations of $Gal(\overline{Q}/Q)$ are in 1-1 correspondence with automorphic representations of $GL(n,A)$.

2. This correspondence predicts that the invariants characterizing the n-dimensional representations of $Gal(\overline{Q}/Q)$ *resp.* $GL(n,A)$ should correspond to each other. The invariants at Galois sides are the eigenvalues of Frobenius conjugacy classes Fr_p in $Gal(\overline{Q}/Q)$. The non-trivial implication is that in the case of l-adic representations the latter must be algebraic numbers. The ground states of the representations of $Gl(n,R)$ are in turn eigen states of so called Hecke operators $H_{p,k}$, $k=1,..,n$ acting in group algebra of $Gl(n,R)$. The eigenvalues of Hecke operators for the ground states of representations must correspond to the eigenvalues of Frobenius elements if Langlands correspondence holds true.

3. The characterization of the K-valued representations of reductive groups in terms of Weil group W_F associated with the algebraic extension K/F allows to characterize the representations in terms of homomorphisms of Weil group to the Langlands dual $G_L(F)$ of $G(F)$.

10.2.1 Correspondence between n-dimensional representations of $Gal(\overline{F}/F)$ and representations of $GL(n,A_F)$ in the space of functions in $GL(n,F)\backslash GL(n,A_F)$

The starting point is that the maximal abelian subgroup $Gal(Q^{ab}/Q)$ of the Galois group of algebraic closure of rationals is isomorphic to the infinite product $\hat{Z} = \prod_p Z_p^{\times}$, where $Z_p^{\times}$ consists of invertible p-adic integers [22].

By introducing the ring of adeles one can transform this result to a slightly different form. Adeles are defined as collections $((f_p)_{p\in P}, f_\infty)$, P denotes primes, $f_p \in Q_p$, and $f_\infty \in R$, such that $f_p \in Z_p$ for all p for all but finitely many primes p. It is easy to convince oneself that one has $A_Q = (\hat{Z} \otimes_Z Q) \times R$ and $Q^{\times}\backslash A_Q = \hat{Z} \times (R/Z)$. The basic statement of abelian class field theory is that abelian Galois group is isomorphic to the group of connected components of $F^{\times}\backslash A_F^{\times}$.

This statement can be transformed to the following suggestive statement:

1) *1-dimensional representations of $Gal(\overline{F}/F)$ correspond to representations of $GL(1, A_F)$ in the space of functions defined in $GL(1, F)\backslash GL(1, A_F)$.*

The basic conjecture of Langlands was that this generalizes to n-dimensional representations of $Gal(\overline{F}/F)$.

2) *The n-dimensional representations of $Gal(\overline{F}/F)$ correspond to representations of $GL(n, A_F)$ in the space of functions defined in $GL(n, F)\backslash GL(n, A_F)$.*

This relation has become known as Langlands correspondence.

It is interesting to relate this approach to that discussed in this chapter.

1. In TGD framework adeles do not seem natural although p-adic number fields and l-adic representations have a natural place also here. The new view about numbers is of course an essentially new element allowing geometric interpretation.

2. The irreducible representations of $Gal(\overline{F}, F)$ are assumed to reduce to those for its finite subgroup G. If $Gal(\overline{F}, F)$ is identifiable as S_∞, finite dimensional representations cannot correspond to ordinary unitary representations since, by argument to be represented later, their dimension is of order order $n \to \infty$ at least. Finite Galois groups can be however interpreted as a sub-group of outer automorphisms defining a sub-factor of $Gal(\overline{Q}, Q)$ interpreted as HFF. Outer automorphisms result at the limit $n \to \infty$ from a diagonal imbedding of finite Galois group to its n^{th} Cartesian power acting as automorphisms in S_∞. At the limit $n \to \infty$ the imbedding does not define inner automorphisms anymore. Physicist would interpret the situation as a spontaneous symmetry breaking.

3. These representations have a natural extension to representations of $Gl(n, F)$ and of general reductive groups if also realized as point-wise symmetries of sub-factors of HFF. Continuous groups correspond to outer automorphisms of group algebra of S_∞ not inducible from outer automorphisms of S_{infty}. That finite Galois groups and Lie groups act in the same representation space should provide completely new insights to the understanding of Langlands correspondence.

4. The l-adic representations of $Gal(\overline{Q}/Q)$ could however change the situation. The representations of finite permutation groups in R and in p-adic number fields $p < n$ are more complex and actually not well-understood [45]. In the case of elliptic curves [22] (say $y^2 = x^3 + ax + b$, a, b rational numbers with $4a^3 + 27b^2 \neq 0$) so called first etale cohomology group is Q_l^2 and thus 2-dimensional and it is possible to have 2-dimensional representations $Gal(\overline{Q}/Q) \to GL(2, Q_l)$. More generally, l-adic representations σ of of $Gal(\overline{F}/F) \to GL(n, \overline{Q_l})$ is assumed to satisfy the condition that there exists a finite extension $E \subset \overline{Q}_l$ such that σ factors through a homomorphism to $GL(n, E)$.

   Assuming $Gal(\overline{Q}/Q) = S_\infty$, one can ask whether l-adic or adelic representations and the representations defined by outer automorphisms of sub-factors might be two alternative manners to state the same thing.

Frobenius automorphism

Frobenius automorphism is one of the basic notions in Langlands correspondence. Consider a field extension K/F and a prime ideal v of F (or prime p in case of ordinary integers). v decomposes into a product of prime ideals of K: $v = \prod w_k$ if v is unramified and power of this if not. Consider unramified case and pick one w_k and call it simply w. Frobenius automorphisms Fr_v is by definition the generator of the the Galois group $Gal(K/w, F/v)$, which reduces to Z/nZ for some n.

Since the decomposition group $D_w \subset Gal(K/F)$ by definition maps the ideal w to itself and preserves F point-wise, the elements of D_w act like the elements of $Gal(O_K/w, O_F/v)$ (O_X denotes integers of X). Therefore there exists a natural homomorphism $D_w : Gal(K/F) \to Gal(O_K/w, O_F/v)$ ($= Z/nZ$ for some n). If the inertia group I_w identified as the kernel of the homomorphism is trivial then the Frobenius automorphism Fr_v, which by definition generates $Gal(O_K/w, O_F/v)$, can be regarded as an element of D_w and $Gal(K/F)$. Only the conjugacy class of this element is fixed since any w_k can be chosen. The significance of the result is that the eigenvalues of Fr_p define invariants characterizing the representations of $Gal(K/F)$. The notion of Frobenius element can be generalized also to the case of $Gal(\overline{Q}/Q)$ [22]. The representations can be also l-adic being defined in $GL(n, E_l)$ where E_l is extension of Q_l. In this case the eigenvalues must be algebraic numbers so that they make sense as complex numbers.

Two examples discussed in [22] help to make the notion more concrete.

1. For the extensions of finite fields $F = G(p,1)$ Frobenius automorphism corresponds to $x \rightarrow x^p$ leaving elements of F invariant.

2. All extensions of Q having abelian Galois group correspond to so called cyclotomic extensions defined by polynomials $P_N(x) = x^N + 1$. They have Galois group $(Z/NZ)^{\times}$ consisting of integers $k < n$ which do not divide n and the degree of extension is $\phi(N) = |Z/NZ^{\times}|$, where $\phi(n)$ is Euler function counting the integers $n < N$ which do not divide N. Prime p is unramified only if it does not divide n so that the number of "bad primes" is finite. The Frobenius equivalence class Fr_p in $Gal(K/F)$ acts as raising to p^{th} power so that the Fr_p corresponds to integer $p \ mod \ n$.

Automorphic representations and automorphic functions

In the following I want to demonstrate that I have at least tried to do my home lessons by trying to reproduce the description of [22] for the route from automorphic adelic representations of $GL(2,R)$ to automorphic functions defined in upper half-plane.

1. Characterization of the representation

The representations of $GL(2,Q)$ are constructed in the space of smooth bounded functions $GL(2,Q)\backslash GL(2,A) \rightarrow C$ or equivalently in the space of $GL(2,Q)$ left-invariant functions in $GL(2,A)$. A denotes adeles and $GL(2,A)$ acts as right translations in this space. The argument generalizes to arbitrary number field F and its algebraic closure $\overline{F}$.

1. Automorphic representations are characterized by a choice of compact subgroup K of $GL(2,A)$. The motivating idea is the central role of double coset decompositions $G = K_1AK_2$, where K_i are compact subgroups and A denotes the space of double cosets K_1gK_2 in general representation theory. In the recent case the compact group $K_2 \equiv K$ is expressible as a product $K = \prod_p K_p \times O_2$. For each unramified prime p one has $K_p = GL(2,Z_p)$. For ramified primes K_p consists of $SL(2,Z_p)$ matrices with $c \in p^{n_p}Z_p$. Here p^{n_p} is the divisor of conductor N corresponding to p. K-finiteness condition states that the right action of K on f generates a finite-dimensional vector space.

2. The representation functions are eigen functions of the Casimir operator C of $gl(2,R)$ with eigenvalue ρ so that irreducible representations of $gl(2,R)$ are obtained. An explicit representation of Casimir operator is given by
$$C = \frac{X_0^2}{4} + X_+X - +X_-X_+ \ ,$$
where one has
$$X_0\left(\begin{array}{cc} 0 & i \\ -i & 0 \end{array}\right) \ , \left(\begin{array}{cc} 1 & \mp i \\ \mp i & -1 \end{array}\right) \ .$$

3. The center $A^{\times}$ of $GL(2,A)$ consists of $A^{\times}$ multiples of identity matrix and it is assumed $f(gz) = \chi(z)f(g)$, where $\chi : A^{\times} \rightarrow C$ is a character providing a multiplicative representation of $A^{\times}$.

4. Also the so called cuspidality condition
$$\int_{Q\backslash NA} f(\left(\begin{array}{cc} 1 & u \\ 0 & 1 \end{array}\right) g)du = 0$$
is satisfied [22]. Note that the integration measure is adelic. Note that the transformations appearing in integrand are an adelic generalization of the 1-parameter subgroup of Lorentz transformations leaving invariant light-like vector. The condition implies that the modular functions defined by the representation vanish at cusps at the boundaries of fundamental domains representing copies $H_u/\Gamma_0(N)$ where N is the conductor. The "basic" cusp corresponds to $\tau = i\infty$ for the "basic" copy of the fundamental domain.

   The groups $gl(2,R)$, $O(2)$ and $GL(2,Q_p)$ act non-trivially in these representations and it can be shown that a direct sum of irreps of $GL(2,A_F) \times gl(2,R)$ results with each irrep occurring only once. These representations are known as cuspidal automorphic representations.

2. From adeles to $\Gamma_0(N)\backslash SL(2,R)$

The path from adeles to the modular forms in upper half plane involves many twists.

1. By so called central approximation theorem the group $GL(2,Q)\backslash GL(2,A)/K$ is isomorphic to the group $\Gamma_0(N)\backslash GL_+(2,R)$, where N is conductor [22]. The group $\Gamma_0(N)\subset SL(2,Z)$ consists of matrices

$$\left(\begin{array}{cc} a & b \\ c & d \end{array}\right) \quad , \quad c \ mod \ N = 0 .$$

$_+$ refers to positive determinant. Note that $\Gamma_0(N)$ contains as a subgroup congruence subgroup $\Gamma_0(N)$ consisting of matrices, which are unit matrices modulo N. Congruence subgroup is a normal subgroup of $SL(2,Z)$ so that also $SL(2,Z)/\Gamma(N)$ is group. Physically $\Gamma(N)$ would be rather interesting alternative for $\Gamma_0(N)$ as a compact subgroup and the replacement $K_p = \Gamma_0(p^{k_p}) \rightarrow \Gamma(p^{k_p})$ of p-adic groups adelic decomposition is expected to guarantee this.

2. Central character condition together with assumptions about the action of K implies that the smooth functions in the original space are completely determined by their restrictions to $\Gamma_0(N)\backslash SL(2,R)$ so that one gets rid of the adeles.

3. From $\Gamma_0(N)\backslash SL(2,R)$ to upper half-plane $H_u = SL(2,R)/SO(2)$

The representations of $(gl(2,C),O(2))$ come in four categories corresponding to principal series, discrete series, the limits of discrete series, and finite-dimensional representations [22]. For the discrete series representation π giving square integrable representation in $SL(2,R)$ one has $\rho = k(k-1)/4$, where $k>1$ is integer. As sl_2 module, π_∞ is direct sum of irreducible Verma modules with highest weight $-k$ and lowest weight k. The former module is generated by a unique, up to a scalar, highest weight vector v_∞ such that

$$X_0 v_\infty = -k v_\infty \quad , \quad X_+ v_\infty = 0 \quad .$$

The latter module is in turn generated by the lowest weight vector

$$\left(\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array}\right) v_\infty \quad .$$

This means that entire module is generated from the ground state v_∞, and one can focus to the function ϕ_π on $\Gamma_0(N)\backslash SL(2,R)$ corresponding to this vector. The goal is to assign to this function $SO(2)$ invariant function defined in the upper half-plane $H_u = SL(2,R)/SO(2)$, whose points can be parameterized by the numbers $\tau = (a+bi)/(c+di)$ determined by $SL(2,R)$ elements. The function $f_\pi(g) = \phi_\pi(g)(ci+d)^k$ indeed is SO(2) invariant since the phase $exp(ik\phi)$ resulting in $SO(2)$ rotation by ϕ is compensated by the phase resulting from $(ci+d)$ factor. This function is not anymore $\Gamma_0(N)$ invariant but transforms as

$$f_\pi((a\tau+b)/(c\tau+d)) = (c\tau+d)^k f_\pi(\tau)$$

under the action of $\Gamma_0(N)$ The highest weight condition $X_+ v_\infty$ implies that f is holomorphic function of τ. Such functions are known as modular forms of weight k and level N. It would seem that the replacement of $\Gamma_0(N)$ suggested by physical arguments would only replace $H_u/\Gamma_0(N)$ with $H_u/\Gamma(N)$.

f_π can be expanded as power series in the variable $q = exp(2\pi\tau)$ to give

$$f_\pi(q) \quad = \quad \sum_{n=0}^{\infty} a_n q^n \quad . \tag{10.2.1}$$

Cuspidality condition means that f_π vanishes at the cusps of the fundamental domain of the action of $\Gamma_0(N)$ on H_u. In particular, it vanishes at $q=0$ which which corresponds to $\tau = -\infty$. This implies $a_0 = 0$. This function contains all information about automorphic representation.

Hecke operators

Spherical Hecke algebra (which must be distinguished from non-commutative Hecke algebra associated with braids) can be defined as algebra of $GL(2,Z_p)$ bi-invariant functions on $GL(2,Q_p)$ with respect to convolution product. This algebra is isomorphic to the polynomial algebra in two generators $H_{1,p}$ and $H_{2,p}$ and the ground states v_p of automorphic representations are eigenstates of these operators. The normalizations can be chosen so that the second eigenvalue equals to unity. Second eigenvalue must be an algebraic number. The eigenvalues of Hecke operators $H_{p,1}$ correspond to the coefficients a_p of the q-expansion of automorphic function f_π so that f_π is completely determined once these coefficients carrying number theoretic information are known [22].

The action of Hecke operators induces an action on the modular function in the upper half-plane so that Hecke operators have also representation as what is known as classical Hecke operators. The existence of this representation suggests that adelic representations might not be absolutely necessary for the realization of Langlands program.

10.2.2 Some remarks about the representations of $Gl(n)$ and of more general reductive groups

The simplest representations of $Gl(n,R)$ have the property that the Borel group B of upper diagonal matrices is mapped to diagonal matrices consisting of character ξ which decomposes to a product of characters χ_k associated with diagonal elements b_k of B defining homomorphism

$$b_k \rightarrow sgn(b)^{m(k)}|b_k|^{ia_k}$$

to unit circle if a_k is real. Also more general, non-unitary, characters can be allowed. The representation itself satisfies the condition $f(bg) = \chi(b)f(g)$. Thus n complex parameters a_k defining a reducible representation of $C^\times$ characterize the irreducible representation.

In the case of $GL(2,R)$ one can consider also genuinely two-dimensional discrete series representations characterized by only single continuous parameter and the previous example represented just this case. These representations are square integrable in the subgroup $SL(2,R)$. Their origin is related to the fact that the algebraic closure of R is 2-dimensional. The so called Weil group W_R which is semi-direct product of complex conjugation operation with $C^\times$ codes for this number theoretically. The 2-dimensional representations correspond to irreducible 2-dimensional representations of W_R in terms of diagonal matrices of $Gl(2,C)$.

In the case of $GL(n,R)$ the representation is characterized by integers n_k: $\sum n_k = n$ characterizing the dimensions $n_k = 1,2$ of the representations of W_R. For $Gl(n,C)$ one has $n_k = 1$ since Weil group W_C is obviously trivial in this case.

In the case of a general reductive Lie group G the homomorphisms of W_R to the Langlands dual G_L of G defined by replacing the roots of the root lattice with their duals characterize the automorphic representations of G.

The notion of Weil group allows also to understand the general structure of the representations of $GL(n,F)$ in $GL(n,K)$, where F is p-adic number field and K its extension. In this case Weil group is a semi-direct product of Galois group of $Gal(K/F)$ and multiplicative group $K^\times$. A very rich structure results since an infinite number of extensions exists and the dimensions of discrete series representations.

The deep property of the characterization of representations in terms of Weyl group is functoriality. If one knows the homomorphisms $W_F \rightarrow G$ and $G \rightarrow H$ then the composite homomorphism defines an automorphic representation of H. This means that irreps of G can be passed to those of H by homomorphism [20].

10.3 TGD inspired view about Langlands program

In this section a general TGD inspired vision about Langlands program is described. The fusion of real and various p-adic physics based on the generalization of the number concept, the notion of number theoretic braid, hyper-finite-factors of type II_1 and their sub-factors, and the notion of infinite prime, lead to a new view about how to represent finite Galois groups and how to unify the number theoretic and geometric Langlands programs.

10.3.1 What is the Galois group of algebraic closure of rationals?

Galois group is essentially the permutation group for the roots of an irreducible polynomial. It is a a subgroup of symmetric group S_n, where n is the degree of polynomial. One can also imagine the notion of Galois group $Gal(\overline{Q}/Q)$ for the algebraic closure of rationals but the concretization of this notion is not easy.

$Gal(\overline{Q}/Q)$ as infinite permutation group?

The maximal abelian subroup of $Gal(\overline{Q}/Q)$, which is obtained by dividing with the normal subgroup of even permutations, is identifiable as a product of multiplicative groups $Z_p^{\times}$ of invertible p-adic integers $n = n_0 + pZ$, $n_0 \in \{1, ..p-1\}$ for all p-adic primes and can be understood reasonably via its isomorphism to the product $\hat{Z} = \prod_p Z_p$ of multiplicative groups Z_p of invertible p-adic integers, one factor for each prime p [21, 22, 20].

Adeles [30] are identified as the subring of $(\hat{Z} \otimes_Z Q) \times R$ containing only elements for which the elements of Q_p belong to Z_p except for a finite number of primes so that the number obtained can be always represented as a product of element of $\hat{Z}$ and point of circle R/Z: $A = \hat{Z} \times R/Z$. Adeles define a multiplicative group $A^{\times}$ of ideles and $GL(1, A)$ allow to construct representations $Gal(Q^{ab}/Q)$.

It is much more difficult to get grasp on $Gal(\overline{Q}/Q)$. The basic idea of Langlands program is that one should try to understand $Gal(\overline{Q}/Q)$ through its representations rather than directly. The natural hope is that n-dimensional representations of $Gal(\overline{Q}/Q)$ could be realized in $GL(n, A)$.

1. $Gal(\overline{Q}/Q)$ as infinite symmetric group?

One could however be stubborn and try a different approach based on the direct identification $Gal(\overline{Q}/Q)$. The naive idea is that $Gal(\overline{Q}/Q)$ could in some sense be the Galois group of a polynomial of infinite degree. Of course, for mathematical reasons also a rational function defined as a ratio of this kind of polynomials could be considered so that the Galois group could be assigned to both zeros and poles of this function. In the generic case this group would be an infinite symmetric group S_∞ for an infinite number of objects containing only permutations for subsets containing a finite number of objects. This group could be seen as the first guess for $Gal(\overline{Q}/Q)$.

S_∞ can be defined by generators e_m representing permutation of $m^{th}$and $(m+1)^{th}$ object satisfying the conditions

$$\begin{aligned} e_m e_m &= e_n e_m \quad \text{for} \quad |m-n| > 1 , \\ e_n e_{n+1} e_n &= e_n e_{n+1} e_n e_{n+1} \quad \text{for} \quad n = 1, ..., n-2 \ , \\ e_n^2 &= 1 \ . \end{aligned} \tag{10.3.-1}$$

By the definition S_∞ can be expected to possess the basic properties of finite-dimensional permutation groups. Conjugacy classes, and thus also irreducible unitary representations, should be in one-one correspondence with partitions of n objects at the limit $n \to \infty$. Group algebra defined by complex functions in S_∞ gives rise to the unitary complex number based representations and the smallest dimensions of the irreducible representations are of order n and are thus infinite for S_∞. For representations based on real and p-adic number based variants of group algebra situation is not so simple but it is not clear whether finite dimensional representations are possible.

S_n and obviously also S_∞ allows an endless number of realizations since it can act as permutations of all kinds of objects. Factors of a Cartesian and tensor power are the most obvious possibilities for the objects in question. For instance, S_n allows a representation as elements of rotation group $SO(n)$ permuting orthonormalized unit vectors e_i with components $(e_i)^k = \delta_i^k$. This induces also a realization as spinor rotations in spinor space of dimension $D = 2^{d/2}$.

2. Group algebra of S_∞ as HFF

The highly non-trivial fact that the group algebra of S_∞ is hyper-finite factor of type II_1 (HFF) [49] suggests a representation of permutations as permutations of tensor factors of HFF interpreted as an infinite power of finite-dimensional Clifford algebra. The minimal choice for the finite-dimensional Clifford algebra is $M^2(C)$. In fermionic Fock space representation of infinite-dimensional Clifford algebra e_i would induce the transformation $(b^\dagger_{m,i}, b^\dagger_{m,i+1}) \to (b^\dagger_{m,i+1}, b^\dagger_{m,i})$. If the index m is lacking, the representation would reduce to the exchange of fermions and representation would be abelian.

3. Projective representations of S_∞ as representations of braid group B_∞

S_n can be extended to braid group B_n by giving up the condition $e_i^2 = 1$ for the generating permutations of the symmetric group. Generating permutations are represented now as homotopies exchanging the neighboring strands of braid so that repeated exchange of neighboring strands induces a sequence of twists by π. Projective representations of S_∞ could be interpreted as representations of B_∞. Note that odd and even generators commute mutually and for unitary representations either of them can be diagonalized and are represented as phases $exp(i\phi)$ for braid group. If $exp(i\phi)$ is not a root of unity this gives effectively a polynomial algebra and the polynomials subalgebras of these phases might provide representations for the Hecke operators also forming commutative polynomial algebras.

The additional flexibility brought in by braiding would transform Galois group to a group analogous to homotopy group and could provide a connection with knot and link theory [36, 37] and topological quantum field theories in general [35]. Finite quantum Galois groups would generate braidings and a connection with the geometric Langlands program where Galois groups are replaced with homotopy groups becomes suggestive [22, 23].

4. What does one mean with S_∞?

There is also the question about the meaning of S_∞. The hierarchy of infinite primes suggests that there is an entire infinity of infinities in number theoretical sense. After all, any group can be formally regarded as a permutation group. A possible interpretation would be in terms of algebraic closure of rationals and algebraic closures for an infinite hierarchy of polynomials to which infinite primes can be mapped. The question concerns the interpretation of these higher Galois groups and HFFs. Could one regard these as local variants of S_∞ and does this hierarchy give all algebraic groups, in particular algebraic subgroups of Lie groups, as Galois groups so that almost all of group theory would reduce to number theory even at this level?

The group algebra of Galois group of algebraic closure of rationals as hyper-finite factor of type II_1

The most natural framework for constructing unitary irreducible representations of Galois group is its group algebra. In the recent case this group algebra would be that for S_∞ or B_∞ if braids are allowed. What puts bells ringing is that the group algebra of S_∞ is a hyper-finite factor of type II_1 isomorphic as a von Neumann algebra to the infinite-dimensional Clifford algebra [49], which in turn is the basic structures of quantum TGD whose localized version might imply entire quantum TGD. The very close relationship with the braid group makes it obvious that same holds true for corresponding braid group B_∞. Indeed, the group algebra of an infinite discrete group defines under very general conditions HFF. One of these conditions is so called amenability [47]. This correspondence gives hopes of understanding the Langlands correspondence between representations of discrete Galois groups and the representations of $GL(n, F)$ (more generally representations of reductive groups).

Thus it seems that configuration space spinors (fermionic Fock space) could naturally define a finite-dimensional spinor representation of finite-dimensional Galois groups associated with the number theoretical braids. Inclusions $\mathcal{N} \subset \mathcal{M}$ of hyper-finite factors realize the notion of finite measurement resolution and give rise to finite dimensional representations of finite groups G leaving elements of $\mathcal{N}$ invariant. An attractive idea is that these groups are identifiable as Galois groups.

The identification of the action of G on $\mathcal{M}$ as homomorphism $G \rightarrow Aut(\mathcal{M})$ poses strong conditions on it. This is discussed in the thesis of Jones [54] which introduces three algebraic invariants for the actions of finite group in hyperfinite-factors of type II_1, denoted by $\mathcal{M}$ in the sequel. In general the action reduces to inner automorphism of $\mathcal{M}$ for some normal subgroup $H \subset G$: this group is one of the three invariants of G action. In general one has projective representation for H so that one has $u_{h_1}u_{h_2} = \mu(h_1, h_2)u_{h_1h_2}$, where $\mu(h_1)$ is a phase factor which satisfies cocyle conditions coming from associativity.

1. The simplest action is just a unitary group representation for which $g \in G$ is mapped to a unitary operator u_g in $\mathcal{M}$ acting in $\mathcal{M}$ via adjoint action $m \rightarrow u_g m u_g^\dagger = Ad(u_g)m$. In this case one has $H = G$. In this case the fixed point algebra does not however define a factor and there is no natural reduction of the representations of $Gal(\overline{Q}/Q)$ to a finite subgroup.

2. The exact opposite of this situation outer action of G mean $H = \{e\}$. All these actions are conjugate to each other. This gives gives rise to two kinds of sub-factors and two kinds of representations of G. Both actions of Galois group could be realized either in the group or braid algebra of $Gal(\overline{Q}/Q)$ or in infinite dimensional Clifford algebra. In neither case the action be inner automorphic action $u \rightarrow gug^{\dagger}$ as one might have naively expected. This is crucial for circumventing the difficulty caused by the fact that $Gal(\overline{Q}/Q)$ identified as S_{∞} allows no finite-dimensional complex representation.

3. The first sub-factor is $\mathcal{M}^G \subset \mathcal{M}$ corresponding, where the action of G on $\mathcal{M}$ is outer. Outer action defines a fixed point algebra for all finite groups G. For $D = \mathcal{M} : \mathcal{N} < 4$ only finite subgroups $G \subset SU(2)$ would be represented in this manner. The index identifiable as the fractal dimension of quantum Clifford algebra having $\mathcal{N}$ as non-abelian coefficients is $D = 4cos^2(\pi/n)$. One can speak about quantal representation of Galois group. The image of Galois group would be a finite subgroup of $SU(2)$ acting as spinor rotations of quantum Clifford algebra (and quantum spinors) regarded as a module with respect to the included algebra invariant under inner automorphisms. These representations would naturally correspond to 2-dimensional representations having very special role for the simple reason that the algebraic closure of reals is 2-dimensional.

4. Second sub-factor is isomorphic to $\mathcal{M}^G \subset (\mathcal{M} \otimes L(H))^G$. Here $L(H)$ is the space of linear operators acting in a finite-dimensional representation space H of a unitary irreducible representation of G. The action of G is a tensor product of outer action and adjoint action. The index of the inclusion is $dim(H)^2 \geq 1$ [55] so that the representation of Galois group can be said to be classical (non-fractal).

5. The obvious question is whether and in what sense the outer automorphisms represent Galois subgroups. According to [54] the automorphisms belong to the completion of the group of inner automorphisms of HFF. Identifying HFF as group algebra of S_{∞}, the interpretation would be that outer automorphisms are obtained as diagonal embeddings of Galois group to $S_n \times S_n \times ....$ If one includes only a finite number of these factors the outcome is an inner automorphisms so that for all finite approximations inner automorphisms are in question. At the limit one obtains an automorphisms which does not belong to S_{∞} since it contains only finite permutations. This identification is consistent with the identification of the outer automorphisms as diagonal embedding of G to an infinite tensor power of sub-Clifford algebra of Cl_{∞}.

This picture is physically very appealing since it means that the ordering of the strands of braid does not matter in this picture. Also the reduction of the braid to a finite number theoretical braid at space-time level could be interpreted in terms of the periodicity at quantum level. From the point of view of physicist this symmetry breaking would be analogous to a spontaneous symmetry breaking above some length scale L. The cutoff length scale L would correspond to the number N of braids to which finite Galois group G acts and corresponds also to some p-adic length scale.

One might hope that the emergence of finite groups in the inclusions of hyper-finite factors could throw light into the mysterious looking finding that the representations of finite Galois groups and unitary infinite-dimensional automorphic representations of $GL(n, R)$ are correlated by the connection between the eigenvalues of Frobenius element Fr_p on Galois side and eigenvalues of commuting Hecke operators on automorphic side. The challenge would be to show that the action of Fr_p as outer automorphism of group algebra of S_{∞} or B_{∞} corresponds to Hecke algebra action on configuration space spinor fields or in modular degrees of freedom associated with partonic 2-surface.

Could there exist a universal rational function having $Gal(\overline{Q}/Q)$ as the Galois group of its zeros/poles?

The reader who is not fascinated by the rather speculative idea about a universal rational function having $Gal(\overline{Q}/Q)$ as a permutation group of its zeros and poles can safely skip this subsection since it will not be needed anywhere else in this chapter.

1. Taking the idea about permutation group of roots of a polynomial of infinite order seriously, one could require that the analytic function defining the Galois group should behave like a polynomial or a rational function with rational coefficients in the sense that the function should have an

everywhere converging expansion in terms of products over an infinite number of factors $z - z_i$ corresponding to the zeros of the numerator and possible denominator of a rational function. The roots z_i would define an extension of rationals giving rise to the entire algebraic closure of rationals. This is a tall order and the function in question should be number theoretically very special.

2. One can speculate even further. TGD has inspired the conjecture that the non-trivial zeros $s_n = 1/2 + iy_n$ of Riemann zeta [25] (assuming Riemann hypothesis) are algebraic numbers and that also the numbers p^{s_n}, where p is any prime, and thus local zeta functions serving as multiplicative building blocks of ζ have the same property [E8]. The story would be perfect if these algebraic numbers would span the algebraic closure of rationals.

   The symmetrized version of Riemann zeta defined as $\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$ satisfying the functional equation $\xi(s) = \xi(1-s)$ and having only the trivial zeros could appear as a building block of the rational function in question. The function

$$f(s) = \frac{\xi(s)}{\xi(s+1)} \times \frac{s-1}{s}$$

   has non-trivial zeros s_n of ζ as zeros and their negatives as $-s_n$ as poles. There are no other zeros since trivial zeros as well as the zeros at $s = 0$ and $s = 1$ are eliminated. Using Stirling formula one finds that $\xi(s)$ grows as s^s for real values of $s \to \infty$. The growths of the numerator and denominator compensate each other at this limit so that the function approaches constant equal to one for $Re(s) \to \infty$.

   If $f(s)$ indeed behaves as a rational function whose product expansion converges everywhere it can be expressed in terms of its zeros and poles as

$$\begin{aligned} f(s) &= \prod_{n>0} A_n(s) \ , \\ A_n &= \frac{(s-s_n)(s-\overline{s}_n)}{(1+s-s_n)(1+s-\overline{s}_n)} \ . \end{aligned} \tag{10.3.-1}$$

   The product expansion seems to converge for any finite value of s since the terms A_n approach unity for large values of $|s_n| = |1/2 + iy_n|$. $f(s)$ has $s_n = 1/2 + iy_n$ indeed has zeros and $s_n = -1/2 + iy_n$ as poles.

3. This proposal might of course be quite too simplistic. For instance, one might argue that the phase factors p^{iy} associated with the non-trivial zeros give only roots of unity multiplied by Gaussian integers. One can however imagine more complex functions obtained by forming products of $f(s)$ with its shifted variants $f(s + \Delta)$ with algebraic shift Δ in, say, the interval $[-1/2, 1/2]$. Some kind of limiting procedure using a product of this kind of functions might give the desired universal function.

10.3.2 Physical representations of Galois groups

It would be highly desirable to have concrete physical realizations for the action of finite Galois groups. TGD indeed provides two kinds of realizations of this kind. For both options there are good hopes about the unification of number theoretical and geometric Galois programs obtained by replacing permutations with braiding homotopies and by discretization of continuous situation to a finite number theoretic braids having finite Galois groups as automorphisms.

Number theoretical braids and the representations of finite Galois groups as outer automorphisms of braid group algebra

Number theoretical braids [E1, C1, C2] are in a central role in the formulation of quantum TGD based on general philosophical ideas which might apply to both physics and mathematical cognition and, one might hope, also to a good mathematics.

An attractive idea inspired by the notion of the number theoretical braid is that the symmetric group S_n might act on roots of a polynomial represented by the strands of braid and could thus be replaced by braid group.

The basic philosophy underlying quantum TGD is the notion of finite resolution, both the finite resolution of quantum measurement and finite cognitive resolution [C1, C2]. The basic implication is discretization at space-time level and finite-dimensionality of all mathematical structures which can be represented in the physical world. At space-time level the discretization means that the data involved with the definition of S-matrix comes from a subset of a discrete set of points in the intersection of real and p-adic variants of partonic 2-surface obeying same algebraic equations. Note that a finite number of braids could be enough to code for the information needed to reconstruct the entire partonic 2-surface if it is given by polynomial or rational function having coefficients as algebraic numbers. Entire configuration space of 3-surfaces would be discretized in this picture. Also the reduction of the infinite braid to a finite one would conform with the spontaneous symmetry breaking S_∞ to diagonally imbedded finite Galois group imbedded diagonally.

1. Two objections

Langlands correspondence assumes the existence of finite-dimensional representations of $Gal(\overline{Q}/Q)$. In the recent situation this encourages the idea that the restrictions of mathematical cognition allow to realize only the representations of $Gal(\overline{Q}/Q)$ reducing in some sense to representations for finite Galois groups. There are two counter arguments against the idea.

1. It is good to start from a simple abelian situation. The abelianization of $G(\overline{A}/Q)$ must give rise to multiplicative group of adeles defined as $\hat{Z} = \prod_p Z_p^\times$ where $Z_p^\times$ corresponds to the multiplicative group of invertible p-adic integers consisting of p-adic integers having p-adic norm equal to one. This group results as the inverse limit containing the information about subgroup inclusion hierarchies resulting as sequences $Z^\times/(1+pZ)^\times \subset Z^\times/(1+p^2Z)^\times \subset ..$ and expressed in terms factor groups of multiplicative group of invertible p-adic integers. Z_∞/A_∞ must give the group $\prod_p Z_p^\times$ as maximal abelian subgroup of Galois group. All smaller abelian subgroups of S_∞ would correspond to the products of subgroups of $\hat{Z}^\times$ coming as $Z_p^\times/(1+p^nZ)^\times$. Representations of finite cyclic Galois groups would be obtained by representing trivially the product of a commutator group with a subgroup of $\hat{Z}$. Thus one would obtain finite subgroups of the maximal abelian Galois group at the level of representations as effective Galois groups. The representations would be of course one-dimensional.

   One might hope that the representations of finite Galois groups could result by a reduction of the representations of S_∞ to $G = S_\infty/H$ where H is normal subgroup of S_∞. Schreier-Ulam theorem [43] however implies that the only normal subgroup of S_∞ is the alternating subgroup A_∞. Since the braid group B_∞ as a special case reduces to S_∞ there is no hope of obtaining finite-dimensional representations except abelian ones.

2. The identification of $Gal(\overline{Q}/Q) = S_\infty$ is not consistent with the finite-dimensionality in the case of complex representations. The irreducible unitary representations of S_n are in one-one correspondence with partitions of n objects. The direct numerical inspection based on the formula for the dimension of the irreducible representation of S_n in terms of Yang tableau [44] suggests that the partitions for which the number r of summands differs from $r = 1$ or $r = n$ (1-dimensional representations) quite generally have dimensions which are at least of order n. If d-dimensional representations corresponds to representations in $GL(d, C)$, this means that important representations correspond to dimensions $d \to \infty$ for S_∞.

Both these arguments would suggest that Langlands program is consistent with the identification $Gal(\overline{F}, F) = S_\infty$ only if the representations of $Gal(\overline{Q}, Q)$ reduce to those for finite Galois subgroups via some kind of symmetry breaking.

2. Diagonal imbedding of finite Galois group to S_∞ as a solution of problems

The idea is to imbed the Galois group acting as inner automorphisms diagonally to the m-fold Cartesian power of S_n imbedded to S_∞. The limit $m \to \infty$ gives rise to outer automorphic action since the resulting group would not be contained in S_∞. Physicist might prefer to speak about number theoretic symmetry breaking $Gal(\overline{Q}/Q) \to G$ implying that the representations are irreducible only in finite Galois subgroups of $Gal(\overline{Q}/Q)$. The action of finite Galois group G is indeed analogous to that of global gauge transformation group which belongs to the completion of the group of local gauge transformations. Note that G is necessarily finite.

About the detailed definition of number theoretic braids

The work with hyper-finite factors of type II_1 (HFFs) combined with experimental input led to the notion of hierarchy of Planck constants interpreted in terms of dark matter [A9]. The hierarchy is realized via a generalization of the notion of imbedding space obtained by gluing infinite number of its variants along common lower-dimensional quantum critical sub-manifolds. These variants of imbedding space are characterized by discrete subgroups of SU(2) acting in M^4 and CP_2 degrees of freedom as either symmetry groups or homotopy groups of covering. Among other things this picture implies a general model of fractional quantum Hall effect.

The identification of number theoretic braids

To specify number theoretical criticality one must specify some physically preferred coordinates for $M^4 \times CP_2$ or at least $\delta M^4_\pm \times CP_2$. Number theoretical criticality requires that braid belongs to the algebraic intersection of real and p-adic variants of the partonic 2-surface so that number theoretical criticality reduces to a finite number of conditions. This is however not strong enough condition and one must specify further physical conditions.

1. What are the preferred coordinates for H?

What are the preferred coordinates of M^4 and CP_2 in which algebraicity of the points is required is not completely clear. The isometries of these spaces must be involved in the identification as well as the choice of quantization axes for given CD. In [E4] I have discussed the natural preferred coordinates of M^4 and CP_2.

1. For M^4 linear M^4 coordinates chosen in such manner that $M^2 \times E^2$ decomposition fixing quantization axes is respected are very natural. This restricts the allowed Lorentz transformations to Lorentz boosts in M^2 and rotations in E^2 and the identification of M^2 as hyper-complex plane fixes time coordinate uniquely. E^2 coordinates are fixed apart from the action of $SO(2)$ rotation. The rationalization of trigonometric functions of angle variables allows angles associated with Pythagorean triangles as number theoretically simplest ones.

2. The case of CP_2 is not so easy. The most obvious guess in the case of CP_2 the coordinates corresponds to complex coordinates of CP_2 transforming linearly under $U(2)$. The condition that color isospin rotations act as phase multiplications fixes the complex coordinates uniquely. Also the complex coordinates transforming linearly under $SO(3)$ rotations are natural choice for S^2 ($r_M = constant$ sphere at $\delta M^4_\pm$).

3. Another manner to deal with CP_2 is to apply number M^8-H duality. In M^8 CP_2 corresponds to E^4 and the situation reduces to linear one and $SO(4)$ isometries help to fix preferred coordinate axis by decomposing E^4 as $E^4 = E^2 \times E^2$. Coordinates are fixed apart the action of the commuting $SO(2)$ sub-groups acting in the planes E^2. It is not clear whether the images of algebraic points of E^4 at space-time surface are mapped to algebraic points of CP_2.

2. The identification of number theoretic braids

It took some years to end up with a unique identification of number theoretic braids [A6, F12]. As a matter fact, there are several alternative identifications and it seems that all of them are needed. Consider first just braids without the attribute 'number theoretical'.

1. Braids can be identified as lifts of the projections of X^3_l to the quantum critical sub-manifolds M^2 or S^2_I, $i = I, II$, and in the generic case consist of 1-dimensional strands in X^3_l These sub-manifolds are obviously in the same role as the plane to which the braid is projected to obtain a braid diagram.

2. Braid points are always quantum critical against the change of Planck constant so that TQFT like theory characterizes the freedom remaining intact at quantum criticality. Quantum criticality in this sense need not have anything to do with the quantum criticality in the sense that the second variation of Kähler action vanishes -at least for the variations representing dynamical symmetries in the sense that only the inner product $\int(\partial L_D/\partial h^k_\alpha)\delta h^k d^4x$ (L_D denotes modified Dirac Lagrangian) without the vanishing of the integrand. This criticality leads to a generalization of the conceptual framework of Thom's catastrophe theory [A6].

3. It is not clear whether these three braids form some kind of trinity so that one of them is enough to formulate the theory or whether all of them are needed. Note also that one has quantum superposition over CDs corresponding to different choices of M^2 and the pair formed by S^2_I and S^2_{II} (note that the spheres are not independent if both appear). Quantum measurement however selects one of these choices since it defines the choice of quantization axes.

4. One can consider also more general definition. The extrema of Kähler magnetic field strength $\epsilon^{\alpha\beta}J_{\alpha\beta}$ at X^2 define in natural manner a discrete set of points defining the nodes of symplectic triangulation. This set of extremals is same for all deformations of X^3_l allowed in the functional integral over symplectic group although the positions of points change. For preferred symplectically invariant light-like coordinate of X^3_l braid results. Also now geodesic spheres and M^2 would define the counterpart of the plane to which the braids are projected.

Number theoretic braids would be braids which are number theoretically critical. This means that the points of braid in preferred coordinates are algebraic points so that they can be regarded as being shared by real partonic 2-surface and its p-adic counterpart obeying same algebraic equations. The phase transitions between number fields would mean leakage via these 2-surfaces playing the role of back of a book along which real and p-adic physics representing the pages of a book are glued together. The transformation of intention to action would represent basic example of this kind of leakage and number theoretic criticality could be decisive feature of living matter. For number theoretic braids at X^3_l whose real and p-adic variants obey same algebraic equations, only subset of algebraic points is common to real and p-adic pages of the book so that discretization of braid strand is unavoidable.

Representation of finite Galois groups as outer automorphism groups of HFFs

Any finite group G has a representation as outer automorphisms of a hyper-finite factor of type II_1 (briefly HFF in the sequel) and this automorphism defines sub-factor $\mathcal{N} \subset \mathcal{M}$ with a finite value of index $\mathcal{M} : \mathcal{N}$ [48]. Hence a promising idea is that finite Galois groups act as outer automorphisms of the associated hyper-finite factor of type II_1.

More precisely, sub-factors (containing Jones inclusions as a special case) $\mathcal{N} \subset \mathcal{M}$ are characterized by finite groups G acting on elements of $\mathcal{M}$ as outer automorphisms and leave the elements of $\mathcal{N}$ invariant whereas finite Galois group associated with the field extension K/L act as automorphisms of K and leave elements of L invariant. For finite groups the action as outer automorphisms is unique apart from a conjugation in von Neumann algebra. Hence the natural idea is that the finite subgroups of $Gal(\overline{Q}/Q)$ have outer automorphism action in group algebra of $Gal(\overline{Q}/Q)$ and that the hierarchies of inclusions provide a representation for the hierarchies of algebraic extensions. Amusingly, the notion of Jones inclusion was originally inspired by the analogy with field extensions [48]!

It must be emphasized that the groups defining sub-factors can be extremely general and can represent much more than number theoretical information understood in the narrow sense of the word. Even if one requires that the inclusion is determined by outer automorphism action of group G uniquely, one finds that any amenable, in particular compact [47], group defines a unique sub-factor by outer action [48]. It seems that practically any group works if uniqueness condition is given up.

The TGD inspired physical interpretation is that compact groups would serve as effective gauge groups defining measurement resolution by determining the measured quantum numbers. Hence the physical states differing by the action of $\mathcal{N}$ elements which are G singlets would not be indistinguishable from each other in the resolution used. The physical states would transform according to the finite-dimensional representations in the resolution defined by G.

The possibility of Lie groups as groups defining inclusions raises the question whether hyper-finite factors of type II_1 could mimic any gauge theory and one might think of interpreting gauge groups as Galois groups of the algebraic structure of this kind of theories. Also Kac-Moody algebras emerge naturally in this framework as will be discussed, and could also have an interpretation as Galois algebras for number theoretical dynamical systems obeying dynamics dictated by conformal field theory. The infinite hierarchy of infinite rationals in turn suggests a hierarchy of groups S_∞ so that even algebraic variants of Lie groups could be interpreted as Galois groups. These arguments would suggest that HFFs might be kind of Universal Math Machines able to mimic any respectable mathematical structure.

Number theoretic braids and unification of geometric and number theoretic Langlands programs

The notion of number theoretic braid has become central in the attempts to fuse real physics and p-adic physics to single coherent whole. Number theoretic braid leads to the discretization of quantum physics by replacing the stringy amplitudes defined over curves of partonic 2-surface with amplitudes involving only data coded by points of number theoretic braid. The discretization of quantum physics could have counterpart at the level of geometric Langlands program [22, 31], whose discrete version would correspond to number theoretic Galois groups associated with the points of number theoretic braid. The extension to braid group would mean that the global homotopic information is not lost.

1. Number theoretic braids belong to the intersection of real and p-adic partonic surface

The points of number theoretic braid belong to the intersection of the real and p-adic variant of partonic 2-surface consisting of rationals and algebraic points in the extension used for p-adic numbers. The points of braid have same projection on an algebraic point of the geodesic sphere of $S^2 \subset CP_2$ belonging to the algebraic extension of rationals considered (the reader willing to understand the details can consult [C1]).

The points of braid are obtained as solutions of polynomial equation and thus one can assign to them a Galois group permuting the points of the braid. In this case finite Galois group could be realized as left or right translation or conjugation in S_∞ or in braid group.

To make the notion of number theoretic braid more concrete, suppose that the complex coordinate w of $\delta M^4_\pm$ is expressible as a polynomial of the complex coordinate z of CP_2 geodesic sphere and the radial light-like coordinate r of $\delta M^4_\pm$ is obtained as a solution of polynomial equation $P(r, z, w) = 0$. By substituting w as a polynomial $w = Q(z, r)$ of z and r this gives polynomial equation $P(r, z, Q(z, r)) = 0$ for r for a given value of z. Only real roots can be accepted. Local Galois group (in a sense different as it is used normally in literature) associated with the algebraic point of S^2 defining the number theoretical braid is thus well defined.

If the partonic 2-surface involves all roots of an irreducible polynomial, one indeed obtains a braid for each point of the geodesic sphere $S^2 \subset CP_2$. In this case the action of Galois group is naturally a braid group action realized as the action on induced spinor fields and configuration space spinors.

The choice of the points of braid as points common to the real and p-adic partonic 2-surfaces would be unique so that the obstacle created by the fact that the finite Galois group as function of point of S^2 fluctuates wildly (when some roots become rational Galois group changes dramatically: the simplest example is provided by $y - x^2 = 0$ for which Galois group is Z_2 when y is not a square of rational and trivial group if y is rational).

2. Modified Dirac operator assigns to partonic 2-surface a unique prime p which could define l-adic representations of Galois group

The overall scaling of the eigenvalue spectrum of the modified Dirac operator assigns to the partonic surface a unique p-adic prime p which physically corresponds to the p-adic length scale which appears in the discrete coupling constant evolution [C1, C4]. One can solve the roots of the the resulting polynomial also in the p-adic number field associated with the partonic 2-surface by the modified Dirac equation and find the Galois group of the extension involved. The p-adic Galois group, known as local Galois group in literature, could be assigned to the p-adic variant of partonic surface and would have naturally l-adic representation, most naturally in the p-adic variant of the group algebra of S_∞ or B_∞ or equivalently in the p-adic variant of infinite-dimensional Clifford algebra. There are however physical reasons to believe that infinite-dimensional Clifford algebra does not depend on number field. Restriction to an algebraic number based group algebra therefore suggests itself. Hence, if one requires that the representations involve only algebraic numbers, these representation spaces might be regarded as equivalent.

3. Problems

There are however problems.

1. The triviality of the action of Galois group on the entire partonic 2-surface seems to destroy the hopes about genuine representations of Galois group.
2. For a given partonic 2-surface there are several number theoretic braids since there are several algebraic points of geodesic sphere S^2 at which braids are projected. What happens if the Galois groups are different? What Galois group should one choose?

A possible solution to both problems is to assign to each braid its own piece X^2_k of the partonic 2-surface X^2 such that the deformations X^2 can be non-trivial only in X^2_k. This means separation of modular degrees of freedom to those assignable to X^2_k and to "center of mass" modular degrees of freedom assignable to the boundaries between X^2_k. Only the piece X^2_k associated with the k^{th} braid would be affected non-trivially by the Galois group of braid. The modular invariance of the conformal field theory however requires that the entire quantum state is modular invariant under the modular group of X^2. The analog of color confinement would take place in modular degrees of freedom. Note that the region containing braid must contain single handle at least in order to allow representations of $SL(2,C)$ (or $Sp(2g,Z)$ for genus g).

As already explained, in the general case only the invariance under the subgroup $\Gamma_0(N)$ [29] of the modular group $SL(2,Z)$ can be assumed for automorphic representations of $GL(2,R)$ [24, 22, 20]. This is due to the fact that there is a finite set of primes (prime ideals in the algebra of integers), which are ramified [24]. Ramification means that their decomposition to a product of prime ideals of the algebraic extension of Q contains higher powers of these prime ideals: $p \rightarrow (\prod_k P_k)^e$ with $e > 1$. The congruence group is fixed by the integer $N = \prod_k p^{n_k}$ known as conductor coding the set of exceptional primes which are ramified.

The construction of modular forms in terms of representations of $SL(2,R)$ suggests that it is possible to replace $\Gamma_0(N)$ by the congruence subgroup $\Gamma(N)$, which is normal subgroup of $SL(2,R)$ so that $G_1 = SL(2,Z)/\Gamma$ is group. This would allow to assign to individual braid regions carrying single handle well-defined G_1 quantum numbers in such a manner that entire state would be G_1 singlet.

Physically this means that the separate regions of the partonic 2-surface each containing one braid strand cannot correspond to quantum states with full modular invariance. Elementary particle vacuum functionals [F1] defined in the moduli space of conformal equivalence classes of partonic 2-surface must however be modular invariant, and the analog of color confinement in modular degrees of freedom would take place.

Hierarchy of Planck constants and dark matter and generalization of imbedding space

Second hierarchy of candidates for Galois groups is based on the generalization of the notion of the imbedding space $H = M^4 \times CP_2$, or rather the spaces $H_\pm = M^4_\pm \times CP_2$ defining future and past light-cones inside H [A9]. This generalization is inspired by the quantization of Planck constant explaining dark matter as a hierarchy of macroscopically quantum coherent phases and by the requirement that sub-factors have a geometric representation at the level of the imbedding space and space-time (quantum-classical correspondence).

Galois groups could also correspond to finite groups $G_a \times G_b \subset SU(2) \times SU(2) \subset$SL(2,C)$\times SU(3)$. These groups act as covering symmetries for the sectors of the imbedding space, which can be regarded as singular $H_\pm = M^4_\pm \times CP_2 \rightarrow H_\pm/G_a \times G_b$ bundles containing orbifold points (fixed points of $G_a \times G_b$ or either of them. The copies of H with same G_a or G_b are glued together along $M^4_\pm$ or CP_2 factor and along common orbifold points left fixed by G_b or G_a. The group $G_a \times G_b$ plays both the role of both Galois group and homotopy group.

There are good reasons to expect that both these Galois groups and those associated with number theoretic braids play a profound role in quantum TGD based description of dark matter as macroscopically quantum coherent phases. For instance, G_a would appear as symmetry group of dark matter part of bio-molecules in TGD inspired biology [18].

Question about representations of finite groups

John Baez made an interesting question in n-Category-Cafe [57]. The question reads as follows:

Is every representation of every finite group definable on the field Q^{ab} obtained by taking the field Q of rational numbers and by adding all possible roots of unity?

Since every finite group can appear as Galois group the question translates to the question whether one can represent all possible Galois groups using matrices with elements in Q^{ab}.

This form of question has an interesting relation to Langlands program. By Langlands conjecture the representations of the Galois group of algebraic closure of rationals can be realized in the space of functions defined in $GL(n,F)\backslash GL(n,Gal(Q^{ab}/Q))$, where $Gal(Q^{ab}/Q)$ is the maximal Abelian subgroup of the Galois group of the algebraic closure of rationals. Thus one has group algebra associated with the matrix group for which matrix elements have values in $Gal(Q^{ab}/Q)$. Something by several orders of more complex than matrices having values in Q^{ab}.

Suppose that Galois group of algebraic numbers can be regarded as the permutation group S_∞ of infinite number of objects generated by permutations for finite numbers of objects and that its physically interesting representations reduce to the representations of finite Galois groups G with element $g \in G$ represented as infinite product $g \times g \times ...$ belonging to the completion of S_∞ and thus to the completion of its group algebra identifiable as hyper-finite factor of type II_1. This would mean number theoretic local gauge invariance in the sense that all elements of S_∞ would leave physical states invariant whereas G would correspond to global gauge transformations. These tensor factors would have as space-time correlates number theoretical braids allowing to represent the action of G.

What this has then to do with John's question and Langlands program? S_∞ contains any finite group G as a subgroup. If all the representations of finite-dimensional Galois groups could be realized as representations in $Gl(n,Q^{ab})$, same would hold true also for the proposed symmetry breaking representations of the completion of S_∞ reducing to the representations of finite Galois groups. There would be an obvious analogy with Langlands program using functions defined in the space $Gl(n,Q)\backslash Gl(n,Gal(Q^{ab}/Q))$. Be as it may, mathematicians are able to work with incredibly abstract objects! A highly respectful sigh is in order!

10.3.3 What could be the TGD counterpart for the automorphic representations?

The key question in the following is whether quantum TGD could act as a general math machine allowing to realize any finite-dimensional manifold and corresponding function space in terms of configuration space spinor fields and whether also braided representations of Galois groups accompanying the braiding could be associated naturally with this kind of representations.

Some general remarks

Before getting to the basic idea some general remarks are in order.

1. Configuration space spinor fields would certainly transform according to a finite-dimensional and therefore non-unitary representation of $SL(2,C)$ which is certainly the most natural group involved and should relate to the fact that Galois groups representable as subgroups of SU(2) acting as rotations of 3-dimensional space correspond to sub-factors with $\mathcal{M}:\mathcal{N}\leq 4$.

2. Also larger Lie groups can be considered and diagonal imbeddings of Galois groups would be naturally accompanied by diagonal imbeddings of compact and also non-compact groups acting on the decomposition of infinite-dimensional Clifford algebra Cl_∞ to an infinite tensor power of finite-dimensional sub-Clifford algebra of form $M(2,C)^n$.

3. The basic difference between Galois group representation and corresponding Lie group representations is that the automorphisms in the case of discrete groups are automorphisms of S_∞ or B_∞ whereas for Lie groups the automorphisms are in general automorphisms of group algebra of S_∞ or B_∞. This could allow to understand the correspondence between discrete groups and Lie groups naturally.

4. Unitary automorphic representations are infinite-dimensional and require group algebra of $GL(n,F)$. Therefore configuration space spinors - to be distinguished from configuration space spinor fields- cannot realize them. Configuration space spinor field might allow the realization of these infinite-dimensional representations if groups themselves allow a finite-dimensional geometric realization of groups. Are this kind of realizations possible? This is the key question.

Could TGD Universe act as a universal math machine?

The questions are following. Could one find a representations of both Lie groups and their linear and non-linear representation spaces -and even more - of any manifold representable as a sub-manifold of some linear space in terms of braid points at partonic 2-surfaces X^2? What about various kinds of projective spaces and coset spaces? Can one construct representations of corresponding function spaces in terms of configuration space spinor fields? Can one build representations of parameter groups of Lie groups as braided representations defined by the orbits of braid points in X_l^3? Note that this would assign to the representations of closed paths in the group manifold a representation of braid group and Galois group of the braid and might make it easier to understand the Langlands correspondence.

A professional mathematician - if she still continues reading - might regard the following argument as rather pathetic poor man's argument but I want to be honest and demonstrate my stupidity openly.

1. The n braid points represent points of $\delta H = \delta M^4_\pm \times CP_2$ so that braid points represent a point of $7n$-dimensional space $\delta H^n/S_n$. $\delta M^4_\pm$ corresponds to E^3 with origin removed but $E^{2n}/S_n = C^n/S_n$ can be represented as a sub-manifold of $\delta M^4_\pm$. This allows to almost-represent both real and complex linear spaces. E^2 has a unique identification based on $M^4 = M^2 \times E_2$ decomposition required by the choice of quantization axis. One can also represent the spaces $(CP_2)^n/S_n$ in this manner.

2. The first - and really serious - problem is caused by the identification of the points obtained by permuting the n coordinates: this is of course what makes possible the braiding since braid group is the fundamental group of $(X^2)^n$. Could the quantum numbers at the braid points act as markers distinguishing between them so that one would effectively have E^{2n}? Could the fact that the representing points are those of imbedding space rather than X^2 be of significance? Second - less serious - problem is that the finite size of CD allows to represent only a finite region of E^2. On the other hand, ideal mathematician is a non-existing species and even non-ideal mathematician can imagine the limit at which the size of CD becomes infinite.

3. Matrix groups can be represented as sub-manifolds of linear spaces defined by the general linear group $Gl(n,R)$ and $Gl(n,C)$. In the p-adic pages of the imbedding space one can realize also the p-adic variants of general linear groups. Hence it is possible to imbed any real (complex) Lie group to E^{2n} (C^n), if n is chosen large enough.

4. Configuration space spinor fields restricted to the linear representations spaces or to the group itself represented in this manner would allow to realize as a special case various function spaces, in particular groups algebras. If configuration space spinor fields satisfy additional symmetries, projective spaces and various coset spaces can be realized as effective spaces. For instance CP_2 could be realized effectively as $SU(3)/U(2)$ by requiring $U(2)$ invariance of the configuration space spinor fields in $SU(3)$ or as C^3/Z by requiring that configuration space spinor field is scale invariant. Projective spaces might be also realized more concretely as imbeddings to $(CP_2)^n$.

5. The action of group element $g = exp(Xt)$ belonging to a one-parameter sub-group of a non-compact linear group in a real (complex) linear representation space of dimension m could be realized in a subspace of E^{2n}, $m < 2n$ (C^n, $m \leq n$), as a flow in X_l^3 taking the initial configuration of points of representation space to the final configuration. Braid strands - the orbits of points p_i defining the point p of the representation manifold under the action of one-parameter subgroup- would correspond to the points $exp(Xu)(p)$, $0 \leq u \leq t$. Similar representation would work also in the group itself represented in a similar manner.

6. Braiding in X_l^3 would induce a braided representation for the action of the one parameter subgroup. This representation is not quite the same thing as the automorphic representation since braiding is involved. Also trivial braid group representation is possible if the representation can be selected freely rather than being determined by the transformation properties of fermionic oscillator operator basis in the braiding.

7. An important prerequisite for math machine property is that the wave function in the space of light-like 3-surfaces with fixed ends can be chosen freely. This is the case since the degrees of freedom associate with the interior of light-like 3-surface X_l^3 correspond to zero modes assignable

to Kac-Moody symmetries [B2, E1]. Dicretization seems however necessary since functional integral in these degrees of freedom is not-well defined even in the real sense and even less so p-adically. This conforms with the fact that real world mathematical representations are always discrete. Quantum classical correspondence suggests the dynamics represented by X^3_l correlates with the quantum numbers assigned with X^2 so that Boolean statements represented in terms of Fermionic Fock states would be in one-one correspondence with these wave functions.

Besides representing mathematical structures this kind of math machine would be able to perform mathematical deductions. The fermionic part of the state zero energy state could be interpreted as a quantum super-position of Boolean statement $A_i \rightarrow B_i$ representing various instances of the general rule $A \rightarrow B$. Only the statements consistent with fundamental conservation laws would be possible. Quantum measurements performed for both positive and negative energy parts of the state would produce statements. Performing the measurement of the observable $O(A \rightarrow B)$ would produce from a given state a zero energy state representing statement $A \rightarrow B$. If the measurement of observable $O(C \rightarrow D)$ affects this state then the statement $(A \rightarrow B) \rightarrow (C \rightarrow D)$ cannot hold true. For $A = B$ the situation reduces to simpler logic where one tests truth value of statements of form $A \rightarrow B$. By increasing the number of instances in the quantum states generalizations of the rule can be tested.

10.3.4 Super-conformal invariance, modular invariance, and Langlands program

The geometric Langlands program [22, 23] deals with function fields, in particular the field of complex rational analytic functions on 2-dimensional surfaces. The sheaves in the moduli spaces of conformal blocks characterizing the n-point functions of conformal field theory replaces automorphic functions coding both arithmetic data and characterizing the modular representations of $GL(n)$ in number theoretic Langlands program [22]. These moduli spaces are labelled both by moduli characterizing the conformal equivalence class of 2-surface, in particular the positions of punctures, in TGD framework the positions of strands of number theoretic braids, as well as the moduli related to the Kac-Moody group involved.

Transition to function fields in TGD framework

According to [22] conformal field theories provide a very promising framework for understanding geometric Langlands correspondence.

1. That the function fields on 2-D complex surfaces would be in a completely unique role mathematically fits nicely with the 2-dimensionality of partons and well-defined stringy character of anticommutation relations for induced spinor fields. According to [22] there are not even conjectures about higher dimensional function fields.

2. There are very direct connections between hyper-finite factors of type II_1 and topological QFTs [36, 35], and conformal field theories. For instance, according to the review article [48] Ocneacnu has show that Jones inclusions correspond in one-one manner to topological quantum field theories and TGD can indeed be regarded as almost topological quantum field theory (metric is brought in by the light-likeness of partonic 3-surfaces). Furthermore, Connes has shown that the decomposition of the hierarchies of tensor powers $\mathcal{M} \otimes_{\mathcal{N}} .... \otimes_{\mathcal{N}} M$ as left and right modules to representations of lower tensor powers directly to fusion rules expressible in terms of 4-point functions of conformal field theories [48].

In TGD framework the transition from number fields to function fields would not be very dramatic.

1. Suppose that the representations of $SL(n, R)$ occurring in number theoretic Langlands program can indeed be realized in the moduli space for conformal equivalence classes of partonic 2-surface (or, by previous arguments, moduli space for regions of them with fixed boundaries). This means that representations of local Galois groups associated with number theoretic braids would involve global data about entire partonic 2-surface. This is physically very important since it otherwise discretization would lead to a loss of the information about dimension of partonic 2-surfaces.

2. In the case of geometric Langlands program this moduli space would be extended to the moduli space for n-point functions of conformal field theory defined at these 2-surfaces containing the original moduli space as a subspace. Of course, the extension could be present also in the number theoretic case. Thus it seems that number theoretic and geometric Langlands programs would utilize basic structures and would differ only in the sense that single braid would be replaced by several braids in the geometric case.

3. In TGD Kac-Moody algebras would be also present as well as the so called super-canonical algebra [C1] related to the isometries of "the world of classical worlds" (the space of light-like 3-surfaces) with generators transforming according to the irreducible representations of rotation group $SO(3)$ and color group $SU(3)$. It must be emphasized that TGD view about conformal symmetry generalizes that of string models since light-like 3-surfaces (orbits of partons) are the basic dynamical objects [C1].

What about more general reductive groups?

Langlands correspondence is conjectured to apply to all reductive Lie groups. The question is whether there is room for them in TGD Universe. There are good hopes.

1. Pairs formed by finite Galois groups and Lie groups containing them and defining sub-factors

Any amenable (in particular compact Lie) group acting as outer automorphism of $\mathcal{M}$ defines a unique sub-factor $\mathcal{N} \subset \mathcal{M}$ as a group leaving the elements of $\mathcal{N}$ invariant. The representations of discrete subgroups of compact groups extended to representations of the latter would define natural candidates for Langlands correspondence and would expand the repertoire of the Galois groups representable in terms of unique factors. If one gives up the uniqueness condition for the sub-factor, one can expect that almost any Lie group can define a sub-factor.

2. McKay correspondences and Langlands correspondence

The so called McKay correspondence assigns to the finite subgroups of SU(2) extended Dynkin diagrams of ADE type Kac-Moody algebras. McKay correspondence also generalizes to the discrete subgroups of other compact Lie groups [52]. The obvious question is how closely this correspondence between finite groups and Lie groups relates with Langlands correspondence.

The principal graphs representing concisely the fusion rules for Connes tensor products of $\mathcal{M}$ regarded as $\mathcal{N}$ bi-module are represented by the Dynkin diagrams of ADE type Lie groups for $\mathcal{M} : \mathcal{N} < 4$ (not all of them appear). For index $\mathcal{M} : \mathcal{N} = 4$ extended ADE type Dynkin diagrams labelling Kac-Moody algebras are assigned with these representations.

I have proposed that TGD Universe is able to emulate almost any ADE type gauge theory and conformal field theory involving ADE type Kac-Moody symmetry and represented somewhat misty ideas about how to construct representations of ADE type gauge groups and Kac-Moody groups using many particle states at the sheets of multiple coverings $H \rightarrow H/G_a \times G_b$ realizing the idea about hierarchy of dark matters already mentioned. Also vertex operator construction also distinguishes ADE type Kac-Moody algebras in a special position.

It is possible to considerably refine this conjecture picture by starting from the observation that the set of generating elements for Lie algebra corresponds to a union of triplets $\{J_i^{\pm}, J_i^3\}$, $i = 1, ..., n$ generating SU(2) sub-algebras. Here n is the dimension of the Cartan sub-algebra. The non-commutativity of quantum Clifford algebra suggests that Connes tensor product can induce deformations of algebraic structures so that ADE Lie algebra could result as a kind of deformation of a direct sum of commuting SU(2) Lie (Kac-Moody) algebras associated with a Connes tensor product. The physical interpretation might in terms of a formation of a bound state. The finite depth of $\mathcal{N}$ would mean that this mechanism leads to ADE Lie algebra for an n-fold tensor power, which then becomes a repetitive structure in tensor powers. The repetitive structure would conform with the diagonal imbedding of Galois groups giving rise to a representation in terms of outer automorphisms.

This picture encourages the guess that it is possible to represent the action of Galois groups on number theoretic braids as action of subgroups of dynamically generated ADE type groups on configuration space spinors. The connection between the representations of finite groups and reductive Lie groups would result from the natural extension of the representations of finite groups to those of Lie groups.

*3. What about Langlands correspondence for Kac-Moody groups?*vm

The appearance of also Kac-Moody algebras raises the question whether Langlands correspondence could generalize also to the level of Kac-Moody groups or algebras and whether it could be easier to understand the Langlands correspondence for function fields in terms of Kac-Moody groups as the transition from global to local occurring in both cases suggests.

Could Langlands duality for groups reduce to super-symmetry?

Langlands program involves dualities and the general structure of TGD suggests that there is a wide spectrum of these dualities.

1. A very fundamental duality would be between infinite-dimensional Clifford algebra and group algebra of S_∞ or of braid group B_∞. For instance, one can ask could it be possible to map this group algebra to the union of the moduli spaces of conformal equivalence classes of partonic 2-surfaces. HFFs consists of bounded operators of a separable Hilbert space. Therefore they are expected to have very many avatars: for instance there is an infinite number sub-factors isomorphic to the factor. This seems to mean infinite number of manners to represent Galois groups reflected as dualities.

2. Langlands program involves the duality between reducible Lie groups G and its Langlands dual having dual root lattices. The interpretation for this duality in terms of electric-magnetic duality is suggested by Witten [31]. TGD suggests an alternative interpretation. The super symmetry aspect of super-conformal symmetry suggests that bosonic and fermionic representations of Galois groups could be very closely related. In particular, the representations in terms of configuration space spinors and in terms of modular degrees of freedom of partonic 2-surface could be in some sense dual to each other. Rotation groups have a natural action on configuration space spinors whereas symplectic groups have a natural action in the moduli spaces of partonic 2-surfaces of given genus possessing symplectic and Kähler structure. Langlands correspondence indeed relates $SO(2g+1,R)$ realized as rotations of configuration space spinors and $Sp(2g,C)$ realized as transformations in modular degrees of freedom. Hence one might indeed wonder whether super-symmetry could be behind the Langlands correspondence.

10.3.5 What is the role of infinite primes?

Infinite primes primes at the lowest level of the hierarchy can be represented as polynomials and as rational functions at higher levels. These in turn define rational function fields. Physical states correspond in general to infinite rationals which reduce to unit in real sense but have arbitrarily complex number theoretical anatomy [E3, 16, 19].

Does infinite prime characterize the l-adic representation of Galois group associated with given partonic 2-surface

Consider first the lowest level of hierarchy of infinite primes [E3]. Infinite primes at the lowest level of hierarchy are in a well-defined sense composites of finite primes and correspond to states of super-symmetric arithmetic quantum field theory. The physical interpretation of primes appearing as composites of infinite prime is as characterizing of the p-adic prime p assigned by the modified Dirac action to partonic 2-surfaces associated with a given 3-surface [A6, C1].

This p-adic prime could naturally correspond to the possible prime associated with so called l-adic representations of the Galois group(s) associated with the p-adic counterpart of the partonic 2-surface. Also the Galois groups associated with the real partonic 2-surface could be represented in this manner. The generalization of moduli space of conformal equivalence classes must be generalized to its p-adic variant. I have proposed this generalization in context of p-adic mass calculations [F1].

It should be possible to identify configuration space spinors associated with real and p-adic sectors if anti-commutations relations for the fermionic oscillator operators make sense in any number field (that is involve only rational or algebraic numbers). Physically this seems to be the only sensible option.

Could one assign Galois groups to the extensions of infinite rationals?

A natural question is whether one could generalize the intuitions from finite number theory to the level of infinite primes, integers, and rationals and construct Galois groups and there representations for them. This might allow alternative very number theoretical approach to the geometric Langlands duality.

1. The notion of infinite prime suggests that there is entire hierarchy of infinite permutation groups such that the N_∞ at given level is defined as the product of all infinite integers at that level. Any group is a permutation group in formal sense. Could this mean that the hierarchy of infinite primes could allow to interpret the infinite algebraic sub-groups of Lie groups as Galois groups? If so one would have a unification of group theory and number theory.

2. An interesting question concerns the interpretation of the counterpart of hyper-finite factors of type II_1 at the the higher levels of hierarchy of infinite primes. Could they relate to a hierarchy of local algebras defined by HFF? Could these local algebras be interpreted in terms of direct integrals of HFFs so that nothing essentially new would result from von Neumann algebra point of view? Would this be a correlate for the fact that finite primes would be the irreducible building block of all infinite primes at the higher levels of the hierarchy?

3. The transition from number fields to function fields is very much analogous to the replacement of group with a local gauge group or algebra with local algebra. I have proposed that this kind of local variant based on multiplication by of HFF by hyper-octonion algebra could be the fundamental algebraic structure from which quantum TGD emerges. The connection with infinite primes would suggest that there is infinite hierarchy of localizations corresponding to the hierarchy of space-time sheets.

4. Perhaps it is worth of mentioning that the order of S_∞ is formally $N_\infty = \lim_{n\to\infty} n!$. This integer is very large in real sense but zero in p-adic sense for all primes. Interestingly, the numbers $N_\infty/n + n$ behave like normal integers in p-adic sense and also number theoretically whereas the numbers $N_\infty/n + 1$ behave as primes for all values of n. Could this have some deeper meaning?

Could infinite rationals allow representations of Galois groups?

One can also ask whether infinite primes could provide representations for Galois groups. For instance, the decomposition of infinite prime to primes (or prime ideals) assignable to the extension of rationals is expected to make sense and would have clear physical interpretation. Also (hyper-)quaternionic and (hyper-)octonionic primes can be considered and I have proposed explicit number theoretic interpretation of the symmetries of standard model in terms of these primes. The decomposition of partonic primes to hyper-octonionic primes could relate to the decomposition of parton to regions, one for each number theoretic braid.

There are arguments supporting the view that infinite primes label the ground states of super-conformal representations [C1, E3]. The question is whether infinite primes could allow to realize the action of Galois groups. Rationality of infinite primes would imply that the invariance of ground states of super-conformal representations under the braid realization of $Gal(\overline{Q}/Q)$ of finite Galois groups. The infinite prime as a whole could indeed be invariant but the primes in the decomposition to a product of primes in algebraic extension of rationals need not be so. This kind of decompositions of infinite prime characterizing parton could correspond to the above described decomposition of partonic 2-surface to regions X^2_k at which Galois groups act non-trivially. It could also be that only infinite integers are rational whereas the infinite primes decomposing them are hyper-octonionic. This would physically correspond to the decomposition of color singlet hadron to colored partons [E3].

10.3.6 Could Langlands correspondence, McKay correspondence and Jones inclusions relate to each other?

The understanding of Langlands correspondence for general reductive Lie groups in TGD framework seems to require some physical mechanism allowing the emergence of these groups in TGD based physics. The physical idea would be that quantum dynamics of TGD is able to emulate the dynamics of any gauge theory or even stringy dynamics of conformal field theory having Kac-Moody type symmetry and that this emulation relies on quantum deformations induced by finite measurement resolution described in terms of Jones inclusions of sub-factors characterized by group G leaving elements of sub-factor invariant. Finite measurement resolution would would result simply from the fact that only quantum numbers defined by the Cartan algebra of G are measured.

There are good reasons to expect that infinite Clifford algebra has the capacity needed to realize representations of an arbitrary Lie group. It is indeed known that that any quantum group characterized by quantum parameter which is root of unity or positive real number can be assigned to Jones inclusion [48]. For $q = 1$ this would gives ordinary Lie groups. In fact, all amenable groups define unique sub-factor and compact Lie groups are amenable ones.

It was so called McKay correspondence [52] which originally stimulated the idea about TGD as an analog of Universal Turing machine able to mimic both ADE type gauge theories and theories with ADE type Kac-Moody symmetry algebra. This correspondence and its generalization might also provide understanding about how general reductive groups emerge. In the following I try to cheat the reader to believe that the tensor product of representations of SU(2) Lie algebras for Connes tensor powers of $\mathcal{M}$ could induce ADE type Lie algebras as quantum deformations for the direct sum of n copies of $SU(2)$ algebras This argument generalizes also to the case of other compact Lie groups.

About McKay correspondence

McKay correspondence [52] relates discrete finite subgroups of $SU(2)$ ADE groups. A simple description of the correspondences is as follows [52].

1. Consider the irreps of a discrete subgroup $G \subset SU(2)$ which correspond to irreps of G and can be obtained by restricting irreducible representations of $SU(2)$ to those of G. The irreducible representations of $SU(2)$ define the nodes of the graph.

2. Define the lines of graph by forming a tensor product of any of the representations appearing in the diagram with a doublet representation which is always present unless the subgroup is 2-element group. The tensor product regarded as that for $SU(2)$ representations gives representations $j - 1/2$, and $j + 1/2$ which one can decompose to irreducibles of G so that a branching of the graph can occur. Only branching to two branches occurs for subgroups yielding extended ADE diagrams. For the linear portions of the diagram the spins of corresponding SU(2) representations increase linearly as $.., j, j + 1/2, j + 1, ...$

   One obtains extended Dynkin diagrams of ADE series representing also Kac-Moody algebras giving A_n, D_n, E_6, E_7, E_8. Also A_∞ and $A_{-\infty,\infty}$ are obtained in case that subgroups are infinite. The Dynkin diagrams of non-simply laced groups B_n ($SO(2n+1)$), C_n (symplectic group $Sp(2n)$ and quaternionic group $Sp(n)$), and exceptional groups G_2 and F_4 are not obtained.

ADE Dynkin diagrams labelling Lie groups instead of Kac-Moody algebras and having one node less, do not appear in this context but appear in the classification of Jones inclusions for $\mathcal{M} : \mathcal{N} < 4$. As a matter fact, ADE type Dynkin diagrams appear in very many contexts as one can learn from John Baez's This Week's Finds [53].

1. The classification of integral lattices in Rn having a basis of vectors whose length squared equals 2

2. The classification of simply laced semisimple Lie groups.

3. The classification of finite sub-groups of the 3-dimensional rotation group.

4. The classification of simple singularities . In TGD framework these singularities could be assigned to origin for orbifold CP_2/G, $G \subset SU(2)$.

5. The classification of tame quivers.

Principal graphs for Connes tensor powers $\mathcal{M}$

The thought provoking findings are following.

1. The so called principal graphs characterizing $\mathcal{M} : \mathcal{N} = 4$ Jones inclusions for $G = SU(2)$ are extended Dynkin diagrams characterizing ADE type affine (Kac-Moody) algebras. D_n is possible only for $n \geq 4$.

2. $\mathcal{M} : \mathcal{N} < 4$ Jones inclusions correspond to ordinary ADE type diagrams for a subset of simply laced Lie groups (all roots have same length) A_n $(SU(n))$, D_{2n} $(SO(2n))$, and E_6 and E_8. Thus D_{2n+1} $(SO(2n+2))$ and E_7 are not allowed. For instance, for $G = S_3$ the principal graph is not D_3 Dynkin diagram.

The conceptual background behind principal diagram is necessary if one wants to understand the relationship with McKay correspondence.

1. The hierarchy of higher commutations defines an invariant of Jones inclusion $\mathcal{N} \subset \mathcal{M}$. Denoting by $\mathcal{N}'$ the commutant of $\mathcal{N}$ one has sequences of horizontal inclusions defined as $C = \mathcal{N}' \cap \mathcal{N} \subset \mathcal{N}' \cap \mathcal{M} \subset \mathcal{N}' \cap \mathcal{M}^1 \subset ...$ and $C = \mathcal{M}' \cap \mathcal{M} \subset \mathcal{M}' \cap \mathcal{M}^1 \subset ....$ There is also a sequence of vertical inclusions $\mathcal{M}' \cap \mathcal{M}^k \subset \mathcal{N}' \cap \mathcal{M}^k$. This hierarchy defines a hierarchy of Temperley-Lieb algebras [50] assignable to a finite hierarchy of braids. The commutants in the hierarchy are direct sums of finite-dimensional matrix algebras (irreducible representations) and the inclusion hierarchy can be described in terms of decomposition of irreps of k^{th} level to irreps of $(k-1)^{th}$ level irreps. These decomposition can be described in terms of Bratteli diagrams [52, 51].

2. The information provided by infinite Bratteli diagram can be coded by a much simpler bi-partite diagram having a preferred vertex. For instance, the number of $2k$-loops starting from it tells the dimension of k^{th} level algebra. This diagram is known as principal graph.

Principal graph emerges also as a concise description of the fusion rules for Connes tensor powers of $\mathcal{M}$.

1. It is natural to decompose the Connes tensor powers [52] $\mathcal{M}_k = \mathcal{M} \otimes_{\mathcal{N}} ... \otimes_{\mathcal{N}} \mathcal{M}$ to irreducible $\mathcal{M}-\mathcal{M}$, $\mathcal{N}-\mathcal{M}$, $\mathcal{M}-\mathcal{N}$, or $\mathcal{N}-\mathcal{N}$ bi-modules. If $\mathcal{M} : \mathcal{N}$ is finite this decomposition involves only finite number of terms. The graphical representation of these decompositions gives rise to Bratteli diagram.

2. If $\mathcal{N}$ has finite depth the information provided by Bratteli diagram can be represented in nutshell using principal graph. The edges of this bipartite graph connect $\mathcal{M}-\mathcal{N}$ vertices to vertices describing irreducible $\mathcal{N}-\mathcal{N}$ representations resulting in the decomposition of $\mathcal{M}-\mathcal{N}$ irreducibles. If this graph is finite, $\mathcal{N}$ is said to have finite depth.

A mechanism assigning to tensor powers Jones inclusions ADE type gauge groups and Kac-Moody algebras

The proposal made for the first time in [A9] is that in $\mathcal{M} : \mathcal{N} < 4$ case it is possible to construct ADE representations of gauge groups or quantum groups and in $\mathcal{M} : \mathcal{N} = 4$ using the additional degeneracy of states implied by the multiple-sheeted cover $H \rightarrow H/G_a \times G_b$ associated with space-time correlates of Jones inclusions. Either G_a or G_b would correspond to G. In the following this mechanism is articulated in a more refined manner by utilizing the general properties of generators of Lie-algebras understood now as a minimal set of elements of algebra from which the entire algebra can be obtained by repeated commutation operator (I have often used " Lie algebra generator" as an synonym for "Lie algebra element"). This set is finite also for Kac-Moody algebras.

1. Two observations

The explanation to be discussed relies on two observations.

1. McKay correspondence for subgroups of G $(\mathcal{M} : \mathcal{N} = 4)$ *resp.* its variants $(\mathcal{M} : \mathcal{N} < 4)$ and its counterpart for Jones inclusions means that finite-dimensional irreducible representations of allowed $G \subset SU(2)$ label both the Cartan algebra generators and the Lie (Kac-Moody) algebra generators of t_+ and t_- in the decomposition $g = h \oplus t_+ \oplus t_-$, where h is the Lie algebra of maximal compact subgroup.

2. Second observation is related to the generators of Lie-algebras and their quantum counterparts (see Appendix for the explicit formulas for the generators of various algebras considered). The observation is that each Cartan algebra generator of Lie- and quantum group algebras, corresponds to a triplet of generators defining an SU(2) sub-algebra. The Cartan algebra of affine algebra contains besides Lie group Cartan algebra also a derivation d identifiable as an infinitesimal scaling operator L_0 measuring the conformal weight of the Kac-Moody generators. d is exceptional in that it does not give rise to a triplet. It corresponds to the preferred node added to the Dynkin diagram to get the extended Dynkin diagram.

2. Is ADE algebra generated as a quantum deformation of tensor powers of $SU(2)$ Lie algebras representations?

The ADE type symmetry groups could result as an effect of finite quantum resolution described by inclusions of HFFs in TGD inspired quantum measurement theory.

1. The description of finite resolution typically leads to quantization since complex rays of state space are replaced as $\mathcal{N}$ rays. Hence operators, which would commute for an ideal resolution cease to do so. Therefore the algebra $SU(2)\otimes ... \otimes SU(2)$ characterized by n mutually commuting triplets, where n is the number of copies of $SU(2)$ algebra in the original situation and identifiable as quantum algebra appearing in $\mathcal{M}$ tensor powers with $\mathcal{M}$ interpreted as $\mathcal{N}$ module, could suffer quantum deformation to a simple Lie algebra with $3n$ Cartan algebra generators. Also a deformation to a quantum group could occur as a consequence.

2. This argument makes sense also for discrete groups $G \subset SU(2)$ since the representations of G realized in terms of configuration space spinors extend to the representations of SU(2) naturally.

3. Arbitrarily high tensor powers of $\mathcal{M}$ are possible and one can wonder why only finite-dimensional Lie algebra results. The fact that $\mathcal{N}$ has finite depth as a sub-factor means that the tensor products in tensor powers of $\mathcal{N}$ are representable by a finite Dynkin diagram. Finite depth could thus mean that there is a periodicity involved: the kn tensor powers decomposes to representations of a Lie algebra with $3n$ Cartan algebra generators. Thus the additional requirement would be that the number of tensor powers of $\mathcal{M}$ is multiple of n.

3. Space-time correlate for the tensor powers $\mathcal{M} \otimes_{\mathcal{N}} ... \otimes_{\mathcal{N}} \mathcal{M}$

By quantum classical correspondence there should exist space-time correlate for the formation of tensor powers of $\mathcal{M}$ regarded as $\mathcal{N}$ module. A concrete space-time realization for this kind of situation in TGD would be based on n-fold cyclic covering of H implied by the $H \rightarrow H/G_a \times G_b$ bundle structure in the case of say G_b. The sheets of the cyclic covering would correspond to various factors in the n-fold tensor power of $SU(2)$ and one would obtain a Lie algebra, affine algebra or its quantum counterpart with n Cartan algebra generators in the process naturally. The number n for space-time sheets would be also a space-time correlate for the finite depth of $\mathcal{N}$ as a factor.

Configuration space spinors could provide fermionic representations of $G \subset SU(2)$. The Dynkin diagram characterizing tensor products of representations of $G \subset SU(2)$ with doublet representation suggests that tensor products of doublet representations associated with n sheets of the covering could realize the Dynkin diagram.

Singlet representation in the Dynkin diagram associated with irreps of G would not give rise to an SU(2) sub-algebra in ADE Lie algebra and would correspond to the scaling generator. For ordinary Dynkin diagram representing gauge group algebra scaling operator would be absent and therefore also the exceptional node. Thus the difference between $(\mathcal{M} : \mathcal{N} = 4)$ and $(\mathcal{M} : \mathcal{N} < 4)$ cases would be that in the Kac-Moody group would reduce to gauge group $\mathcal{M} : \mathcal{N} < 4$ because Kac-Moody central charge k and therefore also Virasoro central charge resulting in Sugawara construction would vanish.

4. Do finite subgroups of $SU(2)$ play some role also in $\mathcal{M} : \mathcal{N} = 4$ case?

One can ask wonder the possible interpretation for the appearance of extended Dynkin diagrams in $(\mathcal{M} : \mathcal{N} = 4)$ case. Do finite subgroups $G \subset SU(2)$ associated with extended Dynkin diagrams appear also in this case. The formal analog for $H \rightarrow G_a \times G_b$ bundle structure would be $H \rightarrow H/G_a \times SU(2)$. This would mean that the geodesic sphere of CP_2 would define the fiber. The notion of number theoretic braid meaning a selection of a discrete subset of algebraic points of the geodesic sphere of CP_2 suggests that $SU(2)$ actually reduces to its subgroup G also in this case.

5. Why Kac-Moody central charge can be non-vanishing only for $\mathcal{M} : \mathcal{N} = 4$?

From the physical point of view the vanishing of Kac-Moody central charge for $\mathcal{M} : \mathcal{N} < 4$ is easy to understand. If parton corresponds to a homologically non-trivial geodesic sphere, space-time surface typically represents a string like object so that the generation of Kac-Moody central extension would relate directly to the homological non-triviality of partons. For instance, cosmic strings are string like objects of form $X^2 \times Y^2$, where X^2 is minimal surface of M^2 and Y^2 is a holomorphic sub-manifold of CP_2 reducing to a homologically non-trivial geodesic sphere in the simplest situation. A conjecture that deserves to be shown wrong is that central charge k is proportional/equal to the absolute value of the homology (Kähler magnetic) charge h.

6. More general situation

McKay correspondence generalizes also to the case of subgroups of higher-dimensional Lie groups [52]. The argument above makes sense also for discrete subgroups of more general compact Lie groups H since also they define unique sub-factors. In this case, algebras having Cartan algebra with nk generators, where n is the dimension of Cartan algebra of H, would emerge in the process. Thus there are reasons to believe that TGD could emulate practically any dynamics having gauge group or Kac-Moody type symmetry. An interesting question concerns the interpretation of non-ADE type principal graphs associated with subgroups of SU(2).

7. Flavor groups of hadron physics as a support for HFF?

The deformation assigning to an n-fold tensor power of representations of Lie group G with k-dimensional Cartan algebra a representation of a Lie group with nk-dimensional Cartan algebra could be also seen as a dynamically generated symmetry. If quantum measurement is characterized by the choice of Lie group G defining measured quantum numbers and defining Jones inclusion characterizing the measurement resolution, the measurement process itself would generate these dynamical symmetries. Interestingly, the flavor symmetry groups of hadron physics cannot be justified from the structure of the standard model having only electro-weak and color group as fundamental symmetries. In TGD framework flavor group $SU(n)$ could emerge naturally as a fusion of n quark doublets to form a representation of $SU(n)$.

Conformal representations of braid group and a possible further generalization of McKay correspondence

Physically especially interesting representations of braid group and associated Temperley-Lieb-Jones algebras (TLJ) are representations provided by the n-point functions of conformal field theories studied in [56]. The action of the generator of braid group on n-point function corresponds to a duality transformation of old-fashioned string model (or crossing) represented as a monodromy relating corresponding conformal blocks. This effect can be calculated. Since the index $r = \mathcal{M} : \mathcal{N}$ appears as a parameter in TLJ algebra, the formulas expressing the behavior of n-point functions under the duality transformation reveal also the value of index which might not be easy to calculate otherwise.

Note that in TGD framework the arguments of n-point function would correspond to the strands of the number theoretic braid and thus to the points of the geodesic sphere S^2 associated with the light-cone boundary $\delta M^4_{\pm}$. The projection to the geodesic sphere of CP_2 projection would be same for all these strands.

WZW model for group G and Kac-Moody central charge k quantum phase is discussed in [56]. The non-triviality of braiding boils to the fact that quantum group G_q defines the effect of braiding operation. Quantum phase is given as $q = exp(i\pi/(k + C(G))$, where $C(G)$ is the value of Casimir operator in adjoint representation. The action of the braid group generator reduces to the unitary matrix relating the basis defined by the tensor product of representations of G_q to the basis obtained by application of a generator of the braid group. For n-point functions of primary fields belonging to a representation D of G, index is the square of the quantum dimension $d_q(D)$ of the corresponding representation of G_q. Hence each primary field correspond to its own inclusion of HFF, which corresponds to $n \rightarrow \infty$-point function.

The result could have been guessed as the dimension of quantum Clifford algebra emerging naturally in inclusion when HFF is represented as an infinite tensor power of $M(d(D), C)$. For $j = 1/2$ representation of $SU(2)$ standard Jones inclusions with $r < 4$ are obtained. The resulting inclusion is irreducible ($\mathcal{N}' \cap \mathcal{M} = C$, where $\mathcal{N}'$ is the commutator of $\mathcal{N}'$). Using the representation of HFF as infinite tensor power of $M(2, C)$ the result would not be so easy to understand.

The mathematical challenge would be to understand how the representations HFF as an infinite tensor power of $M(n, C)$ relate to each other for different values of n. It might be possible to understand the relationship between different infinite tensor power representations of HFF by representing $M(n_1, C)$ as a sub-algebra of a tensor power of a finite tensor power of $M(n_2, C)$. Perhaps a detailed construction of the maps between representations of HFF as infinite tensor power of $M(n, C)$ for various values of n could reveal further generalizations of McKay correspondence.

10.3.7 Technical questions related to Hecke algebra and Frobenius element

Frobenius elements

Frobenius element Fr_p is mapped to a conjugacy class of Galois group using the decomposition of prime p to prime ideals in the algebraic extension K/F.

1. At the level of braid group Frobenius element Fr_p corresponds to some conjugacy class of Galois group acting imbedded to S_n (only the conjugacy equivalence class is fixed) and thus can be mapped to an element of the braid group. Hence it seems possible to assign to Fr_p an element of infinitely cyclic subgroup of the braid group.

2. One can always reduce in given representation the element of given conjugacy class to a diagonal matrix so that it is possible to chose the representatives of Fr_p to be commuting operators. These operators would act as a spinor rotation on quantum Clifford algebra elements defined by Jones inclusion and identifiable as element of some cyclic group of the group G defining the sub-factor via the diagonal embedding.

3. Fr_p for a given finite Galois group G should have representation as an element of braid group to which G is imbedded as a subgroup. It is possible to chose the representatives of Fr_p so that they commute. Could one chose them in such a manner that they belong to the commuting subgroup defined by even (odd) generators e_i? The choice of representatives for Fr_p for various Galois groups must be also consistent with the hierarchies of intermediate extensions of rationals associated with given extension and characterized by subgroups of Galois group for the extension.

How the action of commutative Hecke algebra is realized in hyper-finite factor and braid group?

One can also ask how to imbed Hecke algebra to the braid algebra. Hecke algebra for a given value of prime p and group $GL(n, R)$ is a polynomial algebra in Hecke algebra generators. There is a fundamental difference between Hecke algebra and Frobenius element Fr_p in the sense that Fr_p has finite order as an element of finite Galois group whereas Hecke algebra elements do not except possibly for representations. This means that Hecke algebra cannot have a representation in a finite Galois groups.

Situation is different for braid algebra generators since they do not satisfy the condition $e_i^2 = 1$ and odd and even generators of braid algebra commute. The powers of Hecke algebra generators would correspond to the powers of basic braiding operation identified as a π twist of neighboring strands. For unitary representations eigenvalues of e_i are phase factors. Therefore Hecke algebra might be realized using odd or even commuting sub-algebra of braid algebra and this could allow to deduce the Frobenius-Hecke correspondence directly from the representations of braid group. The basic questions are following.

1. Is it possible to represent Hecke algebra as a subalgebra of braid group algebra in some natural manner? Could the infinite cyclic group generated by braid group image of Fr_p belong represent element of Hecke algebra fixed by the Langlands correspondence? If this were the case then the eigenvalues of Frobenius element Fr_p of Galois group would correspond to the eigen values of Hecke algebra generators in the manner dictated by Langlands correspondence.

2. Hecke operators $H_{p,i}$, $i = 1, .., n$ commute and expressible as two-side cosets in group $GL(n, Q_p)$. This group acts in $\mathcal{M}$ and the action could be made rather explicit by using a proper representations of $\mathcal{M}$ (note however that physical situation can quite well distinguish between various representations). Does the action of the Hecke sub-algebra fixed by Hecke-Frobenius correspondence co-incide with the action of Frobenius element Fr_p identified as an element of braid sub-group associated with some cyclic subgroup of the Galois group identified as a group defining the sub-factor?

10.4 Appendix

10.4.1 Hecke algebra and Temperley-Lieb algebra

Braid group is accompanied by several algebras. For Hecke algebra, which is particular case of braid algebra, one has

$$\begin{aligned} e_{n+1}e_ne_{n+1} &= e_ne_{n+1}e_n \ , \\ e_n^2 &= (t-1)e_n + t \ . \end{aligned} \tag{10.4.0}$$

The algebra reduces to that for symmetric group for $t = 1$.

Hecke algebra can be regarded as a discrete analog of Kac Moody algebra or loop algebra with G replaced by S_n. This suggests a connection with Kac-Moody algebras and imbedding of Galois groups to Kac-Moody group. $t = p^n$ corresponds to a finite field. Fractal dimension $t = \mathcal{M} : \mathcal{N}$ relates naturally to braid group representations: fractal dimension of quantum quaternions might be appropriate interpretation. t=1 gives symmetric group. Infinite braid group could be seen as a quantum variant of Galois group for algebraic closure of rationals.

Temperley-Lieb algebra assignable with Jones inclusions of hyper-finite factors of type II_1 with $\mathcal{M} : \mathcal{N} < 4$ is given by the relations

$$\begin{aligned} e_{n+1}e_nen+1 &= e_{n+1} \\ e_ne_{n+1}e_n &= e_n \ , \\ e_n^2 &= te_n \ , \ \ ,t = -\sqrt{\mathcal{M}:\mathcal{N}} = -2cos(\pi/n) \ \ ,n = 3,4,... \end{aligned} \tag{10.4.-1}$$

The conditions involving three generators differ from those for braid group algebra since e_n are now proportional to projection operators. An alternative form of this algebra is given by

$$\begin{aligned} e_{n+1}e_nen+1 &= te_{n+1} \\ e_ne_{n+1}e_n &= te_n \ , \\ e_n^2 &= e_n = e_n^* \ , \ \ ,t = -\sqrt{\mathcal{M}:\mathcal{N}} = -2cos(\pi/n) \ \ ,n = 3,4,... \end{aligned} \tag{10.4.-2}$$

This representation reduces to that for Temperley-Lieb algebra with obvious normalization of projection operators. These algebras are somewhat analogous to function fields but the value of coordinate is fixed to some particular values. An analogous discretization for function fields corresponds to a formation of number theoretical braids.

10.4.2 Some examples of bi-algebras and quantum groups

The appendix summarizes briefly the simplest bi- and Hopf algebras and some basic constructions related to quantum groups.

Simplest bi-algebras

Let $k(x_1,..,x_n)$ denote the free algebra of polynomials in variables x_i with coefficients in field k. x_i can be regarded as points of a set. The algebra $Hom(k(x_1,...,x_n),A)$ of algebra homomorphisms $k(x_1,...,x_n)\rightarrow A$ can be identified as A^n since by the homomorphism property the images $f(x_i)$ of the generators $x_1,...x_n$ determined the homomorphism completely. Any commutative algebra A can be identified as the $Hom(k[x],A)$ with a particular homomorphism corresponding to a line in A determined uniquely by an element of A.

The matrix algebra $M(2)$ can be defined as the polynomial algebra $k(a,b,c,d)$. Matrix multiplication can be represented universally as an algebra morphism Δ from from $M_2 = k(a,b,c,d)$ to $M_2^{\otimes 2} = k(a',a'',b',b'',c',c'',d',d'')$ to $k(a,b,c,d)$ in matrix form as

$$\Delta\left(\begin{array}{cc} a & b\\ c & d\end{array}\right) = \left(\begin{array}{cc} a' & b'\\ c' & d'\end{array}\right)\left(\begin{array}{cc} a'' & b''\\ c'' & d''\end{array}\right) \ .$$

This morphism induces algebra multiplication in the matrix algebra $M_2(A)$ for any commutative algebra A.

$M(2)$, $GL(2)$ and $SL(2)$ provide standard examples about bi-algebras. $SL(2)$ can be defined as a commutative algebra by dividing free polynomial algebra $k(a,b,c,d)$ spanned by the generators a,b,c,d by the ideal $det-1 = ad-bc-1=0$ expressing that the determinant of the matrix is one. In the matrix representation μ and η are defined in obvious manner and μ gives powers of the matrix

$$A = \left(\begin{array}{cc} a & b\\ c & d\end{array}\right) \ .$$

Δ, counit ϵ, and antipode S can be written in case of $SL(2)$ as

$$\left(\begin{array}{cc} \Delta(a) & \Delta(b)\\ \Delta(c) & \Delta(d)\end{array}\right) = \left(\begin{array}{cc} a & b\\ c & d\end{array}\right)\otimes\left(\begin{array}{cc} a & b\\ c & d\end{array}\right) \ ,$$

$$\left(\begin{array}{cc} \epsilon(a) & \epsilon(b)\\ \epsilon(c) & \epsilon(d)\end{array}\right) = \left(\begin{array}{cc} 1 & 0\\ 0 & 1\end{array}\right) \ .$$

$$S\left(\begin{array}{cc} a & b\\ c & d\end{array}\right) = (ad-bc)^{-1}\left(\begin{array}{cc} d & -b\\ -c & a\end{array}\right) \ .$$

Note that matrix representation is only an economical manner to summarize the action of Δ on the generators a,b,c,d of the algebra. For instance, one has $\Delta(a) = a\rightarrow a\otimes a + b\otimes c$. The resulting algebra is both commutative and co-commutative.

$SL(2)_q$ can be defined as a Hopf algebra by dividing the free algebra generated by elements a,b,c,d by the relations

$$\begin{array}{ll} ba = qab \ , & db = qbd \ ,\\ ca = qac \ , & dc = qcd \ ,\\ bc = cb \ , & ad-da = (q^{-1}-1)bc \ , \end{array}$$

and the relation

$$det_q = ad - q^{-1}bc = 1$$

stating that the quantum determinant of $SL(2)_q$ matrix is one.

μ,η,Δ,ϵ are defined as in the case of $SL(2)$. Antipode S is defined by

$$S\left(\begin{array}{cc} a & b\\ c & d\end{array}\right) = det_q^{-1}\left(\begin{array}{cc} d & -qb\\ -q^{-1}c & a\end{array}\right) \ .$$

The relations above guarantee that it defines quantum inverse of A. For q an n^{th} root of unity, $S^{2n} = id$ holds true which signals that these parameter values are somehow exceptional. This result is completely general.

Given an algebra, the R point of $SL_q(2)$ is defined as a four-tuple (A,B,C,D) in R^4 satisfying the relations defining the point of $SL_q(2)$. One can say that R-points provide representations of the universal quantum algebra $SL_q(2)$.

Quantum group $U_q(sl(2))$

Quantum group $U_q(sl(2))$ or rather, quantum enveloping algebra of $sl(2)$, can be constructed by applying Drinfeld's quantum double construction (to avoid confusion note that the quantum Hopf algebra associated with $SL(2)$ is the quantum analog of a commutative algebra generated by powers of a 2×2 matrix of unit determinant).

The commutation relations of $sl(2)$ read as

$$[X_+, X_-] = H \ , \quad [H, X_\pm] = \pm 2X_\pm \ . \tag{10.4.-1}$$

$U_q(sl(2))$ allows co-algebra structure given by

$$\begin{aligned} &\Delta(J) = J\otimes 1 + 1\otimes J \ , \quad S(J) = -J \ , \quad \epsilon(J) = 0 \ , \quad J = X_\pm, H \ , \\ &S(1) = 1 \ , \qquad\qquad\qquad \epsilon(1) = 1 \ . \end{aligned} \tag{10.4.0}$$

The enveloping algebras of Borel algebras $U(B_\pm)$ generated by $\{1, X_+, H\}$ $\{1, X_-, hH\}$ define the Hopf algebra H and its dual $H^\star$ in Drinfeld's construction. h could be called Planck's constant vanishes at the classical limit. Note that $H^\star$ reduces to $\{1, X_-\}$ at this limit. Quantum deformation parameter q is given by $exp(2h)$. The duality map $\star : H \to H^\star$ reads as

$$\begin{array}{l} a \to a^\star \ , \quad ab = (ab)^\star = b^\star a^\star \ , \\ 1 \to 1 \ , \qquad H \to H^\star = hH \ , \qquad X_+ \to (X_+)^\star = hX_- \ . \end{array} \tag{10.4.1}$$

The commutation relations of $U_q(sl(2)$ read as

$$[X_+, X_-] = \frac{q^H - q^{-H}}{q - q^{-1}} \ , \quad [H, X_\pm] = \pm 2X_\pm \ . \tag{10.4.2}$$

Co-product Δ, antipode S, and co-unit ϵ differ from those $U(sl(2))$ only in the case of $X_\pm$:

$$\begin{aligned} &\Delta(X_\pm) = X_\pm \otimes q^{H/2} + q^{-H/2}\otimes X_\pm \ , \\ &S(X_\pm) = -q^{\pm 1}X_\pm \ . \end{aligned} \tag{10.4.3}$$

When q is not a root of unity, the universal R-matrix is given by

$$R = q^{\frac{H\otimes H}{2}} \sum_{n=0}^{\infty} \frac{(1-q^{-2})^n}{[n]_q!} q^{\frac{n(1-n)}{2}} q^{\frac{nH}{2}} X_+^n \otimes q^{-\frac{nH}{2}} X_-^n \ . \tag{10.4.4}$$

When q is m:th root of unity the q-factorial $[n]_q!$ vanishes for $n \geq m$ and the expansion does not make sense.

For q not a root of unity the representation theory of quantum groups is essentially the same as of ordinary groups. When q is m^{th} root of unity, the situation changes. For $l = m = 2n$ n^{th} powers of generators span together with the Casimir operator a sub-algebra commuting with the whole algebra providing additional numbers characterizing the representations. For $l = m = 2n+1$ same happens for m^{th} powers of Lie-algebra generators. The generic representations are not fully reducible anymore. In the case of $U_q(sl(2))$ irreducibility occurs for spins $n < l$ only. Under certain conditions on q it is possible to decouple the higher representations from the theory. Physically the reduction of the number of representations to a finite number means a symmetry analogous to a gauge symmetry. The phenomenon resembles the occurrence of null vectors in the case of Virasoro and Kac Moody representations and there indeed is a deep connection between quantum groups and Kac-Moody algebras [39].

One can wonder what is the precise relationship between $U_q(sl(2)$ and $SL_q(2)$ which both are quantum groups using loose terminology. The relationship is duality. This means the existence of a morphism $x \rightarrow \Psi(x)$ $M_q(2) \rightarrow U_q^\star$ defined by a bilinear form $\langle u, x\rangle = \Psi(x)(u)$ on $U_q \times M_q(2)$, which is bi-algebra morphism. This means that the conditions

$$\langle uv, x\rangle = \langle u \otimes v, \Delta(x)\rangle \ , \quad \langle u, xy\rangle = \langle \Delta(u), x \otimes y\rangle \ ,$$

$$\langle 1, x\rangle = \epsilon(x) \ , \qquad\qquad \langle u, 1\rangle = \epsilon(u)$$

are satisfied. It is enough to find $\Psi(x)$ for the generators $x = A, B, C, D$ of $M_q(2)$ and show that the duality conditions are satisfied. The representation

$$\rho(E) = \left(\begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array}\right) \ , \quad \rho(F) = \left(\begin{array}{cc} 0 & 0 \\ 1 & 0 \end{array}\right) \ , \quad \rho(K = q^H) = \left(\begin{array}{cc} q & 0 \\ 0 & q^{-1} \end{array}\right) \ ,$$

extended to a representation

$$\rho(u) = \left(\begin{array}{cc} A(u) & B(u) \\ C(u) & D(u) \end{array}\right)$$

of arbitrary element u of $U_q(sl(2)$ defines for elements in $U_q^\star$. It is easy to guess that $A(u), B(u), C(u), D(u)$, which can be regarded as elements of $U_q^\star$, can be regarded also as R points that is images of the generators a, b, c, d of $SL_q(2)$ under an algebra morphism $SL_q(2) \rightarrow U_q^\star$.

General semisimple quantum group

The Drinfeld's construction of quantum groups applies to arbitrary semi-simple Lie algebra and is discussed in detail in [39]. The construction relies on the use of Cartan matrix.

Quite generally, Cartan matrix $A = \{a_{ij}\}$ is $n \times n$ matrix satisfying the following conditions:
i) A is indecomposable, that is does not reduce to a direct sum of matrices.
ii) $a_{ij} \leq 0$ holds true for $i < j$.
iii) $a_{ij} = 0$ is equivalent with $a_{ij} = 0$.
A can be normalized so that the diagonal components satisfy $a_{ii} = 2$.
The generators e_i, f_i, k_i satisfying the commutations relations

$$\begin{array}{cc} k_ik_j = k_jk_i \ , & k_ie_j = q_i^{a_{ij}} e_jk_i \ , \\ k_if_j = q_i^{-a_{ij}} e_jk_i \ , & e_if_j - f_je_i = \delta_{ij} \frac{k_i - k_i^{-1}}{q_i - q_i^{-1}} \ , \end{array} \tag{10.4.5}$$

and so called Serre relations

$$\begin{array}{c} \sum_{l=0}^{1-a_{ij}} (-1)^l \left[\begin{array}{c} 1-a_{ij} \\ l \end{array}\right]_{q_i} e_i^{1-a_{ij}-l} e_j e_i^l = 0, \ \ i \neq j \ , \\ \sum_{l=0}^{1-a_{ij}} (-1)^l \left[\begin{array}{c} 1-a_{ij} \\ l \end{array}\right]_{q_i} f_i^{1-a_{ij}-l} f_j f_i^l = 0 \ , \ \ i \neq j \ . \end{array} \tag{10.4.6}$$

Here $q_i = q^{D_i}$ where one has $D_ia_{ij} = a_{ij}D_i$. $D_i = 1$ is the simplest choice in this case.

Comultiplication is given by

$$\Delta(k_i) = k_i \otimes k_i \ , \tag{10.4.7}$$
$$\Delta(e_i) = e_i \otimes k_i + 1 \otimes e_i \ , \tag{10.4.8}$$
$$\Delta(f_i) = f_i \otimes 1 + k_i^{-1} \otimes 1 \ . \tag{10.4.9}$$
$$\tag{10.4.10}$$

The action of antipode S is defined as

$$S(e_i) = -e_ik_i^{-1} \ , \quad S(f_i) = -k_if_i \ , \quad S(k_i) = -k_i^{-1} \ . \tag{10.4.11}$$

Quantum affine algebras

The construction of Drinfeld and Jimbo generalizes also to the case of untwisted affine Lie algebras, which are in one-one correspondence with semisimple Lie algebras. The representations of quantum deformed affine algebras define corresponding deformations of Kac-Moody algebras. In the following only the basic formulas are summarized and the reader not familiar with the formalism can consult a more detailed treatment can be found in [39].

1. Affine algebras

The Cartan matrix A is said to be of affine type if the conditions $det(A) = 0$ and $a_{ij}a_{ji} \geq 4$ (no summation) hold true. There always exists a diagonal matrix D such that $B = DA$ is symmetric and defines symmetric bilinear degenerate metric on the affine Lie algebra.

The Dynkin diagrams of affine algebra of rank l have $l+1$ vertices (so that Cartan matrix has one null eigenvector). The diagrams of semisimple Lie-algebras are sub-diagrams of affine algebras. From the $(l+1) \times (l+1)$ Cartan matrix of an untwisted affine algebra $\hat{A}$ one can recover the $l \times l$ Cartan matrix of A by dropping away 0:th row and column.

For instance, the algebra A_1^1, which is affine counterpart of $SL(2)$, has Cartan matrix a_{ij}

$$A = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$$

with a vanishing determinant.

Quite generally, in untwisted case quantum algebra $U_q(\hat{G}_l)$ as $3(l+1)$ generators e_i, f_i, k_i $(i = 0, 1, .., l)$ satisfying the relations of Eq. 10.4.6 for Cartan matrix of $\mathcal{G}^{(1)}$. Affine quantum group is obtained by adding to $U_q(\hat{\mathcal{G}}_l)$ a derivation d satisfying the relations

$$[d, e_i] = \delta_{i0}e_i \ , \quad [d, f_i] = \delta_{i0}f_i, \quad [d, k_i] = 0 \ . \tag{10.4.12}$$

with comultiplication $\Delta(d) = d \otimes 1 + 1 \otimes d$.

2. Kac Moody algebras

The undeformed extension $\hat{\mathcal{G}}_l$ associated with the affine Cartan matrix $\mathcal{G}_l^{(1)}$ is the Kac Moody algebra associated with the group G obtained as the central extension of the corresponding loop algebra. The loop algebra is defined as

$$L(\mathcal{G}) = \mathcal{G} \otimes C\left[t, t^{-1}\right] \ , \tag{10.4.13}$$

where $C\left[t, t^{-1}\right]$ is the algebra of Laurent polynomials with complex coefficients. The Lie bracket is

$$[x \times P, y \otimes Q] = [x, y] \otimes PQ \ . \tag{10.4.14}$$

The non-degenerate bilinear symmetric form $(,)$ in $\mathcal{G}_l$ induces corresponding form in $L(\mathcal{G}_l)$ as $(x \otimes P, y \otimes Q) = (x, y)PQ$.

A two-cocycle on $L(\mathcal{G}_l)$ is defined as

$$\Psi(a, b) = Res(\frac{da}{dt}, b) \ , \tag{10.4.15}$$

where the residue of a Laurent is defined as $Res(\sum_n a_n t^n) = a_{-1}$. The two-cocycle satisfies the conditions

$$\begin{gathered} \Psi(a, b) = -\Psi(b, a) \ , \\ \Psi([a, b], c) + \Psi([b, c], a) + \Psi([c, a], b) = 0 \ . \end{gathered} \tag{10.4.15}$$

The two-cocycle defines the central extension of loop algebra $L(\mathcal{G}_l)$ to Kac Moody algebra $L(\mathcal{G}_l) \otimes Cc$, where c is a new central element commuting with the loop algebra. The new bracket is defined as

$[,]+\Psi(,)c$. The algebra $\tilde{L}(\mathcal{G}_l)$ is defined by adding the derivation d which acts as td/dt measuring the conformal weight.

The standard basis for Kac Moody algebra and corresponding commutation relations are given by

$$\begin{aligned} J_n^x &= x \otimes t^n \ , \\ [J_n^x, J_m^y] &= J_{n+m}^{[x,y]} + n\delta_{m+n,0}c \ . \end{aligned} \tag{10.4.15}$$

The finite dimensional irreducible representations of G defined representations of Kac Moody algebra with a vanishing central extension $c = 0$. The highest weight representations are characterized by highest weight vector $|v\rangle$ such that

$$\begin{aligned} J_n^x|v\rangle &= 0, \ \ n > 0 \ , \\ c|v\rangle &= k|v\rangle \ . \end{aligned} \tag{10.4.15}$$

3. Quantum affine algebras

Drinfeld has constructed the quantum affine extension $U_q(\mathcal{G}_l)$ using quantum double construction. The construction of generators uses almost the same basic formulas as the construction of semi-simple algebras. The construction involves the automorphism $D_t : U_q(\tilde{\mathcal{G}}_l) \otimes C\left[t, t^{-1}\right] \rightarrow U_q(\tilde{\mathcal{G}}_l) \otimes C\left[t, t^{-1}\right]$ given by

$$\begin{array}{ll} D_t(e_i) = t^{\delta_{i0}} e_i \ , & D_t(f_i) = t^{\delta_{i0}} f_i \ , \\ D_t(k_i) = k_i & D_t(d) = d \ , \end{array} \tag{10.4.16}$$

and the co-product

$$\Delta_t(a) = (D_t \otimes 1)\Delta(a) \ , \quad \Delta_t^{op}(a) = (D_t \otimes 1)\Delta^{op}(a) \ , \tag{10.4.17}$$

where the $\Delta(a)$ is the co-product defined by the same general formula as applying in the case of semi-simple Lie algebras. The universal R-matrix is given by

$$\mathcal{R}(t) = (D_t \otimes 1)\mathcal{R} \ , \tag{10.4.18}$$

and satisfies the equations

$$\begin{aligned} &\mathcal{R}(t)\Delta_t(a) = \Delta_t^{op}(a)\mathcal{R} \ , \\ &(\Delta_z \otimes id)\mathcal{R}(u) = \mathcal{R}_{13}(zu)\mathcal{R}_{23}(u) \ , \\ &(id \otimes \Delta_u)\mathcal{R}(zu) = \mathcal{R}_{13}(z)\mathcal{R}_{12}(zu) \ , \\ &\mathcal{R}_{12}(t)\mathcal{R}_{13}(tw)\mathcal{R}_{23}(w) = \mathcal{R}_{23}(w)\mathcal{R}_{13}(tw)\mathcal{R}_{12}(t) \ . \end{aligned} \tag{10.4.19}$$

The infinite-dimensional representations of affine algebra give representations of Kac-Moody algebra when one restricts the consideration to generations e_i, f_i, k_i, $i > 0$.

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Appendix A

Appendix

A-1 Basic properties of CP_2

A-1.1 CP_2 as a manifold

CP_2, the complex projective space of two complex dimensions, is obtained by identifying the points of complex 3-space C^3 under the projective equivalence

$$(z^1, z^2, z^3) \equiv \lambda(z^1, z^2, z^3) \ . \tag{A-1.1}$$

Here λ is any non-zero complex number. Note that CP_2 can also regarded as the coset space $SU(3)/U(2)$. The pair z^i/z^j for fixed j and $z^i \neq 0$ defines a complex coordinate chart for CP_2. As j runs from 1 to 3 one obtains an atlas of three charts covering CP_2, the charts being holomorphically related to each other (e.g. CP_2 is a complex manifold). The points $z^3 \neq 0$ form a subset of CP_2 homoeomorphic to R^4 and the points with $z^3 = 0$ a set homeomorphic to S^2. Therefore CP_2 is obtained by "adding the 2-sphere at infinity to R^4".

Besides the standard complex coordinates $\xi^i = z^i/z^3$, $i = 1, 2$ the coordinates of Eguchi and Freund [2] will be used and their relation to the complex coordinates is given by

$$\begin{aligned} \xi^1 &= z + it \ , \\ \xi^2 &= x + iy \ . \end{aligned} \tag{A-1.1}$$

These are related to the "spherical coordinates" via the equations

$$\begin{aligned} \xi^1 &= rexp(i\frac{(\Psi+\Phi)}{2})cos(\frac{\Theta}{2}) \ , \\ \xi^2 &= rexp(i\frac{(\Psi-\Phi)}{2})sin(\frac{\Theta}{2}) \ . \end{aligned} \tag{A-1.1}$$

The ranges of the variables r, Θ, Φ, Ψ are $[0, \infty], [0, \pi], [0, 4\pi], [0, 2\pi]$ respectively.

Considered as a real four-manifold CP_2 is compact and simply connected, with Euler number 3, Pontryagin number 3 and second Betti number $b = 1$.

A-1.2 Metric and Kähler structures of CP_2

In order to obtain a natural metric for CP_2, observe that CP_2 can be thought of as a set of the orbits of the isometries $z^i \rightarrow exp(i\alpha)z^i$ on the sphere S^5: $\sum z^i \bar{z}^i = R^2$. The metric of CP_2 is obtained by projecting the metric of S^5 orthogonally to the orbits of the isometries. Therefore the distance between the points of CP_2 is that between the representative orbits on S^5. The line element has the following form in the complex coordinates

$$ds^2 = g_{a\bar{b}} d\xi^a d\bar{\xi}^b \ , \tag{A-1.2}$$

where the Hermitian, in fact Kähler, metric $g_{a\bar{b}}$ is defined by

$$g_{a\bar{b}} = R^2 \partial_a \partial_{\bar{b}} K \ , \tag{A-1.3}$$

where the function K, Kähler function, is defined as

$$\begin{aligned} K &= log(F) \ , \\ F &= 1 + r^2 \ . \end{aligned} \tag{A-1.3}$$

The representation of the metric is given by

$$\frac{ds^2}{R^2} = \frac{(dr^2 + r^2\sigma_3^2)}{F^2} + \frac{r^2(\sigma_1^2 + \sigma_2^2)}{F} \ , \tag{A-1.4}$$

where the quantities σ_i are defined as

$$\begin{aligned} r^2\sigma_1 &= Im(\xi^1 d\xi^2 - \xi^2 d\xi^1) \ , \\ r^2\sigma_2 &= -Re(\xi^1 d\xi^2 - \xi^2 d\xi^1) \ , \\ r^2\sigma_3 &= -Im(\xi^1 d\bar{\xi}^1 + \xi^2 d\bar{\xi}^2) \ . \end{aligned} \tag{A-1.3}$$

The vierbein forms, which satisfy the defining relation

$$s_{kl} = R^2 \sum_A e_k^A e_l^A \ , \tag{A-1.4}$$

are given by

$$\begin{aligned} e^0 &= \tfrac{dr}{F} \ , \quad & e^1 &= \tfrac{r\sigma_1}{\sqrt{F}} \ , \\ e^2 &= \tfrac{r\sigma_2}{\sqrt{F}} \ , \quad & e^3 &= \tfrac{r\sigma_3}{F} \ . \end{aligned} \tag{A-1.5}$$

The explicit representations of vierbein vectors are given by

$$\begin{aligned} e^0 &= \tfrac{dr}{F} \ , \quad & e^1 &= \tfrac{r(sin\Theta cos\Psi d\Phi + sin\Psi d\Theta)}{2\sqrt{F}} \ , \\ e^2 &= \tfrac{r(sin\Theta sin\Psi d\Phi - cos\Psi d\Theta)}{2\sqrt{F}} \ , \quad & e^3 &= \tfrac{r(d\Psi + cos\Theta d\Phi)}{2F} \ . \end{aligned} \tag{A-1.5}$$

The explicit representation of the line element is given by the expression

$$ds^2/R^2 = dr^2/F^2 + (r^2/4F^2)(d\Psi + cos\Theta d\Phi)^2 + (r^2/4F)(d\Theta^2 + sin^2\Theta d\Phi^2) \ . \tag{A-1.5}$$

The vierbein connection satisfying the defining relation

$$de^A = -V_B^A \wedge e^B \ , \tag{A-1.6}$$

is given by

$$\begin{array}{lllllll} V_{01} & = & -\frac{e^1}{r} \ , & V_{23} & = & \frac{e^1}{r} \ , \\ V_{02} & = & -\frac{e^2}{r} \ , & V_{31} & = & \frac{e^2}{r} \ , \\ V_{03} & = & (r-\frac{1}{r})e^3 \ , & V_{12} & = & (2r+\frac{1}{r})e^3 \ . \end{array} \qquad \text{(A-1.7)}$$

The representation of the covariantly constant curvature tensor is given by

$$\begin{array}{lllllll} R_{01} & = & e^0\wedge e^1 - e^2\wedge e^3 \ , & R_{23} & = & e^0\wedge e^1 - e^2\wedge e^3 \ , \\ R_{02} & = & e^0\wedge e^2 - e^3\wedge e^1 \ , & R_{31} & = & -e^0\wedge e^2 + e^3\wedge e^1 \ , \\ R_{03} & = & 4e^0\wedge e^3 + 2e^1\wedge e^2 \ , & R_{12} & = & 2e^0\wedge e^3 + 4e^1\wedge e^2 \ . \end{array} \qquad \text{(A-1.8)}$$

Metric defines a real, covariantly constant, and therefore closed 2-form J

$$J = -ig_{a\bar{b}}d\xi^a d\bar{\xi}^b \ , \qquad \text{(A-1.9)}$$

the so called Kähler form. Kähler form J defines in CP_2 a symplectic structure because it satisfies the condition

$$J^k_{\ r}J^{rl} = -s^{kl} \ . \qquad \text{(A-1.10)}$$

The form J is integer valued and by its covariant constancy satisfies free Maxwell equations. Hence it can be regarded as a curvature form of a $U(1)$ gauge potential B carrying a magnetic charge of unit $1/2g$ (g denotes the gauge coupling). Locally one has therefore

$$J = dB \ , \qquad \text{(A-1.11)}$$

where B is the so called Kähler potential, which is not defined globally since J describes magnetic monopole.

It should be noticed that the magnetic flux of J through a 2-surface in CP_2 is proportional to its homology equivalence class, which is integer valued. The explicit representations of J and B are given by

$$\begin{array}{lll} B & = & 2re^3 \ , \\ J & = & 2(e^0\wedge e^3 + e^1\wedge e^2) = \frac{r}{F^2}dr\wedge(d\Psi + cos\Theta d\Phi) + \frac{r^2}{2F}sin\Theta d\Theta d\Phi \ . \end{array} \qquad \text{(A-1.10)}$$

The vielbein curvature form and Kähler form are covariantly constant and have in the complex coordinates only components of type (1,1).

Useful coordinates for CP_2 are the so called canonical coordinates in which Kähler potential and Kähler form have very simple expressions

$$\begin{array}{lll} B & = & \sum_{k=1,2} P_k dQ_k \ , \\ J & = & \sum_{k=1,2} dP_k\wedge dQ_k \ . \end{array} \qquad \text{(A-1.10)}$$

The relationship of the canonical coordinates to the "spherical" coordinates is given by the equations

$$\begin{array}{lll} P_1 & = & -\frac{1}{1+r^2} \ , \\ P_2 & = & \frac{r^2cos\Theta}{2(1+r^2)} \ , \\ Q_1 & = & \Psi \ , \\ Q_2 & = & \Phi \ . \end{array} \qquad \text{(A-1.8)}$$

A-1.3 Spinors in CP_2

CP_2 doesn't allow spinor structure in the conventional sense [5]. However, the coupling of the spinors to a half odd multiple of the Kähler potential leads to a respectable spinor structure. Because the delicacies associated with the spinor structure of CP_2 play a fundamental role in TGD, the arguments of Hawking are repeated here.

To see how the space can fail to have an ordinary spinor structure consider the parallel transport of the vierbein in a simply connected space M. The parallel propagation around a closed curve with a base point x leads to a rotated vierbein at x: $e^A = R^A_B e^B$ and one can associate to each closed path an element of $SO(4)$.

Consider now a one-parameter family of closed curves $\gamma(v) : v \in (0,1)$ with the same base point x and $\gamma(0)$ and $\gamma(1)$ trivial paths. Clearly these paths define a sphere S^2 in M and the element $R^A_B(v)$ defines a closed path in $SO(4)$. When the sphere S^2 is contractible to a point e.g., homologically trivial, the path in $SO(4)$ is also contractible to a point and therefore represents a trivial element of the homotopy group $\Pi_1(SO(4)) = Z_2$.

For a homologically nontrivial 2-surface S^2 the associated path in $SO(4)$ can be homotopically nontrivial and therefore corresponds to a nonclosed path in the covering group Spin(4) (leading from the matrix 1 to -1 in the matrix representation). Assume this is the case.

Assume now that the space allows spinor structure. Then one can parallelly propagate also spinors and by the above construction associate a closed path of Spin(4) to the surface S^2. Now, however this path corresponds to a lift of the corresponding $SO(4)$ path and cannot be closed. Thus one ends up with a contradiction.

From the preceding argument it is clear that one could compensate the nonallowed -1- factor associated with the parallel transport of the spinor around the sphere S^2 by coupling it to a gauge potential in such a way that in the parallel transport the gauge potential introduces a compensating -1-factor. For a $U(1)$ gauge potential this factor is given by the exponential $exp(i2\Phi)$, where Φ is the the magnetic flux through the surface. This factor has the value -1 provided the $U(1)$ potential carries half odd multiple of Dirac charge $1/2g$. In case of CP_2 the required gauge potential is half odd multiple of the Kähler potential B defined previously. In the case of $M^4 \times CP_2$ one can in addition couple the spinor components with different chiralities independently to an odd multiple of $B/2$.

A-1.4 Geodesic submanifolds of CP_2

Geodesic submanifolds are defined as submanifolds having common geodesic lines with the imbedding space. As a consequence the second fundamental form of the geodesic manifold vanishes, which means that the tangent vectors h^k_α (understood as vectors of H) are covariantly constant quantities with respect to the covariant derivative taking into account that the tangent vectors are vectors both with respect to H and X^4.

In [3] a general characterization of the geodesic submanifolds for an arbitrary symmetric space G/H is given. Geodesic submanifolds are in 1-1-correspondence with the so called Lie triple systems of the Lie-algebra g of the group G. The Lie triple system t is defined as a subspace of g characterized by the closedness property with respect to double commutation

$$[X,[Y,Z]] \in t \ \text{ for } \ X,Y,Z \in t \ . \tag{A-1.9}$$

$SU(3)$ allows, besides geodesic lines, two nonequivalent (not isometry related) geodesic spheres. This is understood by observing that $SU(3)$ allows two nonequivalent $SU(2)$ algebras corresponding to subgroups $SO(3)$ (orthogonal 3×3 matrices) and the usual isospin group $SU(2)$. By taking any subset of two generators from these algebras, one obtains a Lie triple system and by exponentiating this system, one obtains a 2-dimensional geodesic submanifold of CP_2.

Standard representatives for the geodesic spheres of CP_2 are given by the equations

$$S^2_I \ : \ \xi^1 = \bar{\xi}^2 \ \text{ or equivalently } \ (\Theta = \pi/2, \Psi = 0) \ ,$$

$$S^2_{II} \ : \ \xi^1 = \xi^2 \ \text{ or equivalently } \ (\Theta = \pi/2, \Phi = 0) \ .$$

The non-equivalence of these sub-manifolds is clear from the fact that isometries act as holomorphic transformations in CP_2. The vanishing of the second fundamental form is also easy to verify. The first geodesic manifold is homologically trivial: in fact, the induced Kähler form vanishes identically for S^2_I. S^2_{II} is homologically nontrivial and the flux of the Kähler form gives its homology equivalence class.

A-2 Identification of the electro-weak couplings

The delicacies of the spinor structure of CP_2 make it a unique candidate for space S. First, the coupling of the spinors to the $U(1)$ gauge potential defined by the Kähler structure provides the missing $U(1)$ factor in the gauge group. Secondly, it is possible to couple different H-chiralities independently to a half odd multiple of the Kähler potential. Thus the hopes of obtaining a correct spectrum for the electromagnetic charge are considerable. In the following it will be demonstrated that the couplings of the induced spinor connection are indeed those of the GWS model [4] and in particular that the right handed neutrinos decouple completely from the electro-weak interactions.

To begin with, recall that the space H allows to define three different chiralities for spinors. Spinors with fixed H-chirality $e = \pm 1$, CP_2-chirality l, r and M^4-chirality L, R are defined by the condition

$$\begin{aligned} \Gamma\Psi &= e\Psi \ , \\ e &= \pm 1 \ , \end{aligned} \tag{A-2.0}$$

where Γ denotes the matrix $\Gamma_9 = \gamma_5 \times \gamma_5$, $1 \times \gamma_5$ and $\gamma_5 \times 1$ respectively. Clearly, for a fixed H-chirality CP_2- and M^4-chiralities are correlated.

The spinors with H-chirality $e = \pm 1$ can be identified as quark and lepton like spinors respectively. The separate conservation of baryon and lepton numbers can be understood as a consequence of generalized chiral invariance if this identification is accepted. For the spinors with a definite H-chirality one can identify the vielbein group of CP_2 as the electro-weak group: $SO(4) = SU(2)_L \times SU(2)_R$.

The covariant derivatives are defined by the spinorial connection

$$A = V + \frac{B}{2}(n_+ 1_+ + n_- 1_-) \ . \tag{A-2.1}$$

Here V and B denote the projections of the vielbein and Kähler gauge potentials respectively and $1_{+(-)}$ projects to the spinor H-chirality $+(-)$. The integers $n_\pm$ are odd from the requirement of a respectable spinor structure.

The explicit representation of the vielbein connection V and of B are given by the equations

$$\begin{aligned} V_{01} &= -\frac{e^1}{r} \ , & V_{23} &= \frac{e^1}{r} \ , \\ V_{02} &= -\frac{e^2}{r} \ , & V_{31} &= \frac{e^2}{r} \ , \\ V_{03} &= (r - \frac{1}{r})e^3 \ , & V_{12} &= (2r + \frac{1}{r})e^3 \ , \end{aligned} \tag{A-2.2}$$

and

$$B = 2re^3 \ , \tag{A-2.3}$$

respectively. The explicit representation of the vielbein is not needed here.

Let us first show that the charged part of the spinor connection couples purely left handedly. Identifying Σ^0_3 and Σ^1_2 as the diagonal (neutral) Lie-algebra generators of $SO(4)$, one finds that the charged part of the spinor connection is given by

$$A_{ch} = 2V_{23}I^1_L + 2V_{13}I^2_L \ , \tag{A-2.4}$$

where one have defined

$$
\begin{aligned}
I_L^1 &= \frac{(\Sigma_{01} - \Sigma_{23})}{2} \ , \\
I_L^2 &= \frac{(\Sigma_{02} - \Sigma_{13})}{2} \ .
\end{aligned}
\tag{A-2.4}
$$

A_{ch} is clearly left handed so that one can perform the identification

$$
W^{\pm} = \frac{2(e^1 \pm ie^2)}{r} \ , \tag{A-2.5}
$$

where $W^{\pm}$ denotes the charged intermediate vector boson.

Consider next the identification of the neutral gauge bosons γ and Z^0 as appropriate linear combinations of the two functionally independent quantities

$$
\begin{aligned}
X &= re^3 \ , \\
Y &= \frac{e^3}{r} \ ,
\end{aligned}
\tag{A-2.5}
$$

appearing in the neutral part of the spinor connection. We show first that the mere requirement that photon couples vectorially implies the basic coupling structure of the GWS model leaving only the value of Weinberg angle undetermined.

To begin with let us define

$$
\begin{aligned}
\bar{\gamma} &= aX + bY \ , \\
\bar{Z}^0 &= cX + dY \ ,
\end{aligned}
\tag{A-2.5}
$$

where the normalization condition

$$
ad - bc = 1 \ ,
$$

is satisfied. The physical fields γ and Z^0 are related to $\bar{\gamma}$ and $\bar{Z}^0$ by simple normalization factors.

Expressing the neutral part of the spinor connection in term of these fields one obtains

$$
\begin{aligned}
A_{nc} &= [(c+d)2\Sigma_{03} + (2d-c)2\Sigma_{12} + d(n_+1_+ + n_-1_-)]\bar{\gamma} \\
&+ [(a-b)2\Sigma_{03} + (a-2b)2\Sigma_{12} - b(n_+1_+ + n_-1_-)]\bar{Z}^0 \ .
\end{aligned}
\tag{A-2.4}
$$

Identifying Σ_{12} and $\Sigma_{03} = 1 \times \gamma_5 \Sigma_{12}$ as vectorial and axial Lie-algebra generators, respectively, the requirement that γ couples vectorially leads to the condition

$$
c = -d \ . \tag{A-2.5}
$$

Using this result plus previous equations, one obtains for the neutral part of the connection the expression

$$
A_{nc} = \gamma Q_{em} + Z^0(I_L^3 - sin^2\theta_W Q_{em}) \ . \tag{A-2.6}
$$

Here the electromagnetic charge Q_{em} and the weak isospin are defined by

$$
\begin{aligned}
Q_{em} &= \Sigma^{12} + \frac{(n_+1_+ + n_-1_-)}{6} \ , \\
I_L^3 &= \frac{(\Sigma^{12} - \Sigma^{03})}{2} \ .
\end{aligned}
\tag{A-2.6}
$$

The fields γ and Z^0 are defined via the relations

$$\begin{aligned}\gamma &= 6d\bar{\gamma} = \frac{6}{(a+b)}(aX+bY) \ , \\ Z^0 &= 4(a+b)\bar{Z}^0 = 4(X-Y) \ . \end{aligned} \quad \text{(A-2.6)}$$

The value of the Weinberg angle is given by

$$sin^2\theta_W = \frac{3b}{2(a+b)} \ , \quad \text{(A-2.7)}$$

and is not fixed completely. Observe that right handed neutrinos decouple completely from the electro-weak interactions.

The determination of the value of Weinberg angle is a dynamical problem. The angle is completely fixed once the YM action is fixed by requiring that action contains no cross term of type γZ^0. Pure symmetry non-broken electro-weak YM action leads to a definite value for the Weinberg angle. One can however add a symmetry breaking term proportional to Kähler action and this changes the value of the Weinberg angle.

To evaluate the value of the Weinberg angle one can express the neutral part F_{nc} of the induced gauge field as

$$F_{nc} = 2R_{03}\Sigma^{03} + 2R_{12}\Sigma^{12} + J(n_+1_+ + n_-1_-) \ , \quad \text{(A-2.8)}$$

where one has

$$\begin{aligned}R_{03} &= 2(2e^0\wedge e^3 + e^1\wedge e^2) \ , \\ R_{12} &= 2(e^0\wedge e^3 + 2e^1\wedge e^2) \ , \\ J &= 2(e^0\wedge e^3 + e^1\wedge e^2) \ , \end{aligned} \quad \text{(A-2.7)}$$

in terms of the fields γ and Z^0 (photon and Z- boson)

$$F_{nc} = \gamma Q_{em} + Z^0(I_L^3 - sin^2\theta_W Q_{em}) \ . \quad \text{(A-2.8)}$$

Evaluating the expressions above one obtains for γ and Z^0 the expressions

$$\begin{aligned}\gamma &= 3J - sin^2\theta_W R_{03} \ , \\ Z^0 &= 2R_{03} \ . \end{aligned} \quad \text{(A-2.8)}$$

For the Kähler field one obtains

$$J = \frac{1}{3}(\gamma + sin^2\theta_W Z^0) \ . \quad \text{(A-2.9)}$$

Expressing the neutral part of the symmetry broken YM action

$$\begin{aligned}L_{ew} &= L_{sym} + fJ^{\alpha\beta}J_{\alpha\beta} \ , \\ L_{sym} &= \frac{1}{4g^2}Tr(F^{\alpha\beta}F_{\alpha\beta}) \ , \end{aligned} \quad \text{(A-2.9)}$$

where the trace is taken in spinor representation, in terms of γ and Z^0 one obtains for the coefficient X of the γZ^0 cross term (this coefficient must vanish) the expression

$$\begin{aligned} X &= -\frac{K}{2g^2} + \frac{fp}{18} \ , \\ K &= Tr\left[Q_{em}(I_L^3 - sin^2\theta_W Q_{em})\right] \ , \end{aligned} \tag{A-2.9}$$

In the general case the value of the coefficient K is given by

$$K = \sum_i \left[-\frac{(18 + 2n_i^2)sin^2\theta_W}{9} \right] \ , \tag{A-2.10}$$

where the sum is over the spinor chiralities, which appear as elementary fermions and n_i is the integer describing the coupling of the spinor field to the the Kähler potential. The cross term vanishes provided the value of the Weinberg angle is given by

$$sin^2\theta_W = \frac{9\sum_i 1}{(fg^2 + 2\sum_i(18 + n_i^2))} \ . \tag{A-2.11}$$

In the scenario where both leptons and quarks are elementary fermions the value of the Weinberg angle is given by

$$sin^2\theta_W = \frac{9}{(\frac{fg^2}{2} + 28)} \ . \tag{A-2.12}$$

The bare value of the Weinberg angle is 9/28 in this scenario, which is quite close to the typical value 9/24 of GUTs [6].

A-2.1 Discrete symmetries

The treatment of discrete symmetries C, P, and T is based on the following requirements:
a) Symmetries must be realized as purely geometric transformations.
b) Transformation properties of the field variables should be essentially the same as in the conventional quantum field theories [1].

The action of the reflection P on spinors of is given by

$$\Psi \rightarrow P\Psi = \gamma^0 \otimes \gamma^0 \Psi \ . \tag{A-2.13}$$

in the representation of the gamma matrices for which γ^0 is diagonal. It should be noticed that W and Z^0 bosons break parity symmetry as they should since their charge matrices do not commute with the matrix of P.

The guess that a complex conjugation in CP_2 is associated with T transformation of the physicist turns out to be correct. One can verify by a direct calculation that pure Dirac action is invariant under T realized according to

$$\begin{aligned} m^k &\rightarrow T(M^k) \ , \\ \xi^k &\rightarrow \bar{\xi}^k \ , \\ \Psi &\rightarrow \gamma^1\gamma^3 \otimes 1\Psi \ . \end{aligned} \tag{A-2.12}$$

The operation bearing closest resemblance to the ordinary charge conjugation corresponds geometrically to complex conjugation in CP_2:

$$\begin{aligned} \xi^k &\rightarrow \bar{\xi}^k \ , \\ \Psi &\rightarrow \Psi^\dagger\gamma^2\gamma^0 \otimes 1 \ . \end{aligned} \tag{A-2.12}$$

As one might have expected symmetries CP and T are exact symmetries of the pure Dirac action.

A-3 Space-time surfaces with vanishing em, Z^0, Kähler, or W fields

In the sequel it is shown that space-times for which either em, Z^0, or Kähler field vanishes decompose into regions characterized by six vacuum parameters: two of these quantum numbers (ω_1 and ω_2) are frequency type parameters, two (k_1 and k_2) are wave vector like quantum numbers, two of the quantum numbers (n_1 and n_2) are integers. The parameters ω_i and n_i will be referred as electric and magnetic quantum numbers. The existence of these quantum numbers is not a feature of these solutions alone but represents a much more general phenomenon differentiating in a clear cut manner between TGD and Maxwell's electrodynamics.

The simplest manner to avoid surface Kähler charges and discontinuities or infinities in the derivatives of CP_2 coordinates on the common boundary of two neighboring regions with different vacuum quantum numbers is topological field quantization, 3-space decomposes into disjoint topological field quanta, 3-surfaces having outer boundaries with possibly macroscopic size.

If one requires that space-time surface is an extremal of Kähler action and has a 2-dimensional CP_2 projection, only vacuum extremals and space-time surfaces for which CP_2 projection is a geodesic sphere, are allowed. Homologically non-trivial geodesic sphere correspond to vanishing W fields and homologically non-trivial sphere to non-vanishing W fields but vanishing γ and Z^0. For vacuum extremals all electro-weak gauge fields are in general non-vanishing although the net gauge field has U(1) holonomy.

A-3.1 Em neutral space-times

Em and Z^0 neutral spacetimes are especially interesting space-times as far as applications of TGD are considered. Consider first the electromagnetically neutral space-times. Using spherical coordinates (r,Θ,Ψ,Φ) for CP_2, the expression of Kähler form reads as

$$\begin{aligned} J &= \frac{r}{F^2}dr\wedge(d\Psi+cos(\Theta)d\Phi)+\frac{r^2}{2F}sin(\Theta)d\Theta\wedge d\Phi \ , \\ F &= 1+r^2 \ . \end{aligned} \tag{A-3.0}$$

The general expression of electromagnetic field reads as

$$\begin{aligned} F_{em} &= (3+2p)\frac{r}{F^2}dr\wedge(d\Psi+cos(\Theta)d\Phi)+(3+p)\frac{r^2}{2F}sin(\Theta)d\Theta\wedge d\Phi \ , \\ p &= sin^2(\Theta_W) \ , \end{aligned} \tag{A-3.0}$$

where Θ_W denotes Weinberg angle.

The vanishing of the electromagnetic fields is guaranteed, when the conditions

$$\begin{aligned} \Psi &= k\Phi \ , \\ (3+2p)\frac{1}{r^2F}(d(r^2)/d\Theta)(k+cos(\Theta)) &+ (3+p)sin(\Theta)=0 \ , \end{aligned} \tag{A-3.0}$$

hold true. The conditions imply that CP_2 projection of the electromagnetically neutral space-time is 2-dimensional. Solving the differential equation one obtains

$$\begin{aligned} r &= \sqrt{\frac{X}{1-X}} \ , \\ X &= D\left[|\frac{(k+u}{C}|\right]^{\epsilon} \ , \\ u &\equiv cos(\Theta) \ , \ C=k+cos(\Theta_0) \ , \ D=\frac{r_0^2}{1+r_0^2} \ , \ \epsilon=\frac{3+p}{3+2p} \ , \end{aligned} \tag{A-3.-1}$$

where C and D are integration constants. $0 \leq X \leq 1$ is required by the reality of r. $r = 0$ would correspond to $X = 0$ giving $u = -k$ achieved only for $|k| \leq 1$ and $r = \infty$ to $X = 1$ giving $|u + k| = [(1 + r_0^2)/r_0^2)]^{(3+2p)/(3+p)}$ achieved only for

$$sign(u + k) \times [\frac{1 + r_0^2}{r_0^2}]^{\frac{3+2p}{3+p}} \leq k + 1 \ ,$$

where $sign(x)$ denotes the sign of x.

Under rather general conditions the coordinates Ψ and Φ can be written in the form

$$\begin{aligned} \Psi &= \omega_2 m^0 + k_2 m^3 + n_2 \phi + \text{Fourier expansion} \ , \\ \Phi &= \omega_1 m^0 + k_1 m^3 + n_1 \phi + \text{Fourier expansion} \ . \end{aligned} \tag{A-3.-1}$$

m^0,m^3 and ϕ denote the coordinate variables of the cylindrical M^4 coordinates) so that one has $k = \omega_2/\omega_1 = n_2/n_1 = k_2/k_1$. The regions of the space-time surface with given values of the vacuum parameters ω_i,k_i and n_i and m and C are bounded by the surfaces at which the electromagnetically neutral imbeddings become ill-defined, say by $r > 0$ or $r < \infty$ surfaces.

The space-time surface decomposes into regions characterized by different values of the vacuum parameters r_0 and Θ_0. At $r = \infty$ surfaces n_2,ω_2 and m can change since all values of Ψ correspond to the same point of CP_2: at $r = 0$ surfaces also n_1 and ω_1 can change since all values of Φ correspond to same point of CP_2, too. If $r = 0$ or $r = \infty$ is not in the allowed range space-time surface develops a boundary.

This implies what might be called topological quantization since in general it is not possible to find a smooth global imbedding for, say a constant magnetic field. Although global imbedding exists it decomposes into regions with different values of the vacuum parameters and the coordinate u in general possesses discontinuous derivative at $r = 0$ and $r = \infty$ surfaces. A possible manner to avoid edges of space-time is to allow field quantization so that 3-space (and field) decomposes into disjoint quanta, which can be regarded as structurally stable units a 3-space (and of the gauge field). This doesn't exclude partial join along boundaries for neighboring field quanta provided some additional conditions guaranteing the absence of edges are satisfied.

The vanishing of the electromagnetic fields implies that the condition

$$\Omega \equiv \frac{\omega_2}{n_2} - \frac{\omega_1}{n_1} = 0 \ , \tag{A-3.0}$$

is satisfied. In particular, the ratio ω_2/ω_1 is rational number for the electromagnetically neutral regions of space-time surface. The change of the parameter n_1 and n_2 (ω_1 and ω_2) in general generates magnetic field and therefore these integers will be referred to as magnetic (electric) quantum numbers.

The expression for the Kähler form and Z^0 field of the electromagnetically neutral space-time surface will be needed in sequel and is given by

$$\begin{aligned} J &= -\frac{p}{3+2p} X du \wedge d\Phi \ , \\ Z^0 &= -\frac{6}{p} J \ . \end{aligned} \tag{A-3.0}$$

The components of the electromagnetic field generated by varying vacuum parameters are proportional to the components of the Kähler field: in particular, the magnetic field is parallel to the Kähler magnetic field. The generation of a long range Z^0 vacuum field is a purely TGD based feature not encountered in the standard gauge theories.

The effective form of the CP_2 metric is given by

$$\begin{aligned} ds^2_{eff} &= (s_{rr}(\frac{dr}{d\Theta})^2 + s_{\Theta\Theta}) d\Theta^2 + (s_{\Phi\Phi} + 2k s_{\Phi\Psi}) d\Phi^2 = \frac{R^2}{4}[s^{eff}_{\Theta\Theta} d\Theta^2 + s^{eff}_{\Phi\Phi} d\Phi^2] \ , \\ s^{eff}_{\Theta\Theta} &= X \times \left[\frac{\epsilon^2(1-u^2)}{(k+u)^2} \times \frac{1}{1-X} + 1 - X\right] \ , \\ s^{eff}_{\Phi\Phi} &= X \times \left[(1-X)(k+u)^2 + 1 - u^2\right] \ , \end{aligned} \tag{A-3.-1}$$

and is useful in the construction of electromagnetically neutral imbedding of, say Schwartchild metric. Note however that in general these imbeddings are not extremals of Kähler action.

A-3.2 Space-times with vanishing Z^0 or Kähler fields

The results just derived generalize to the Z^0 neutral case as such. The only modification is the replacement of the parameter ϵ with $\epsilon = 1/2$ as becomes clear by considering the condition stating that Z^0 field vanishes identically. Also the relationship $F_{em} = 3J = -\frac{3}{4}\frac{r^2}{F}du \wedge d\Phi$ is useful.

Also the generalization to the case of vacuum extremals is straightforward and corresponds to $\epsilon = 1, p = 0$ in the formula for em neutral space-times. In this case classical em and Z^0 fields are proportional to each other:

$$\begin{aligned} Z^0 &= 2e^0 \wedge e^3 = \frac{r}{F^2}(k+u)\frac{\partial r}{\partial u}du \wedge d\Phi = (k+u)du \wedge d\Phi \ , \\ r &= \sqrt{\frac{X}{1-X}} \ , \quad X = D|k+u| \ , \\ \gamma &= -\frac{p}{2}Z^0 \ . \end{aligned} \tag{A-3.-2}$$

For vanishing value of Weinberg angle ($p = 0$) em field vanishes and only Z^0 field remains as a long range gauge field. Vacuum extremals for which long range Z^0 field vanishes but em field is non-vanishing are not possible.

A-3.3 Induced gauge fields for space-times for which CP$_2$ projection is a geodesic sphere

For space-time sheets for which CP$_2$ projection is $r = \infty$ homologically non-trivial geodesic sphere of CP_2 one has

$$\gamma = (\frac{3}{4} - \frac{sin^2(\theta_W)}{2})Z^0 \simeq \frac{5Z^0}{8} \ .$$

The induced W fields vanish in this case and they vanish also for all geodesic sphere obtained by $SU(3)$ rotation.

For homologically trivial geodesic sphere a standard representative is obtained by using for the phase angles of standard complex CP_2 coordinates constant values. In this case induced em, Z^0, and Kähler fields vanish but induced W fields are non-vanishing. This holds also for surfaces obtained by color rotation. Hence one can say that for non-vacuum extremals with 2-D CP$_2$ projection color rotations and weak symmetries commute.

A-4 Second variation of the Kähler action

The Kähler action is apart from a multiplicative constant defined by the Lagrangian density

$$L = J^{\alpha\beta}J_{\alpha\beta}\sqrt{g} \ , \tag{A-4.1}$$

and depends on the imbedding space coordinates only through the induced metric and Kähler form. In order to calculate the second variation of the Kähler action one can use "covariantization" trick made possible by the covariant constancy of the imbedding space metric and Kähler form. Calculate second variation by treating components of the metric and Kähler form as a constant so that the action depends effectively only on the derivatives of the imbedding space coordinates and replace ordinary derivatives of the deformation with the covariant derivatives in the resulting expression for the second variation.

$$\begin{aligned} \partial_\alpha \delta h^k &\rightarrow D_\alpha \delta h^k \\ &= \partial_\alpha \delta h^k + \{ {}^{\ k}_{l\ m} \} \partial_\alpha h^m \delta h^l \ . \end{aligned} \tag{A-4.1}$$

The first variation of the Maxwell term is given by the expression

$$\delta_1 L = 2[T^{\alpha\beta}\delta_1 g_{\alpha\beta} + J^{\alpha\beta}\delta_1 J_{\alpha\beta}]\sqrt{g} \ , \qquad \text{(A-4.2)}$$

where the canonical energy momentum tensor $T^{\alpha\beta}$ is given by

$$T^{\alpha\beta} = J^{\alpha\nu}J^{\beta}_{\ \nu} - (1/4)g^{\alpha\beta}J^{\mu\nu}J_{\mu\nu} \ . \qquad \text{(A-4.3)}$$

and is traceless by Weyl invariance.

Second variation is obtained by differentiating first variation and decomposes into three terms

$$\delta_2 L = \delta_2^a L + \delta_2^b L + \delta_2^c L \ . \qquad \text{(A-4.4)}$$

The first term is given by the expression

$$\begin{aligned} \delta_2^a L &= [T^{\alpha\beta}\delta_2 g_{\alpha\beta} + J^{\alpha\beta}\delta_2 J_{\alpha\beta} \\ &+ (T^{\alpha\beta}\delta_1 g_{\alpha\beta} + J^{\alpha\beta}\delta_1 J_{\alpha\beta})g^{\mu\nu}\delta_1 g_{\mu\nu}]\sqrt{g} \ . \end{aligned} \qquad \text{(A-4.4)}$$

The second term is given by

$$\begin{aligned} \delta_2^b L &= [(\partial T^{\alpha\beta}/\partial g_{\mu\nu})\delta_1 g_{\alpha\beta}\delta_1 g_{\mu\nu} \\ &+ 2(\partial T^{\alpha\beta}/\partial J_{\mu\nu})\delta_1 g_{\alpha\beta}\delta_1 J_{\mu\nu}]\sqrt{g} \ . \end{aligned} \qquad \text{(A-4.4)}$$

The partial derivatives of the energy momentum tensor appearing in the expression are given by

$$\begin{aligned} \partial T^{\alpha\beta}/\partial g_{\mu\nu} &= -g^{\alpha\mu}T^{\beta\nu} + K^{\alpha\nu}g^{\beta\mu} - \frac{1}{2}K^{\mu\nu}g^{\alpha\beta} + J^{\alpha\nu}J^{\beta\mu} \ , \\ K^{\alpha\beta} &= J^{\alpha\nu}J^{\beta}_{\ \nu} \ . \end{aligned} \qquad \text{(A-4.4)}$$

$$\partial T^{\alpha\beta}/\partial J_{\mu\nu} = 2[g^{\alpha\mu}J^{\beta\nu} - g^{\alpha\beta}J^{\mu\nu}/4] \ . \qquad \text{(A-4.5)}$$

The third term is given by the expression

$$\begin{aligned} \delta_2^c L &= [(\partial J^{\alpha\beta}/\partial J_{\mu\nu})\delta_1 J_{\alpha\beta}\delta_1 J_{\mu\nu}]\sqrt{g} \ , \\ \partial J^{\alpha\beta}/\partial J_{\mu\nu} &= g^{\alpha\mu}g^{\beta\nu} \ . \end{aligned} \qquad \text{(A-4.5)}$$

Expressing the first term in terms of the coordinate variations one obtains

$$\delta_2^a L = 2[T^{\alpha\beta}h^{\perp}_{kl} + J^{\alpha\beta}J^{\perp}_{kl}]D_\alpha\delta_1 h^k D_\beta\delta_1 h^l\sqrt{g} \ , \qquad \text{(A-4.6)}$$

where $h^{\perp}_{kl}$ and $J^{\perp}_{kl}$ are the projections of the imbedding space metric and Kähler form to the orthogonal complement of the tangent space of X^4

$$\begin{aligned} h^{\perp}_{kl} &= h_{kl} - g^{\mu\nu}h_{kr}h_{ls}\partial_\mu h^r\partial_\nu h^s \ , \\ J^{\perp}_{kl} &= h^{\perp}_{kr}h^{\perp}_{ls}J^{rs} \ , \end{aligned} \qquad \text{(A-4.6)}$$

so that $\delta_2^a L$ vanishes for four-dimensional $Diff$ deformations parallel to X^4. This term vanishes also, when the induced Kähler form vanishes.

The contribution of the second term to the second variation is given by the expression

$$
\begin{aligned}
\delta_2^b L &= 4[(-g^{\alpha\mu}T^{\beta\nu} + K^{\alpha\nu}g^{\beta\mu} - \frac{1}{2}K^{\mu\nu}g^{\alpha\beta} + J^{\alpha\nu}J^{\beta\mu})h_{kr}h_{ls} \\
&+ 2(g^{\alpha\mu}J^{\beta\nu} - g^{\alpha\beta}J^{\mu\nu}/4)h_{ks}J_{lr}]\partial_\alpha h^k \partial_\beta h^l D_\mu \delta_1 h^r D_\nu \delta_1 h^s \sqrt{g} \ .
\end{aligned} \tag{A-4.5}
$$

Also this term is non-vanishing only provided the induced Kähler field is nontrivial.

The third term is given by the expression

$$
\delta_2^c L = [g^{\alpha\mu}g^{\beta\nu}J_{kr}J_{ls}]\partial_\alpha h^k \partial_\beta h^l D_\mu \delta_1 h^r D_\nu \delta_1 h^s \sqrt{g} \ . \tag{A-4.6}
$$

This term is the only term, which is nontrivial for the vacuum extremals with vanishing Kähler field and also in this case the variation is nontrivial for CP_2 coordinates only.

The second variation for the Kähler Lagrangian can be written in the following general form

$$
\delta^2 L_{intX^4} = I^{\alpha\beta}_{kl} D_\alpha \delta h^k D_\beta \delta h^l \ , \tag{A-4.6}
$$

where the general expressions for the tensor $I^{\alpha\beta}_{kl}$ reads as

$$
I^{\alpha\beta}_{kl} = \partial_{\partial_\alpha h^k}\partial_{\partial_\beta h^l} L \ . \tag{A-4.6}
$$

The explicit expression for the tensors $I^{\alpha\beta}_{kl}$ can be read from the expressions for $\delta L_2^i, i = a, b, c$ and $\delta_2 L_{CS}$ respectively.

The general form of the variational equations satisfied by the second variation in the interior of X^4 reads as

$$
D_\alpha(I^{\alpha\beta}_{kl} D_\beta \delta h^l) = 0 \ . \tag{A-4.7}
$$

On the boundary the variational equations read

$$
I^{n\beta}_{kl} D_\beta \delta h^l = 0 \ . \tag{A-4.8}
$$

These equations are satisfied on a dynamically generated boundary only. These equations are not satisfied on the intersection of the four-surface with the surfaces $a = \sqrt{(m^0)^2 - r_M^2} \rightarrow \infty$ and $a = 0$ (light cone boundary).

The expression for the second variation of the action reduces to a mere boundary term resulting from the intersections of the four-surface with $a \rightarrow \infty$ and $a = 0$ surfaces, when X^4 corresponds to a submanifold of light cone and reads

$$
\delta^2 S = \int_{a=0}^{a=\infty} I^{n\beta}_{kl} \delta h^k D_\beta \delta h^l d^3x. \tag{A-4.8}
$$

The general expressions for the tensor I suggests that only non-vanishing contribution to the second variation comes from the boundary of the light cone.

A-5 p-Adic numbers

p-Adic numbers (p is prime: 2,3,5,...) can be regarded as a completion of the rational numbers using a norm, which is different from the ordinary norm of real numbers [8]. p-Adic numbers are representable as power expansion of the prime number p of form:

$$x = \sum_{k \geq k_0} x(k)p^k, \; x(k) = 0, ..., p-1 \; . \tag{A-5.1}$$

The norm of a p-adic number is given by

$$|x| = p^{-k_0(x)} \; . \tag{A-5.2}$$

Here $k_0(x)$ is the lowest power in the expansion of the p-adic number. The norm differs drastically from the norm of the ordinary real numbers since it depends on the lowest pinary digit of the p-adic number only. Arbitrarily high powers in the expansion are possible since the norm of the p-adic number is finite also for numbers, which are infinite with respect to the ordinary norm. A convenient representation for p-adic numbers is in the form

$$x = p^{k_0}\varepsilon(x) \; , \tag{A-5.3}$$

where $\varepsilon(x) = k + ...$ with $0 < k < p$, is p-adic number with unit norm and analogous to the phase factor $exp(i\phi)$ of a complex number.

The distance function $d(x, y) = |x - y|_p$ defined by the p-adic norm possesses a very general property called ultra-metricity:

$$d(x, z) \leq max\{d(x, y), d(y, z)\} \; . \tag{A-5.4}$$

The properties of the distance function make it possible to decompose R_p into a union of disjoint sets using the criterion that x and y belong to same class if the distance between x and y satisfies the condition

$$d(x, y) \leq D \; . \tag{A-5.5}$$

This division of the metric space into classes has following properties:

a) Distances between the members of two different classes X and Y do not depend on the choice of points x and y inside classes. One can therefore speak about distance function between classes.

b) Distances of points x and y inside single class are smaller than distances between different classes.

c) Classes form a hierarchical tree.

Notice that the concept of the ultra-metricity emerged in physics from the models for spin glasses and is believed to have also applications in biology [10]. The emergence of p-adic topology as the topology of the effective space-time would make ultra-metricity property basic feature of physics.

A-6 Canonical correspondence between p-adic and real numbers

There exists a natural continuous map $Id : R_p \rightarrow R_+$ from p-adic numbers to non-negative real numbers given by the "pinary" expansion of the real number for $x \in R$ and $y \in R_p$ this correspondence reads

$$
\begin{aligned}
y &= \sum_{k>N} y_k p^k \rightarrow x = \sum_{k<N} y_k p^{-k} \quad , \\
y_k &\in \{0,1,..,p-1\} \quad .
\end{aligned}
\tag{A-6.0}
$$

This map is continuous as one easily finds out. There is however a little difficulty associated with the definition of the inverse map since the pinary expansion like also decimal expansion is not unique ($1 = 0.999...$) for the real numbers x, which allow pinary expansion with finite number of pinary digits

$$
\begin{aligned}
x &= \sum_{k=N_0}^{N} x_k p^{-k} \quad , \\
x &= \sum_{k=N_0}^{N-1} x_k p^{-k} + (x_N - 1)p^{-N} + (p-1)p^{-N-1} \sum_{k=0,..} p^{-k} \quad .
\end{aligned}
\tag{A-6.-1}
$$

The p-adic images associated with these expansions are different

$$
\begin{aligned}
y_1 &= \sum_{k=N_0}^{N} x_k p^{k} \quad , \\
y_2 &= \sum_{k=N_0}^{N-1} x_k p^{k} + (x_N - 1)p^{N} + (p-1)p^{N+1} \sum_{k=0,..} p^{k} \\
&= y_1 + (x_N - 1)p^N - p^{N+1} \quad ,
\end{aligned}
\tag{A-6.-2}
$$

so that the inverse map is either two-valued for p-adic numbers having expansion with finite pinary digits or single valued and discontinuous and non-surjective if one makes pinary expansion unique by choosing the one with finite pinary digits. The finite pinary digit expansion is a natural choice since in the numerical work one always must use a pinary cutoff on the real axis.

What about the p-adic counterpart of the negative real numbers? It seems that in the applications this correspondence is not needed since canonical identification is used only in the direction $R_p \rightarrow R$ to map the predictions of p-adic probability calculus and statistics to real numbers (in particular, p-adic entanglement entropy must be mapped to its real counterpart). This means that also the inverse of the canonical identification is not needed in the applications. At tge space time level the p-adics and reals relate via common rationals. p-Adic effective topology is expected to be a good approximation only within some length scale range which means infrared and UV cutoffs.

The topology induced by the canonical identification t map in the set of positive real numbers differs from the ordinary topology. The difference is easily understood by interpreting the p-adic norm as a norm in the set of the real numbers. The norm is constant in each interval $[p^k, p^{k+1})$ (see Fig. A-6) and is equal to the usual real norm at the points $x = p^k$: the usual linear norm is replaced with a piecewise constant norm. This means that p-adic topology is coarser than the usual real topology and the higher the value of p is, the coarser the resulting topology is above a given length scale. This hierarchical ordering of the p-adic topologies will be a central feature as far as the proposed applications of the p-adic numbers are considered.

Ordinary continuity implies p-adic continuity since the norm induced from the p-adic topology is rougher than the ordinary norm. p-Adic continuity implies ordinary continuity from right as is clear already from the properties of the p-adic norm (the graph of the norm is indeed continuous from right). This feature is one clear signature of the p-adic topology.

The linear structure of the p-adic numbers induces a corresponding structure in the set of the non-negative real numbers and p-adic linearity in general differs from the ordinary concept of linearity. For example, p-adic sum is equal to real sum only provided the summands have no common pinary digits. Furthermore, the condition $x +_p y < max\{x, y\}$ holds in general for the p-adic sum of the

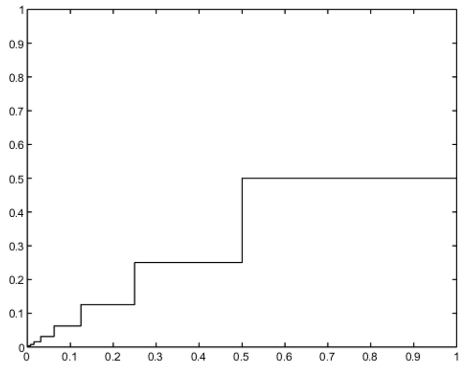

Figure A.1: The real norm induced by canonical identification from 2-adic norm.

real numbers. p-Adic multiplication is equivalent with the ordinary multiplication only provided that either of the members of the product is power of p. Moreover one has $x \times_p y < x \times y$ in general. The p-Adic negative -1_p associated with p-adic unit 1 is given by $(-1)_p = \sum_k (p-1)p^k$ and defines p-adic negative for each real number x. An interesting possibility is that p-adic linearity might replace the ordinary linearity in some strongly nonlinear systems so these systems would look simple in the p-adic topology.

Canonical identification is in a key role in the successful predictions of the elementary particle masses. Canonical identification makes also possible to understand the connection between p-adic and real probabilities. These suggests that canonical identification is involved with some deeper mathematical structure. The following inequalities hold true:

$$\begin{array}{rcl} (x+y)_R & \leq & x_R + y_R \ , \\ |x|_p|y|_R \leq (xy)_R & \leq & x_R y_R \ , \end{array} \tag{A-6.-2}$$

where $|x|_p$ denotes p-adic norm. These inequalities can be generalized to the case of $(R_p)^n$ (a linear vector space over the p-adic numbers).

$$\begin{array}{rcl} (x+y)_R & \leq & x_R + y_R \ , \\ |\lambda|_p|y|_R \leq (\lambda y)_R & \leq & \lambda_R y_R \ , \end{array} \tag{A-6.-2}$$

where the norm of the vector $x \in T_p^n$ is defined in some manner. The case of Euclidian space suggests the definition

$$(x_R)^2 = (\sum_n x_n^2)_R \ . \tag{A-6.-1}$$

These inequalities resemble those satisfied by the vector norm. The only difference is the failure of linearity in the sense that the norm of a scaled vector is not obtained by scaling the norm of the original vector. Ordinary situation prevails only if the scaling corresponds to a power of p.

These observations suggests that the concept of a normed space or Banach space might have a generalization and physically the generalization might apply to the description of some nonlinear systems. The nonlinearity would be concentrated in the nonlinear behavior of the norm under scaling.

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